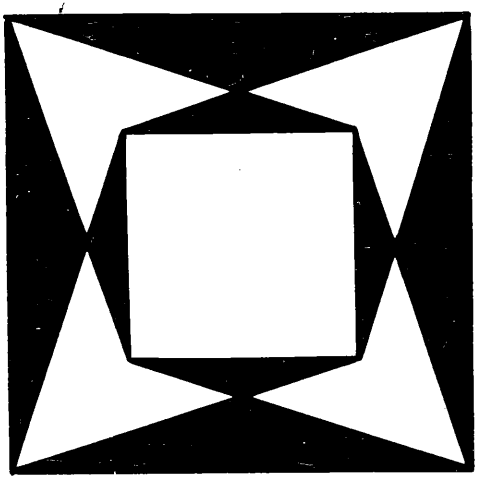
IRISH MATHEMATICAL SOCIETY



NEWSLETTER

NUMBER 15 DECEMBER 1985

THE IRISH MATHEMATICAL SOCIETY

NEWSLETTER

EDITOR

Pat Fitzpatrick

ASSOCIATE EDITOR

Martin Stynes

The aim of the *Newsletter* is to inform Society members about the activities of the Society and also about items of general mathematical interest. It appears three times each year: March, September and December. Deadline for copy is six weeks prior to publication date.

The *Newsletter* also seeks articles of mathematical interest written in an expository manner. All areas of mathematics are welcome, pure and applied, old and new.

Detailed instructions relating to the preparation of manuscripts may be found on the inside back cover.

Correspondence relating to the *Newsletter* should be sent to:

Irish Mathematical Society Newsletter,
Department of Mathematics,
University College,
Cork.

CONTENTS

NEWS AND ANNOUNCEMENTS

DUBLIN INSTITUTE FOR ADVANCED STUDIES

10, Burlington Road, Dublin 4

Mathematical Symposium

Thursday 19 December, 2.30 - 5.00

Friday 20 December, 10.00 - 5.00

IRISH MATHEMATICAL SOCIETY

General Meeting

Friday 20 December, 12.15 pm

at

Dublin Institute for Advanced Studies

NOTICE

The John L. Synge Award

The Mathematics Division of the Royal Society of Canada has decided to raise funds to establish an award for outstanding contributions to the Mathematical Sciences, to be awarded at regular intervals, not necessarily to a Fellow of the Society.

There is such an award in Physics and in Chemistry; both of these awards are named after Rutherford, they are awarded every year if suitable candidates are available, and each of them consists of a Medal and a cash award of $1,500.

We hope that the Mathematics award will be of the same nature but all depends on our success in raising funds. In any case the award will be named in honour of John L. Synge, who was elected to the Royal Society of Canada more than 50 years ago and who was the first recipient of the Henry M. Tory Gold Medal of the Royal Society of Canada.

Prof. Synge, now resident in Dublin, Ireland, was for many years Professor of Applied Mathematics at University of Toronto.

Persons and organizations are invited to contribute to the proposed John L. Synge award. Cheques should be made payable to:

The Royal Society of Canada, John L. Synge Award

and sent to Professor Israel Halperin, Chairman of the John L. Synge Awards Committee, Department of Mathematics, University of Toronto, Toronto, Ontario M5S 1A1, Canada.

Income Tax Deductible (in Canada) Receipts will be given by The Royal Society of Canada.

CONFERENCE ANNOUNCEMENT

PRELIMINARY ANNOUNCEMENT

CONFERENCE

ON

ASPECTS OF ANALYSIS

****

A conference, spread over two days, will be held at University College, Cork, in mid-May, 1986. It should appeal to workers in OPERATOR THEORY and FUNCTION THEORY.

The registration fee will be IR£10.00 (ten Irish punts). Assistance in arranging accommodation will be provided, if required.

Further information may be obtained from Professor Finbarr Holland, Department of Mathematics, University College, Cork, Ireland.

****

AN INTRODUCTION TO SPARSE MATRICES

Derek O'Connor

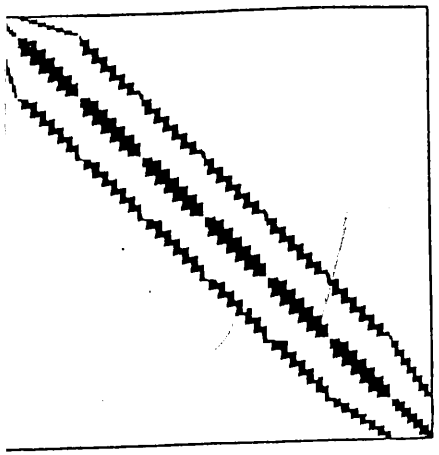
1. INTRODUCTION

There is no strict definition of a sparse matrix, but generally speaking it is an nxn matrix which has O(n) non-zero elements. A full or dense matrix has $O(n^2)$ non-zero elements.

Sparse matrices occur naturally in the solution of many practical problems, e.g. electrical, gas, and water distribution systems; civil and mechanical engineering (structural analysis); production and financial planning (inventory control and portfolio selection); national and local government operations (income tax analysis and scheduling of fire and ambulance services); economics (Input-Output analysis). At a more theoretical level sparse matrices arise in Graph Theory, Linear Programming, Finite Element Methods, and the solution of ordinary and partial differential equations. Duff's excellent survey article [6] contains a long list of application areas, with references.

The interest in sparse matrices comes about because of the need to solve on a computer, linear equations or linear optimization problems that have many thousands of variables (possibly millions) but whose coefficients are mostly zeros. Generally, this sparsity can be exploited, giving large savings in computer time and storage. Indeed, were it not possible to exploit sparsity then many important problems could not be solved on present or future computers.

The sparse matrices that arise in practice are not only sparse but are also highly structured, i.e. the non-zeros form very definite patterns. Figures 1.1 to 1.4 below are taken from Duff [7] and show some typical sparse matrices. This structure can be exploited also, giving even greater savings



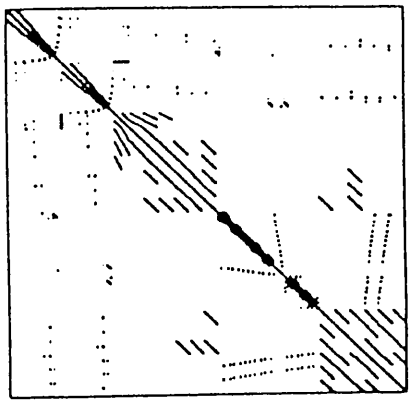
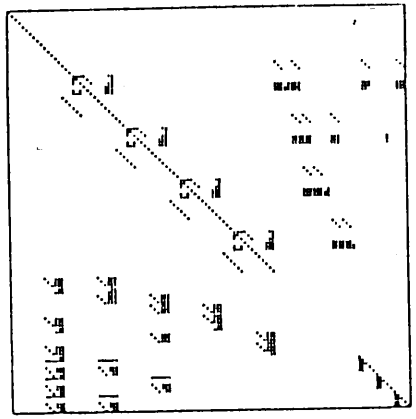


Figure 1.1: Matrix of Order 147 Figure 1.2: Matrix of Order 503



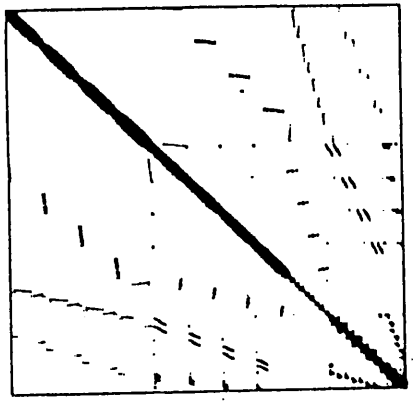


Figure 1.3: Matrix of Order 113 Figure 1.4: Matrix of Order 1005

in space and time.

An outline of this paper is as follows: Section 2 briefly outlines the history and goals of sparse matrix research; Section 3 discusses the standard schemes for sparse matrices in digital computers; Section 4 discusses direct and iterative methods for solving sets of sparse linear equations; Section 5 considers the special but very important class of symmetric positive definite matrices; Section 6 is the conclusion with a brief discussion of extensions to eigenvalue, least squares and optimization problems.

2. THE HISTORY AND GOALS OF SPARSE MATRIX RESEARCH

The interest in sparse matrices seems to have started in the late 1950s when researchers in linear programming and electrical power system analysis began to solve realistic problems on computers. It was noticed that when real problems were modelled by systems of linear equations, the resulting matrices were sparse with highly structured non-zero patterns. It was also noticed that these matrices were large and that the sparsity would need to be exploited if these problems were to be solved on the rather small and slow computers available at that time. Sparse matrices became increasingly important in the 1960s and the first conference on sparse matrices was held at the IBM Research Center, Yorktown Heights, in 1968 [24]. Since then there have been at least six international conferences, with published proceedings, devoted entirely to sparse matrices. The number of papers on sparse matrices is very large. Duff's survey paper [6] lists over 600 references up to the end of 1975 and a casual check of journals since then shows that this number is growing steadily. To date there are at least three textbooks on sparse matrices, while most textbooks on numerical analysis and data structures contain sections or chapters on the subject.

There are two complementary goals in sparse matrix research: (1) reduce the amount of computer memory needed to store sparse matrices, and (2) reduce the amount of computation time needed to solve problems involving sparse matrices. It is a general rule of thumb in computing that SPACE x TIME = CONSTANT, i.e. to save storage space more computation time must be spent and *vice-versa*. We will see that in solving sparse matrix problems it is often possible to make a simultaneous reduction in the amount of time and space required for the dense matrix case.

Some may wonder why it is important to reduce or conserve computer time and space when computers are getting larger and faster at lower costs each year (you can now buy a microcomputer for $5,000 which is as powerful as a mainframe which cost $ millions in the early '60s). Here are three reasons why conserving space and time is important:

* There is no computer available today that can store a 1000 x 1000 matrix in core. Many engineering and business problems have from 100,000 to 25,000,000 variables. Problems of this size are routinely solved but not without great care being taken to reduce their storage and time requirements.

* Sparse linear equation solvers are often a small but critical part of a much larger algorithm. As such, they may be used many thousands of times in solving a single problem. Any inefficiency in the equation solver will be greatly magnified and thus degrade the performance of the otherwise good, larger algorithm.

* Microcomputers are widely used today and in general they have small memories and slow processors. If sparsity techniques are used then these machines can solve realistic problems in the order of 100-1000 variables.

Thus it seems that, despite the increasing size and speed of computers, there is still - and perhaps always will be - a need

to conserve computer time and storage.

3. SPARSE MATRIX STORAGE SCHEMES

We now discuss some general schemes for compactly storing sparse matrices. These schemes should not be viewed in isolation from the algorithms which use them. In general, they must be tailored to suit the problem and the algorithm used to solve the problem. Thus the algorithm and the data structure (storage scheme) are intimately linked.

We start with the standard scheme for dense matrices and then discuss the three main methods for sparse matrices. For simplicity, we consider only square matrices.

3.1 Dense Matrix Storage

The memory of a digital computer can be regarded as a one-dimensional array or vector because each cell or word in memory is indexed or addressed by a single number, called the machine address.

A 2-dimensional array or matrix must be stored as a one-dimensional array and this is done by placing the columns (or rows) one after another in memory. This is shown in Fig. 3.1.

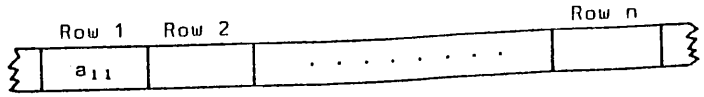


Figure 3.1: A Matrix Stored Row by Row

To access an element a_{ij} of a matrix A we must calculate its position relative to the location of the first element a_{11} of A. Thus,

$loc(a_{ij}) = loc(a_{11}) + n(i-1) + j-1$ Stored row by row, or

$loc(a_{ij}) = loc(a_{11}) + n(j-1) + i-1$ Stored column by column.

It is important to note that these calculations assume that all n^2 elements of the matrix are present and in a predetermined place relative to the first element. Also note that two extra cells are needed to store $loc(a_{11})$ and n and that each access requires one multiplication and two additions. This is called the 'overhead' of this storage scheme.

3.2 Static Sparse Matrix Storage

The dense scheme in the last section is appropriate if the number of non-zeros is high, because very little space is wasted in storing zeros. It should be remembered however, that with current computer technology, we are limited to storing matrices of order 300-500 in dense or full form.

In many practical applications, involving matrices of high order ($n > 500$), there may be only 2 to 20 non-zeros per row. Even if we could store such matrices in dense form we would be wasting most of the allocated space in storing zeros. Worse still, most of the arithmetic calculations would involve addition and multiplication of zeros, and these are *null* calculations, i.e. require no actual computation.

Most storage schemes for sparse matrices store only the non-zeros along with indexing information for each non-zero. This indexing information must be stored explicitly with each non-zero because the non-zeros will not be in predetermined positions relative to the first element of the matrix. This is in contrast to the dense scheme where no indexing information is stored (it is calculated).

We now describe three schemes for storing sparse matrices that are static, i.e. the number and locations of the non-zeros do not change during calculations. If this is not the case then we assume their number and locations at any point in a calculation can be determined before the calculation begins.

Storage Scheme 1:

Consider the 6 x 6 sparse matrix shown in Fig. 3.2.

$$A = \begin{vmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 5 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

	1	2	3	4	5	6	7	8	9	10
VAL	2	1	4	3	6	3	8	5	9	1
COL	1	4	3	1	2	6	4	1	5	6
ROW	1	1	2	3	4	4	5	6	7	6

Figure 3.2: A Sparse Matrix and Storage Scheme 1

If the matrix has t non-zeros then we use 3 arrays of length t to store the non-zeros (VAL), the row position (ROW) and the column position (COL). This scheme requires 3t cells instead of n^2 for the dense scheme. Thus in a 1000 x 1000 matrix with a non-zero density ($t/n^2 \times 100\%$) = 5%, we would save $1000^2 - 3 \times 50{,}000 = 850{,}000$ cells.

Storage Scheme 2:

This scheme is obtained from the first scheme by observing that some of the row index numbers are repeated and hence redundant. This is because the elements of any single row are stored contiguously in a block of memory cells. As a result we need only know where each row block begins. This scheme is shown in Fig. 3.3 for the matrix A above.

	1	2	3	4	5	6	7	8	9	10
VAL	2	1	4	3	6	3	8	5	9	1
COL	1	4	3	1	2	6	4	1	5	6
ROW START	1	3	4	7	8	10				

Figure 3.3: A Sparse Matrix and Storage Scheme 2

This scheme requires two arrays of length t and one array of length n, giving a total of 2t+n cells. An alternative method is to store the number of non-zeros in each row instead of the row-start positions. This alternative scheme is used in the program given in the appendix.

Storage Scheme 3:

The third scheme saves storage by *packing* each row and column index of the first scheme into one cell, using the following calculation:

$$N_{ij} = n \times (i-1) + j$$

To retrieve or *unpack* the values of i and j we use the following calculation:

$$i = [(N_{ij}-1)/n] + 1,$$

$$j = N_{ij} - n \times (i-1).$$

This scheme requires only 2t cells but it involves more computation to pack and unpack the indices. An example of this storage scheme is shown in Fig. 3.4 again using the matrix A.

	1	2	3	4	5	6	7	8	9	10
VAL	2	1	4	3	6	3	8	5	9	1
N_{ij}	1	4	9	13	14	18	22	25	29	36

Figure 3.4: A Sparse Matrix and Storage Scheme 3

3.3 Dynamic Sparse Matrix Storage

In many algorithms involving matrices the number of non-zeros varies during the course of the computation. In addition to this the locations of newly created non-zeros or zeros are not known in advance. These two difficulties require that we be able to insert and delete elements of the matrix data structure (storage scheme).

The static storage schemes above do not allow efficient implementation of the 'insert' and 'delete' operations. This is because all non-zero elements are stored contiguously and inserting a new element in its appropriate place requires a movement of half the elements, on average.

The most common way of storing sparse matrices whose elements vary dynamically is to use some form of *linked list* scheme. Linked lists are fundamental to, and widely used in, computer science. In the context of sparse matrix storage, a linked list scheme requires that for each non-zero, we store its value (a_{ij}), its indices (i,j), and the address in memory (m) of the next non-zero element in the same row or column. This is in contrast to the static scheme in which the value and the indices are stored, and the dense scheme in which only the value is stored. The advantage of the linked list scheme is that each non-zero can be stored anywhere in memory, i.e. the non-zeros do not need to be stored contiguously, as in the static and dense schemes. Hence, inserting or deleting a non-zero does not disturb the other non-zeros.

To implement a linked list storage scheme we need to 'create' new storage cells and 'destroy' old cells as the number of non-zeros varies. We will not discuss where new storage cells come from or what is done with old cells except to say that these are 'house-keeping' functions which are handled by a 'memory manager'. Such memory managers may be part of the language (e.g. in Pascal) or may need to be programmed by the user/designer (e.g. in FORTRAN). Suffice it to say that the

memory manager must efficiently recycle old cells and make available new cells. For a good discussion of memory management see Knuth [14] or Horowitz and Sahni [12].

3.4 Operations on Sparse Matrices

Writing software to perform the standard matrix operations is easy when matrices are stored in dense form. All scientific programming languages allow the user to declare matrices as 2-dimensional arrays and to perform arithmetic operations on the individual matrix elements in a direct way. Thus in Pascal, the sum of two nxn real matrices A, B, is computed as follows:

```
Var A, B, C : Array [1..n, 1..n] of Real;
        i, j : Integer;
begin
    for i := 1 to n do
        for j := 1 to n do
            C[i,j] := A[i,j] + B[i,j]
end;
```

The above program segment is essentially a direct translation of the mathematical definition of matrix addition. Furthermore, it is clear from the program that matrix addition is being performed. Writing the equivalent program for sparse matrices compactly stored is a much more difficult job. Also, reading and understanding such a program is difficult because the matrices are no longer represented in a direct and obvious way. These comments are true in general for sparse matrix software and for this reason such software is written only by those who have a sufficient knowledge of both mathematics and computer science. A list of the best-known sparse matrix software is given in the appendix.

3.5 Literature on Sparse Matrix Storage

The literature on sparse matrix storage schemes is scattered throughout books and journals in engineering, numerical linear algebra, computer science and operations research. The book by Jennings [13], and the survey article by Pooch and Nieder [15] are good starting points. Also, many computer science texts on Data Structures contain sections on sparse matrix storage (see [12], [14] and [23]).

4. SOLVING SPARSE LINEAR EQUATIONS

We now consider the solution of the equation

$$Ax = b,$$

where x, b are vectors in R^n and A is an n x n sparse matrix.

The algorithms for solving this equation fall into two broad categories, viz., Direct and Iterative. Iterative algorithms (Gauss-Seidel, Jacobi, etc.) are easy to implement (program) but may suffer from slow convergence. Direct algorithms (Gaussian Elimination, LUP Decomposition, etc.) give a solution in a finite number of steps but are difficult to implement, especially when the sparsity of A is exploited.

4.1 Iterative Algorithms

Iterative algorithms require that the equation Ax = b be transformed into the *'Fixed Point'* form

$$x = Cx + d = T(x),$$

which is then solved by *successive approximations* using the iteration formula

$$x^{(k+1)} = Cx^{(k)} + d, \quad x^{(0)} = 0.$$

This iteration formula (or algorithm) generates a sequence of

ectors $(x^{(1)}, x^{(2)}, \ldots, x^{(k)}, \ldots)$, which, under suitable onditions on C, converges to x, the solution of Ax = b.

This general iterative algorithm has very nice features nd it is easy to implement sparse versions of it. These eatures are:

1. Transforming A to C is simple and if A is sparse then C is sparse.
2. The matrix C does not change during the iteration process and hence an efficient static data structure can be used.
3. Programs for iterative algorithms tend to be short and simple.
4. Roundoff error is not generally a problem.

'he main difficulty with iterative algorithms is that they may ıave very slow linear convergence. This is why direct methods ;end to be preferred. Nonetheless iterative methods are use- 'ul for extremely large problems or where good initial solut- ons are available. In fact it is good practice to 'polish')r refine a direct solution with one or two iterations of an ıterative algorithm (see Rice [17] and Forsythe and Moler [9]).

The program shown in the appendix is a straightforward ORTRAN implementation of the Gauss-Seidel iterative method sing the static sparse storage scheme No. 2 given above. It an be seen that the heart of the program requires only 10 ines of code (lines 30-39). This program was run on a Z80, -bit, 64 K Byte microcomputer and solved a 1200 variable prob- em in 2.5 minutes. The A matrix had 3 non-zeros per row and 6 iterations were required to get 7-digit accuracy. Thus we ee that even very slow and small microcomputers can solve ealistic problems.

4.2 Direct Algorithms

Direct algorithms usually use some variant of the classic Gaussian Elimination method. This method transforms the system $Ax = b$ into $Ux = y$, where U is upper-triangular. $x = U^{-1}y$ is then found by *back-substitution*.

A more general statement of the Gaussian Elimination Algorithm is as follows:

LU - Decomposition to Solve $Ax = b$

Step 1: Decompose (Factorize) A, i.e. find L and U such that $LU = A$, where U is upper triangular and L is unit lower triangular.

Step 2: Solve $LUx = b$ as follows:

a. Solve $Ly = b$ by Forward-Substitution.
b. Solve $Ux = y$ by Back-Substitution.

Roundoff errors can accumulate in Step 1, and if A is *ill-conditioned* these errors will be greatly magnified. To control these errors, some form of *pivoting* (row or column interchanges or both) is necessary. Hence we normally obtain an LU factorization of PA or PAQ where P and Q are *permutation matrices*. These permutation matrices are determined *during the factorization*.

Exploiting the sparsity of the A matrix is difficult when using direct algorithms. This is because L and U may be dense, even though A is sparse. This is called the *fill-in problem*. Fill-in can be controlled by permuting rows and columns but these permutations may affect the pivoting scheme and hence increase the roundoff. Thus, in choosing P and Q we must balance the competing requirements of roundoff-error and fill-in control. An additional complication is that P and Q cannot be determined before the factorization process begins. Therefore we cannot predict where fill-in will occur. Thus we are

forced to use a *dynamic* storage scheme which allocates storage as fill-in occurs.

5. SPARSE, SYMMETRIC POSITIVE DEFINITE MATRICES

Symmetric positive definite (SPD) matrices do not suffer from the problems outlined in Section 4 above. This is fortuitous because many real, physical systems are modelled by very large, sparse SPD matrices.

5.1 Some Computational Properties of SPD Matrices

1. Gaussian Elimination becomes Cholesky's Method and yields the triangular factorization

$$A = LL^t,$$

where L is lower triangular with positive diagonal elements.

2. No pivoting is required to control roundoff errors, i.e. the diagonal elements, a_{ii}, are stable pivots.

3. A symmetric permutation of A is symmetric, positive definite, i.e. if A is SPD then PAP^t is SPD, where P is a permutation matrix.

Properties 2 and 3 allow us to find a P which reduces fill-in *before* we begin the factorizing, without worrying about pivoting to control roundoff error. Thus we can predict fill-in and allocate storage accordingly.

We can now break a direct algorithm for SPD matrices into four independent steps as shown below in Fig. 5.1.

The ability to break the direct algorithm into four independent steps has enormous practical implications. We are free to look for the best possible ordering P to minimize fill-in.

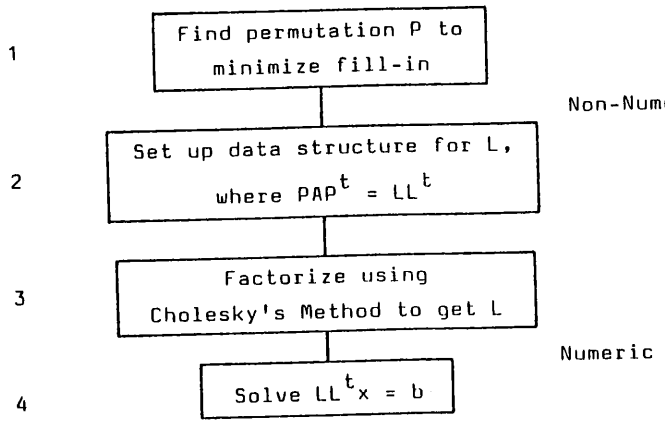


Figure 5.1: Direct Algorithm for Sparse, SPD Systems

A static storage scheme can be used because the factorization will cause only that fill-in predicted in Step 1. Different problems, with the same non-zero structure can be solved using only the numeric Steps 3 and 4. Thus the cost of the non-numeric Steps 1 and 2 can be amortized over a set of problems. The software for each step can be designed and developed independently. This helps to 'modularize' the complete program and yields a more reliable, useful, and versatile software package.

5.2 Ordering an SPD Matrix to Reduce Fill-In

The focus of much research in the last 10 years has been on finding P to minimize fill-in. The problem of finding a good ordering or permutation P is called the *Ordering Problem*. A very thorough exposition is given in the book by George and Liu [11].

We illustrate the ordering problem with the 6 x 6 symmetric matrix shown in Fig. 5.2(a), where the non-zeros are indicated by asterisks. It is assumed that the matrix is positive definite and hence pivoting on the diagonal elements will not cause roundoff problems.

Using standard Gaussian Elimination, variable 1 is eliminated from rows 2,...,6 by subtracting suitable multiples of row 1 from rows 2,...,6. Variable 2 is eliminated from rows 3,...,6 by subtracting suitable multiples of row 2 from rows 3,...,6. Eventually the upper triangular matrix U is obtained, which is shown in Fig. 5.2(b). We note that the number of non-zeros in A and U are 16 and 21 respectively. In general, for this type of matrix, the numbers of non-zeros will increase from 3n-2 to $(n^2+n)/2$, using the same pivot order. Now, if we symmetrically permute the A matrix so that row 1 becomes row 6, var 1 becomes var 6, etc., then we get the matrix PAP^t shown in Fig. 5.2(c). Performing the elimination process on this matrix gives the matrix U' shown in Fig. 5.2(d), which has only 2n-1 non-zeros. Thus the number of additional non-zeros generated by the elimination process has been drastically reduced.

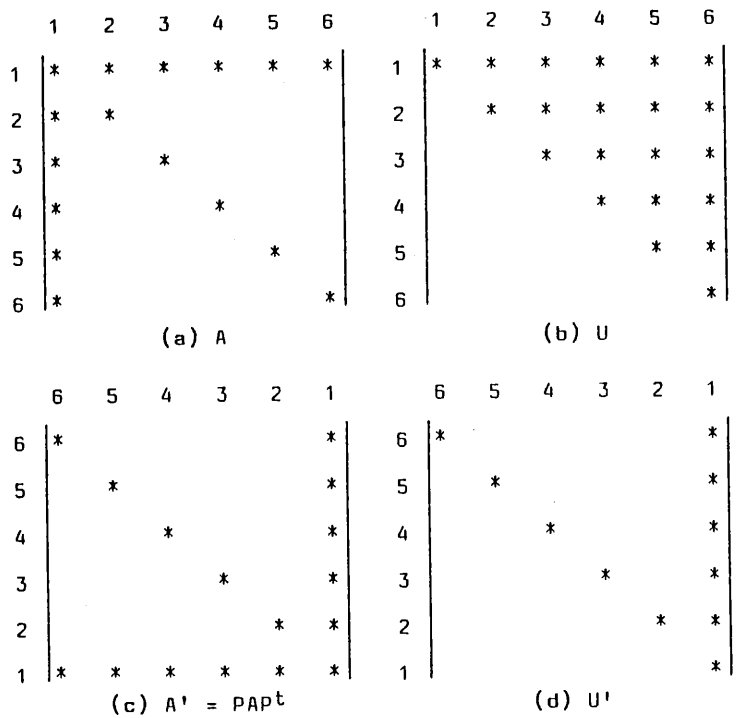


Figure 5.2: Elimination with Different Pivot Orders

The example above shows that a judicious ordering of pivots (choice of P) leads to no increase in the number of non-zeros during elimination. In general it is not possible to find a P so that no fill-in occurs. It is possible to find a P to minimize fill-in, but even this is difficult because it has been shown [19] that this problem is *NP-Complete* (see [10]) and so is essentially intractable.

Although we cannot hope to minimize fill-in there are practical algorithms that tend to reduce fill-in. At present there are four general types of ordering algorithms:

1. Band and Envelope Methods.
2. Mimimum Degree Methods.
3. Quotient Tree Methods.
4. Dissection Methods.

We will not describe the methods because they require a good knowledge of graph algorithms and theory. The first method is designed for band-like SPD matrices; the second for general SPD matrices; and the third and fourth for SPD matrices arising in finite element problems. Figs 5.3, 5.4 and 5.5 show the effects of the first two methods on a 35 x 35 SPD matrix. These figures are reproduced from [11] which gives a complete description and analysis of all four methods.

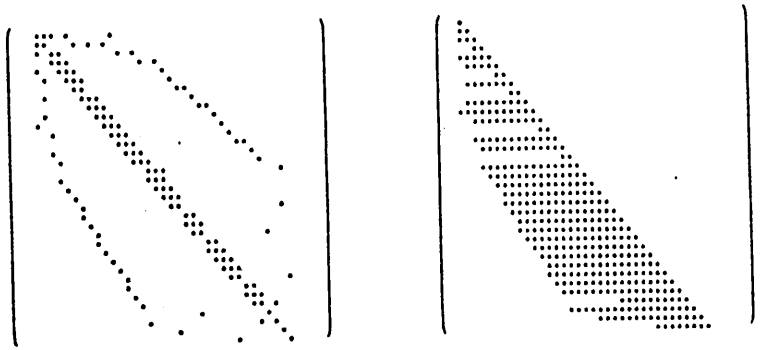


Figure 5.3: Unordered Matrix A and its Factor L

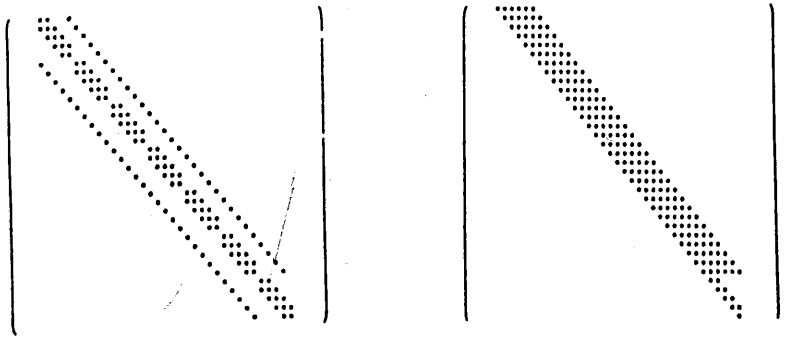


Figure 5.4: Band-Ordered Matrix A' and its Factor L'

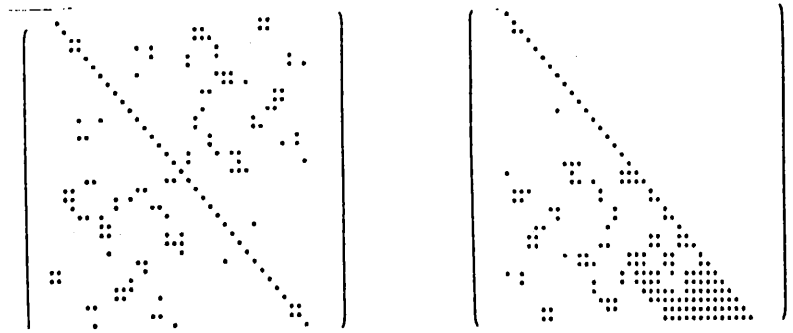


Figure 5.5: Minimum Degree-Ordered Matrix A" and its Factor L"

6. EXTENSIONS AND CONCLUSION

In this paper we have outlined the problems in storing and solving large sparse sets of linear equations. We have seen that substantial reductions in both storage and computation time can be achieved by the proper application of sparsity techniques. Many real problems give rise to matrices that are large and sparse, and these problems cannot be solved by the naive application of standard (dense) matrix methods. Hence it is important that both theoretical and applied scientists be aware of the large body of research, experience, and software that exists in this area of applied mathematics.

Although 'sparse matrices' may seem a rather narrow and specialized topic, it has developed into a very active and broad area of research, with contributions from chemists, computer scientists, engineers, mathematicians, operations researchers and physicists. It draws on topics such as linear algebra, numerical analysis, graph theory, combinatorics, data structures and software design. Research in the area ranges from the theoretical (see [3]) to the practical (see [7]).

The general techniques of sparse matrix storage and factorization are also applied to eigenvalue problems, least squares and linear programming. Special techniques for linear programs of the minimum-cost-flow type have been particularly successful (see [5]), with Barr and Turner [4] solving a network flow problem containing 25,000,000 variables. Sparse matrix techniques are also important in differential, integral, and non-linear equation-solving and in optimization problems. This is because most solution methods use some form of local or piecewise linearization which gives rise to sparse matrices.

Software for sparse matrix problems is not easy to design and develop. In general a sparse matrix subroutine in FORTRAN is 3 to 10 times longer than its dense equivalent and is difficult to understand and modify. For this reason we have not discussed sparse matrix software in any detail. Instead we give a list of some of the available software packages in the appendix.

APPENDIX

Sources of Sparse Matrix Software

Most general collections of mathematical software contain some sparse matrix routines. However, the best and most up-to-date software comes from those institutions and universities that have active research programs in sparse matrices and related areas. These are indicated by an (*) below.

1. ACM - Association of Computing Machinery, publishes *Trans. on Mathematical Software* and the *Collected Algorithms from ACM*. All software is available in machine-readable form from IMSL below.

2. AERE (*) - Atomic Energy Research Establishment, Harwell, Didcot, Oxfordshire, England. One of the leading places for sparse matrix research and software. Reports published by H.M. Stationery Office.

3. Boeing Computer Services Co., Seattle, Washington 98124.

4. IMSL - International Mathematical and Statistical Library, Inc., NBC Building, 7500 Bellaire Blvd, Houston, Texas 77036. This company develops, maintains and sells the general IMSL library and the Collected Algorithms from ACM. The IMSL library is available on UCD's DEC20 and IBM machines.

5. NAG - National Algorithms Group, Oxford University. Available on UCD's IBM machine,

6. SPARSPAK (*) - Prof. A. George, University of Waterloo, Waterloo, Ontario, Canada.

7. Yale Sparse Matrix Package (*) - Prof. S. Eisenstat, Yale University, New Haven, Connecticut 06520.

```
C=======================================================================
      PROGRAM SSOLVE
C=======================================================================
C This is a driver program and problem generator to test the Gauss-Seidel
C Method for Sparse Matrices
C
C
C-----------------------------------------------------------------------
C
      INTEGER IDXCOL(2400),NONZER(1200),
     +        N, KOUNT, SCREEN, ANSWER
      REAL CMAT(2400), D(1200), X(1200),
     +     EPSIL, A, B, C
      DATA SCREEN /5/

C     WRITE (SCREEN,900)
  900 FORMAT (' INPUT A, B, C, EPSIL (4F5.0), N (I4) :')
      READ (SCREEN,800)A, B, C, EPSIL, N
  800 FORMAT (4F5.0,I4)
      WRITE (SCREEN, 800)A, B, C, EPSIL, N

C-----START CONSTRUCTION OF BAND MATRIX TEST PROBLEM-------------------
C a(i1,i) = A, a(i,i) = B, a(i+1,i) = C
C

      CMAT(1) = -C/B
      IDXCOL(1) = 2
      NONZER(1) = 1
      D(1) = 1.0/B
      IPOINT = 2

C     NM1 = N-1

C     DO 10 IROW = 2, NM1
         CMAT(IPOINT) = -A/B
         IDXCOL(IPOINT) = IROW - 1
         IPOINT = IPOINT + 1
         CMAT(IPOINT) = -C/B
         IDXCOL(IPOINT) = IROW + 1
         IPOINT = IPOINT + 1
         D(IROW) = 1.0/B
         NONZER(IROW) = 2
   10 CONTINUE
      CMAT(IPOINT) = -A/B
      IDXCOL(IPOINT) = N-1
      NONZER(N) = 1
      D(N) = 1.0/B

C-----------------------------------------------------------------------
C
      CALL SEIDEL(CMAT, D, IDXCOL, NONZER, N, EPSIL, X, KOUNT)

C
```



```
      WRITE (SCREEN, 920) N, KOUNT, EPSIL
  920 FORMAT (' N = ', I4,' NO. ITERS. = ', I5,' EPSIL = ', F10.8)
C
C-----WRITE OUT SOLUTION VECTOR----------------------------------------
C
      WRITE (SCREEN, 930) (I, X(I), I=1,N)
  930 FORMAT (' ',5(I4,2X,F10.4))
      STOP
      END

C
C=======================================================================
      SUBROUTINE SEIDEL (CMAT, D, IDXCOL, NONZER, N, EPSIL, X, KOUNT)
C=======================================================================
C This routine solves the linear system of equations x = Cx + d, using the
C Gauss Seidel Method.  The matrix C is sparse and is stored to take advan-
C tage of this sparsity.
C-----------------------------------------------------------------------
C
      INTEGER IDXCOL(2400),NONZER(1200),
     +        N, KOUNT, IPOINT, IROW, JCOL, NUMNZI, SCREEN
      REAL CMAT(2400), D(1200), X(1200),
     +     EPSIL, MAXOLD, MAXNEW, MAXDIF, DELTAX, RELERR, INFNTY
      DOUBLE PRECISION SUM

C     INFNTY = 1.0E20
      KOUNT = 0
      MAXOLD = 1.0

C     DO 5 IROW = 1, N
        X(IROW) = D(IROW)
    5 CONTINUE
C
C-----START OF MAIN ITERATION LOOP -------------------------------------
C
   10 KOUNT = KOUNT + 1
      MAXDIF = -INFNTY
      MAXNEW = -INFNTY
      IPOINT = 1

C     DO 200 IROW = 1, N
         SUM = 0.0
         NUMNZI = NONZER(IROW)
         DO 100 J=1, NUMNZI
            JCOL = IDXCOL(IPOINT)
            SUM = SUM + CMAT(IPOINT) * X(JCOL)
            IPOINT = IPOINT + 1
  100    CONTINUE
C
C-----------CALCULATE INFINITY NORMS-----------------------------------
C
         XNEW = SUM + D(IROW)
         DELTAX = ABS( XNEW - X(IROW) )
         IF( DELTAX .GT. MAXDIF ) MAXDIF = DELTAX
         ABXNEW = ABS(XNEW)
         IF( ABXNEW .GT. MAXNEW ) MAXNEW = ABXNEW
         X(IROW) = XNEW
  200 CONTINUE
```

```
C
C-------CONVERGENCE CHECK-------------------------------------------------
C
      RELERR = MAXDIF/MAXOLD
      MAXOLD = MAXNEW
      WRITE (5,900) KOUNT,MAXDIF,MAXOLD,RELERR
  900 FORMAT (' ITER. NO. =', I4,5X, 3F20.7)
      IF(RELERR .GT. EPSIL) GOTO 10
C
      RETURN
      END
C
C============================END OF SPARSE SEIDEL=======================
```

REFERENCES

1. BRAMELLER, A. and HAMAM, Y.M.
'Sparsity: Its Practical Application to Systems Analysis', Pitman, 1976.

2. BUNCH, J.R. and ROSE, D.J. (Eds)
'Sparse Matrix Computations', Academic Press, 1976.

3. BRUALDI, R. and SCHNEIDER, H. (Eds)
'Linear Algebra and its Applications', Vol. 34, Dec. 1980.

4. BARR, R.S. and TURNER, J.S.
"Microdata File Merging Through Large-Scale Network Technology", *Mathematical Programming Study*, 15, 1-22, North-Holland, 1981.

5. BRADLEY, G.H., BROWN, G.G. and GRAVES, G.W.
"Design and Implementation of Large-Scale Transshipment Algorithms", *Management Science*, Vol. 24, No. 1 (1977).

6. DUFF, I.S.
"A Survey of Sparse Matrix Research", *Proc. IEEE*, Vol. 65 (1977) 500-535.

7. DUFF, I.S. (Ed.)
'Sparse Matrices and Their Uses', Academic Press, 1981.

8. DUFF, I.S. and STEWART, G.Q. (Eds)
'Sparse Matrix Proceedings 1978, SIAM Publications, 1979.

9. FORSYTHE, G. and MOLER, C.
'Computer Solution of Linear Algebraic Systems', Prentice-Hall, 1967.

10. GAREY, M. and JOHNSON, D.
'Computers and Intractability: A Guide to the Theory of NP-Completeness', Freeman, 1979.

11. GEORGE, A. and LIU, J.W.
'Computer Solution of Large Sparse Positive Definite Systems', Prentice-Hall, 1981.

12. HOROWITZ, E. and SAHNI, S.
'Fundamentals of Data Structures in Pascal', Pitman, 1984.

13. JENNINGS, A.
'Matrix Computations for Engineers and Scientists', Wiley, 1977.

14. KNUTH, D.E.
'The Art of Computer Programming', Vol. 1, 2nd ed., Addison-Wesley, 1973.

15. POOCH, U. and NIEDER, A.
"A Survey of Indexing Techniques for Sparse Matrices", *'Computing Surveys*, Vol. 5, No. 2, 1973.

16. REID, J.K.
'Large Sparse Sets of Linear Equations', Academic Press, 1971.

17. RICE, J.
'Numerical Methods, Software, and Analysis', McGraw-Hill, 1983.

18. ROSE, D.J.
"A Graph-Theoretic Study of the Numerical Solution of Sparse Positive Definite Systems of Linear Equations",

in READ, R. (ed.), *'Graph Theory and Computing'*, Academic Press, 1973.

19. ROSE, D.J. and TARJAN, R.E.
"Algorithmic Aspects of Vertex Elimination on Directed Graphs", *SIAM Journal on Applied Mathematics*, Vol. 34 (1978) 176-197.

20. ROSE, D.J. and WILLOUGHBY, R.A. (Eds)
'Sparse Matrices and their Applications', Plenum Press, 1972.

21. TARJAN, R.E.
"Graph Theory and Gaussian Elimination", in [2] (1976) 3-22.

22. VARGA, R.S.
'Matrix Iterative Analysis', Prentice-Hall, 1962.

23. WELSH, J., ELDER, J. and BUSTARD, D.
'Sequential Program Structures', Prentice-Hall, 1984.

24. WILLOUGHBY, R. (Ed.)
'Sparse Matrix Proceedings', IBM Research, 1968.

Management Information Systems Department,
University College, Dublin

AN INTRODUCTION TO NONSTANDARD ANALYSIS

Christopher Thompson

1. Introduction

This note describes the axiomatic approach to nonstandard analysis developed by Nelson and illustrates it by proving the fundamental theorem of algebra and a form of the spectral theorem in finite dimensions. It is based in part on a talk given at the DIAS in April, 1985.

2. The Axioms

We shall be working in a mathematical universe that contains all the usual familiar objects (e.g. numbers 0, 1, $\sqrt{2}$, π etc., sets $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ etc., function spaces ℓ^2, $C[0,1]$ etc.) and in addition contains new and unfamiliar objects such as infinitely large natural numbers and infinitely small positive real numbers. To ensure the presence of the familiar objects we adopt the usual axioms of set theory, for example the Zermelo-Fraenkel axioms together with the axiom of choice. To make visible the unfamiliar objects we adopt a new undefined unary predicate *standard* and axioms (I), (S) and (T) to govern its use. The resulting theory is called *internal set theory* (IST) and is due to Nelson [7]. Just as the binary predicate "$\in$" of ZFC has the informal interpretation "is a member of" (although strictly speaking it is undefined and therefore meaningless), so also has the unary predicate "standard" of IST an informal meaning: 'x standard' has the interpretation 'x is a familiar object of classical mathematics'. It is a consequence of the axioms that 0, 1, $\sqrt{2}$, π, $\mathbb{N}$, $\mathbb{R}$ etc. are indeed standard, as we shall see.

A formula of IST may or may not contain the predicate "standard". If it does then it is called an *external* formula, otherwise it is called *internal*. Speaking informally, internal

formulas are those that make sense to a classical mathematician. We shall use a system of abbreviations illustrated by

$$\forall^{st}x \quad \text{for} \quad \forall x \ (x \text{ standard}) \Rightarrow$$

$$\exists^{st}x \quad \text{for} \quad \exists x \ (x \text{ standard}) \wedge$$

$$\forall^{fin}x \quad \text{for} \quad \forall x \ (x \text{ finite}) \Rightarrow .$$

Here, "x finite" means that there is a bijection of x with $\{m \in \mathbb{N} : m < n\}$ for some natural number n.

The axioms of IST are the axioms of ZFC and three new axioms called *transfer* (T), *idealisation* (I) and *standardization* (S).

TRANSFER. *Let* A(x) *be an internal formula with free variable* x *and no other free variable:* A(x) *may contain constants but they must be standard. Then*

$$(T) \qquad \forall^{st}x \ A(x) \Rightarrow \forall x \ A(x).$$

The transfer axiom implies that all "classical" objects are standard. To see this, observe that (T) is equivalent to

$$\exists x \ A(x) \Rightarrow \exists^{st}x \ A(x). \qquad (1)$$

Consequently if there is a unique x such that A(x) then that x must be standard. In particular, taking A(x) to be "x is a complete ordered field", we deduce that $\mathbb{R}$ is standard. In a similar way any uniquely specified classical object such as 0, 1, $\sqrt{2}$, $\mathbb{N}$, $\mathbb{C}$ is standard.

IDEALISATION. *Let* B(x,y) *be an internal formula with free variables* x,y *and possibly other free variables. Then*

$$(I) \quad \forall^{st\,fin}z \, \exists x \, \forall y \in z \ \ B(x,y) \Leftrightarrow \exists x \, \forall^{st}y \ \ B(x,y).$$

The idealisation axiom implies the existence of infinitely large natural numbers and non-zero infinitesimal real numbers. To see this we need some definitions. If $x \in \mathbb{R}$ or $\mathbb{C}$ then

x is *infinitesimal* ($x \approx 0$) if $\forall^{st}n \in \mathbb{N} \ |x| < 1/n$,

x is *unlimited* ($|x| \approx +\infty$) if $\forall^{st}n \in \mathbb{N} \ |x| > n$,

x is *limited* ($|x| << +\infty$) if $\exists^{st}n \in \mathbb{N} \ \ |x| \leq n$.

We call x and y infinitely close (and write $x \approx y$) if $|x-y|$ is infinitesimal. If we let B(x,y) be $x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y < x$ then from (I) we deduce that

$$\exists x \in \mathbb{N} \ \forall^{st}y \in \mathbb{N} \ \ y < x.$$

This says that $\mathbb{N}$ has an unlimited element. Let us fix on one such element and call it ω. Then $\omega \in \mathbb{R}$ and therefore $1/\omega \in \mathbb{R}$. It is easy to see that $1/\omega$ is positive and infinitesimal.

STANDARDISATION. *Let* C(z) *be a formula, internal or external, with free variable* z *and possibly other free variables. Then*

$$(S) \quad \forall^{st}x \, \exists^{st}y \, \forall^{st}z \ \ (z \in y \Leftrightarrow z \in x \wedge C(z)).$$

In words: given any standard set x and any property C, there is a standard set y whose standard elements are exactly those standard elements of x that satisfy C. The need for this axiom arises because in IST it is illegal to use an external formula C to form a new set y from a given set x by letting $y = \{z \in x : C(z)\}$; for unless C is a formula of ZFC (i.e. unless C is internal) there is no axiom to permit the formation of y. For example let x be $\mathbb{R}$ and let C(z) be "$z \approx 0$". Then we cannot form the set $\{z \in \mathbb{R} : z \approx 0\}$ within IST; the set y in (S) in this case is $\{0\}$.

Fortunately there are two consequences of (S) that are much easier to grasp and are sufficient for our purposes. We call

the first of these the *standard part* property (SP).

(SP) *Every limited real number is infinitely close to a standard real number.*

For example, $\sqrt{2} + 1/\omega$ is limited and is infinitely close to $\sqrt{2}$ which is standard. If x is limited and $x \approx y$ where y is standard then y is unique and is called the *standard part* of x, written $y = {}^{o}x$. A similar property to (SP) is easily seen to hold for complex numbers by considering real and imaginary parts.

The second consequence of (S) is called *External Induction* (EI).

(EI) *Let $A(x)$ be any formula, internal or external, with x as free variable and perhaps other free variables. Suppose that $A(0)$ and for all standard natural numbers n, if $A(n)$ then $A(n+1)$. Then for all standard natural numbers n we have $A(n)$.*

We can use this to prove that if z and w are limited complex numbers and $z \approx w$ then $z^n \approx w^n$ for all standard n. The induction step is accomplished by noting that if $z_1 \approx w_1$ and $z_2 \approx w_2$ (all limited) then $z_1 z_2 \approx w_1 w_2$. This is because $z_1 = w_1 + \varepsilon_1$ and $z_2 = w_2 + \varepsilon_2$ where ε_1 and ε_2 are infinitesimal and therefore

$$z_1 z_2 - w_1 w_2 = \varepsilon_1 w_2 + \varepsilon_2 w_1 + \varepsilon_1 \varepsilon_2$$

which is clearly infinitesimal. We can extend the result to standard polynomials by a second application of (EI). Let P be a standard polynomial. Then its degree is a standard natural number and its coefficients are standard complex numbers. If z and w are limited and $z \approx w$ then $P(x) \approx P(w)$.

3. The Fundamental Theorem of Algebra

The fundamental theorem of algebra states that every non-constant polynomial with complex coefficients has at least one complex root. The following proof is based on a classical one ([3], pp. 53-55), but the availability of infinitesimals greatly simplifies the technical details.

Let $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ be any complex polynomial with $n \geq 1$ and $a_n \neq 0$. We must prove that P has a root. By transfer we may suppose that P is standard. Then n and all the a_j are standard.

(i) We first prove that $|P(z)|$ attains its minimum at some standard point β in $\mathbb{C}$. Observe that if $|z| \approx +\infty$ then $|P(z)| \approx +\infty$. The reason is that if $|z| \approx +\infty$ then

$$\frac{P(z)}{z^n} = a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \approx a_n,$$

and a_n is standard and non-zero. Now let $\omega \in \mathbb{N}$ satisfy $\omega \approx +\infty$ and let F be defined by

$$F = \{(m + in)/\omega : m,n \in \mathbb{Z},\ |m|, |n| \leq \omega^2\}.$$

Because F is a finite set, the minimum of $|P(z)|$ as z runs through F is attained at some point α of F. By the remark above $|\alpha| << +\infty$ and so by (SP), $\alpha \approx \beta$ for some standard β in $\mathbb{C}$. We will prove that $|P(z)| \geq |P(\beta)|$ for all z. By transfer, it suffices to prove this for all standard z. If z is standard then $z \approx$ some point ζ of F, and therefore

$$|P(z)| \approx |P(\zeta)| \geq |P(\alpha)| \approx |P(\beta)|.$$

Since the extreme numbers are standard, it follows that $|P(z)| \geq |P(\beta)|$.

(ii) It is an elementary fact that every equation of the form $z^n = c$ has a solution in $\mathbb{C}$. It suffices to write $c = re^{i\theta}$ and let $z = r^{1/n}e^{i\theta/n}$.

(iii) We now prove that $P(\beta) = 0$. This is done by examining $P(\beta+h)$ when $h \in \mathbb{C}$ and $|h|$ is small. Applying the binomial theorem to $(\beta+h)^j$ we obtain

$$P(\beta+h) = P(\beta) + b_1h + b_2h + \ldots + b_nh^n \quad (b_n=a_n\neq 0)$$

where the b_j are standard complex numbers. Let b_m be the first non-zero coefficient. Then

$$P(\beta+h) = P(\beta) + b_mh^m(1+Q(h)), \qquad (3)$$

where $Q(h)$ is a standard polynomial with no constant term; $Q(0) = 0$. Now let h be a solution of the equation

$$z^m = -\frac{1}{\omega^m}\frac{P(\beta)}{b_m}.$$

Then $h \approx 0$ and so $Q(h) \approx Q(0) = 0$ and hence $|Q(h)| < 1$. From (3) we have

$$P(\beta+h) = \left(1 - \frac{1}{\omega^m}\right)P(\beta) - \frac{1}{\omega^m}P(\beta)Q(h)$$

and if $P(\beta) \neq 0$ then we have the contradiction

$$|P(\beta+h)| \leq \left(1 - \frac{1}{\omega^m}\right)|P(\beta)| + \frac{1}{\omega^m}|P(\beta)||Q(h)| < |P(\beta)|.$$

Thus $P(\beta) = 0$ and the proof is complete.

Remarks. This proof is elementary in that it avoids the notions of compactness, continuity and complex integration. It is also constructive, at least in spirit, because the argument by contradiction is held back until the end. In practice we have no unlimited natural number ω to help us, but for any

specific polynomial, a suitably large standard value of ω will make the mesh F fine enough to catch the zeros of F approximately.

4. The Spectral Theorem

The following result is a version of the Jordan normal form theorem. The proof is due to Lutz and Goze [6] and it is so natural that it deserves to be widely known.

THEOREM. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$, where λ_j has multiplicity m_j, $j=1, \ldots, p$. Then there is a basis $u_1, u_2, \ldots, u_n$ for $\mathbb{C}^n$ such that*

$$M(f,(u_j)) = \begin{bmatrix} A_1 & & 0 \\ & A_2 & \\ & \ddots & \\ 0 & & A_p \end{bmatrix}, \quad A_j = \begin{bmatrix} \lambda_j & * & * & * \\ & \lambda_j & & \\ & & \ddots & * \\ 0 & & & \lambda_j \end{bmatrix}$$

where A_j is an $m_j \times m_j$ matrix.

COROLLARY. *There is a direct sum decomposition $\mathbb{C}^n = F_1 \oplus F_2 \oplus \ldots \oplus F_p$ such that $\dim(F_j) = m_j$, $f(F_j) \subseteq F_j$ and $f|_{F_j} = \lambda_j I + N_j$ where N_j is nilpotent.*

PROOF. If f has no repeated eigenvalues so that $m_j = 1$ for all j, then the result is classical and elementary. The difficulty comes from the possibility that some eigenvalues may be repeated. We can get rid of such degeneracies by an infinitesimal perturbation and this is the key idea of the proof, which is in three parts.

(i) Choose a "Good" Infinitesimal Perturbation g of f

Choose a linear map $g : \mathbb{C}^n \to \mathbb{C}^n$ such that $g(x) \approx f(x)$ for all limited x, the eigenvalues of $\mu_1, \mu_2, \ldots, \mu_n$ g are distinct and each $\mu_j \approx$ some λ_i. This can be done by making infinitesimal changes in a matrix representing f. Since the eigenvalues of g are distinct, there is a basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$ consisting of eigenvectors of g. We normalise so that $|v_j| = 1$ (where $|x|$ denotes the Euclidean norm of x).

(ii) Get a "Good" Matrix Representation of g

Group all μ_j which are infinitely close, and group the corresponding v_j. After relabelling we get

$$\mu_1 \approx \mu_2 \approx \ldots \approx \mu_{m_1} \approx \lambda_1 \; ; \; \mu_{m_1+1} \approx \ldots \approx \mu_{m_1+m_2} \approx \lambda_2 ; \text{ etc.}$$

Define subspaces G_j by

$$G_1 = \text{span}\{v_1, v_2, \ldots, v_{m_1}\}, \quad G_2 = \text{span}\{v_{m_1+1}, \ldots, v_{m_1+m_2}\},$$

etc.

Then clearly $\dim G_j = m_j$, $g(G_j) \subseteq G_j$ and $\mathbb{C}^n = G_1 \oplus G_2 \oplus \ldots \oplus G_p$. In each G_j, use the Gram-Schmidt process to obtain an orthonormal basis (w_i) from (v_i). We get

$$G_1 = \text{span}\{w_1, w_2, \ldots, w_{m_1}\}, \; G_2 = \text{span}\{w_{m_1+1}, \ldots, w_{m_1+m_2}\},$$

etc., and $\langle w_i, w_j \rangle = \delta_{ij}$ within each subspace. By the nature of the Gram-Schmidt process

$$M(g,(w_i)) = \begin{bmatrix} B_1 & & \\ & B_2 & \\ & & \ddots \\ & & & B_p \end{bmatrix}, \quad B_1 = \begin{bmatrix} \mu_1 & * & \cdots & * \\ & \mu_2 & \ddots & \vdots \\ & & & \mu_{m_1} \end{bmatrix} \text{ etc.}$$

(iii) Deduce a "Good" Matrix Representation for f

For each j, w_j is limited because $|w_j| = 1$ and therefore $w_j \approx u_j$ for some standard $u_j \in \mathbb{C}^n$. Define subspaces $F_1 F_2, \ldots, F_p$ by

$$F_1 = \text{span}\{u_1, u_2, \ldots, u_{m_1}\}, \quad F_2 = \text{span}\{u_{m_1+1}, \ldots, u_{m_1+m_2}\},$$

etc., so that in a certain sense $F_j \approx G_j$. We pass now from a nonstandard situation:

$$\mathbb{C}^n = G_1 \oplus G_2 \oplus \ldots \oplus G_p, \quad g(G_j) \subseteq G_j, \; \dim G_j = m_j, \text{ and}$$

$$M(g,(w_i)) = \text{diag}[B_1, B_2, \ldots, B_p],$$

to a standard one. It is easy to believe and not difficult to prove that

$$\mathbb{C}^n = F_1 \oplus F_2 \oplus \ldots \oplus F_p, \quad f(F_j) \subseteq F_j, \quad \dim F_j = m_j, \text{ and}$$

$$M(f,(u_i)) = \text{diag}[A_1, A_2, \ldots, A_p]$$

where A_j is standard and $A_j \approx B_j$ in the sense that corresponding entries are infinitely close, so that

$$A_j = \begin{bmatrix} \lambda_j & * & * & \cdots & * \\ & \lambda_j & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & & \lambda_j \end{bmatrix}$$

5. Concluding Remarks

Nonstandard analysis is also of use in providing new results and in providing the framework for mathematical modelling of physical processes where different orders of magnitude are involved. Two recent examples are moiré patterns [4] and the theory of singular perturbations of ordinary differential equations [6], [1], [9].

It should be mentioned that the original, constructive approach to nonstandard analysis due to Robinson [8] is also flourishing today especially in the fields of functional analysis, measure theory, stochastic differential equations and optimal control [2], [5].

References

1. CARTIER, P.
"Perturbations Singulières des Équations Différentielles et Analyse Non-Standard", Séminaire Bourbaki No. 580, *Astérisque* 92-93 (1982) 21-44.

2. CUTLAND, N.J.
"Nonstandard Measure Theory and its Applications", *Bull. London Math Soc.*, 15 (1983) 529-589.

3. DIENES, P.
'The Taylor Series', Dover, New York, 1957.

4. HARTHONG, J.
"Le Moiré", *C.R. Acad. Sci. Paris*, Sér. A-B 290 (1980), No. 19, A877-A879.

5. HURD, A. (Editor).
'Nonstandard Analysis - Recent Developments', Lecture Notes in Mathematics 983 (Springer, Berlin, 1981).

6. LUTZ, R. and GOZE, M.
'Nonstandard Analysis. A Practical Guide with Applications', Lecture Notes in Mathematics 881 (Springer, Berlin, 1981).

7. NELSON, E.
"Internal Set Theory: A New Approach to Nonstandard Analysis", *Bull. Amer. Math. Soc.*, 83 (1977) 1165-1198.

8. ROBINSON, A.
'Non-Standard Analysis', 2nd Ed, American Elsevier, N.Y. 1974.

9. ZVONKIN, A. and SHUBIN, M.A.
"Non-Standard Analysis and Singular Perturbations of Ordinary Differential Equations", *Russian Math. Surveys*, 39:2 (1984) 69-131.

Faculty of Mathematical Studies,
The University,
Southampton.

INTEGRAL MEANS OF UNIVALENT FUNCTIONS - A FRAGMENT

Finbarr Holland

Introduction

A notable event in the theory of the class S of normalised univalent functions on the open unit disc U occurred last summer (1984) when de Branges [3] settled the long outstanding conjecture of Bieberbach [2], to the effect that if

$$f(z) = z + \sum_2^\infty a_n z^n, \quad z \in U,$$

belongs to S, then

$$|a_n| \leq n, \text{ for } n = 2,3, \ldots, \qquad (B)$$

with equality holding for one n, and so for all n, if and only if f is of the form $f(z) = k(e^{i\theta}z)$ for some real θ, where k denotes Koebe's function, i.e.

$$k(z) = z/(1-z)^2 = \sum_1^\infty nz^n, \ z \in U.$$

Not only did de Branges settle this conjecture, but he settled Robertson's conjecture [8] for the coefficients of *odd* functions in S, which was known to imply Bieberbach's, and a still stronger conjecture - Milin's [7] - into the bargain. We recall these: given f in S write

$$f(z) = z\exp(2\sum_1^\infty \gamma_n z^n), \quad z \in U,$$

and define the odd univalent function g in S as follows:

$$g(z) = \sqrt{f(z^2)} = z\exp(\sum_1^\infty \gamma_n z^{2n}) = \sum_1^\infty c_n z^n.$$

Robertson conjectured that

$$\sum_0^n |c_{2k+1}|^2 \leq n + 1, \quad n = 0,1,2, \ldots\ldots \qquad (R)$$

Millin conjectured that, if

$$\Delta_n = \sum_1^n (k|\gamma_k|^2 - 1/k), \quad n = 1,2, \ldots$$

then

$$\sum_1^n (n + 1 - k)(k|\gamma_k|^2 - 1/k)_1^n = \sum_1^n \Delta_j \leq 0, \quad n = 1,2, \ldots \qquad (M)$$

Since (in the notation just introduced) the inequalities that follow were known to hold, Bieberbach's conjecture was affirmed once Robertson's was, and Robertson's once Milin's was. Thanks to de Branges, who settled (M) affirmatively, all three are true.

$$|a_n| \leq \sum_0^{n-1} |c_{2k+1}|^2, \quad n = 2,3, \ldots.$$

$$\sum_0^n |c_{2k+1}|^2 \leq (n+1)\exp(\sum_1^n \Delta_j/(n+1)), \quad n = 1,2, \ldots \qquad (ML)$$

The first of these follows by expressing the coefficients of $f(z^2)$ in terms of those of $(g(z))^2$ and applying Schwarz's inequality. The second lies deeper, and is a special case of one of the celebrated Milin-Lebedev inequalities [7], which we will enunciate shortly.

The First Integral Mean

Given an analytic function h on U, and $0 < p < \infty$, we write

$$I_p(r,h) = \int_0^{2\pi} |h(re^{i\theta})|^p d\theta, \quad 0 \leq r < 1.$$

In a major assault on the Bieberbach conjecture, Littlewood [6] showed in 1925 that

$$\sup\{I_1(r,f) : f \in S\} \leq r/(1 - r), \quad 0 \leq r < 1, \qquad (L)$$

and deduced from this that $|a_n| < en$, for $n = 2,3, \ldots$

Almost fifty years were to elapse before Baernstein succeeded in sharpening inequality (L): amongst other remarkable things he showed in [1] that, for any $p > 0$,

$$\sup\{I_p(r,f) : f \in S\} = I_p(r,k), \quad 0 \leq r < 1. \qquad (B_p)$$

In particular, then, (B_1) is equivalent to the statement that

$$I_1(r,f) \leq r/(1 - r^2), \quad 0 \leq r < 1,$$

if $f \in S$.

Here we point out that the latter inequality is a simple consequence of (R), a fact which appears to have gone unnoticed until now. To see this, let $f \in S$. Keeping the notation as before, and bearing in mind that g is odd, we have, for $0 \leq r < 1$,

$$\begin{aligned} I_1(r^2,f) &= \int_0^{2\pi} |f(r^2e^{2i\theta})|d\theta = \int_0^{2\pi} |g(re^{i\theta})|^2 \, d\theta \\ &= 2\pi\sum_1^\infty |c_n|^2 r^{2n} = 2\pi r^2 \sum_0^\infty |c_{2k+1}|^2 r^{4k} \\ &= 2\pi r^2(1 - r^4)\sum_0^\infty \Big(\sum_0^k |c_{2j+1}|^2\Big) r^{4k} \\ &\leq 2\pi r^2(1 - r^4)\sum_0^\infty (k + 1)r^{4k} \\ &= 2\pi r^2/(1 - r^4) \\ &= I_1(r^2,k), \end{aligned}$$

using (R). It is clear that equality holds for some r in 1) if and only if f is the Koebe function composed possibly th a rotation.

Integral Means for $p \geq 2$

Bieberbach's inequality (B) coupled with Parseval's identy tells us that

$$\sup\{I_2(r,f) : f \in S\} = I_2(r,k), \quad 0 \leq r < 1.$$

What about the other means? Can (B_p) be deduced from efficient inequalities for other values of p as well? In e remainder of this article we will answer these questions firmatively for the means I_p with $p > 2$. The approach is e same as the one adopted in the previous section: we first rive sharp coefficient estimates for the auxiliary function $(z)/z)^{p/2}$; these are provided by the coefficients of $(z)/z)^{p/2}$. Hayman, who had often raised this question at cent conferences, announced this result in the course of his cture on the Fitzgerald-Pommerenke version [4] of de Branges' oof of (B) at the One-Day Function Theory Conference in verpool in September, 1984; but gave no indication of the oof. We will show that it is a consequence of (M) and some nor adaptations of the general Milin-Lebedev inequalities, ich we proceed to state.

Let

$$(k(z)/z)^{p/2} = (1 - z)^{-p} = \sum_1^\infty d_n(p)z^n, \qquad z \in U,$$

d set

$$(f(z)/z)^{p/2} = \exp\Big(p\sum_1^\infty \gamma_n z^n\Big) = \sum_1^\infty a_n(p)z^n,$$

ere here and from now on $f \in S$.

Then (see inequality (2.37) on p. 37 of [7]) for every 1 and any $p > 0$

$$|a_n(p)| \leq d_n(p)\exp\Big(p\sum_{k=1}^n d_{n-k}(p-1)\Delta_k/2d_n(p)\Big). \qquad (ML_p(1))$$

Consider the sequence of real numbers

$$\sum_{k=1}^{n} d_{n-k}(p)\Delta_k, \quad n = 1,2, \ldots$$

These are the coefficients in the power series expansion of the product $(1-z)^{-p}\Sigma\Delta_n z^n$, and so of the product

$$(1 - z)^{-(p-1)}\sum_{1}^{\infty}(\sum_{1}^{n}\Delta_k)z^n.$$

It is easy to see that the coefficients of the first factor in the last displayed product are non-negative if $p \geq 1$. By (M), the coefficients of the second factor are non-positive. Hence

$$\sum_{k=1}^{n} d_{n-k}(p)\Delta_k \leq 0, \quad n = 1,2, \ldots$$

if $p \geq 1$.

Returning to $(ML_p(1))$, we now see that for every $n \geq 1$ and any $p \geq 2$

$$|a_n(p)| \leq d_n(p) \qquad (H_p)$$

Hence, for $0 \leq r < 1$,

$$I_p(r,f) = \int_0^{2\pi} |f(re^{i\theta})|^p d\theta = 2\pi r^p \sum_0^{\infty} |a_n(p)|^2 r^{2n}$$

$$\leq 2\pi r^p \sum_0^{\infty} (d_n(p))^2 r^{2n}$$

$$= I_p(r,k)$$

which gives (B_p) for $p \geq 2$.

The Integral Means for $1 < p < 2$

Our results for the remaining means are incomplete; to derive them, we recall another of the Milin-Lebedev inequalities (see formula (2.33) on p. 35 in [7]):

$$\sum_{k=0}^{n} |a_k(p)|^2/d_k(p) \leq d_n(p+1)\exp(p\sum_{k=1}^{n} d_{n-k}(p)\Delta_k/d_n(p+1)).$$

$(ML_p(2))$

As we observed in the previous section, the argument of the exponential on the right-hand side of this inequality is non-positive for $p \geq 1$. Hence we can infer that for every $n \geq 1$ and any $p \geq 1$

$$\sum_{k=0}^{n} |a_k(p)|^2/d_k(p) \leq d_n(p+1). \qquad (*)$$

(In passing, we note that (*) implies (H_p), for the range $1 < p < 2$, in an average sense:

$$(\sum_0^n |a_k(p)|)^2 \leq (\sum_0^n |a_k(p)|^2/d_k(p))(\sum_0^n d_k(p))$$

$$\leq (d_k(p+1))^2 = (\sum_0^n d_k(p))^2$$

i.e.

$$\sum_0^n |a_k(p)| \leq \sum_0^n d_k(p), \quad n = 0,1, \ldots).$$

As a simple consequence of (*), we deduce that

$$\sum_0^{\infty} |a_n(p)|^2 r^n/d_n(p) = (1 - r)\sum_0^{\infty}(\sum_0^n |a_k(p)|^2/d_k(p))r^n$$

$$\leq (1 - r)\sum_0^{\infty} d_n(p+1)r^n$$

$$= (1 - r)^{-p}.$$

But for $p > 1$

$$1/d_n(p) = \Gamma(n+1)\Gamma(p)/\Gamma(n+p)$$

$$= (p-1)\int_0^1 t^n(1-t)^{p-2}dt$$

and

$$I_p(r,f) = 2\pi r^p \sum_0^\infty |a_n(p)|^2 r^{2n}, \quad 0 \le r < 1.$$

Hence combining these facts we see that

$$\int_0^r I_p(t,f)(r^2-t^2)^{p-2}t^{1-p}dt = 2\pi\sum_0^\infty |a_n(p)|^2 \int_0^r t^{2n+1}(r^2-t^2)^{p-2}dt$$

$$= \pi r^{2p-2}\sum_0^\infty |a_n(p)|^2 r^{2n}/(p-1)d_n(p)$$

$$\le \pi r^{2p-2}/(p-1)(1-r^2)^p$$

$$= \int_0^r I_p(t,k)(r^2-t^2)^{p-2}t^{1-p}dt$$

if $0 \le r < 1$ and $p > 1$.

In particular, if $1 < p < 2$ and $0 \le r < 1$, then

$$\int_0^r I_p(t,f)(r^2-t^2)^{p-2}t^{1-p}dt \le \int_0^r I_p(t,k)(r^2-t^2)^{p-2}t^{1-p}dt,$$

which is the closest we can come to (B_p) for this range of p.

The Integral Means for $0 < p < 1$

Something similar holds for p in $(0,1)$. Indeed, if we utilise another of the Milin-Lebedev inequalities - this time formula (2.36) on p. 36 of [7] - we find that for any $p > 0$

$$\sum_0^\infty |a_k(p)|^2 r^{2k}/d_k(p) \le \exp(p\sum_1^\infty k|\gamma_k|^2 r^{2k})$$

$$= \exp(p(1-r^2)^2\sum_1^\infty(\sum_1^n(n+1-k)k|\gamma_k|^2)r^{2n})$$

$$\le \exp(p(1-r^2)^2\sum_1^\infty(\sum_1^n(n+1-k)/k)r^{2n})$$

$$= \exp(p\sum_1^\infty r^{2n}/n)$$

$$= (1-r^2)^{-p},$$

on using (M) again. Equality holds if and only if $f = k$, apart possibly from a rotation.

This inequality is a weak substitute for (B_p) in case $0 < p < 1$. Together with Schwarz's inequality it implies that

$$\sum_0^\infty |a_k(p)|r^k \le (1-r)^{-p}, \quad 0 \le r < 1,$$

which can be viewed as (H_p) in an average sense. We remark too that it forces

$$\sum_0^\infty |a_k(p)|^2/(k+1)d_k(p) \le 1/(1-p)$$

with equality holding only when $f = k$.

Concluding Remarks

It remains open whether (M) implies (B_p) for p in $(0,1) \cup (1,2)$. The implication would follow if the following inequality were true:

$$\sum_0^n |a_k(p)|^2 \le \sum_0^n (d_k(p))^2, \quad n = 0,1,2, \ldots$$

This holds true for $p = 1$ and for all $p \ge 2$. It is surely true for $1 < p < 2$, but I do not see how to prove it. It is even possible that it holds for the remaining values of p as well.

(When the first draft of this note was finished, Hayman very kindly sent me a copy of his joint work [5] with Hummel, in which (H_p) is also proved, for $p \geq 2$, in substantially the same way as that outlined above, the major difference being that a stronger inequality than (M), also obtained by de Branges, is used to show that the argument of the exponential in $(ML_p(1))$ is non-positive when $p \geq 2$. They mention too that Grinzpan and Aharonov have apparently made the same observation, and point out that (H_p) is false for $0 < p < 2$.)

REFERENCES

1. BAERNSTEIN II, A.
"Integral Means, Univalent Functions and Circular Symmetrization", *Acta Math.*, 133 (1974) 139-169.

2. BIEBERBACH, L.
"Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln". *S.B. preuss. Akad. Wiss.* (1916) 940-945.

3. de BRANGES, L.
"A Proof of the Bieberbach Conjecture", *Acta Math.*, 154 (1985) 137-152.

4. FITZGERALD, C.H. and POMMERENKE, Ch.
"The de Branges Theorem on Univalent Functions", *Trans. Amer. Math. Soc.*, to appear.

5. HAYMAN, W.K. and HUMMEL, J.A.
"Coefficients of Powers of Univalent Functions", preprint.

6. LITTLEWOOD, J.E.
"On Inequalities in the Theory of Functions", *Proc. London Math. Soc.*, 23 (1925) 481-519.

7. MILIN, I.M.
"Univalent Functions and Orthonormal Systems", English Translation, *Amer. Math. Soc.*, Providence, R.I. (1977).

8. ROBERTSON, M.S.
"A Remark on the Odd Schlicht Functions", *Bull. Amer. Math. Soc.*, (1936) 366-370.

Department of Mathematics,
University College,
Cork.

A NOTE ON A COMMENT OF AXLER AND SHAPIRO

Mícheál Ó Searcóid

In a recent paper of Axler and Shapiro [1], the following comment occurs:

> "Purely C* theorems should have C* proofs, not proofs that rely on Hilbert space ... . One of our favorite examples of this principle is the following proposition: If B is a C*-algebra and T ∈ B is left invertible, then T*T is invertible. This is fairly easy to prove for operators on Hilbert space (and thus we get a proof for arbitrary C*-algebras ...). A purely C* proof is more difficult (and more interesting) to discover, but once you find the purely C* proof you are likely to be convinced that it's the right proof."

Axler and Shapiro leave the carrot dangling there, and the temptation to look for a C* proof is compelling. We found two quite different ones which form the subject of the present note.

For the non-specialist, we start with two definitions and an explanation.

Firstly, the definitions: A unital Banach algebra B with involution * is called a B*-algebra if the property $||x^*x|| = ||x||^2$ holds for each $x \in B$. If H is a Hilbert space and $\mathcal{B}(H)$ is the algebra of bounded linear operators on H, and if A is a closed subalgebra of $\mathcal{B}(H)$ with the property that $T \in A$ implies $T^* \in A$, then A is called a C*-algebra.

Secondly, the explanation: That every C*-algebra is a B*-algebra is elementary. That the converse is also true, i.e. that every B*-algebra can be represented as a C*-algebra of operators on some Hilbert space, is a much more difficult theorem. Many theorems about operators on Hilbert space are automatically true for elements of B*-algebras because of this representation, and the theorem in question is one of them. We should also mention that some analysts dispense altogether with the term B*-algebra, but that we prefer to retain it when we are pretending we do not know that all B*-algebras are, in effect, C*-algebras. The task in hand, then, is to find a B* proof of the above-mentioned theorem.

Let us first examine the special case for operators on Hilbert space. A proof using polar decomposition is the only one which springs immediately to mind:

TAKE 1. Let $T \in \mathcal{B}(H)$ be left invertible.
Let $T = UP$ be the polar decomposition of T, where U is a partial isometry and P is the unique positive operator such that $P^2 = T^*T$. Since T is left invertible, so is P, and so also T*T. Being hermitian, T*T is therefore invertible.

Short and sweet, but it uses polar decomposition. The existence of a positive square root for each positive operator is usually presented as a consequence of the spectral theorem, and to invoke such a theorem in the present context is rather like using a pile driver to crack an almond. Whilst it is true that square roots and polar decomposition can be presented without reference to the spectral theorem (see Halmos [3], p.64), there is still too much work involved for our liking; in any case the proof, as it stands, will not go over to B*-algebras since, although square roots can be manufactured in them, polar decomposition is not always possible.

We soon found the following elementary operator proof:

TAKE 2. Let $T \in \mathcal{B}(H)$ be left invertible.
Then T has closed range, since ST = I and $Tx_n \to y \Rightarrow y = TSy$.
Also T* is right invertible, so T*(H) = H.
Therefore $T^*T(H) = T^*(\mathcal{N}(T^*)^{\perp}) = T^*(H) = H$ where $\mathcal{N}$

stands for null space. Whence T*T is right invertible, and, being hermitian, is invertible.

We are quite satisfied with that proof. Moreover, although it is by no means obvious, there is a way of mimicking it to produce a B* proof. (Actually, we discovered the B* argument first.) The proof requires the following B* folk lemma (see, for example, Goodearl [2], p. 148).

LEMMA A. Let A be a B*-algebra and $q = q^2 \in A$. Then there exists $p = p^* = p^2 \in A$ such that $pA = qA$.

Proof: Firstly, $q - q^*$ is normal, so that the B*-subalgebra of A generated by $q - q^*$ is commutative and can be represented as the algebra B of continuous complex functions on some compact Hausdorff space. Suppose $q - q^*$ corresponds to $g \in B$. The range of $g\bar{g}$, and therefore also $\sigma_A((q - q^*)(q^* - q))$, is contained in the non-negative real axis.
Put $x = 1 + (q - q^*)(q^* - q)$. Then $x = x^*$ is invertible.
Also $qx = qq^*q = xq$, so each of x and x^{-1} commutes with both q and q^*.
Put $p = qq^*x^{-1}$.
Then $p^* = x^{-1}qq^* = p$.
and $p^2 = x^{-1}qq^*qq^*x^{-1} = x^{-1}xqq^*x^{-1} = p$.
Also $qp = p$ and $pq = qq^*x^{-1}q = qq^*qx^{-1} = qxx^{-1} = q$.
Hence $pA = qA$.

TAKE 3. Let A be a B*-algebra and let $x \in A$ be left invertible.
Suppose $yx = 1$; then $xy = q$ is an idempotent. Let $p = p^* = p^2 \in A$ be such that $qA = pA$.
Then $xA = xyxA \subseteq xyA \subseteq xA$, whence $xA = pA$, $x = px$, and $x^*p = x^*$; so $x^*xA = x^*pA = x^*A = A$, yielding x^*x invertible.

That's easy when you see it, but it could take a while to find. Perhaps it is similar to the proof which Axler and Shapiro describe as "difficult to discover". The lemma, in particular, involves some trickery; it also involves a highly non-trivial representation theorem. Our second proof, in contrast, is surprisingly elementary, no less so than Take 2 above; neither was it difficult to discover!

Firstly, we remark that there is a well-known elementary proof of the fact that every hermitian element of a B*-algebra has real spectrum. For the sake of completeness we include it here:

LEMMA B. Let A be a B*-algebra and let $x = x^* \in A$. Then $\sigma(x) \subset R$.

Proof: Suppose $a,b \in R$ and $a + ib \in \sigma(x)$.
Then $a + ib + ic \in \sigma(x + ic)$ for every $c \in R$.
So $a^2 + (b + c)^2 = |a + ib + ic|^2 \leq ||x + ic||^2$

$$= ||(x + ic)(x - ic)|| \quad (B^* \text{ property})$$
$$= ||x^2 + c^2||$$
$$\leq ||x^2|| + c^2.$$

Hence $a^2 + b^2 + 2bc \leq ||x^2||$ for all $c \in R$.

It follows that $b = 0$.

Whereas that lemma uses the lovely classical trick of the dummy variable, our proof of the theorem uses no trick, only the elementary fact that the boundary of the set of invertible elements of a Banach algebra is contained in the set of two-sided topological divisors of zero.

TAKE 4. Let A be a B*-algebra and $x \in A$.

σ_ℓ will denote left spectrum and $\partial\sigma$ boundary of spectrum.

$0 \in \sigma(x^*x) \Rightarrow 0 \in \partial\sigma(x^*x)$ (by Lemma B)

$\Rightarrow \exists\ (z_n)_{n\in N} \subset A$, with $||z_n||=1$ $(n\in N)$

such that $x^*xz_n \to 0$.

$\Rightarrow (xz_n)^*xz_n \to 0$

$\Rightarrow xz_n \to 0$ (by B* property)

$\Rightarrow 0 \in \sigma_\ell(x)$.

Well, you couldn't get much easier than that. We are tempted to say that it's "the right proof", but there may well be a dozen others with equal claim. Who knows?

References

1. AXLER, S, and SHAPIRO, J.
"Putnam's Theorem, Alexander's Spectral Area Estimate, and VMO", *Math. Ann.*, 183 (1985) 161-183.

2. GOODEARL, K.R.
'Notes on Real and Complex C-Algebras'*, Shiva Publishing Ltd, 1982.

3. HALMOS, P.
'A Hilbert Space Problem Book', 2nd ed., Springer-Verlag, 1982.

Department of Mathematics,
University College Cork.

PLAYING THE NUMBERS GAME IN NEW YORK

Gabrielle Kelly

When I arrived in New York I was struck by the curious phenomenon of lines of people outside very small newsagents and tobacconists. It was widespread. All over Manhattan from 216th St to 14th St, from west side to east side there were snake lines of people on the pavement. They were Hispanics, blacks, whites and Chinese. They ranged from street people and bag ladies to white-coated MDs and business executives. Were they lining up to be rubber-stamped? Did I need some more identification other than my two university IDs, my social security number, my telephone and computer numbers, bank numbers and alien card number? I then saw the magic word - Lotto. Ah! The numbers game I thought. Run by the Mafia I thought. But what about those MDs? Over the next couple of days I was advised by bus drivers, at the hardware store and by my doorman to go out and buy my Lotto number. Surely, I said, not braving to reveal my ignorance. Every four-year old in New York obviously knew all about it. Strangers called to me in the street - got your Lotto number yet?

I enquired from my colleagues. A $41 million prize had accumulated and the draw was to be in two days time - August 23rd. Some of my colleagues were also buying tickets. A ststistician went on TV to declare to the public "the bigger the prize, the bigger the payoff!" I could not believe my ears. On the average I thought ..., in the long run. The front page of the *New York Times*, August 22, revealed all.

The New York Lotto 48

Select any combination of six numbers between 1 and 48.
Enter as often as you like. Minimum purchase 2 tickets.
Tickets are 50 cents each. You win:

First Prize : all your 6 numbers are drawn.

Second Prize: 5 of your 6 numbers are drawn.

Third Prize : 4 of your six numbers are drawn.

Fourth Prize: 3 of your 6 numbers are drawn and of your remaining 3 numbers one matches a supplemental number that is drawn after the 6 main numbers. The supplemental number is used only in figuring fourth prize.

Under the law that created the New York State Lottery nine years ago, 45% of the revenue from ticket sales is used to finance education, with 40% going into the prize pot. The remaining 15% goes to administering the lottery. The 40% that is prize money is split as follows: 50% for the 1st prize, 11% for the second prize, 28% for the third prize and 11% for the fourth. Prize money is split if there are joint winners. If there are no winners the money is carried over to the next drawing in the appropriate prize category. The reason for the Lotto fever this August was that there had been no first prize winners in the previous seven drawings. With the money rolled over, it was predicted that there would be a total of $47 million in prize money, with $41 million of that going to 1st prize.

To Play or Not to Play: That is the Question

There are 72 million possible combinations of 6 numbers. If I buy two tickets, my odds are 6.1 million to 1 of winning a prize.

Assuming random choice of numbers, lottery officials were predicting 6 first prize winners. Taking *only* first prize of $41 million into account shared among 6, I computed my average winnings to be approximately a dime, based on a purchase of two tickets. A dime is a positive number! A gambling game with

average payoff positive is almost unknown. But the average means the long run, i.e. repeated plays of the game. And to quote Keynes: "In the long run we are all dead." My chances, I knew, of winning first prize were 1 in 72 million. I spent my $1 on a nice cool beer. But there are many experts.

More Words to the Wise

"Your lucky number enclosed in a privacy folder", read the Lucky Green Number Machine on 42nd St. "All numbers randomly selected by a computer (experts say random is best)."

The Happy Players Club suggests numbers to play based on "probability and past pattern performance". The Lottery Advantage Newsletter (run by a former stockbroker) provides readers with a variety of charts to determine, for example, which numbers tend to win together or which numbers are hot and which are cold. Advice abounded: "Don't choose a number less than 32 - many people use birthdays and months to select their numbers and you have to share the prize." A less empirical approach was offered by the astrologers. "Luck", according to Mr Martin, a Long Island astrologer and lottery specialist who advises clients, is determined in part by the "metaphysical essence of the trine aspects at the time of birth", and people born under the Sagittarius sun sign, such as Mayor Koch and Woody Allen, tend to be naturally blessed.

If You Win ...

The State Lottery director advised: "Get a lawyer, hire an accountant, write a will, and get an unlisted telephone number."

Meanwhile ...

On August 22nd, tickets were selling at the rate of 20,000 a minute as people waited in line at 3,903 ticket outlets throughout the State. Lines started at 4.00 a.m. and some people

waited as long as three hours. One came with $1,100 to buy tickets for himself and everyone on his block. Executives of companies bought tickets for employees - a summer bonus.

In Case You Missed It:

14-17-22-23-30-47

Supplementary Number : 33

What They Won

First Prize : $13,666,666.66.
Three tickets shared $41 million.

Second Prize: $2,611.50.
578 tickets had five of the numbers.

Third Prize : $79.50.
48,052 tickets had four of the numbers.

Fourth Prize : $23.00.
65,037 tickets had three of the numbers plus the supplementary number.

Total number of ticket sales: 36 million approximately.

One first prize ticket was owned by a pool of 21 factory workers, most of them recent immigrants. Everyone was delighted with this. After taxes, they got $36,000 a year for the next 20 years.

Postscript

De Moivre, corresponding with Lord Carpenter concerning his work 'The Doctrine of Chances', wrote: "this is not to promote play but rather that people knowing the correct odds in play will become more chaste in their gambling habits". Times have not changed.

Reference

New York Times, August 22 - August 26, 1985.

School of Public Health (Biostatistics),
CUNY,
New York, N.Y. 10032.

Congratulations!

To Phil Rippon (OU), our Problem Page Editor, on being awarded a Junior Whitehead Prize by the London Mathematical Society in June 1985.

DRAWING THE LINE AT THE FOURTH DIMENSION*

There are three dimensions of space and one of time - that makes the world four dimensional, as every schoolchild knows. But two Austrian physicists have come up with an ingenious way of physically measuring the number of dimensions of space and time, and they calculate with something less than four: 3.99999947 to be exact.

Just an error? Well, not necessarily. Space-time *could* have a fractional dimension, and it could be something more or less than four. So perhaps Anton Zeilinger of the Vienna Atominstitut and Karl Svozil of the Institute for Theoretical Physics in the Technical University of Vienna are on to something.

What Zeilinger and Svozil have done is to apply the concept of "fractals" to space-time. Fractals, developed by French mathematician Benoit B. Mandelbrot, are objects like coastlines or mountain ranges whose exact lengths, or areas, or whatever, are impossible to measure in principle. Thus for example, a geographer might pace out the arc of a bay as 1,000 metres. But suppose he were to measure round every rock — the length would be greater - say 1,500 metres. And then around every grain -- greater still; and so on.

Similar effects occur in pure mathematics. For example a line can be defined from A to B in this way: first draw a zig-zag from A to B. Then on each straight portion of the zig-zag, replace the straight line by a miniature version of the zig-zag. Now you have a zig-zag zig-zag. Take each miniature straight portion of the zig-zag zig-zag and replace it by a tiny zig-zag. As this process is continued *ad infinitum*, the result - which is surely composed of lines, and so one-

Reprinted from the *Guardian*, 1 August 1985, with the permission of the editor.

dimensional, begins to fill in a whole area between A and B. But areas are two-dimensional. So is the infinitely zig-zagged line "really" one- or two-dimensional?

To solve questions such as that, mathematicians sometimes redefine terms - and here an early 20th century mathematician, Felix Hausdorff, provided the redefinition. If you measure the "length" (the sum of little lengths in the line) of an infinitely zig-zagged line, Hausdorff argued, the answer will be infinity. And if you measure its "area" (the sum of little areas in the line), the answer will be zero - as mathematical lines are infinitely thin.

But what if you measure something in between length and area - a fractional power, between 1 and 2, of the distance between points in the line, Hausdorff asked. Now, Hausdorff showed, the answer comes out to be neither infinite nor zero, but finite, for one precise power of distance. It is this power that enables a measure of the "size" of the set of points (just as length and area measure the size of more familiar sets). The value of the power required to produce a finite result is now called the "Hausdorff dimension" of a set.

Fractals are then lines, surfaces or what have you that twist and turn so much their Hausdorff dimensional is fractional.

So Zeilinger and Svozil asked a simple but profound question: what really is the Hausdorff dimension of space-time? It could be more than four, if the collection of points which we can reach with our particles - accessible space-time - twists and turns in some other space in which we are embedded (just as the zig-zag line twists on the page). This is not unreasonable to suppose, for particle physicists are now being driven to consider 10 and 12-dimensional spaces, in order to explain the properties of the elementary particles, of which our familiar space-time is only a part.

Or, the Hausdorff dimension of space-time could be less than four, if we are embedded in strictly 4-dimensional space-time, all the points of that space-time are not accessible to us. Werner Heisenberg, for example, had the idea that space-time might be granular - that there might be steps in space and time.

But how to measure the Hausdorff dimension of the world? Zeilinger and Svozil have an answer to that. In the world of high-energy particle physics, particles are spread out in the "wave-particles" of quantum mechanics - and as such they probe every cranny of space-time. By reinterpreting certain very precise measurements on the magnetic moment of the electron as estimates of the Hausdorff dimension, Zeilinger and Svozil come out with their value of 3.99999947.

This is so close to 4 that it may be questioned whether the result is real. But it may just indicate the beginnings of the detection of granularity in space-time. For the measurements are limited by the quantum "wavelength" of the particles used. Experimentally, the waves cannot penetrate "crannies" or discontinuities shorter than their wavelength. And wavelength decreases with energy. The energy of the magnetic moment experiment was low, and so was perhaps like the geographer pacing the coastline with a long step.

The requirement must be now for higher energy tests (though the higher the energy, the more difficult precise measurements become). Ideally, the tests should be done at energies approaching the levels where theorists expect the 10 or 12 dimensional twists and turns to appear. Sadly, however, such energy regions are astronomically high and probably inaccessible, and we may have to content ourselves with Zeilinger and Svozil's 3.99999947.

Reference: *Phys. Rev. Lett.*, Vol. 54 P, 2553 (1985).

TEACHING NOTES

EIGENVECTORS AND DIAGONALIZATION

P.D. Barry

Can we motivate or encounter naturally eigenvectors of matrices, especially as occurring in the diagonalization of a quadratic form, or must they be produced apparently out of the blue?

When deriving the equation of an ellipse E in standard form we select the axes of coordinates to be along the major and minor axes of the ellipse. Thus if

$$ax^2 + 2hxy + by^2 = 1 \qquad \text{(i)}$$

represents an ellipse with centre at the origin O, to convert it to standard form we need to map its major and minor axes onto the x' and y' axes. If $P_0 = (x_0, y_0) \neq O$ is a point on either the major or the minor axis, we then seek a linear transformation

$$\begin{aligned} x &= a_{1,1}x' + a_{1,2}y' \\ y &= a_{2,1}x' + a_{2,2}y' \end{aligned} \qquad \text{(ii)}$$

which maps the line OP_0 to the x'-axis; in matrix form (ii) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} . \qquad \text{(iii)}$$

As then

$$\begin{aligned} x_0 &= a_{1,1}x_0' \\ y_0 &= a_{2,1}x_0' \end{aligned}$$

for some $x_0' \neq 0$, we have

$$a_{1,1} = jx_0, \quad a_{2,1} = jy_0$$

for some $j \neq 0$. This gives us the two coefficients in the first column of the two-by-two transformation matrix in (iii).

We note too that if $Q_0 = (-y_0, x_0)$, then OQ_0 is perpendicular to OP_0 and so will lie along the other axis of the ellipse. Thus if our linear transformation is to preserve perpendicularity, as a rotation or an axial symmetry would, we will have the line OQ_0 mapping to the y'-axis. Then

$$-y_0 = a_{1,2}y_0'$$
$$x_0 = a_{2,2}y_0'$$

for some $y_0' \neq 0$ and so we have

$$a_{1,2} = -ky_0, \quad a_{2,2} = kx_0$$

for some $k \neq 0$. This gives us the two coefficients in the second column of the transformation matrix in (iii), which thus has been shown to have the form

$$\begin{pmatrix} jx_0 & -ky_0 \\ jy_0 & kx_0 \end{pmatrix}. \qquad \text{(iv)}$$

It still remains to locate P_0, and we recall the manifest property of a point at the end of a diameter of E that it maximizes or minimizes the distance of a point P on E from the centre O. We thus seek a point at which $x^2 + y^2$ is maximized or minimized, subject to the condition (i). Recalling the method of Lagrange multipliers, we wish to locate a point $(x,y) = (x_0,y_0) \neq (0,0)$ at which

$$F(x,y) = x^2 + y^2 + \mu(ax^2 + 2hxy + by^2 - 1)$$

satisfies

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0.$$

This gives

$$\mu(ax + hy) + x = 0,$$
$$\mu(hx + by) + y = 0,$$

that is,

$$ax + hy = \lambda x,$$
$$hx + by = \lambda y,$$

where $\lambda = -1/\mu$, at the point $(x,y) = (x_0,y_0) \neq (0,0)$. Thus

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is an eigenvector of the matrix

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix}, \qquad \text{(v)}$$

corresponding to the eigenvalue λ. This eigenvector can now be inserted in the first column in (iv) and so in the first column in the transformation matrix in (iii), and a similar conclusion for

$$\begin{pmatrix} -y_0 \\ x_0 \end{pmatrix}$$

gives the second column in (iv) and (iii).

We have hitherto presented this as converting the expression on the left-hand side of (i) to standard form. In matrix notation it can be written as

$$(x,y)\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and this exhibits the matrix (v) for which we have been encountering eigenvectors. Our work can also be presented as diagonalizing this matrix.

The above algebra and geometry can surely be asked of anyone at this level of linear algebra, but perhaps Lagrange multipliers seem a late topic in some unappetising course on calculus. Supposing more elementarily that E has been parameter-

ized, with variable t, to find P we need

$$\tfrac{1}{2}\frac{d}{dt}(x^2 + y^2) = x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \qquad \text{(vi)}$$

where, on differentiating (i),

$$(ax + hy)\frac{dx}{dt} + (hx + by)\frac{dy}{dt} = 0. \qquad \text{(vii)}$$

Now (vi) and (vii) have a non-trivial solution in dx/dt, dy/dt if

$$\det \begin{pmatrix} ax + hy & hx + by \\ x & y \end{pmatrix} = 0,$$

which is a condition that

$$\begin{aligned} 1(ax + hy) - \lambda x &= 0 \\ 1(hx + by) - \lambda y &= 0 \end{aligned} \qquad \text{(viii)}$$

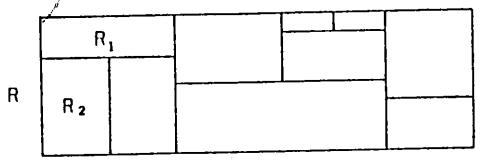
have a solution in λ. In (vi) and (vii) we are looking for a solution (x,y) which is not (0,0). Then (viii) brings in the eigenvectors of (v), so with some loss of immediacy we can omit Lagrange multipliers and still reach our objective.

Department of Mathematics,
University College,
Cork.

PROBLEM PAGE

The first problem this time is 'going around' at the moment. I heard it from several different people within a period of a week, and it has a remarkable solution.

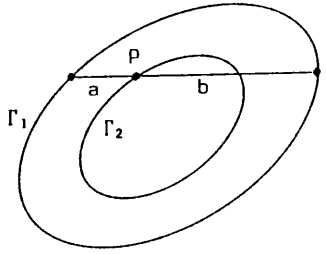
1. A rectangle R is partitioned into a finite number of rectangles $R_1, R_2, \ldots, R_n$, each of which has the property that at least one side is of integer length.



Prove that R has the same property.

The next problem came from Jim Clunie who learnt it from Tom Willmore.

2. A rod moves so that its endpoints lie on a convex curve Γ_1 in $\mathbb{R}^2$ and a point P, which divides the rod into lengths a and b, then describes a closed curve Γ_2.



Prove that the region lying between Γ_1 and Γ_2 has area πab.

Now for the solutions to two earlier problems.

1. Let A_1, A_2, A_3, A_4 be 3x3 complex matrices and let

$$M = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_4 & A_1 & A_2 & A_3 \\ A_3 & A_4 & A_1 & A_2 \\ A_2 & A_3 & A_4 & A_1 \end{bmatrix}.$$

Express det M as the product of four 3x3 determinants.

This problem was sent in by Finbarr Holland and also solved elegantly by Allan Solomon as follows.

Let

$$M(\lambda) = A_1 + \lambda A_2 + \lambda^2 A_3 + \lambda^3 A_4, \qquad \lambda \in C.$$

Then

$$\det M = \prod_{\lambda \in \Phi} \det M(\lambda), \text{ where } \Phi = \{1, i, -1, -i\}.$$

To see this, note that

$$M = I \otimes A_1 + T \otimes A_2 + T^2 \otimes A_3 + T^3 \otimes A_4,$$

where

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

and ' $\otimes$ ' denotes the tensor product.

The matrix T generates $\mathbb{Z}_4$ and has eigenvalues 1, i, -1, -i. Thus there is a 4x4 matrix R_0 such that

$$R_0 T R_0^{-1} = \text{diag}\{1, i, -1, -i\} = D,$$

say, and so if $R = R_0 \otimes I$ then

$$RMR^{-1} = I \otimes A_1 + D \otimes A_2 + D^2 \otimes A_3 + D^3 \otimes A_4 = \begin{bmatrix} M(1) & 0 & 0 & 0 \\ 0 & M(i) & 0 & 0 \\ 0 & 0 & M(-1) & 0 \\ 0 & 0 & 0 & M(-i) \end{bmatrix}.$$

Hence

$$\det M = \det RMR^{-1} = \prod_{\lambda \in \Phi} \det M(\lambda),$$

as required.

Allan points out that this generalises to

$$M = \sum_{n=1}^{N} T^{n-1} \otimes A_n,$$

where T generates $\mathbb{Z}_N$. Also, if $\{A_i\}$ generates a Lie algebra $\mathcal{G}$, then $\{T^k \otimes A : k \in \mathbb{Z}, A \in \mathcal{G}\}$ generates a graded Lie algebra. In an article in 'Group Theoretical Methods in Theoretical Physics' (academic Press 1977), he employed this algebra (with $\mathcal{G} = SU(2)$) to give a new solution to the Ising model on a cyclic lattice of N points.

2. Prove or give a counterexample to the following statement:

if $a_n \geq 0$, for $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} a_n < \infty$ then

$$\sum_{n=3}^{\infty} a_n^{(1 - \frac{1}{\log n})} < \infty.$$

In fact this statement is true. Indeed, for any term a_n such that $n \geq e^4$ and

$$a_n \leq \frac{1}{n^2},$$

we have

$$a_n^{(1 - \frac{1}{\log n})} \leq (\tfrac{1}{n^2})^{(1 - \frac{1}{\log n})} \leq \frac{1}{n^{3/2}}.$$

On the other hand, if

$$a_n \geq \frac{1}{n^2},$$

then

$$a_n^{-\frac{1}{\log n}} \leq (n^2)^{\frac{1}{\log n}} = e^2,$$

and so

$$a_n^{1 - \frac{1}{\log n}} \leq e^2 a_n.$$

Since Σa_n and $\Sigma \frac{1}{n^{3/2}}$ are both convergent, the desired result follows.

Phil Rippon

BOOKS RECEIVED

"NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS. COLLÈGE DE FRANCE SEMINAR VOLUME VII"

Edited by *H. Brezis* and *J.L. Lions*

Published by *Pitman Publishing*, London, 1985, 292 pp.
Stg £16.50. ISBN 0-273-08679-0

This book contains the texts of selected lectures delivered at a weekly seminar held at the Collège de France. It includes contributions by leading experts from various centres on recent results in nonlinear functional analysis and partial differential equations. The emphasis is laid on applications to numerous fields including control theory, theoretical physics, fluid mechanics, free boundary value problems, dynamical systems, numerical analysis and engineering. The book will be of particular interest to postgraduate students and specialists in these areas.

"ENNIO DE GIORGI COLLOQUIUM"

Edited by *Paul Krée*

Published by *Pitman Publishing*, London, 1985, 169 pp.
Stg £14.50. ISBN 0-273-08680-4

This research note includes sixteen papers reporting mathematical research in France and Italy related to the work of Ennio de Giorgi.

In July 1983, Professor Ennio de Giorgi was awarded the title 'Doctor Honoris Causa' by the Council of the Université de Paris VI. The very profound and influential nature of his

work meant that this award had a considerable impact on the international scientific community, particularly in France and Italy. As a result, a French-Italian colloquium on de Giorgi's work was held at the H. Poincaré Institute in November 1983.

De Giorgi's work was concentrated mainly on six areas: evolution problems, minimal surfaces, regularity of solution of second order partial differential equations, analytic solutions of partial differential equations with constant coefficients, Gamma-convergence theory and connected problems, and hyperbolic equations with discontinuous coefficients with respect to the time variable.

BOOK REVIEWS

"STATISTICAL METHODS IN RESEARCH AND PRODUCTION" (4th Edn)

Edited by *Owen L. Davies* and *Peter L. Goldsmith*

Published by *Longman*, 1984, xii + 478 pp. £10.95 (paperback)

"In Japanese firms, statistics is everywhere .....";
"... statistical charts at loading docks and throughout the factory floor";
"Last year Japanese Industry carried out over 1 million statistical experiments";
"Japan has an annual National Statistics Day";
"Train-schedules in Tokyo railway stations are stem and leaf displays";
"American industry is now investing in massive training and education programs in Statistics, Quality Control and Reliability Methods".

- from a presentation by Blanton Godfrey (Bell Labs) at the 1984 American Statistical Association meeting in Philadelphia.

Most people nowadays accept that Japanese consumer goods are of high quality, and much of this progress in quality has been attributed to the work of W. Edwards Deming, the American statistician who has been a consultant to Japanese industry since 1945. The statistical methods that have been of such value to Japanese industry are essentially developments of the basic methods presented so competently in this book.

The object of the book, stated in the opening lines of the introduction, is "to facilitate a better understanding and a wider use of statistical methods by staff engaged in Research and Production, particularly in the chemical industry". The book originated as a 'company book' written by a team of authors

within Imperical Chemical Industries, the first edition appearing in 1947. Revised and enlarged editions appeared in 1949 and 1957. The current edition (available in paper covers) is essentially the fourth edition published in 1972, but with corrections made and some references updated. This book is excellent, and should be available to every engineer working in production or research in industry. Because of its practical orientation, the book will continue to be used as a methods textbook by Statistics students.

The Chapter headings are:

1. Introduction;
2. Frequency Distributions;
3. Averages and Measures of Dispersion;
4. Statistical Inference;
5. Statistical Tests; Choosing the Number of Observations;
6. Analysis of Variance;
7. Linear Relationships between Two Variables;
8. Multiple and Curvilinear Regression;
9. Frequency Data and Contingency Tables;
10. Control Charts;
11. Sampling and Specifications;
12. Simulation.

I found many attractive features in the book:

* An excellent introduction to Significance testing (Sections 4.5 - 4.8) which covers the t-test, sign test, rank-sum test and F-test in just six pages, and still manages to find space for such sensible comments as: "... it is wrong to associate different levels of statistical significance with different levels of practical importance".

* A useful explanation of the conclusions to be drawn in Multiple Comparisons (pp. 150 - 153).

* A good discussion of the distinction between Functional and Regression Relationships (p. 181).

* A treatment of Pure Error and Lack of Fit when fitting a Regression with replicated observations (Section 7.32).

* A good introduction to Multiple Linear Regression (Chapter 8) which begins with a realistic example rather than a definition of the Regression model, and goes on to discuss Model Selection, Transformation of Variables, Analysis of Residuals, Curvilinear Regression and Nonlinear Estimation. Not surprisingly, this is the longest chapter, and in fact Chapters 6, 7 and 8 account for over one-third of the book.

* A chapter on Simulation, which although it is quite short, will by its inclusion in a Statistical Methods book underline the importance of Simulation.

* A useful collection of statistical tables together with examples illustrating their use.

* Excellent layout and print.

It is virtually impossible to find nothing to complain about in a book such as this, and my complaints are:

- The diagrammatic representation of confidence limits (Fig. 4.1) is not helpful.

- A rather skimpy coverage (half page) of Dummy Variables in Regression (p. 274).

- In claiming that the sum of ψ^2 variates is also ψ^2, the requirement of independence is not mentioned (p. 318).

- In updating the references, the opportunity of referring to the considerable advances in the Analysis of Frequency Data was not grasped.

- There is the occasional instance of awkward phrasing, such as the following on the construction of Control Charts: "It is also important to take measurements at those points in the process at which any action consequent on points falling outside the control limits would rectify the trouble in the shortest possible time" (p. 341).

However, these shortcomings are minor when measured against the overall excellence of the book.

Patrick D, Bourke,
University College, Cork.

"GENERAL RELATIVITY"

By *Hans Stephani*

Published by *Cambridge University Press*, in paperback, 1985, £13.95 stg. ISBN 0 521 31534 4

This is a translation from the German of a book published in 1977 and republished with amendments and corrections in 1980. One of the two translators (John Stewart) adds comments in several places as footnotes. The bibliography has been brought up to date as far as 1980/81.

Professor Stephani's book is one of the clearest text-books on 'mainstream' general relativity I have seen in a very long time. It is beautifully balanced between the mathematics and the physics of the subject. Although there are only 298

pages of text and it may seem that vast areas of research of the last 30 years have been omitted, nevertheless the reader who wishes to pursue any topic further is given three tiers of references at the end of each section: (i) Text-books, (ii) Monographs and Collected Works, and (iii) Review and Research Articles. Because of this I found I could read the book at a very leisurely pace. Many familiar things struck me with the force they had when I first learned them.

The book is divided into 30 large sections which are grouped into 8 chapters.

After an introductory chapter on Newtonian Mechanics that includes Lagrange's equations and a study of relative accelerations, all expressed in tensor notation, there is a chapter on Riemannian and Semi-Riemannian Geometry. In this chapter "Foundations of Riemannian Geometry" there is a very compact treatment of tensor algebra which treats symmetries of tensors as well as tetrad and spinor components of tensors and then comes the Lie derivative, parallel transport, Fermi-Walker transport, curvature and integral conservation laws. This chapter ends with electrodynamics in geometrical optics, thermodynamics and finally perfect fluids.

Chapters 3 and 4 study Einstein's field equations, the observational tests of the theory, the linearized theory of gravitation and gravitational waves. Einstein's quadruple formula for gravitational radiation is derived. Chapter 5, a mere 40 pages, gives all the techniques that are used to classify the exact solutions of Einstein's field equations. In view of all the research that has been done in this area it is a masterpiece of compression.

In chapters 6 and 7 on Blackholes, Gravitational Collapse and Cosmology, I was disappointed at the few pages given to the Hawking-Penrose singularity theorem, but the reader gets ample compensation in the very readable accounts of spherically symmetric stars and the Schwarzschild black hole.

Chapter 8, called "Non-Einsteinian Theories of Gravitation", deals with topics of current interest for which Professor Stephani could find no room earlier. Possible tests of alternatives to Einstein's theory of gravitation, the PPN formalism and quantum gravity. This last chapter is not up to the standard of the others because the author seems to me to be very unenthusiastic about the subject matter. Section 30 on quantum gravity is very skeptical about the outcome of current research.

An excellent course of lectures could be created from this book for either an honours degree course or M.Sc. It has everything the potential research student ought to know!

Patrick Dolan,
Department of Mathematics,
Imperial College,
London, SW7 2BZ.

"MATHEMATICAL BYWAYS IN AYLING, BEELING AND CEILING"

By *Hugh ApSiman*

Published by *Oxford University Press*, 1984, xii + 97 pp.
Stg £5.95. ISBN 019-853-201-6

This is the first book of a new series called Recreations in Mathematics from O.U.P. The series is aimed at 'all lovers of mathematics' and will include not only new titles, but also translations and reprints of classics.

Mathematical Byways is a book of problems. The problems themselves are of a familiar type, many being concerned with integer-sided triangles and optimising strategies, but the format of the book is rather unusual. Each of the chapters is introduced by a specific problem featuring some aspect of rural life in the three villages of Ayling, Beeling and Ceiling. For example, in Chapter 1 we meet Farmer Able's pretty daughter and are asked to work out how far from the ground is her window-sill if a ladder of length 18 ft 5 ins will just reach it when there is a packing case with cross-section 5 ft x 5 ft directly below the window. Later problems involve, for example, the areas of sheep pens, strategies for sheepdog trials and the shortest possible road system which connects the three villages.

Each of these specific problems is followed directly by the statement of a more general problem, which is solved in detail. The solution to the original problem is then given and each chapter ends with remarks about the 'composer's problem' and possible 'extension problems'. In this way each of the eleven problems in the book is given a very thorough treatment which should certainly be accessible, as the author claims, to a 'properly taught sixteen year old".

In fact the author's solutions are in some ways almost too good! Many readers will not, I fear, try to solve the specific problems, but turn immediately to the general problems (which are sometimes easier to understand) and so proceed inevitably to their solutions. With a more conventional style of format this might have been avoided, and also many more problems of a similar standard could have been included, without reducing the depth of their treatment. Most of the extension problems given look decidedly difficult, though one at least can be settled by a short computer search.

To sum up, this is an interesting book, well illustrated and with only a few misprints, and I enjoyed the treatment of the problems given. However I couldn't help recalling how much more is to be found in the books of puzzles and games by Martin Gardner or, at a more advanced level, in Donald Newman's splendid 'Problem Seminar'.

P.J. Rippon, Faculty of Maths, O.U.

CONFERENCE REPORT

IRISH MECHANICS GROUP CONFERENCE

A two-day conference, sponsored by the National Committee for Theoretical and Applied Mechanics and the Department of Mathematical Physics, U.C.C., on "Current Research in Mechanics" was held at U.C.C. on 25/26 July. Twenty-five Mechanicians and Applied Mathematicians from Ireland, the United States, the United Kingdom and Canada attended. The main speakers were Professor J.E. Fitzgerald, Georgia Tech. and Professor M.M. Carroll, Berkeley. The fourteen lectures, in order of presentation, were as follows:

"Recent Developments in Non-Linear Materials", Professor J.E. Fitzgerald, Georgia Tech.

"Stability in Thermoelasticity", Dr N.H. Scott, East Anglia.

"Edge-Function Methods: Completeness and Convergence", Dr J.J. Grannell, U.C.C.

"Dynamics of Buried Pipelines", Dr P. O'Leary, U.C.G., and Dr S.K. Datta, Boulder (presented by Dr O'Leary).

"Free Vibrations of Orthotropic Plates Using the Edge-Function Method", Dr R.P. Studdert and Dr M.J.A. O'Callaghan, U.C.C. (presented by Dr Studdert).

"On the Wiener-Hopf-Hilbert Method of Matrix Factorization", Dr A.D. Rawlins, Brunel.

"Criterion for Failure in Certain Composites", Dr M.H. Quinlan, U.C.C.

"Bivectors and Jayvectors", Professor M.A. Hayes, U.C.D.

"Some Spatial Decay Estimates in Continuum Dynamics", Professor J.N. Flavin, U.C.G.

"Developments in Rock Mechanics", Professor M.M. Carroll, Berkeley.

"The Edge-Function Method for Fracture Mechanics", Professor P.M. Quinlan, U.C.C.

"The Evolution of Resonant Oscillations", Professor M.P. Mortell, U.C.C. and Mr E.A. Cox, U.C.D. (presented by Professor Mortell).

"T-Matrices for Scattering from Discs", Professor M.F. McCarthy, U.C.G.

"Numerical Modelling of Hydraulic Fracture Propagation", Professor J. Curran, Toronto.

J.J. Grannell,
Mathematical Physics Department,
U.C.C.

THE IRISH MATHEMATICAL SOCIETY

Instructions to Authors

The Irish Mathematical Society seeks articles of mathematical interest for inclusion in the *Newsletter*. All areas of mathematics are welcome, pure and applied, old and new.

In order to facilitate the editorial staff in the compilation of the *Newsletter*, authors are requested to comply with the following instructions when preparing their manuscripts.

1. Manuscripts should be typed on A4-size paper and double-spaced.

2. Pages of the manuscripts should be numbered.

3. Commencement of paragraphs should be clearly indicated, preferably by indenting the first line.

4. Facilities are available for italics and bold-face type, apart from the usual mathematical symbols and Greek letters.

   Words or phrases to be printed in italics should be singly underlined in the manuscript; those to be printed in bold-face type should be indicated by a wavy underline.

5. Diagrams should be prepared on separate sheets of paper (A4-size) in black ink. Two copies of all diagrams should be submitted: the original without lettering, and a copy with lettering attached.

6. Authors should send two copies of their manuscript and keep a third copy as protection against possible loss.

If the above instructions are not adhered to, correct reproduction of a manuscript cannot be guaranteed.

Correspondence relating to the *Newsletter* should be addressed to:

Irish Mathematical Society Newsletter,
Department of Mathematics,
University College,
Cork.