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FREE TOPOLOGICAL GROUPS

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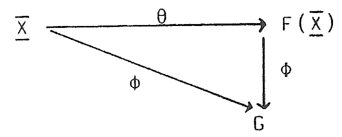
The purpose of this paper is to provide a brief expository sketch of [1].

If $\underline{X}$ is any set, the free group $F(\underline{X})$ is defined abstractly as follows: $F(\underline{X})$ is a group such that if G is any group and if $\phi : \underline{X} \to G$ is any map of $\underline{X}$ into G then there is a homomorphism $\Phi : F(\underline{X}) \to G$ so that the diagram below commutes:



(*)

The embedding θ is fixed and is independent of ϕ and of G.

The existence of $F(\underline{X})$ is assured by the construction described next.

A *word* is a finite sequence $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$ in which x_i is an element of $\underline{X}$ and each $\varepsilon_i = \pm 1$. The product of two words $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ and $y_1^{\delta_1} \dots y_m^{\delta_m}$ is the word $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} y_1^{\delta_1} \dots y_m^{\delta_m}$. The collection W of all words is thus an associative semigroup. The subsemigroup S generated by all words of the form $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ in which $x_1 = x_2 = \dots = x_n$ and

$$\sum_{i=1}^{n} \varepsilon_i = 0$$

leads to the quotient structure W/S, a group $F(\underline{X})$ for which $x_n^{-\varepsilon_n} \dots x_1^{-\varepsilon_1}$ is a representative of the inverse of the element represented by $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$.

If $\underline{X}$ is a topological space, the natural object corresponding to $F(\underline{X})$ is a topological group for which the same diagram (*) obtains and where θ is a fixed topological embedding, G is a topological group and ϕ, Φ are continuous. Since a topological group Γ is completely regular (if x is a point and U is an open set containing x there is a continuous map $f : \Gamma \to [0,1]$ such that $f(x) = 1$, $f(\Gamma \setminus U) = 0$) $\underline{X}$ must be completely regular. The questions of existence and uniqueness are answered by the following theorem which is proved independently in [3] and [4] and can be proved in still another way [1] as outlined below.

__Theorem.__ *If $\underline{X}$ is completely regular space there is a topological group $F(\underline{X})$ a topological embedding $\theta : \underline{X} \to F(\underline{X})$ such that for ϕ a continuous map into the topological group G the diagram (*) commutes and Φ is a continuous homomorphism.*

The proof consists of a number of simple steps:

1. Let $\mathbb{H}_1$ be the group of all quaternions of norm 1.

2. Let $C(\underline{X}, \mathbb{H}_1)$ be the set of all continuous maps $f : \underline{X} \to \mathbb{H}_1$.

3. For each f in $C(\underline{X}, \mathbb{H}_1)$ let $\mathbb{H}_{1f}$ be $\mathbb{H}_1$ and let $\mathcal{H}$ be the Cartesian product $\prod_f \mathbb{H}_{1f}$.

Then $\mathcal{H}$ is a compact topological group and the map $\theta : \underline{X} \to \{f(x)\}$(the *index* is f) is a topological embedding of $\underline{X}$ into $\mathcal{H}$ (here the complete regularity enters for the first time)

4. The group $\mathbb{H}_1$ and the group $\mathcal{R}$ of rotations of $\mathbb{R}^3$ are isomorphic (according to the correspondence: for R in $\mathcal{R}$ and (x,y,z) in $\mathbb{R}^3$ let $xi + yj + zk$ be the corresponding quaternion; then the quaternion corresponding to $R(x,y,z)$ is $g(xi + yj + zk)g'$ for a unique g in $\mathbb{H}_1$ (g' = conjugate of g).

5. The group $\mathcal{R}$ and hence $\mathbb{H}_1$ contains an infinite set $\{g_1, g_2, \ldots\}$ that is *free* [2]. That is, if $g_1^{\varepsilon_1} \ldots g_n^{\varepsilon_n} = 1$ then the word $g_1^{\varepsilon_1} \ldots g_n^{\varepsilon_n}$ is in the subsemigroup S of the semigroup W of words generated by the set $\{g_1, g_2, \ldots\}$.

6. If $\{p_k\}_{k=1}^{n}$ is a set of n different points in $\underline{X}$ there is in $C(\underline{X}, \mathbb{H}_1)$ an f such that $f(p_k) = g_k$. (Here the complete regularity of $\underline{X}$ enters again.)

Thus the group $F_0(\underline{X})$ generated in $\mathcal{H}$ by $\theta(\underline{X})$ is a free and topological group ($F_0(\underline{X})$ (for the *set* $\underline{X}$) in the topology inherited from $\mathcal{H}$). Thus $\mathcal{T}$, the set of all *group topologies* on $F(\underline{X})$, is not empty. The topology $\sup(\mathcal{T})$ makes $F(\underline{X})$ a topological group described in the conclusion of the theorem stated above.

Remarks. 1. The nub of the proof is found in 5 and 6 above. The existence of *some* group topology on $F(\underline{X})$ permits the conclusion (*via* $\sup(\mathcal{T})$) that $F(\underline{X})$ may be topologized to conform to the diagram (*).

2. If $\underline{X}$ is set, $F(\underline{X})$ is unique and if $\underline{X}$ is a completely regular space $F(\underline{X})$ is unique.

3. A topological group G is *free* by definition if whenever $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} = 1$ then the word $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ is in the subsemigroup S in the semigroup W of words constructed from the *set* G. Thus $F_0(\underline{X})$ is a topological group and is free but does not, *a priori*, conform to the requirements of (*) if $\underline{X}$ is a completely regular space. Indeed, if $G = F(\underline{X})$, *the* free topological group, then for (*) to be valid $F(\underline{X})$ must indeed be endowed with the topology $\sup(\mathcal{T})$.

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HYPERBOLIC BEHAVIOUR OF GEODESIC FLOWS*

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INTRODUCTION

Geodesic flows, particularly those on manifolds of negative curvature, have been a rich source in the determination and display of possible types of macroscopic behaviour of motions in dynamical systems. Their study goes back to Hadamard and Poincare who considered the existence of periodic geodesics on some classes of surfaces. Later, in the 1930s Hedlund, Hopf and Morse studied the topological and ergodic properties of the flows on compact surfaces of negative curvature [H]. Already, they recognised the special role of the local instability of trajectories and proved that this was closely linked with the statistical (ergodic) behaviour of the flows.

One of the ways of expressing this local instability is the hyperbolic behaviour of the derivative of the flow. The central idea is that close to any fixed trajectory, the behaviour of neighbouring trajectories resembles the behaviour of trajectories in the neighbourhood of a saddle point singularity. Anosov [A] was the first to give an explicit formulation of hyperbolicity. He then used this condition as a basic assumption to study a class of dynamical systems which are now referred to as *Anosov systems*. The geodesic flow on compact manifolds of negative curvature is a very important example of these flows.

The conditions formulated by Anosov in 1967 are the strongest type of hyperbolic conditions. In 1977, Pesin [P] formulated a weaker set of hyperbolic conditions and studied the dynamical systems satisfying these conditions. Again, the geodesic flows on a class of manifolds without focal points

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