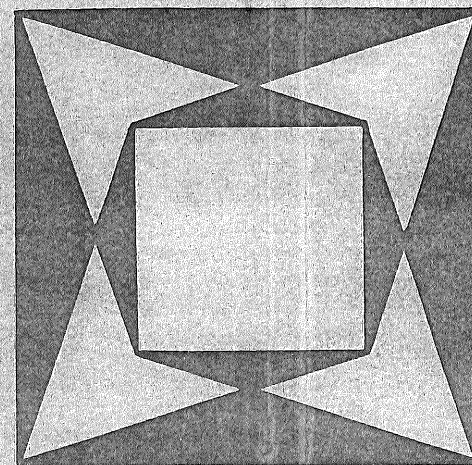


IRISH MATHEMATICAL SOCIETY NEWSLETTER

IRISH MATHEMATICAL SOCIETY



NEWSLETTER

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THE IRISH MATHEMATICAL SOCIETY

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NEWSLETTER

EDITOR

Donal Hurley

The aim of the *Newsletter* is to inform Society members about the activities of the Society and also about items of general mathematical interest. It appears three times each year: March, September and December. Deadline for copy is six weeks prior to publication date.

ASSOCIATE EDITOR

Patrick Fitzpatrick

The *Newsletter* also seeks articles of mathematical interest written in and expository manner. All parts of mathematics are welcome, pure and applied, old and new.

Manuscripts should be typewritten and double-spaced on A4 paper. Authors should send two copies and keep one copy as protection against possible loss. Prepare illustrations carefully on separate sheets of paper in black ink, the original without lettering and a copy with lettering added.

Correspondence relating to the *Newsletter* should be sent to:

Irish Mathematical Society Newsletter,
Department of Mathematics,
University College,
Cork.

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EDITORIAL

This is the final issue of the Newsletter in my term as Editor and I should like to avail of the opportunity to make some comments on the development of the Newsletter since I took over the job.

As you are aware, I've encouraged authors to write in an expository fashion so that a wide range of the readership can easily understand and benefit from reading the articles. There is always a danger that mathematicians will confine their expositions (both written and oral) to a narrow audience of colleagues with similar interests in narrow fields. This tendency runs counter to the development of mathematics which depends very much on the cross fertilization of ideas and techniques from one area to another. If we are interested in our work, it is good to convey this enthusiastically to people working in other fields and so stimulate interaction between various fields of specialization. The specialist research journals and seminars provide opportunities to communicate with colleagues having similar interests; this Newsletter helps mathematicians to reach out to a wider group. To do this, it is clear that the exposition should be less technical and less demanding of background knowledge on those reading. The difficulty involved in presentations of this type is obvious: just how much or how little can one assume? But I do believe that the effort involved in overcoming the difficulty is beneficial to both the presenter and to the readers. The former is awakened to the context of his work within the wider field of all mathematics, while those who read the article gain insight into other areas of mathematics and also perhaps get some ideas about possible solutions to problems in their own areas.

During my term as Editor, I've considered the Newsletter to be a publication for exposition of the type outlined above.

To conclude, I wish to thank all those who contributed articles during the past three years. A special word of thanks is due to those whose assistance was invaluable: Pat Fitzpatrick (Associate Editor), Phil Rippon (Problem Page) and Leslie Brookes (Typist).

Donal Hurley

IRISH MATHEMATICAL SOCIETY

Notice of Ordinary Meeting

Thursday, April 4th, 1985, 12:15 p.m.

at

Dublin Institute for Advanced Studies

Minutes of Ordinary Meeting

Friday, December 21st, 1984

at

Dublin Institute for Advanced Studies

The meeting commenced at 12.15 p.m. with Professor A.G. O'Farrell in the chair. The minutes of the previous meeting were taken as read.

Matters Arising

The question of the post-graduate awards was regarded as a dead letter. No progress had been made in relation to the Young Scientists Award. The Massera Campaign was over. M. Clancy reported that some progress in the talks with the IMTA had occurred. It was proposed that reciprocal membership without voting rights might be obtained at a cost of £1.50. The IMTA still had some misgivings and would consider this proposal at a delegates meeting. This and any subsequent proposals will be considered by the new Committee.

Secretary's Report

This was accepted on a proposal from S. Tobin, seconded

by M. Newell.

Treasurer's Report

This report was circulated and further details were given by G. Enright. The report was accepted on a proposal by F. Holland, seconded by M. Newell. The chairman, Professor O'Farrell, complimented the Treasurer on the orderly fashion in which he had handled the affairs of the Society.

The membership list was circulated and some minor and one major - the omission of NUU from the Institutional list - mistakes were noted. Local representatives were asked to inform the Treasurer, G. Enright, of any changes of address of members. It was noted that a 12% increase in ordinary membership had occurred.

It was proposed that the subscription be increased to £5 per annum to take effect from January 1986. It was hoped that this rate could be held for a significant period of time. S. Tobin was unhappy with an increase just for increase sake and some discussion took place regarding the need for increased reserves. It was hoped that more conferences could be supported in the future. The proposal was formally put by T.T. West, seconded by F. Holland and was accepted. The overseas rate for libraries will be increased *pro rata*.

Orlov and Shcharansky

The basic situation in relation to Orlov and Shcharansky was outlined in the current Newsletter. T.T. West proposed that the Society should ask for their release. This was agreed.

Changes to Constitution and Rules

(i) It was proposed to simplify the procedure of election to membership. This would now be confirmed by the Committee.

- (ii) It was proposed to replace the word "session" by the word "term" in the rule governing the period of consecutive holding of office. This would result in offices being allowed to be held for 3x2 year periods rather than 3x1 year periods as at present.
- (iii) That members 18 or more months in arrears be deemed to have resigned.
- (iv) That the first sentence of paragraph 5 of the rules be deleted.

Items (i) - (iv) were unanimously accepted.

[Text of revised Constitution at the end of this report - Ed.]

Ratification of Members

This was formally proposed by G. Enright and unanimously accepted. It was pointed out by Dr Enright that this would no longer be necessary in view of (i) above. He further suggested, and it was agreed, that the revised Constitution and Rules be published.

Elections

The following were elected unopposed:

| <u>Position</u> | <u>Member</u> | <u>Proposer</u> | <u>Seconder</u> |
|--------------------|----------------|-----------------|-----------------|
| President | M. Newell | A.G. O'Farrell | F. Holland |
| Vice-President | S. Dineen | A.G. O'Farrell | F. Holland |
| Secretary (1 year) | A.G. O'Farrell | G. Enright | S. Dineen |
| Committee Member | T.T. West | F. Holland | S. Dineen |
| | F.G. Gaines | P. Boland | S. Dineen |
| | N. Buttimore | S. Tobin | F. Holland |
| | B. Goldsmith | D. Hurley | P. Boland |

Other Business

- (a) T. Murphy reported that it was hoped to run a conference on the Mathematics of Theoretical Computing.
- (b) There would be a joint meeting with the LMS on Operator Theory in Dublin. Easter 1986 was a possible date.
- (c) It was agreed to query the possibility of reciprocal membership arrangements with the Institute of Mathematics and its Applications.

IRISH MATHEMATICAL SOCIETY

CONSTITUTION

1. The Irish Mathematical Society shall consist of Ordinary and Honorary Members.
2. Any person may apply to the Treasurer for membership by paying one year's membership fee. His admission to membership must then be confirmed by the Committee of the Society. Candidates for honorary membership may be nominated by the Committee only, following a proposal of at least three members of the Society. Nominations for honorary membership must be made at one Ordinary Meeting of the Society and voted upon at the next, a simple majority of the members present being necessary for election.
3. Every Ordinary member shall pay subscription to the funds of the Society at the times and of the amounts specified in the Rules.
4. The Office-Bearers shall consist of a President, a Vice-President, the Secretary, a Treasurer. The office of President or Vice-President may be held in conjunction with any of the other offices.
5. The Committee shall consist of the President, the Vice-President, the Secretary, the Treasurer, and eight additional members. No person may serve as an additional member for more than three years consecutively.
6. There shall normally be at least 2 ordinary meetings in a session.
7. Notice of a motion to repeal or alter part of the Constitution shall be given at one Ordinary Meeting. Written notice of one month shall be given to all members before the next Ordinary Meeting at which the motion shall be voted upon, being carried if it receives the consent of two-thirds of the members present.
8. One month's written notice of a motion to repeal or alter a Rule, or to enact a new Rule, shall be given to all members before the meeting at which it is to be voted upon, the motion being carried if it receives the consent of a simple majority of the members present.
9. All questions not otherwise provided for in the Constitution and Rules shall be decided by a simple majority of members present at a Meeting. Eleven ordinary members shall form a quorum for such business.

RULES

These rules shall be subject to the over-riding authority of the Constitution.

SUBSCRIPTIONS

1. Every Ordinary Member shall pay, on election to membership and during January in each succeeding session, an annual subscription to be determined by the Committee. A change in the annual subscription shall be ratified by a Meeting of the Society.
2. Ordinary members whose subscriptions are more than eighteen months in arrears shall be deemed to have resigned from the Society.

OFFICERS AND COMMITTEE

3. The election of the Office-Bearers and the additional members of Committee shall take place at the first Ordinary Meeting of each session.
4. The term of office of the Office-Bearers and of the Committee shall be two years.
5. On alternate years elections for the following positions will take place (a) President, Vice-President and half of the additional members of the Committee (b) Secretary, Treasurer and one half of the additional members of the Committee.
6. Each session shall commence on the 1st day of October and last until the following 30th of September.
7. The Committee shall meet at least twice during each session, the President to be convenor. Five shall form a quorum.
8. A Secretary shall keep minutes of the Meetings of the Society and of the Committee and shall issue notices of meetings to members resident in Ireland.
9. At the first Ordinary Meeting of each session the Treasurer shall submit a Financial Statement for the previous session, duly audited by two persons appointed by the Committee.

MEMBERSHIP LIST SUPPLEMENT 85-1

14th January 1985

ADDITIONS

Institutional Members

The New University of Ulster (Coleraine)
Dublin Institute of Technology (Kevin St)

Maynooth Students

Therese Shore, Kieran Flanagan, Jacqueline Maher

U.C.D. Students

Henry McLoughlin, Nada Couzi, Annette Pilkington

N.I.H.E. Limerick

Dr M. Burke, Dr M.A. Rahman,
Mr J. Buckley, Mr A. Hegarty

Other Ordinary Members

Mr R. Jordan, Regional Technical College, Carlow
Dr C. Thompson, The University of Southampton, England

AMENDMENTS

Dr J.W. Bruce (ex UCC), Dept of Math., The University, Newcastle-upon-Tyne, England.

Dr P. Rippon (ex UCC), Faculty of Math., Open University, Milton Keynes, England.

Dr E. O'Riordan (ex Waterford), Regional Technical College, Dundalk.

Dr M. Stynes (ex Waterford), Maths Dept, University College Cork.

Dr J. Stynes (ex Waterford), Regional Technical College, Cork.

Mr M. Brennan (ex Athlone), Regional Technical College,
Waterford.

Dr M. Klimek (ex TCD), Maths Dept, University College Dublin.

NEWS AND ANNOUNCEMENTS

ADVANCES IN LINEAR PROGRAMMING

Linear programming is perhaps the most important mathematical technique in use today, at least if importance is judged by any economic or utilitarian measure. By some estimates nearly one-fourth of the scientific computation time of all the computers in the world is devoted to solving linear programming problems. Efficient solutions to these problems can save industry millions of pounds each month. Modern economics and management science depend very much on solutions to linear programming problems.

The first method of solving these problems was devised during World War II by George Danzig (now of Stanford University) in an attempt to resolve the logistical problems of maintaining steady supplies to distant troops subject to the constraints of wartime scarcities. This method is called the simplex method and since its development, enormous economic benefits resulted from its use. In the mid-1950s, the Exxon Corporation used it to improve the blending of petroleum products and saved 2% to 3% of the cost of its blending operations. The application soon spread within the petroleum industry and at the same time other industries began to adopt the method. Today, "packages" of computer programs based on the simplex algorithm are offered commercially to customers who pay sizeable fees for their use.

The algorithm relies on two key ideas: that the solution must be one of the vertices of the polytope of feasible points, and that the sure way to find it is to climb steadily uphill (or downhill) along the edges. The number of vertices is finite, but even in a routine problem, it can be enormous. It is estimated that for the problem of allocating 2,000 limited resources to 2,000 products, the number of vertices of the polytope is of the order of 2^{2000} or 10^{602} . Yet the

simplex algorithm can generally find the optimum solution by examining only about 6,000 of the vertices. In theory, the simplex algorithm is what computer scientists call a "hard" problem, that is, the computations required grow at a rate which depends exponentially on the number of constraints. However, practitioners have found that a good rule of thumb is that the number of calculations increases as a linear function of the constraints.

In the late 1970s L.G. Khachian, working at the computer centre of the Soviet Academy of Sciences, developed an algorithm which proved that linear programming is not really a "hard" problem and provided an alternative computation that can be used in those cases where the simplex algorithm proves too slow. But there were difficulties with this method. Since matrix inverses need to be calculated in the Khachian algorithm, roundoff errors are introduced at an alarming rate, and it is possible that the intrinsic computer error will grow so rapidly that the algorithm will not converge, yielding nothing but nonsense in the end.

Stephen Smale of the University of California at Berkeley demonstrated in 1981 that there is an upper bound to the expected number of vertices that must be checked by the simplex algorithm. His bound is a function of the "size" of the problem, where the "size" is defined as the sum of the number of constraints and the number of items being manufactured/allocated. As the problems become larger Smale's upper bound grows more slowly. Hence, although he has not fully explained why the simplex algorithm has performed as well as it has, he has shown that for extremely large problems, the expected number of vertices investigated is even smaller than the number given by a rule of thumb widely used by mathematicians. His bound is not a guarantee that the simplex algorithm will always work rapidly: the problem is a probabilistic one, although it should apply to the great majority of cases. For an account of Smale's work see [2].

When Danzig invented his algorithm, he reportedly told his colleagues not to worry about its slowness as he believed a more efficient method of solving resource-allocation problems would soon be forthcoming. Recently his hopes were realized. A new algorithm for solving linear programming problems has been developed and it is apparently much faster than the simplex method [1]. The algorithm is due to a 28 year-old Indian-born mathematician, Narendra Karmakar, who works for Bell Laboratories in New Jersey. Unlike the simplex method, Karmakar's algorithm deforms the polytope of feasible solutions as it proceeds, but unfortunately no technical details are available yet in the research literature. Mathematicians are keen to see and analyse the details of the algorithm and several large corporations such as Exxon, American Airlines and A. T. & T. wish to use it in their scheduling and allocation problems.

REFERENCES

1. ANGIER, N.
"Folding the Perfect Corner", *Time*, December 3rd, 1984, Science Column, page 55.
2. GAINES, F.
"Recent Developments in Linear Programming", *Irish Mathematical Society Newsletter*, No. 7 (March 1983), 29-35.

D. Hurley

SUPERBRAIN 1985

The Superbrain Competition is an annual mathematical examination open to all full-time students of University College, Cork, regardless of whether they are students of mathematics or not. The competition arose in 1984 as a result of a challenge from students of Electrical Engineering who claimed that because of their high points entry requirements, they were the best mathematicians in College. The topics of the examination are roughly those of the Honours Leaving Certificate in Mathematics so as not to give an advantage to students taking more advanced courses. However, the level of difficulty is sometimes considerably greater. The examination was set and corrected by Dr D. MacHale of U.C.C.

In its first year, the competition was won by an exceptional fourth year Science student, Stephen Buckley, recent winner of the Travelling Studentship in Mathematical Science, who is now pursuing a Ph.D. degree in mathematics at the University of Chicago. However, the next eight places in the order of merit were filled by students of Electrical Engineering, so they claimed a moral victory. This year the number of entrants was down from 44 to 30 and with the paper probably just a little less difficult than last year, the standard was a good deal higher. Interestingly, the winning mark was almost exactly the same as last year.

The competition was again a triumph for the Electrical Engineers who filled nine of the top ten places. The 1985 Superbrain is James Cunnane (EE4) (second last year) with a score of 71. In second place was Richard Kavanagh (EE,PG) with a score of 63. Third was Patrick Gaffney (EE2) with 61, while John O'Connell (Sc 2) with 57 took fourth place.

This year's contest was a straight fight between students of Electrical Engineering and Science, because students of Civil Engineering, Arts, Commerce and Medicine did not take part. Also, only two girls risked their mathematical reput-

ations. Are these facts signs of the times?

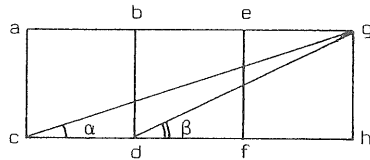
Fittingly, prizes for the top three were donated by Arthur Guinness and Co., in whose products many of the participants have a keen interest.

D. MacHale

Answer all *ten* questions

Answer all *ten* questions

- 6.



7. Given a triangle abc , show how to find a point p such that $|p,a|^2 + |p,b|^2 + |p,c|^2$ is as small as possible.
8. What is the maximum and the minimum number of "Friday the thirteenths" that can occur in any calendar year?
9. By considering

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx,$$

show that $\pi < \frac{22}{7}$.

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PERSONAL ITEMS

Professor B.R. Gelbaum of the State University of New York at Buffalo is a Fulbright Senior Scholar visiting the Mathematics Department of University College, Galway during the period January 1st to April 30th 1985. Professor Gelbaum's interests are in Boolean Algebra, Probability Theory and Stochastic Processes.

Dr T. Hurley of the Mathematics Department, University College, Galway is on leave of absence this academic year at the Australian National University In Canberra.

Professor D. McQuillan of the Mathematics Department, University College, Dublin has been recently elected Dean of the Faculty of Arts at U.C.D.

Dr I.G. O'Muircheartaigh of the Mathematics Department, University College, Galway has been promoted to the rank of Associate Professor of Statistics (within the Department of Mathematics).

**** STOP PRESS ****

Des MacHale's biography "*George Boole: his life and work*" has just appeared. This happy event has caused no little rejoicing in the MacHale household, but there seems to be some confusion: "Why did he write all that stuff about a footballer anyway!?"

LETTER TO THE EDITOR

Stichting Mathematisch Centrum,
Centrum voor Wiskunde en Informatica,
Kruislaan 413,
10985J Amsterdam.

26 November 1984

Dear Sir/Madam,

We should be greatly obliged if you would place a notice in the Bulletin for your members about the project on "grey literature" initiated by the European Mathematical Council.

As the need for greater accessibility of "grey literature" on mathematics is widely felt, it has been proposed by the European Mathematical Council that all European mathematical institutes will send one copy of each of their *informal* publications, such as reports, theses, proceedings of meetings, etc., but not reprints, to the Centrum voor Wiskunde en Informatica (CWI) in Amsterdam. Lists of titles of publications received as part of this project are published by CWI and distributed among the participants.

In order to make these lists as useful a tool as possible, publications are listed by subject. Participants are therefore urgently requested to add, where necessary, classification codes according to the 1980 Mathematics Subject Classification scheme of *Mathematical Reviews* and *Zentralblatt*, as well as an English title and abstract, so that eventually KWOT-indexes may be produced.

A great many institutes in Europe are already taking part in this project. As so far response from institutes in your country has not been very great, we hope to draw their interest to this project through your association. Our correct mailing address for reports, etc., is:

Centrum voor Wisjunde en Informatica,
Library,
P.O. Box 4079,
1009 AB Amsterdam,
The Netherlands.

Thank you very much in advance for your cooperation.

Yours sincerely,

Ms Joke Sterringa

Library

CURVE SMOOTHING USING SPLINES

Don Barry

1. INTRODUCTION

Consider a set of data (x_i, y_i) , $i = 1, 2, \dots, n$ with $0 \leq x_1 < x_2 < \dots < x_n \leq 1$ and $y_i = F(x_i) + e_i$, $i = 1, 2, \dots, n$ where F is a well behaved function of x and the errors $\{e_i\}$ are independently and identically distributed each with mean zero and variance v . F is known as the regression function of Y on X and its estimation from a finite set of observations is one of the central problems in statistics.

The usual parametric approach to regression estimation assumes F to lie in $\text{span}\{\phi_j : 1 \leq j \leq m\}$, the set of linear combinations of the basis functions $\phi_1, \phi_2, \dots, \phi_m$ and then estimates F by the function \hat{F} in $\text{span}\{\phi_j : 1 \leq j \leq m\}$ which minimises the residual sum of squares given by

$$\text{Res. SS} = \sum_{i=1}^n (y_i - \hat{F}(x_i))^2.$$

Classical polynomial regression uses as basis functions

$$\phi_j(x) = x^{j-1}, \quad j = 1, 2, \dots, m.$$

We wish to choose \hat{F} in a larger class of functions containing $\text{span}\{\phi_j : 1 \leq j \leq m\}$ as a subset. We minimise the Res SS plus a penalty corresponding to a measure of the distance of F from $\text{span}\{\phi_j : 1 \leq j \leq m\}$. For example we might choose $\hat{F} \in H^{(2)}$ to minimise

$$\sum_{i=1}^n (y_i - g(x_i))^2 + c \int_0^1 g''(x)^2 dx$$

where $H^{(2)} = \{g : [0, 1] \rightarrow \mathbb{R} \mid g, g' \text{ are absolutely continuous and } \int_0^1 g''(x)^2 dx < \infty\}$. The presence of the penalty imposes smoothness on the estimator. If we cannot make any smoothness ass-

umptions regarding F , then y_i contains information about $F(x_i)$ and none about $F(x)$ for $x \neq x_i$. This makes estimation of F impossible. The constant c is chosen by the user and controls the trade-off between roughness as measured by $\int_0^1 g''(x)^2 dx$ and fidelity to the data as measured by

$$\sum_{i=1}^n (y_i - g(x_i))^2.$$

In what follows we describe the use of roughness penalties in more detail, examine the relationship between choice of penalty and choice of Bayesian analysis, briefly describe the large sample properties of such estimators and finally consider a method for using the data to guide the choice of c .

2. POLYNOMIAL SMOOTHING SPLINES

Consider choosing a function g to minimise

$$\sum_{i=1}^n (y_i - g(x_i))^2 + c \int_0^1 g^{(m)}(x)^2 dx \quad 2.1$$

A unique solution to this minimisation problem exists in the space

$$H^{(m)} = \{g : [0,1] \rightarrow \mathbb{R} \mid g, g', \dots, g^{(m-1)} \text{ are}$$

$$\text{absolutely continuous and } \int_0^1 g^{(m)}(x)^2 dx < \infty\}$$

and we choose our regression estimate \hat{F} to be that solution.

Schoenberg (1964) has shown that \hat{F} lies in the linear space S_m of polynomial splines of degree $2m-1$. S_m consists of all functions $g : [0,1] \rightarrow \mathbb{R}$ such that

- (i) g is a polynomial of degree $2m-1$ in each interval $[x_i, x_{i+1}]$, $i = 1, 2, \dots, n-1$.
- (ii) g is a polynomial of degree $m-1$ in the intervals $[0, x_1]$, $[x_n, 1]$.

(iii) g is continuously differentiable up to order $2m-2$.

It can be shown that given a_1, a_2, \dots, a_n there exists one and only one function $s \in S_m$ such that

$$s(x_i) = a_i \quad i = 1, 2, \dots, n \quad 2.2$$

Let σ_i denote the only element of S_m satisfying

$$\sigma_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

It is easy to see that $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ is a basis for S_m and using this basis the element s of 2.2 is given by

$$s(x) = \sum_{i=1}^n a_i \sigma_i(x).$$

See Rice (1969) Chapter 10 for details.

Using this basis we can rewrite the minimisation problem 2.1 as: choose a_1, a_2, \dots, a_n to minimise

$$\sum_{i=1}^n (y_i - \sum_{r=1}^n a_r \sigma_r(x_i))^2 + c \sum_{r,s} a_r a_s w_{rs} \quad 2.3$$

where $w_{rs} = \int_0^1 \sigma_r^{(m)}(x) \sigma_s^{(m)}(x) dx$. This is now a finite problem and the optimal values for a_1, a_2, \dots, a_n are precisely the values $\hat{F}(x_1), \dots, \hat{F}(x_n)$. We can write 2.3 as

$$(\underline{y} - \underline{\hat{F}})^T (\underline{y} - \underline{\hat{F}}) + c \underline{\hat{F}}^T \underline{\Omega} \underline{\hat{F}}$$

where

$$\underline{y} = (y_1, y_2, \dots, y_n)^T,$$

$$\underline{\hat{F}} = (\hat{F}(x_1), \hat{F}(x_2), \dots, \hat{F}(x_n))^T,$$

$$\underline{\Omega} = (w_{rs}), \quad r, s = 1, 2, \dots, n.$$

The minimising value for \hat{F} is $\underline{\hat{F}} = A \underline{y}$, where $A = (I + c \underline{\Omega})^{-1}$.

The value of c must be chosen by the user and is of vital importance. For $c = \infty$ we must have $\int_0^1 F^{(m)}(x)^2 dx = 0$ which, together with the absolute continuity requirements, implies that F must be a polynomial of degree $m-1$, indeed the usual least squares polynomial of degree $m-1$. For $c = 0$ we can make 2.1 equal to zero by choosing $\hat{F} \in S_m$ satisfying

$$\hat{F}(x_i) = y_i$$

Intermediate values for c involve a trade-off between the smoothness of the estimate as measured by $\int_0^1 F^{(m)}(x)^2 dx$ and the fidelity to the data as measured by $\sum (y_i - \hat{F}(x_i))^2$.

Figure 1 shows some data generated by adding normal errors to the function

$$F(x) = K_1 x^{10} (1-x)^6 + K_2 x^3 (1-x)^{10}$$

where K_1, K_2 are positive constants. The 50 x -values are equally spaced in $[0,1]$. Figure 2 shows the fitted curve obtained using the $m = 1$ roughness penalty and a value for c chosen by cross validation (see Section 5). The true curve is shown in Figure 3 and indicates that the estimate in this case behaves very well.

Figures 4-7 refer to some real data concerning computer repair times. Here X is the number of units to be repaired and Y is the length of the call in minutes. The figures show the fitted curve obtained using $m = 2$ as the value for c increases from 0 to ∞ . It can clearly be seen how increasing c causes the estimate to become smoother and to follow the data less closely.

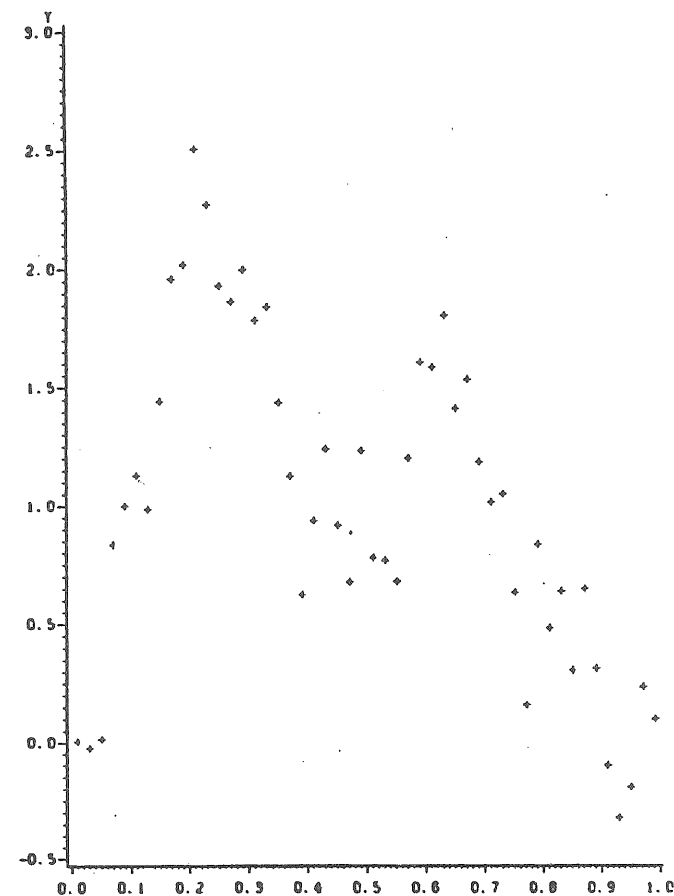


FIGURE 1: Raw Data

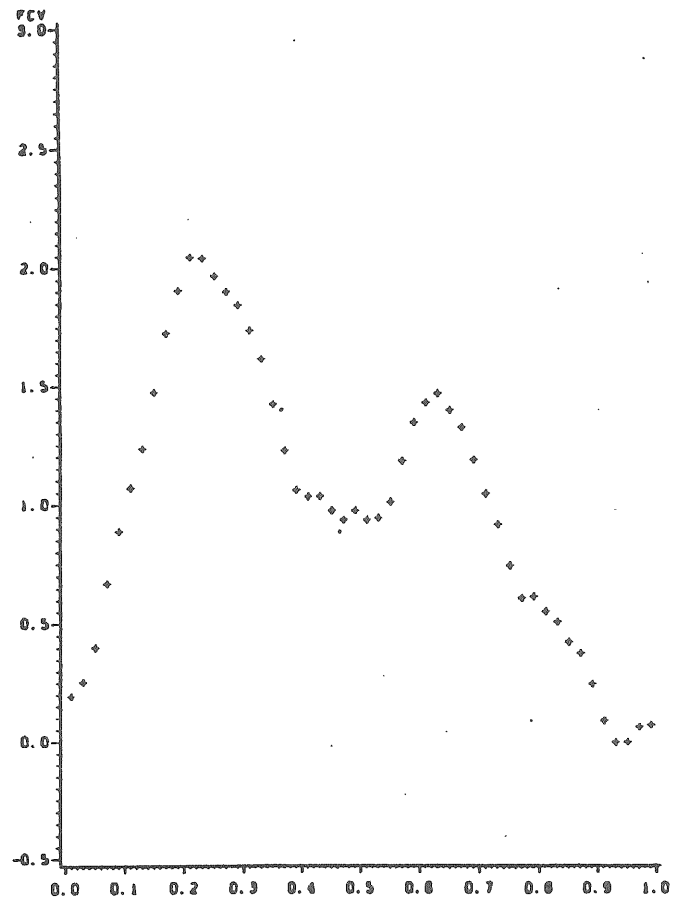


FIGURE 2: Fitted Values

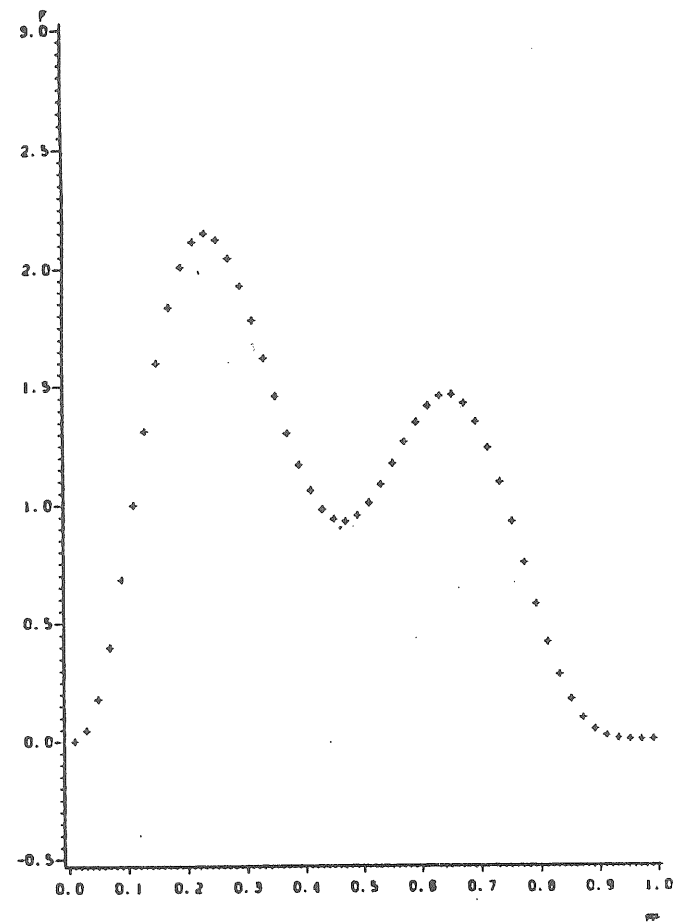


FIGURE 3: $F(x) = k_1 x^{10}(1-x)^6 + k_2 x^3(1-x)^{10}$

$H = 2$

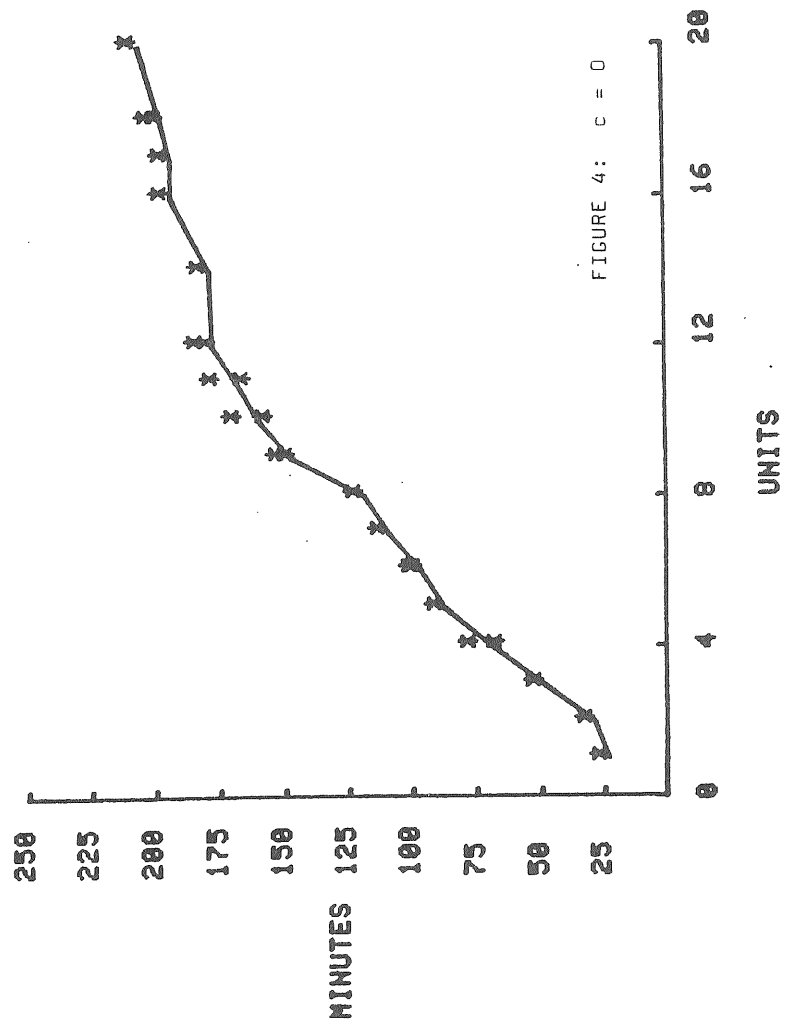


FIGURE 4: $c = 0$

$H = 2$

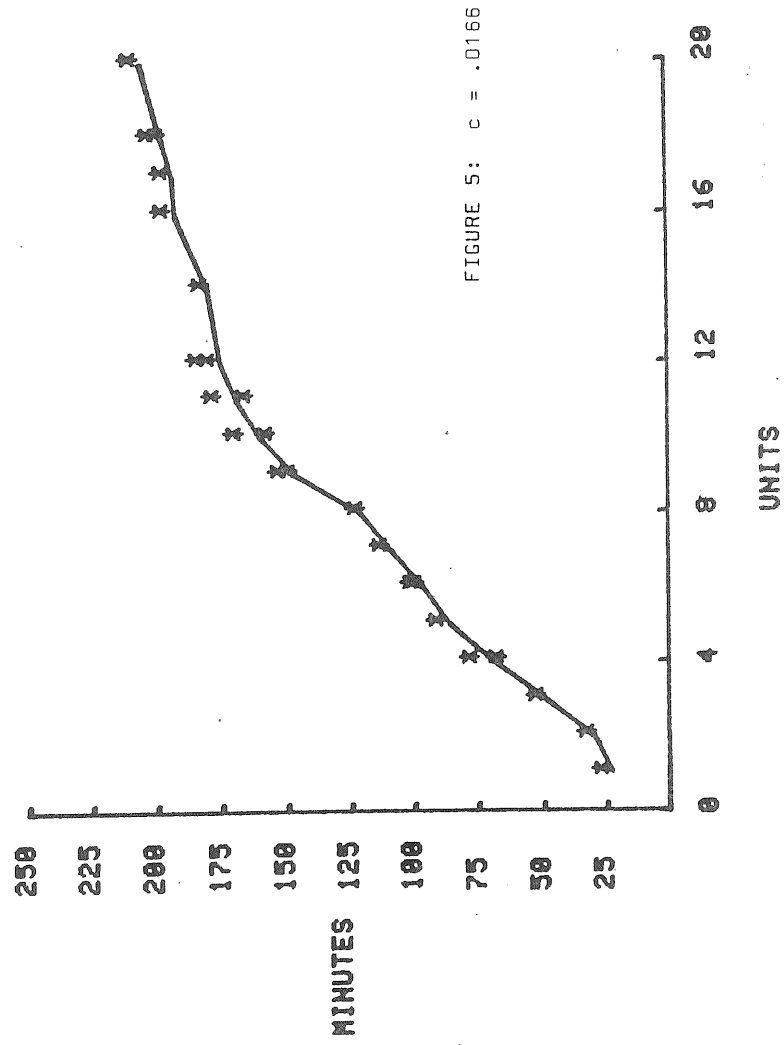


FIGURE 5: $c = .0166$

M = 2

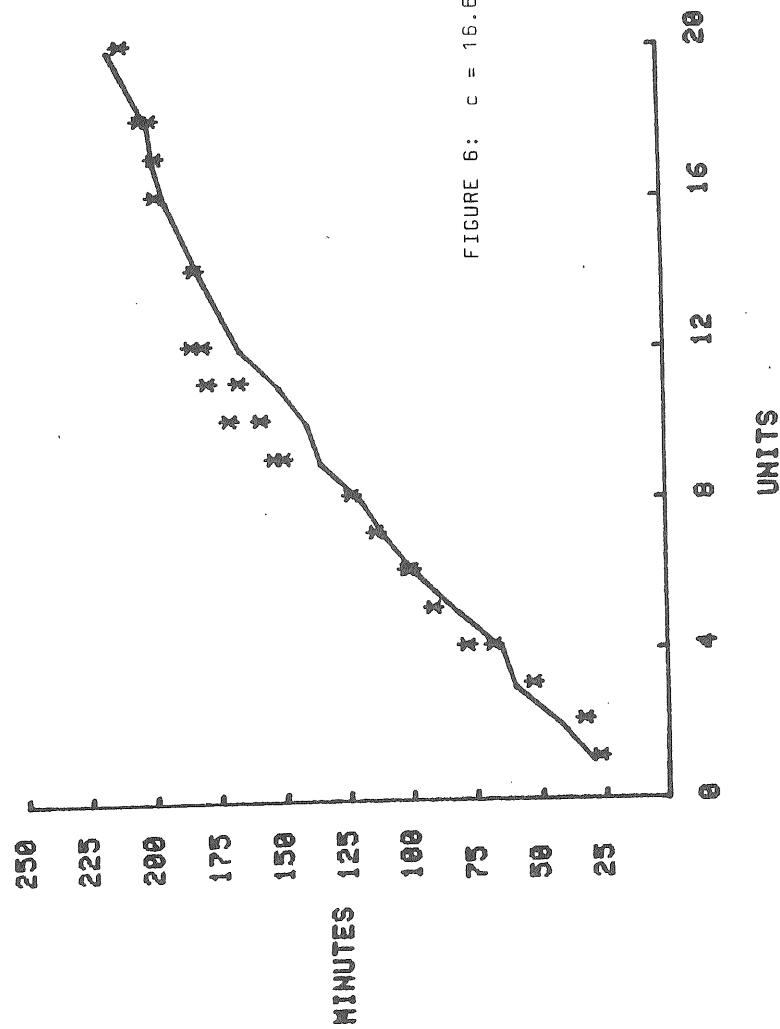


FIGURE 6: $c = 16.6$

M = 2

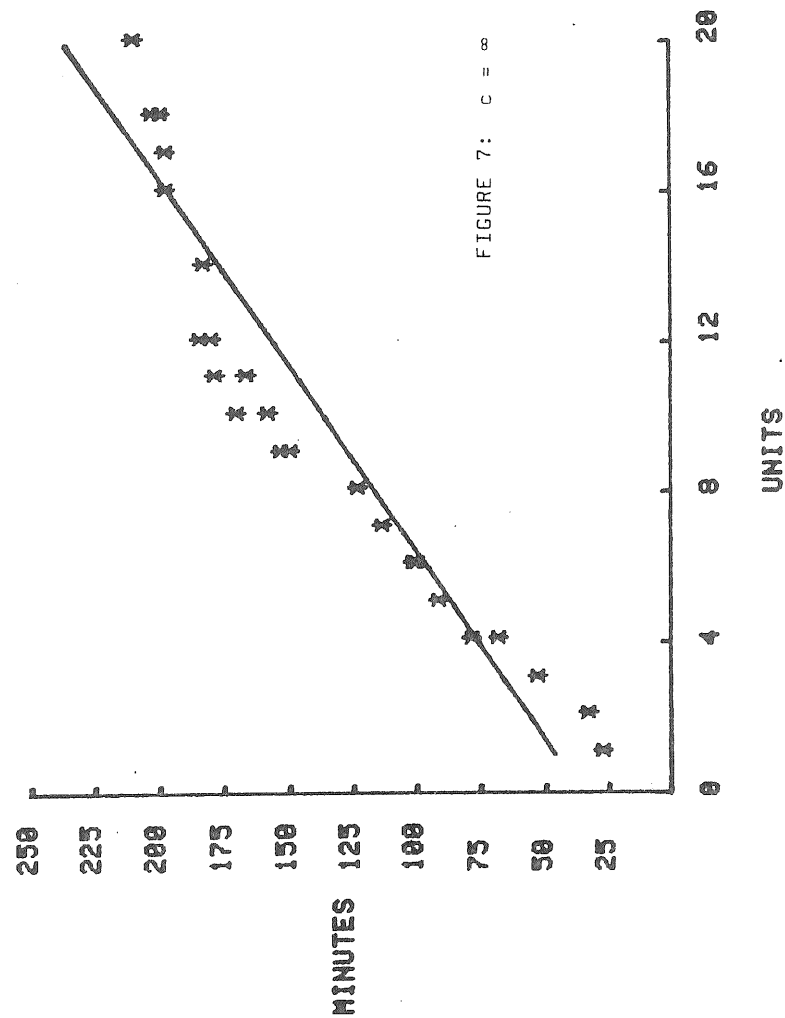


FIGURE 7: $c = 8$

3. THE BAYESIAN CONNECTION

Wahba (1978) proved the following theorem:

Theorem

Assume that $y_i = F(x_i) + e_i$, $i = 1, 2, \dots, n$ where $\{e_i\}$ are independent and identically distributed as $N(0, v)$. Let the prior distribution of $F(x)$, $x \in [0, 1]$ be that of the stochastic process

$$\sum_{j=1}^m \theta_j x^{j-1} + v_1^{\frac{1}{2}} Z(x)$$

where $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_m) \sim N[0, v_0 I]$, $v_1 > 0$ is fixed and $Z(x)$ is an m -fold integrated Wiener process

$$Z(x) = \int_0^1 \frac{(x-u)^{m-1}}{(m-1)!} dW(u)$$

where W is a Brownian motion (see Shepp, 1966). Then $\hat{F}(x; c)$ the minimiser of 2.1 has the property that

$$\hat{F}(x; c) = \lim_{v_0 \rightarrow \infty} E[F(x) \mid y_1, y_2, \dots, y_n]$$

with $v_1 = v/c$ where E is expectation over the posterior distribution of $F(x)$ generated by the above probability model.

Thus the choices of m and c are closely related to the choice of Bayesian prior.

4. ASYMPTOTIC PROPERTIES

Let $\hat{F}(x; c)$ be the estimator corresponding to a particular choice of c . Define

$$R(c) = \sum_{i=1}^n (\hat{F}(x_i; c) - F(x_i))^2$$

i.e. the sum of squared errors if c is used as a smoothing constant. $R(c)$ is a random variable since different y -values produce a different function $\hat{F}(x; c)$ and hence a different

value for $R(c)$. The following theorem shows that if we allow c to increase with n , but not too quickly then $ER(c) \rightarrow 0$ at a fast rate, where expectation is with respect to the normal distribution on the errors.

Theorem

$$ER(c) \leq c \int_0^1 F^{(m)}(x)^2 dx + K(n/c)^{\frac{1}{2m}}$$

where

$$K = v[n \cdot \max(x_{i+1} - x_i)]^{\frac{1}{2m}} \int_0^\infty \frac{dx}{(1+x^{2m})^2}$$

and v is the error variance.

Proof

See Wahba (1978).

Corollary

For $c = O(n^{\frac{1}{2m+1}})$ we have $ER(c) = O(n^{\frac{1}{2m+1}})$.

This is to be compared with $ER(0) = O(n)$ (since $E(y_i - F(x_i))^2 = v$), and shows clearly the benefit of smoothing.

NOTE:

If $\int_0^1 F^{(m)}(x)^2 dx = 0$ then $ER(\infty) = O(1)$.

5. THE CHOICE OF c

Many attempts have been made to use the data to guide the choice of c . We shall describe one such attempt known as cross-validation. The idea underlying cross-validation is that a value of c good for the whole data set should also be good if a single point is removed and that performance can be judged by seeing how well the dropped point is estimated using c as smoothing parameter on the remaining $n-1$ points. Leaving

out each point in turn we would choose c to minimise

$$V_0(c) = \sum_{k=1}^n (y_k - \hat{F}^{[k]}(x_k; c))^2$$

where $\hat{F}^{[k]}(x; c)$ is the estimate of F based on all the data except the k^{th} point. Craven and Wahba (1979) propose that instead of minimising $V_0(c)$ a weighted sum of the form

$$V(c) = \sum_{k=1}^n (y_k - \hat{F}^{[k]}(x_k; c))^2 w_k(c)$$

should be used. They suggest

$$w_k(c) = \left[\frac{1 - a_{kk}(c)}{\frac{1}{n} \text{Trace}(I - A(c))} \right]^2$$

where $A(c)$ is the matrix such that

$$\underline{\hat{F}} = A(c)\underline{y}$$

and $a_{kk}(c)$ is the k^{th} diagonal element of $A(c)$. $a_{kk}(c)$ is the weight given y_k when estimating $F(x_k)$. If it is close to one, the point x_k has few close neighbours and so the error in estimating $F(x_k)$ is unavoidably large and should be down-weighted in measuring the worth of a particular choice for c . In their 1979 paper Craven and Wahba support their advocacy of cross-validation both by theoretical arguments and by means of a simulation study.

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THE BIBERBACH CONJECTURE: A SUMMER DRAMA

Tinlunn Holland

ACT 1

It was Brian Twomey who brought me the news: the Biberbach conjecture was false! More than that: a weaker form of it was untrue! We were redundant!

It transpired that Brian had just been on the 'phone to the extern examiner, who was making his rounds of the National University of Ireland Colleges; he was due in Cork in two days time, on Sunday, June 17, 1984, and rang to confirm his arrival time. Brian had no details, other than that X had cracked it. We would have to await the visit of the extern to learn more.

While trying to digest this piece of startling news, I recalled the story of how Briggs was supposed to have reacted when he congratulated Napier on his discovery of logarithms - it is said that he remained speechless in Napier's presence for fifteen minutes - and wondered how much time I should stand open-mouthed at X when I next met him!

The extern duly arrived. No: he had no further information; but he had no reason to doubt his informant, who had heard it from an impeccable source, and, anyway, X was a reputable mathematician.

During the course of a very heavy work-load of examining, the extern received a 'phone call urging him to adopt a less optimistic stand about X's success; there was some doubt about X being able to realise his hope. We should continue working at it! The prize could still be ours'.

A little later in the month, we heard that X had abandoned all hope of patching up his proof.

Things returned to normal.

ACT 2

On July 15, I went to Lancaster University for the NATO-LMS Advanced Study Institute on Operators and Function Theory. Shortly after arriving, I heard that the Bieberbach conjecture was true, news I greeted with considerable surprise and profound scepticism. I was even more sceptical, when I heard who was credited this time with the proof: Louis de Branges, of all people. But, I said, echoing a comment made by others, his reputation is not untarnished: didn't he claim, with a fanfare of trumpets, to have proved the invariant subspace problem many years ago?

My informant's source was unlikely as well. It was the Russians, Peller and Nikolskii, from the Steklov Institute in Leningrad, who were amongst the distinguished gathering, who brought the news. I was intrigued! My natural curiosity was aroused! Why should the news emerge from the U.S.S.R.? Was it some ingenious plot conjured up by the K.G.B. to discredit American mathematicians? If the claim were true, surely the latter would have been the first to know and the first to announce it? Wouldn't they love to bask in the reflected glory of such an achievement?

I consulted my function theory friends. Yes, they had heard the rumour - for that's all it was as far as we knew at that stage - but knew as little as I did. The rumour travelled like wildfire. People speculated as to whether or not it was true; and wondered about the embarrassment it might cause in America, if it were true. Some of the participants had heard it at another conference. Others had heard de Branges himself deliver a lecture in the Netherlands. Surprisingly enough, neither the few experts in the field of univalent functions, nor the other function theorists present at the conference, however, were aware of the news; they too

were hearing it for the first time.

Bit by bit the story was unfolded: before departing for a visit to Europe, de Branges had circulated a version of his proof to a handful of his colleagues who were specialists in the field of univalent functions; they were, apparently, still examining the proof, which, by all accounts, involved operator theoretic methods. Everybody seemed to think that that fact plus de Branges' reputation for false claims in the past, explained their silence on the matter. Shortly afterwards, de Branges visited the Steklov Institute and presented an account of his work in a series of talks. Members at the Institute were sufficiently impressed by the work that a group of them, including some very eminent function theorists, got together with de Branges and proceeded to remove all reference to operator theory methods and fashioned a proof along traditional lines. The result was a proof of the conjecture that could be outlined in a 13-page typescript. Peller had a copy of this, in Russian only, and was prepared to make it available for circulation to anybody who was interested, and to give a talk on de Branges' proof, if time could be made available in what was a very crowded programme; and there was some doubt about this.

Representations were made to the organisers of the Conference, and it was decided that Peller should give the talk at 6 p.m. on Friday, July 20; the talk would be followed by the banquet.

ACT 3

Excitement was at fever pitch when the moment arrived for Peller to begin. The lecture theatre was packed for the historic occasion; participants from other conferences being run at Lancaster were also present; some non-mathematicians were even there. The majority present were non-experts. It was disappointing that so few experts in the field of univalent

functions were there, although several British mathematicians had been informed of the happenings during the week. I feel most people came out of curiosity; everybody had heard about the Biberbach conjecture, even if they couldn't formulate it exactly and didn't appreciate its importance and relevance; some came to discover "the mistake", and leave early to dress for the banquet. I suspect that most of these remained to the end and were late for the banquet! I was one of them!

Peller gave a short history of the problem and recalled a few familiar inequalities. (So far, so good, I thought.) Next, he introduced a system of first order linear differential equations, and asked us to accept certain properties possessed by the solution. (No problem!) He then appealed to the Loewner theory of univalent functions. (Curse it, I'm not familiar with this, I thought, but what he is using can be verified. I'll hang in there.) This was followed by some very crafty manipulations, in which the role of the system of differential equations became apparent. (Several people left at that stage.) Undaunted, Peller proceeded with his task. Eventually, it emerged that everything hinged on the solution of this system having a certain property. Peller proceeded to discuss the solution. (More people left to prepare for the banquet, others because they had tired minds and simply could not absorb any more. I was fast becoming one of them! But I was determined to see it through.) To clinch the result, Peller introduced the final surprise: the property that he wanted the solution to satisfy fell out from a result of Askey and Gasper on hypergeometric polynomials!! Who would have believed it! I was drained. Peller had spoken dispassionately for one and a half hours, taking great care to present all the important points in the proof. I rushed away exhilarated, to change for the banquet. What an occasion to be present at!

In the days that followed, Peller's talk was discussed, and there was general agreement that the proof was correct,

though most people were unsure about the details, especially the role of the hypergeometric polynomials, which took us all by surprise. As one editor of a respected journal said to me afterwards, if he had received the Askey-Gasper paper for his journal, he would have rejected it out of hand.

(There is a lesson there for all of us: we should never dismiss too lightly what may appear to us to be uninteresting and therefore insignificant, simply because it is out of fashion.)

ACT 4

In September, I attended a one-day Conference in Liverpool, and I heard Hayman give a slightly different version of De Branges' proof, in which he took into account some simplifications due to Fitzgerald and Pommerenke.

There could no longer be any doubt: the Bieberbach conjecture had fallen at the hands of Louis de Branges!

We salute him!

EPILOGUE

What follows is an account of de Branges' proof, based on Peller's talk and Hayman's; I have not seen de Branges' original accounts, but I was privileged to see the 13-page Russian typescript, on which Peller based his lecture, and the informal communication circulated by Fitzgerald and Pommerenke during the late summer. While preparing this, Donal Hurley drew my attention to Fitzgerald's own article in the A.M.S. Notices. Readers desiring more information on the history and solution of the problem are advised to consult this, which contains an up-to-the-minute account of the progress that has been made in the past few months, as well as an extensive bibliography.

S will stand for the class of univalent functions f on the open unit disc D , i.e. $f \in S$ iff f is one-to-one and analytic on D with Taylor series

$$f(z) = \sum_{n=1}^{\infty} a_n(f) z^n \equiv z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad (z \in D)$$

In 1916 Bieberbach conjectured - on the basis of very slim evidence - that

$$(B) \quad A_n = \sup\{|a_n(f)| : f \in S\} = n, \quad n = 2, 3, \dots,$$

with equality only if f is of the form $f(z) = K(\lambda z)$ for some λ , with $|\lambda| = 1$, where

$$K(z) = z/(1-z)^2 = z + \sum_{n=2}^{\infty} n z^n$$

is Koebe's function. This was proved by him for $n = 2$.

Over the years, evidence in support of this conjecture was provided. It was shown to be true for various subclasses of S , for all n ; and verified for small values of n for the full class. Also, stronger conjectures were advanced. Thus, in 1936, Robertson put forward the conjecture that if g is an odd univalent function, then

$$(R) \quad \sum_{k=1}^n |a_k(g)|^2 \leq [(n+1)/2], \quad n = 1, 2, \dots$$

This is stronger than (B), because if $f \in S$, the function g defined by $g(z) = \sqrt{f(z^2)}$, ($z \in D$) belongs to S and is odd, and the coefficients of g and f are related through the convolutions

$$a_n(f) = \sum_{k=1}^{2n} a_k(g) a_{2n-k}(g), \quad n = 1, 2, \dots$$

An easy application of Schwarz's inequality now shows that (R) forces (B).

In 1955 Hayman proved that the limit

$$\lim_{n \rightarrow \infty} \frac{|a_n(f)|}{n}$$

exists and is ≤ 1 , for every $f \in S$. Shortly after, he showed that A_n/n converges to a number ≥ 1 ; and conjectured that the limit was actually equal to 1, expressing the view that this might be easier to prove than (B). (It was this conjecture that X was rumoured to have disproved.)

In the late sixties, Lebedev and Milin derived a powerful inequality for the class S , involving the coefficients of another auxiliary function, viz.,

$$\log[f(z)/z] = 2 \sum_{k=1}^{\infty} \gamma_k(f) z^k \quad (z \in D)$$

and those of g :

$$\sum_{k=1}^{2n} |a_k(g)|^2 \leq n \exp \left[\sum_{k=1}^n (1 - k/n)(k|\gamma_k(f)|^2 - 1/k) \right], \quad n = 1, 2, \dots$$

This brings us to Milin's conjecture, which is that

$$(M) \quad \sum_{k=1}^n (n-k)k|\gamma_k(f)|^2 \leq \sum_{k=1}^n (n-k)/k, \quad n = 1, 2, \dots$$

Clearly, it follows from the previous inequality that $(M) \Rightarrow (R) \Rightarrow (B)$. de Branges' remarkable feat was to establish the validity of (M).

The strategy adopted by de Branges was to prove the inequality for a dense set of functions in S - dense in the sense of uniform convergence on compact subsets of D . Fitzgerald and Pommerenke simplified this part of the proof considerably. Both approaches make use of the theory introduced by Loewner in 1923, establishing the existence of an especially useful dense set, which Loewner himself used to prove that $\sup\{|a_3(f)| : f \in S\} = 3$. For our purposes, it is enough to know that there is a dense subset T in S , such that if $f \in T$,

then there is a continuous function $\lambda: [0, \infty) \rightarrow \partial D$, such that the solution $f(z, t)$ of the partial differential equation

$$\frac{\partial}{\partial t} f(z, t) = \frac{1 + \lambda(t)z}{1 - \lambda(t)z} z \frac{\partial}{\partial z} f(z, t) \quad (z \in D, 0 \leq t < \infty)$$

has the properties that $f(z, 0) = f(z)$ and $e^{-t}f(z, t) \in S$, for all $t \in [0, \infty)$.

Given this, let $f \in T$ and $\lambda: [0, \infty) \rightarrow \partial D$; and let $f(z, t)$ be the corresponding function. Then, for any $t \geq 0$,

$$\log[e^{-t}f(z, t)/z] = 2 \sum \gamma_k(f, t) z^k \equiv 2 \sum \gamma_k z^k,$$

say. Using the differential equation, it is easy to derive the identity

$$1 + 2 \sum_{n=1}^{\infty} \dot{\gamma}_n z^n = [1 + 2 \sum_{n=1}^{\infty} \lambda^n z^n][1 + 2 \sum_{n=1}^{\infty} n \gamma_n z^n] \quad (z \in D),$$

where the dot denotes differentiation with respect to t and the variable t has been suppressed throughout.

Writing

$$b_k \equiv b_k(t) = \sum_{v=1}^n v \gamma_v(t) (\lambda(t))^{-v}, \quad k = 1, 2, \dots$$

and equating coefficients in the above identity, we see that

$$k \gamma_k = \lambda^k (b_k - b_{k-1}), \text{ and } \dot{\gamma}_k = \lambda^k (b_k + b_{k-1} + 1),$$

$$k = 1, 2, \dots$$

Now fix n and consider

$$\phi(t) = \sum_{k=1}^n (k|\gamma_k(t)|^2 - 1/k) \tau_k(t),$$

where $\tau_1, \tau_2, \dots, \tau_n$ are the solutions of the system of differential equations

$$\tau_k - \tau_{k+1} = -\dot{\tau}_k/k - \dot{\tau}_{k+1}/(k+1), \quad k = 1, 2, \dots, n,$$

$$\tau_{n+1} \equiv 0, \text{ and } \tau_k(0) = n + 1 - k, \quad k = 1, 2, \dots, n.$$

We want to show that $\phi(0) \leq 0$. To achieve this, it will be sufficient to show that ϕ is increasing on $[0, \infty)$ and that $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$. We proceed to show that these properties can be detected from the τ_k . Indeed, a fairly straightforward computation shows that

$$\begin{aligned} \phi &= \sum \tau_k [|b_k - b_{k-1}|^2 - 1]/k + 2 \sum \tau_k \operatorname{Re}[k \gamma_k \dot{\gamma}_k] \\ &= \sum \dot{\tau}_k [|b_k - b_{k-1}|^2 - 1]/k + 2 \sum [|b_k|^2 + \operatorname{Re} b_k] (\tau_k - \tau_{k+1}) \\ &= - \sum \dot{\tau}_k |b_k + b_{k+1} + 1|^2/k, \\ &\geq 0, \end{aligned}$$

if $\dot{\tau}_k \leq 0$, for $k = 1, 2, \dots, n$ (The range of summation is from $k=1$ to $k=n$.)

The remarkable fact is that this is true! And this is where the results of Askey and Gasper enter the picture: the $\dot{\tau}_k$ can be expressed as *non-positive* polynomials in e^{-t} ! Thus ϕ is increasing on $[0, \infty)$. It is easy to see that $\tau_k(t) \rightarrow 0$ as $t \rightarrow \infty$, for $k = 1, 2, \dots, n$. Also, the $\gamma_k(f, t)$ can be shown to be bounded with respect to t . Hence $\phi(0) \leq \lim \phi(t) = 0$, and de Branges' proof of the Bieberbach conjecture is complete.

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CAPACITIES, ANALYTIC AND OTHER

Anthony G. O'Tunnell

(1.1) Let E be a compact subset of \mathbb{C} . If f is analytic on $S^2 - E$, then it has the Laurent expansion

$$f = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

near ∞ , where S^2 is the Riemann sphere. The (Ahlfors) *analytic capacity* of E is the non-negative number

$$\gamma(E) = \sup |a_1(f)|$$

where f runs over all functions, analytic on $S^2 - E$, and bounded by 1 in modulus. A compact set E has $\gamma(E) = 0$ if and only if E is *removable* for all bounded analytic functions, i.e. if and only if given U open and $f: U \rightarrow \mathbb{C}$, analytic and bounded, there exists an analytic continuation of f to U . For open sets U , $\gamma(U)$ is defined as

$$\sup \{\gamma(E) : E \subset U, E \text{ compact}\}.$$

For arbitrary sets $A \subset \mathbb{C}$, the *outer analytic capacity* $\gamma^*(A)$ is defined as

$$\inf \{\gamma(U) : A \subset U, U \text{ open}\}.$$

(Readers interested in more details should consult [H] for references.)

(1.2) Analytic capacity plays a key role in the theory of uniform rational approximation (or, what amounts to the same thing, holomorphic approximation) in one variable. Let $\mathcal{O}(E)$ denote the set of functions, holomorphic near E . For X compact in \mathbb{C} , let $R(X)$ denote the set of uniform limits on X of elements of $\mathcal{O}(X)$. Vitushkin showed that a necessary and sufficient condition that all functions continuous on X belong

to $R(X)$ is that

$$\gamma(U \sim X) = \gamma(U)$$

for all open sets U (or, equivalently, for all open discs U). The capacity γ , in combination with another, the *continuous analytic capacity* α , provides a similar resolution (also due to Vitushkin) of the problem of which X have

$$R(X) = \{f : f \text{ is continuous on } X \text{ and analytic on } \text{int } X\}.$$

See [G] for other uses of γ in connection with $R(X)$.

(1.3) There are two important open questions about γ . The first is to give a reasonable "real-variable" characterisation of the γ -null sets. For instance, Vitushkin has conjectured that $\gamma(E) = 0$ if and only if almost all projections of E on lines have outer length zero. Thanks to some work of Havinson, Calderon and others, we know this is true for σ -rectifiable sets, and for those totally unrectifiable sets known to be γ -null [M]. This problem is particularly irritating because the bounded analytic functions are practically the only "reasonable" class of analytic functions for which the null sets lack a real-variable description. For instance, see [C]. The only significant exception are the Smirnov E_p classes, but they do not count, because, when defined, they have the same null sets as γ [H].

The second problem is whether γ is *quasi-subadditive*, i.e. whether there exists a universal constant $\kappa > 0$ such that

$$\gamma(E_1 \cup E_2) \leq \kappa(\gamma(E_1) + \gamma(E_2))$$

whenever E_1 and E_2 are compact in \mathbb{C} . There is a sizeable logjam in uniform holomorphic approximation theory because of this problem. For example, if E is compact, with $\gamma(E) = 0$, and $f: S^2 \rightarrow \mathbb{C}$ is continuous, do there exist functions $f_n: S^2 \rightarrow \mathbb{C}$,

tending uniformly to f on S^2 , holomorphic wherever f is and on a neighbourhood of E ? If γ is quasi-subadditive, the answer is yes. If γ were subadditive, one could define a special topology (the "analytic-fine topology") on \mathbb{C} , finer than the Euclidean topology, that ought to be especially helpful for studying $R(X)$. This topology might provide the real answer to E. Borel's dream of the perfect notion of analytic function.

(1.4) The most penetrating work on the subadditivity problem is in [D]. Davie showed that quasi-subadditivity would follow from the statement:

$$\gamma^*(E \cup F) \leq \gamma(E) + \kappa(E)\gamma(F)$$

wherever E is compact and F is open, where $\kappa(E) > 0$ is independent of F . We know that

$$\gamma(E \cup F) \leq \gamma(E) + \kappa(E)\gamma(F)$$

wherever E and F are compact. It may not seem like much of a gap, but there it is.

In what follows, we shall present another formula for $\gamma(E)$, and use it to cast a little light on the subadditivity problem. It will become clear that subadditivity is just another version of the only "real" problem in analysis, which is how to handle

$$\int_{-1}^1 \frac{f(t)}{t} dt.$$

(2.1) Dolzenko generalised the concept of analytic capacity. Suppose B is a Banach space of functions on \mathbb{C} , such that $\mathcal{D} \hookrightarrow B$, $\mathcal{D} \hookrightarrow B^*$, and the inclusions are continuous. Here $\mathcal{D} = \mathcal{D}(\mathbb{C}, \mathbb{C})$ denotes the space of test functions. We assume that, if B has a predual B_* , then $\mathcal{D} \hookrightarrow B_*$ continuously. Also, we assume $f \in B \Rightarrow \bar{f} \in B$. The analytic B -capacity of a compact $E \subset \mathbb{C}$ is

$$\gamma_B(E) = \sup |a_1(f)|$$

where f runs over all functions in the unit ball of B that are analytic on $S^2 - E$.

Examples are $B = L_p$ (with respect to area measure \mathcal{L}^2), C (for continuous and bounded), $Lip\alpha$, $lip\alpha$, BMO , VMO , C^k (= bounded continuous derivatives up to order k), some weighted L_p spaces, Sobolev spaces, etc.

(2.2) The number $a_1(f)$ equals

$$\frac{1}{2\pi i} \int_{\Gamma} f(\zeta) d\zeta$$

whenever Γ is a rectifiable contour around E , in the usual sense. A more entertaining formula is

$$a_1(f) = -\frac{1}{\pi} \int f(z) \frac{\partial \psi}{\partial \bar{z}} d\mathcal{L}^2(z) = \frac{1}{\pi} \left\langle \psi, \frac{\partial f}{\partial \bar{z}} \right\rangle$$

where $\psi \in \mathcal{D}$ is any test function with $\psi = 1$ on a neighbourhood of E . This follows from Green's formula. It suggests the natural way to generalise γ_B from the Cauchy-Riemann operator to other differential operators.

Let E be a compact subset of \mathbb{R}^d , let B be a Banach space of functions on \mathbb{C} , and let $\mathcal{L}(\mathbb{R}^d, \mathbb{C})$ be the Schwartz space of C^∞ functions from \mathbb{R}^d to \mathbb{C} . Let $L : \mathcal{L}(\mathbb{R}^d, \mathbb{C}) \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{C})$ be a linear differential operator with C^∞ coefficients. Choose $\psi \in \mathcal{D}(\mathbb{R}^d, \mathbb{C})$ with $\psi = 1$ on a neighbourhood of E , and define

$$\gamma_B^L(E) = \sup \left| \int f(x) L^* \psi(x) d\mathcal{L}^d(x) \right|,$$

where f runs over all elements of the unit ball of B which satisfy $Lf = 0$ on $\mathbb{R}^d - E$, in the (weak) sense of distributions. The value of the integral does not depend on the choice of ψ , for such f . This concept embraces those capacities used by

Hedberg, Polking, Bagby, and others [HE] in connection with various approximation problems. The classical Newtonian capacity is $\gamma_{L_\infty}^\Delta$, where Δ is the Laplacian.

(2.3) The technique of the *dual extremal problem* is based on the following fact, which may be proved by using the Hahn-Banach theorem.

Duality Lemma. Let B_0 be a subspace of a Banach space B , and let $A \in B^*$.

- (1) Then $\sup \{ |Af| : f \in B_0, \|f\|_B \leq 1 \} = \text{dist}(A, B_0^\perp)$.
- (2) If B has a predual B_* , if B_0 is B_1^\perp , for some subspace $B_1 \subset B_*$, and if $A \in B_*$, then

$$\sup \{ |Af| : f \in B_0, \|f\|_B \leq 1 \} = \text{dist}(A, B_1).$$

This lemma allows us to turn to an extremal problem in one Banach space into a corresponding problem in the dual, or in the pre-dual (if there is a pre-dual). This technique has been put to good use in the past, but still has plenty of energy left. Our present purpose is to apply it to get formulae for the kind of capacities described above, so as to cast some light on the subadditivity problem.

(2.4) Applying part (1) of the Duality Lemma gives the formula

$$\gamma_B^L(E) = \inf_S \|L^*\psi - S\|_{B_*}$$

where S runs over all elements of B^* such that

$$\left. \begin{array}{l} f \in B \\ Lf = 0 \text{ off } E \end{array} \right\} \Rightarrow Sf = 0.$$

If B has a predual B_* (and $D \leftrightarrow B_*$ is continuous), part (2) gives the nicer formula

$$\gamma_B^L(E) = \inf_\phi \|L^*\psi - L^*\phi\|_{B_*}$$

where ϕ runs over all test functions supported on $\mathbb{R}^d \sim E$. Recalling that ψ is any given test function with $\psi = 1$ near E , we conclude that

$$\gamma_B^L(E) = \inf \{ \|L^*\phi\|_{B_*} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E \}.$$

(2.5) Applying this formula to classical analytic capacity, we get

$$\gamma(E) = \frac{1}{\pi} \inf \{ \left\| \frac{\partial \phi}{\partial \bar{z}} \right\|_{L_1} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E \}.$$

(2.6) Applying it to the analytic capacity associated to $B = L_p$ (the "analytic p -capacity" of Sinanjan), we get

$$\gamma_{L_p}^L(E) = \frac{1}{\pi} \inf \{ \left\| \frac{\partial \phi}{\partial \bar{z}} \right\|_{L_q} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E \}$$

for $1 < p < \infty$, where q is the conjugate index to p . This B has the property that B is mapped continuously to itself by the Beurling transform:

$$(Tf)(z) = \frac{1}{\pi} \int \frac{f(\zeta)}{(\zeta - z)^2} d\mathcal{L}^2(\zeta),$$

where the integral is interpreted as a limit in B norm of principal value integrals of smooth approximation to f . The theory of the continuity properties of this and similar integral operators is known as the Calderon-Zygmund theory $[A, S]$. The operator T has the property that

$$T \frac{\partial \phi}{\partial \bar{z}} = \frac{\partial \phi}{\partial z}$$

for all $\phi \in \mathcal{D}$, so that if T maps $B \rightarrow B$ continuously, we deduce that γ_B is comparable to the real-variable capacity

$$\inf \{ \left\| \frac{\partial \phi}{\partial x} \right\|_{B_*} + \left\| \frac{\partial \phi}{\partial y} \right\|_{B_*} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E \}.$$

Apart from L_p ($1 < p < \infty$), the spaces $Lip\alpha$ and BMO are Beurling-invariant dual spaces, so this argument also applies to their analytic capacities. In all three cases, this real-variable formula gives a proof of quasi-subadditivity. For instance, for BMO we get

$$\gamma_{BMO}(E) \sim \inf \{ \|\nabla\phi\|_{L_1} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E \},$$

where \sim means "is within constant multiplicative bounds of". It makes no difference to restrict to real-valued ϕ , and we get

$$\begin{aligned} \gamma_{BMO}(E) &\sim \inf \{ \|\phi\|_{W^{1,1}} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E \} \\ &= \inf \{ \|h\|_{W^{1,1}} : h \in \mathcal{D}, h = 1 \text{ near } E \} \\ &= \inf \{ \|h\|_{W^{1,1}} : h \in \mathcal{D}, h \geq 1 \text{ near } E \}; \end{aligned}$$

which is obviously subadditive. Here $W^{1,1}$ denotes the Sobolev space of L_1 functions with L_1 distributional derivatives. See [V].

(2.7) This method extends to other hypoelliptic operators. Suppose L^* has an inverse $P : \mathcal{D} \rightarrow \mathcal{E}$ such that $PL\phi = \phi$ whenever $\phi \in \mathcal{D}$. For instance, the Cauchy transform does this for $\frac{\partial}{\partial \bar{z}}$, and, more generally, convolution with a fundamental solution does it for elliptic constant-coefficient L .

Suppose L has order m . Denoting the partial derivative associated to the multi-index j by D_j , we may ask about the continuity properties with respect to B of the operator $D_j P$, for $|j| \leq m$. If all these map B_* continuously into B_* , then $\gamma_B^L(E)$ is comparable to the real-variable capacity

$$\inf \{ \sum_{|j| \leq m} \|D_j \phi\|_{B_*} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E \}.$$

This works for constant-coefficient elliptic operators, with $B = L_p$ ($1 < p < \infty$), $Lip\alpha$, BMO, $Lip(k+\alpha)$, some Sobolev spaces, etc. The associated γ_B^L are then subadditive.

(2.9) If L has real-valued coefficients, then γ_B^L is a real-variable capacity even if B is not Beurling invariant. For instance,

$$\begin{aligned} \gamma_{L_\infty}^\Delta(E) &= \inf \{ \|\Delta\phi\|_{L_1} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E \} \\ &\quad \inf \{ \|\Delta\phi\|_{L_1} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E, \phi \text{ real} \} \\ &\quad \inf \{ \|\Delta\phi\|_{L_1} : \phi \in \mathcal{D}, \phi \geq 1 \text{ near } E, \phi \text{ real} \}. \end{aligned}$$

This is pretty clearly subadditive.

(2.10) The upshot is that among the usual crop of elliptic operators L and dual spaces B , the case $L = \bar{\partial}$ and $B = L_\infty$ is practically the only one we cannot handle with ease. And we cannot handle it at all.

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RAZMYSLOV AND SOLVABILITY

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The exponential growth in the number of active mathematicians in the present era is sometimes illustrated by the remark that there are as many mathematicians alive today as have lived - and died - since classical times. A less picturesque but more interesting indicator of mathematical activity is the rapidity with which well known conjectures and problems, sometimes of long standing, are being resolved. A recent article in the *Newsletter* (No. 11) by David Lewis on the Merkuryev-Suslin Theorem illustrates this point, and the present article (also expository, also concerned with Russian work) provides another example.

INTRODUCTION

Many readers will be familiar with, or at least aware of, the Burnside Problem in group theory, namely: must a group be finite if it is finitely generated and has exponent k ? Having exponent k means that the group elements all satisfy the law $x^k = 1$ and some element has period precisely k . The problem was stated in 1902 [1], and answered negatively in 1968; an outline of developments and a bibliography, may be found in [4] and [3]. The story is by no means complete, and many problems remain open concerning these groups, but one problem concerning solvability has been settled completely by the work of Ju. P. Razmyslov in Moscow.

Let B_k denote the Burnside Variety of all groups satisfying the law $x^k = 1$; let $B_{k,n}$ represent the free group of rank n in B_k (then the n -generator groups of exponent k are just the quotient-groups of $B_{k,n}$). It has been known for many years (> 25) that:

$B_{2,n}$ is finite and abelian
 $B_{3,n}$ is finite and metabelian
 $B_{6,n}$ is finite and solvable, of derived length 3
 $B_{4,n}$ is finite.

Of course $B_{4,n}$ is a finite 2-group, and therefore is solvable - but what is its derived length? What Razmyslov [3] calls the Problem of Hall and Higman, under attack since the 1950s, could be put thus: Is the derived length of $B_{4,n}$ independent of n ?

If this were so, then B_4 would join the varieties B_2 , B_3 and B_6 in being known to be "solvable" in the sense that all groups in these three varieties are solvable, with bounded derived lengths.

A great deal of work on $B_{4,n}$ culminated in the proof by Razmyslov that B_4 is *not* solvable - and this, due to previous work of Gupta and Newman, determined the precise nilpotency class of $B_{4,n}$ which in turn enabled Vaughan-Lee to decide the precise derived length of $B_{4,n}$.

There is, however, much more: Bachmuth and Mochizuki a little earlier had shown that B_5 is not solvable, but Razmyslov has constructed non-solvable groups of exponent p for all primes $p > 3$ and also of exponent 9. A consequence of all this is the following result which we might call:

Razmyslov's Theorem: The Burnside Variety B_k is solvable only when $k = 2, 3$ or 6 .

This is a satisfyingly complete result, although certainly unexpected. The work has been announced and has appeared in Russian sources during the past decade; some of the details have only recently appeared in an English translation by J. Wiegold [3]. As an introduction to the ideas involved we

will explain here a relatively easy way of producing groups of exponent p^2 which are non-solvable when $p > 2$. This is given in [3] as a concession to the readers really, to encourage them to persevere with the far more complex details of exponent 4.

The justification for presenting here what could be read in [3] is that hopefully our account is less Delphic in style than the original - and may perhaps achieve the aim of the original if it encourages study of the entire paper. Furthermore, the use of Lie-ring-theoretic methods has proved to be a most important tool in certain problems of combinatorial group theory; the construction given here is a nice (if not very deep) illustration of its power.

FIRST STAGE - A GROUP OF EXPONENT p^k

Let A_0 be an associative algebra with identity 1, over a field K and let A_0 be generated by an infinite set of non-commuting elements x_1, x_2, x_3, \dots . This means that an element of A_0 has the form $\sum_i k_i \phi_i$ where the sum is finite, $k_i \in K$ and ϕ_i is a product of generators x_j .

In A_0 we introduce the relations

$$x_i w x_i = 0$$

for every word w in A_0 ; note that w may be the empty word. We consider the quotient algebra A say, and we will continue to use the symbols x_i for the images in A of the original generators x_i of A_0 .

Now $(1 + x_i)(1 - x_i) = 1$ in A so the elements

$$g_i = (1 + x_i) \text{ and } g_i^{-1} = (1 - x_i), \quad 1 \leq i$$

generate a group G embedded in A .

We notice now that if $c(y_1, y_2, \dots, y_k)$ is any group commutator in elements y_1, y_2, \dots, y_k we have

$$c(g_1, \dots, g_k) = 1 + c^*(x_1, x_2, \dots, x_k)$$

where $c^*(x_1, x_2, \dots, x_k)$ is the corresponding Lie commutator in A .

For example, if $c(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2$ then

$$c^*(x_1, x_2) = x_1 x_2 - x_2 x_1.$$

Notice also that the group G is locally nilpotent: thus for instance in the subgroup $G(n)$ say generated by g_1, g_2, \dots, g_n if c is any single commutator in the elements g_i which is of weight $n+1$, the corresponding c^* will be a homogeneous polynomial of weight $n+1$ in x_1, x_2, \dots, x_n and each monomial term in c^* will have a repeated x_i and be 0 because of the relations in A ; that is $c = 1$ and so $G(n)$ has nilpotency class n . Of course any finitely generated subgroup of G lies in $G(n)$ for a suitable n .

If we now stipulate that K be a field of characteristic p we have $g_i^p = 1$ for every i , and an element of G may be written as a product of positive powers of the generators g_i .

These generators all have period p , and indeed G now has the property that every element must have period a power of p - but we wish to do more than that, we want every element to satisfy the law $g^{p^k} = 1$: in characteristic p this means that $(g-1)^{p^k} = 0$. We note that if $g \in G$ then

$$g = (1 + x_{i_1})(1 + x_{i_2}) \dots (1 + x_{i_t})$$

and we begin by considering

$$g = (1 + x_1)(1 + x_2) \dots (1 + x_t), \text{ where } t \geq 1.$$

$$\text{Let } D = [(1 + x_1)(1 + x_2) \dots (1 + x_t) - 1]^{p^k}$$

and let $\Delta(x_1, x_2, \dots, x_t)$ be the homogeneous component of maximum weight in the expansion of D : this weight must be t , since terms of higher weight are killed because of repetitions. Of course $D = 0$ when $t < p^k$.

We notice that the homogeneous component of weight $t-1$ in D must be the sum of t separate components, namely:

$$\begin{aligned} &\Delta(x_1, x_2, \dots, x_{t-1}) + \Delta(x_1, x_2, \dots, x_{t-2}, x_t) + \dots + \\ &\Delta(x_2, x_3, \dots, x_t). \end{aligned}$$

Similarly for the homogeneous components of lower weight in D .

Thus finally if we let J be the ideal of A generated by all $\Delta(x_{i_1}, x_{i_2}, \dots, x_{i_s})$ for all $s \geq 1$ and for all possible choices of i_1, i_2, \dots, i_s we have an ideal which must contain $(g-1)^{p^k}$ for all $g \in G$.

Now if we take the quotient algebra $A_2 = A/J$ the image of G in A_2 is a (locally nilpotent and finite) group of exponent p^k .

We might remark by the way that the approach so far is not novel, and similar ideas were used in some earlier papers on groups with exponent 4.

However, we will see that in the particular algebra A which we are going to produce below, this last step is unnecessary; in other words J will be (0) already in A and so G will automatically have exponent p^k .

AN IDENTITY

We digress now to consider an identity which holds in any associative algebra B of dimension s over a field of prime characteristic p .

Consider the symmetric function

$$S_t(y_1, y_2, \dots, y_t) = \sum_{\sigma} y_{1\sigma} y_{2\sigma} \dots y_{t\sigma}$$

where $y_i \in B$ ($1 \leq i \leq t$) and σ runs over all permutations of the set $\{1, 2, \dots, t\}$.

S_t is multilinear in all variables y_i , so if b_1, b_2, \dots, b_s is a basis for B and

$$y_i = \sum_{j=1}^s a_{ij} b_j \quad \text{we get}$$

$$\begin{aligned} S_t(y_1, y_2, \dots, y_t) \\ = \sum_{j_1=1}^s \sum_{j_2=1}^s \dots \sum_{j_t=1}^s a_{1j_1} a_{2j_2} \dots a_{tj_t} S_t(b_{j_1}, b_{j_2}, \dots, b_{j_t}). \end{aligned}$$

Suppose now that we take $t = s(p-1) + 1$. Then in any $S_t(b_{j_1}, b_{j_2}, \dots, b_{j_t})$ some basis element must occur at least p times in the entries b_{j_i} . Suppose, for example, that b_1 occurs $(p + \alpha)$ times. Then $S_t(b_{j_1}, \dots, b_{j_t})$ breaks into a sum of blocks each consisting of $(\alpha + p)!$ identical products: since $(\alpha + p)! \equiv 0 \pmod p$ this means that every $S_t(b_{j_1}, \dots, b_{j_t}) = 0$ and so we see that

$$S_t(y_1, y_2, \dots, y_t) \equiv 0 \text{ is an identity in } B.$$

(We remark that $S_t \equiv 0$ implies $S_{t+m} \equiv 0$, all $m \geq 0$).

We wish to use this result where B is the algebra M of all 2×2 matrices over an infinite field K of characteristic p : then for $t = 4(p-1) + 1$ we have $S_t(y_1, y_2, \dots, y_t) \equiv 0$ in M .

FINAL STAGE

Let us now return to the construction of a non-solvable group. For the algebra A_0 we choose the free algebra, on free generators x_1, x_2, x_3, \dots , in the variety of algebras generated by the matrix algebra M referred to above. A_0 comes furnished with characteristic p ; we construct the quotient algebra A containing the group G as before, $A = A_0/R$ where R is the ideal in A_0 generated by all expressions $x_i w x_i$, w being any (possibly empty) word in A_0 .

Now in the group G (generated by all $g_i = 1 + x_i$, $1 \leq i$ in A) let (u, v) denote the group commutator $u^{-1}v^{-1}uv$. Let $\delta_1 = (g_1, g_2)$, $\delta_2 = ((g_1, g_2), (g_3, g_4))$, $\delta_3 = (\delta_2, ((g_5, g_6), (g_7, g_8)))$ and so on; then δ_k involves 2^k generators and lies in the k th derived subgroup of G . For every $k \geq 1$ there is a corresponding δ_k^* where $\delta_k = 1 + \delta_k^*$; clearly δ_k^* is a homogeneous polynomial in x_1, \dots, x_{2^k} of degree 2^k where no term has a repeated factor x_i . There is a preimage of δ_k^* in A_0 having exactly the same form, call it d_k^* ; the x_i which appear in d_k^* are the free generators of A_0 .

Since M contains the Lie algebra $sl(2, K)$ which is simple when $p > 2$ there is a Lie commutator $\gamma(a_1, \dots, a_{2^k}) \neq 0$ in M , where γ has the same form as d_k^* , for every k ; the mapping $x_j \mapsto a_j$ ($1 \leq j \leq 2^k$) induces a homomorphism of A_0 into M (we might map all other x_j onto 0) which shows that $d_k^* \neq 0$ for any k . The form of d_k^* now shows that it does not lie in the ideal R in A_0 so the image δ_k^* is not 0 in A . Thus finally $\delta_k \neq 1$ in G for any k and so G is non-solvable.

Notice here that we need $p \geq 3$: also that we have yet to show (as we promised) that G has exponent p^2 (where we are now fixing $k = 2$).

What makes this work is the observation that the expression $\Delta(x_1, x_2, \dots, x_t)$ is a sum of terms $S_m(u_1, u_2, \dots, u_m)$ where $m = p^k$ and the u_i are certain monomials in the elements

x_j when $t \geq p^k$; when $t < p^k$ we have $\Delta(x_1, \dots, x_t) = 0$.

This is easy to see - an example will suffice - if, for example, $p^k = 3$ and $t = 5$ we would have

$$\begin{aligned} \Delta(x_1, x_2, x_3, x_4, x_5) &= \sum_{i < j < k} S_3(x_i x_j x_k, x_r, x_s) \\ &+ \sum_{\substack{i < j \\ r < s}} S_3(x_i x_j, x_r x_s, x_k) \end{aligned}$$

where $\{i, j, k, r, s\} = \{1, 2, 3, 4, 5\}$.

Now since A above is in the variety generated by M we have the identity $S_t(y_1, \dots, y_t) = 0$ in A whenever $t \geq 4(p-1) + 1$.

But $p^2 - (4p-3) = (p-2)^2 - 1 \geq 0$ if $p \geq 3$. This means that already in A the relation $(g-1)p^2 = 0$ is satisfied for all g in G .

Thus finally we have arrived at a non-solvable group G of exponent p^2 which is also locally a finite p -group.

FOCAL SCUIR

In consonance with the didactic tendency of this journal, we end with an exercise for the reader: taking 3×3 matrices and applying techniques similar to those used above, construct a non-solvable group of exponent 8.

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FATOU'S THEOREM AND UNIVALENT FUNCTIONS

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1. INTRODUCTION

The purpose of this note is to present some results - old and new - concerning the behaviour of functions analytic and univalent in the unit disc $U = \{z: |z| < 1\}$ as the unit circle $C = \{z: |z| = 1\}$ is approached. Some open questions suggested by these results will also be discussed. The note is based on a lecture given at the December 1984 meeting of the DIAS Mathematical Symposium.

2. ANGULAR LIMITS

We begin with a simple definition. A function f , analytic (holomorphic) in U , is said to have an *angular* (or non-tangential) limit ℓ at a point ζ on C if

$$f(z) \rightarrow \ell$$

as $z \rightarrow \zeta$ inside every symmetric angle with vertex at ζ (as shown in Fig. 1) of opening less than π . This is easily

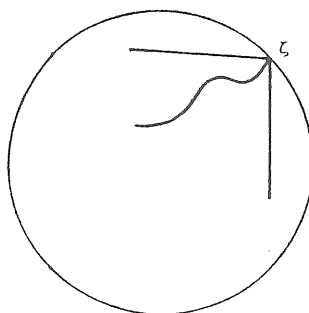


FIGURE 1

shown to be equivalent to the following: f has an angular limit ℓ at ζ if, for every fixed positive K ,

$$f(z) \rightarrow \ell \quad \text{as } z \rightarrow \zeta, \quad z \in \Omega$$

where

$$\Omega = \{z \in U: |\zeta - z| \leq K(1 - |z|)\}$$

We also write, in the usual notation, H^∞ to denote the class of functions that are analytic and bounded in U .

The basic result connecting functions in H^∞ and angular limits is the following well-known theorem of Fatou.

THEOREM 1. (Fatou, 1906)

Let $f \in H^\infty$. Then f has an angular limit at all points $\exp(i\theta)$ on C except possibly for a set of θ of measure zero, that is, angular limits exist almost everywhere on C .

This important result has been generalised in a number of ways, but our interest here is in the fact that the result as stated is sharp in at least two senses. In the first place, given any subset E of C of (linear) measure zero, there is a function f in H^∞ for which the radial limit

$$\lim_{r \rightarrow 1} f(r\zeta)$$

fails to exist for all ζ in E [4], and secondly, if Γ is any curve in U that approaches the point 1 tangentially, there is a function g in H^∞ which does not approach a limit as z approaches any point $e^{i\theta}$ along $e^{i\theta}\Gamma$ [3]. The situation for functions in H^∞ is thus clear-cut: we cannot, in general reduce the size of the exceptional set in Fatou's theorem for such functions nor can we replace angular limit by tangential limit in any uniform sense. To obtain improvements in either of these two directions, therefore, some extra condition must be imposed on our functions, and the extra condition we consider here is that of univalence.

3. UNIVALENT FUNCTIONS

A function f is said to be *univalent* in U if it is analytic and one-to-one on U , that is,

$$f(z_1) = f(z_2), \quad z_1, z_2 \in U \implies z_1 = z_2$$

It is almost immediate that the conclusion of Fatou's theorem holds for *all* univalent functions (and not just for bounded univalent functions). For if f is univalent in U , $f(U)$ cannot be the entire complex plane, so there is a point w in the complement of $f(U)$ and then, by a simple and standard argument [7, pp. 302-3] there is a complex number b such that

$$g(z) = [(f(z) - w)^{\frac{1}{2}} + b]^{-1}$$

is univalent and bounded in U . Then

$$f(z) = w + \left(\frac{1}{g(z)} - b\right)^2, \quad z \in U,$$

and so, if g has an angular limit at a point on C , f has also, unless the limit for g is zero. By a uniqueness theorem of F. and M. Riesz, this can happen on C only at a set of measure zero [2, p. 76]. Hence f has a finite angular limit almost everywhere on C .

The hypothesis of univalence is a highly restrictive one, however, so it is natural to ask (especially with the benefit of hindsight) whether a stronger result than this is true for univalent functions. Fatou's theorem can indeed be strengthened for univalent functions and this was first proved by Beurling in 1946.

THEOREM 2. (Beurling, [1, p. 56])

If f is univalent in U , then f has an angular limit at all points $\exp(i\theta)$ on C except possibly for a set of logarithmic capacity zero.

Beurling's theorem is usually proved (as it is in [1]) for functions in the classical Dirichlet space D , that is, the class of functions analytic in U for which

$$\int \int_U |f'(z)|^2 dx dy \quad (1)$$

is finite, and this is a more general result than Theorem 2. To see this, note first that D contains all bounded univalent functions since, if f is univalent, the integral (1) represents the area of $f(U)$. Hence, by an argument similar to that used at the beginning of this section, if the conclusion of Theorem 2 holds for functions in D , it is valid also for all univalent functions.

The points on C at which angular limits fail to exist thus form a much smaller set for univalent functions than for functions in H^∞ , since sets of logarithmic capacity zero are, in general, much "thinner" than sets of measure zero. It appears to be an open question whether Beurling's theorem is best - possible with regard to the size of the exceptional set but it is certainly easy to show that the exceptional set need not be empty. To do this, it is necessary only to consider a univalent function f which maps U onto the simply connected domain in Fig. 2. Such a function exists by the

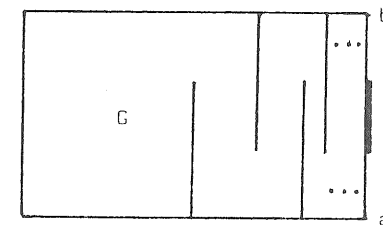


FIGURE 2

Riemann mapping theorem. Note that G has infinitely many vertical slits and that these slits have, as indicated, a fixed overlap. By standard results from the theory of *prime ends* [6, §9.2]

$$\lim_{r \rightarrow 1} f(r\zeta_0)$$

fails to exist where ζ_0 is a point on C which corresponds to the boundary element ab of G .

For univalent functions, therefore, the size of the exceptional set in Fatou's theorem can be reduced. We next turn our attention to the improvements which are possible for univalent functions with regard to the nature of the limit which exists almost everywhere on C . To this end we now introduce the notion of a tangential limit.

4. TANGENTIAL LIMITS

Let ϕ be a decreasing continuous function on $[0,1]$ with $\phi(1) = 0$ for which

$$\frac{1-r}{\phi(r)} \rightarrow 0 \text{ as } r \rightarrow 1. \quad (2)$$

Let $K > 0$, $\theta \in [0, 2\pi]$ and set

$$\Omega(\phi, \theta, K) = \{z \in U: |e^{i\theta} - z| \leq K\phi(r)\}$$

where $r = |z|$. The region Ω makes tangential contact with C at $\exp(i\theta)$; when $\phi(r) = (1-r^2)^{\frac{1}{2}}$, for instance, $\Omega(\phi, \theta, 1)$ is the disc of radius $\frac{1}{2}$ centred at $\frac{1}{2}\exp(i\theta)$.

Definition. For any ϕ satisfying (2), we say that f has a T_ϕ - limit ℓ at $\exp(i\theta)$ on C if, for every positive K ,

$$f(z) \rightarrow \ell \text{ as } z \rightarrow e^{i\theta}, \quad z \in \Omega(\phi, \theta, K)$$

We are now in a position to state our next theorem which is a special case of some recent results of Nagel, Rudin and Shapiro.

THEOREM 3. ([5]).

For $0 < r < 1$, set

$$\phi(r) = (\log \frac{1}{1-r})^{-1},$$

and let $f \in D$, the Dirichlet space. Then f has a T_ϕ - limit at almost all points on C .

Nagel *et al.* also show in [5] that, for certain other kinds of exceptional sets E - intermediate between sets of log-capacity zero and measure zero - every f in D has a T_ϕ - limit at all points $\zeta \in C \setminus E$ where, this time, $\phi(r) = (1-r)^\epsilon$, $0 < \epsilon < 1$, and ϵ depends only on the size of the exceptional set E . All these results of course extend immediately to the full class of univalent functions. In particular, therefore, by Theorem 3, each univalent function has, at almost all points ζ on C , a limit within a region that makes tangential (indeed exponential) contact with C at ζ . This is in sharp contrast with the behaviour of functions in H^∞ , described in Section 2.

The results we have discussed so far all relate to the existence of certain kinds of restricted limits at points on C and one might ask whether, for a univalent f , there must always be some point ζ on C at which f has an *unrestricted limit*, that is, at which

$$\lim_{z \rightarrow \zeta} f(z)$$

exists as $z \rightarrow \zeta$ in any way from inside U . Such a point ζ , however, would correspond to a prime end of $f(U)$ whose *impression* [6, p. 276] consists of a single point and Caratheodory [1, p. 184] has given an example of a bounded simply connected domain G which has no such prime ends. Then, by the Riemann mapping theorem again, there is a univalent function f with $f(U) = G$ and this function cannot have an unrestricted limit at any point on C . There is an interesting subclass of univalent functions, however, the members of which always have unrestricted limits at some points on C .

5. STARLIKE UNIVALENT FUNCTIONS

A univalent function f , with $f(0) = 0$, is said to be *starlike* if the image domain $f(U)$ is starshaped with respect to 0, that is, $f(U)$ contains the line segment $[0, w]$ whenever it contains w . We give two examples of starshaped regions in Fig. 3; note that the region in (b) may have infinitely many slits.

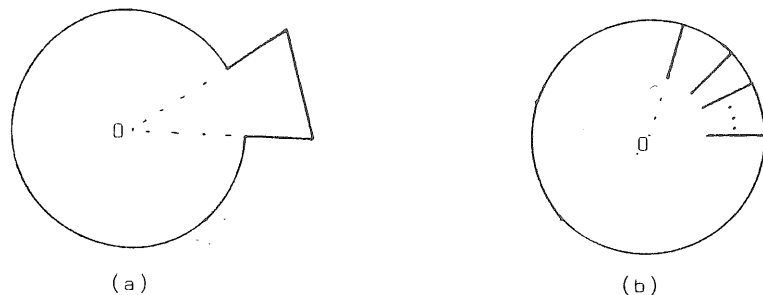


FIGURE 3

If f is starlike and bounded in U , then

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = \hat{f}(\theta) \quad (3)$$

exists for every θ in $[0, 2\pi]$. (Indeed it can be shown that such functions have, for every ϵ in $(0, 1)$, a T_ϕ -limit with $\phi(r) = (1-r)^\epsilon$ at all points on C . Details, the reader will be relieved to learn, to appear elsewhere.) By classical results of Baire on pointwise limits of sequences of continuous functions, it follows from (3) that the set A of points of discontinuity of \hat{f} is a set of the first category. Hence $\theta = [0, 2\pi] \setminus A$, the set of points on which \hat{f} is continuous, is a set of the second category and is thus uncountable and everywhere dense in $[0, 2\pi]$. Next, by the usual Poisson representation formula,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta-t) \hat{f}(t) dt,$$

$$z = re^{i\theta} \in U,$$

where P is the Poisson kernel, and it follows from this, by a standard result, that if f is continuous at $t = \theta_0$,

$$\lim_{z \rightarrow \zeta_0} f(z) = \hat{f}(\theta_0)$$

as $z \rightarrow \zeta_0 = \exp(i\theta_0)$ in any way from inside U . Noting finally that the set of points at which any function is discontinuous is of type F_σ , we have thus proved one part of our concluding theorem; the second part is an easy consequence of (the proof of) [3, Theorem 1].

THEOREM 4.

A subset E of C is the set of points at which some bounded starlike function f does not have unrestricted limits if and only if E is of type F_σ and of first category.

A bounded starlike function thus has unrestricted limits at a set of points on C which may have measure zero but is uncountable and dense on C .

6. SOME OPEN QUESTIONS

A number of questions arise naturally from the results discussed above, and we conclude this note with a brief selection.

- (a) Can we reduce the size of the exceptional set in Theorem 2 or in Theorem 3 for functions in D or for univalent functions?

In this context we note that the existence of an angular limit at a point on C does not imply the existence of a T_ϕ -limit with $\phi(r) = (1-r)^\epsilon$ for any ϵ in $(0, 1)$ at that point either for functions in D or for univalent functions. Details, again, to appear elsewhere.

- (b) Is the conclusion of Theorem 3 sharp, with respect to the type of tangential limit obtained, for star-like functions, for univalent functions or for functions in D ? Is there a function ϕ (satisfying the conditions in Section 4) such that there exists a univalent function f which does not have a T_ϕ - limit at any point on C ?

Answers on a postcard, please.

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BOOK REVIEWS

"FIELD EXTENTIONS AND GALOIS THEORY"

By *Julio R. Bastida*

Published by *Addison-Wesley*, 1984, £41.40 stg.

ISBN 0-201-13521-3

The author begins with a four-page "Historical Introduction" followed by fifteen pages devoted to "Prerequisites" and ten (!) to "Notations". The work proper is divided into four chapters entitled "Preliminaries on Fields and Polynomials", pp. 1-40; "Algebraic Extensions", pp. 41-91; "Galois Theory", pp. 92-211 and finally "Transcendental Extensions", pp. 212-280.

"In this book, Professor Bastida has set forth this classical theory, of field extensions and their Galois groups, with meticulous care and clarity. The treatment is self-contained, at a level accessible to a sufficiently well-motivated graduate student, starting with the most elementary facts about fields and polynomials and proceeding painstakingly, never omitting precise definitions and illustrative examples and problems. The qualified reader will be able to progress rapidly, while securing a firm grasp of the fundamental concepts and of the important phenomena that arise in the theory of fields. Ultimately, the study of this book will provide an intuitively clear and logically exact familiarity with the basic facts of a comprehensive area in the theory of fields. The author has judiciously stopped short (except in exercises) of developed specialized topics important to the various applications of the theory, but we believe he has realized his aim of providing the reader with a sound foundation from which to embark on the study of these more specialized subjects."

The above is an extract from the foreword by Roger Lyndon, and it hardly seems necessary for me to add to it, so I shall just make some comments instead.

Professor Bastida has adopted the term "factorial domain" instead of what I would have considered to be the standard terminology, namely unique factorisation domain or UFD for short. At least "UFD" has the merit of describing (precisely) a property of the domain in question.

The examples given in 3.2.4 - 3.2.8 are well chosen and interesting and illustrate the following facts: There are fields which are not prime, but possess a unique automorphism. There are proper field extensions having trivial Galois group and hence have a unique intermediate field invariant in the top field. There exists a field extension whose Galois group contains infinitely many subgroups having the same field of invariants and making up a chain.

In another instance, a classical example due to Dedekind is used to motivate the introduction of topological notions in studying infinite Galois theory. This didacticism in the author's approach to the subject is very commendable.

The book appears in the series "Encyclopedia of Mathematics and its Applications", however, the treatment is far from encyclopaedic. The notes and suggestions for further reading which follow each chapter indicate it was not the author's intention to provide such coverage. However, it does seem to me that the book is hardly likely to supplant Jacobson's "Lectures in Abstract Algebra" Vol. III in providing a comprehensive introduction to the theory of fields; and Jacobson provides better value for money!

Let me conclude with the following extract which I found rather droll:

"It is well known that some beginners in algebra, with complete disregard for the classical binomial theorem are quite prepared to accept the validity of the equality:

$$(\alpha \pm \beta)^n = \alpha^n \pm \beta^n.$$

As a consequence, they reach a number of interesting conclusions." I wonder what Fermat would have had to say to them?

*James Ward,
Mathematics Department,
University College,
Galway.*

PROBLEM PAGE

It seems much too soon to give the solutions to my last pair of problems so, for now, here are two more. The first was suggested by Finbarr Holland who says that it arises naturally in the theory of edge functions.

1. Consider the 12x12 complex determinant

| | | | | | | | | | | | |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| a ₁ | a ₂ | a ₃ | a ₄ | a ₅ | a ₆ | a ₇ | a ₈ | a ₉ | a ₁₀ | a ₁₁ | a ₁₂ |
| a ₄ | a ₅ | a ₆ | a ₇ | a ₈ | a ₉ | a ₁₀ | a ₁₁ | a ₁₂ | a ₁ | a ₂ | a ₃ |
| a ₇ | a ₈ | a ₉ | a ₁₀ | a ₁₁ | a ₁₂ | a ₁ | a ₂ | a ₃ | a ₄ | a ₅ | a ₆ |
| a ₁₀ | a ₁₁ | a ₁₂ | a ₁ | a ₂ | a ₃ | a ₄ | a ₅ | a ₆ | a ₇ | a ₈ | a ₉ |
| b ₁ | b ₂ | b ₃ | b ₄ | b ₅ | b ₆ | b ₇ | b ₈ | b ₉ | b ₁₀ | b ₁₁ | b ₁₂ |
| b ₄ | b ₅ | b ₆ | b ₇ | b ₈ | b ₉ | b ₁₀ | b ₁₁ | b ₁₂ | b ₁ | b ₂ | b ₃ |
| b ₇ | b ₈ | b ₉ | b ₁₀ | b ₁₁ | b ₁₂ | b ₁ | b ₂ | b ₃ | b ₄ | b ₅ | b ₆ |
| b ₁₀ | b ₁₁ | b ₁₂ | b ₁ | b ₂ | b ₃ | b ₄ | b ₅ | b ₆ | b ₇ | b ₈ | b ₉ |
| c ₁ | c ₂ | c ₃ | c ₄ | c ₅ | c ₆ | c ₇ | c ₈ | c ₉ | c ₁₀ | c ₁₁ | c ₁₂ |
| c ₄ | c ₅ | c ₆ | c ₇ | c ₈ | c ₉ | c ₁₀ | c ₁₁ | c ₁₂ | c ₁ | c ₂ | c ₃ |
| c ₇ | c ₈ | c ₉ | c ₁₀ | c ₁₁ | c ₁₂ | c ₁ | c ₂ | c ₃ | c ₄ | c ₅ | c ₆ |
| c ₁₀ | c ₁₁ | c ₁₂ | c ₁ | c ₂ | c ₃ | c ₄ | c ₅ | c ₆ | c ₇ | c ₈ | c ₉ |

Express this determinant as the product of four 3x3 determinants.

The next problem I heard of from Milne Anderson (University College, London) some years ago.

2. Prove or give a counterexample to the following statement.

If $a_n \geq 0$, for $n = 1, 2, \dots$, and $\sum_{n=1}^{\infty} a_n < \infty$ then

$$\sum_{n=3}^{\infty} a_n \left(1 - \frac{1}{\log n}\right) < \infty.$$

*Phil Rippon,
The Open University,
Milton Keynes.*

CONFERENCE REPORT

IRISH MECHANICS GROUP CONFERENCE

On Wednesday, 19th December last, some thirty Applied Mathematicians and Mechanicians from Ireland, both North and South, together with some colleagues from England and the United States, participated in a one-day Conference held at the School of Theoretical Physics, Dublin Institute for Advanced Studies. The Conference, organised by the Irish Mechanics Group in collaboration with the School of Theoretical Physics, had as its theme "Modern Developments in Mechanics", and was arranged specifically in honour of, and to pay tribute to, Professor Patrick M. Quinlan, B.E., M.Sc., D.Sc., Ph.D., F.I.E.I., M.R.I.A., of the Department of Mathematical Physics, University College, Cork, on the occasion of his sixty-fifth birthday.

The very broad range of topics presented by the seven scheduled speakers at the Conference was, in a way, a tribute to, and representative of, the breadth and depth of the contribution made by Professor Quinlan to the knowledge, understanding and useage of Applied Mathematics and Mechanics here in Ireland, and abroad - not to mention his commitment to and significant involvement in the political and social life of this country over a career as an academic and politician spanning the last forty or so years.

The papers presented to the Meeting ranged from reports on work done by A.D. Norris of Exxon Research, U.S.A., on "Effective Moduli of Composites", through "Numerical Methods in Fracture" by A.P. Parker of North Staffordshire Polytechnic. A.B. Tayler from Oxford addressed the Meeting on "Resonance in a Gear Box" and there then followed a series of papers, all by former students and colleagues of Professor Quinlan. P.F. Hodnett of the National Institute for Higher Education, Limerick, presented a paper on "Computer-Aided Design of Air-Journal Bearings", and J.N. Flavin of University College Galway spoke

on "St Venant's Principle". M.F. McCarthy, also of University College Galway, was scheduled to speak on some recent work done by him on "Scattering by Circular Disks", but, due to family illness, the Meeting was unfortunately deprived of his presentation. No such Conference to honour Professor Quinlan would have been complete without at least some reference to his Edge-Function Method, and this need was admirably met by a presentation by J.J. Grannell of University College Cork on "Progress on the Edge-Function Method", to end the Conference.

Each speaker, in his own unique way, paid tribute to Professor (Paddy) Quinlan and dedicated his paper to honour him of this special occasion. The very interesting results of, and insights into, the recent developments of the speakers' works, together with the congenial atmosphere and excellent Conference facilities afforded by the School of Theoretical Physics, made for a very stimulating and successful Conference.

Later that evening, a celebration dinner in Broc House to honour Paddy and his wife Jane was attended by the Conference participants, together with some former students of Paddy's. After the delightful repast, a further fifteen or so colleagues from the Universities and Higher Institutes joined the group to personally honour Paddy. Jim Flavin of Galway, a former student and close colleague of his, regaled the gathering with a summary of Paddy's career - often with Jim's uniquely humorous anecdotes - and in a most suitable and appropriate manner honoured Paddy on behalf of those present as well as the many former students and friends who had expressed a desire to be associated with the event, but could not themselves be present. A presentation was then made to Paddy and Jane of a piece of Beleek china and an inscribed watercolour print of University College Cork, made in the early days of what was then Queen's College, Cork, with which Paddy has had for so long and continues still to have a deep affectionate association and to the life of which he has made such

a significant and unique contribution.

Michael J.A. O'Callaghan

CONFERENCE ANNOUNCEMENTS

BRITISH MATHEMATICAL COLLOQUIUM

The 37th British Mathematical Colloquium will be held in the University of Cambridge on 2nd, 3rd and 4th April 1985. The principal speakers will be D. Zagier (Bonn), C.L. Feffermann (Princeton), and M. Gromov (Paris).

Speakers from the U.K. include T.O. Hawkes, E.C. Lance, M. Rees, A.D. Gardiner, C.R. Leedham-Green, K.J. Falconer, R.A. Bailey, W. Parry, D.B.A. Epstein, B. Birch, J.M. Hammersley, M. Paterson, R. Brown, J.S. Pym and C. Hooley.

There will be a discussion of Information Technology presented by C.A.R. Hoare, F.R.S., and a display of Computer-Aided Teaching of Applied Mathematics.

The registration fee is £12 for those paying their bill in full by January 31st, 1985; thereafter the fee is £18. For research students of no more than 3 years standing these amounts are halved. The cost of accommodation and meals for the full period is £108.

Application forms and further information are available from the Colloquium secretary, Dr R.C. Mason, Department of Pure Mathematics and Mathematical Statistics, 16 Mill Lane, Cambridge CB2 1SB.

REPRESENTATIONS OF ALGEBRAS

There will be a Symposium on Representations of Algebras at Durham University from 15 July to 25 July 1985. Principal lecturers will be J.L. Alperin (Chicago), M. Auslander (Brandeis), D.J. Benson (Northwestern), P. Gabriel (Zurich), H. Knörrer (Bonn), H. Kraft (Basel), C.M. Ringel (Bielefeld), A.V. Roiter (Kiev).

The meeting is being organised by S. Brenner, M.C.R. Butler and P.J. Webb under the auspices of the LMS and the SERC.

Further particulars are available from Dr S. Brenner, A.M.T.P., The University of Liverpool, P.O. Box 147, Liverpool L69 3BX.

TEACHING MATHEMATICAL MODELLING

The Second International Conference of the Teaching of Mathematical Modelling will be held at Exeter University from 16 July to 19 July, 1985.

The aim of this conference is to provide a forum to discuss how applications can play a vital part in the teaching of mathematics. Mathematics is used extensively in many diverse areas, and this has been reflected to a certain extent in what mathematics we teach and how we teach it. The Conference will bring together those interested in the use of application of mathematics in teaching and will be organised under the following themes: (a) experience with modelling courses, (b) the role of case studies in maths teaching, (c) the impact of micros and (d) methodology.

Further details may be obtained from: Ms S. Williams, Conference Secretary, University of Exeter, School of Education, St Lukes, Exeter. Telephone: 0392-76311 (Ext. 275).

HOMOTOPICAL ALGEBRA

There has recently been particular interest among algebraic topologists in such areas such as abstract homotopy theory, algebraic models of homotopy types and coherence questions. These areas are also attracting interest from topos theorists and category theorists, with a view to application in other areas such as algebraic geometry.

The aim of the Workshop is to bring together workers with interest in this common ground, for lectures and discussions and to allow for participants to learn something of material related to their own major interests.

The workshop will take place at University College of North Wales, Bangor, from 21 July to 26 July, 1985. This date has been chosen because it immediately follows a Category Theory Meeting at the Isle of Thorns, Sussex (for further information contact C.J. Mulvey) and precedes a Durham Symposium on Homotopy Theory (limited numbers, but if interested contact J.D.S. Jones of Warwick University).

The concerns of the Workshop are the following areas and their relationships: Algebraic Homotopy Theory, Homotopy Coherence, Algebraic Models of Homotopy Types, Cohomological Methods, Rational Homotopy Theory, Topos and Sheaf Theory.

Talks at the Workshop will be restricted to those relevant to the main themes. It is expected that discussions will take place on the extensive manuscript of Grothendieck which concerns the area of the Workshop.

Partial support is expected from the London Mathematical Society. For further information contact: Dr T. Porter, Department of Pure Mathematics, University College of North Wales, Bangor, Gwynedd LL57 2UW.

COMBINATORIAL CONFERENCE

The tenth British Combinatorial Conference will be held in the University of Glasgow during the week 22 July to 26 July 1985. The conference is organised jointly by the Mathematics department and the British Combinatorial Committee, and it receives financial support from the London Mathematical Society and the British Council.

The following lectures have been arranged: G. Andrews (*Combinatorics and Ramanujan's 'Lost' Notebook*). J. Beck (*Irregularities of point distributions - a theory on the border of combinatorics, number theory and geometry*). H.J. Beker (*Cryptography*). G.R. Grimmett (*Flows through random networks*). A.J. Hoffman (*On greedy algorithms that work*). J.H. van Lint ($[0,1]^*$ - distance problems in combinatorics). C. St. J.A. Nash-Williams (*Detachments of graphs and generalised Euler trails*). P.D. Seymour (*Graph minors - a survey*). J.L. Tits (*Finite incidence geometries with Coxeter diagrams of affine type*).

There will be special sessions for contributed talks, covering all aspects of combinatorics.

Booking forms and further information can be obtained from the local organiser of the conference: Dr Ian Anderson, Department of Mathematics, University of Glasgow, Glasgow G12 8QW.

GROUPS

This conference is sponsored by the Edinburgh Mathematical Society and the London Mathematical Society and will be held in St Andrews, Scotland, from 27 July to 10 August 1985. During the week 27 July to 3 August the following speakers will each give a course of lectures: Professor S. Bachmuth (California), Professor G. Baumslag (New York), Dr P.M. Neumann (Oxford), Dr J.E. Roseblade (Cambridge), Professor J. Tits (Paris). During the week 3-10 August there will be a programme of seminars and invited lectures by other conference participants.

Information and application forms from Dr C. M. Campbell and Dr E.F. Robertson, Mathematical Institute, University of St Andrews, North Haugh, St Andrews, Fife KY16 9SS, Scotland.

PROTEXT II CONFERENCE

The Second International Conference
on Text Processing Systems

23 - 25 October 1985

and the related event

PROTEXT II

SHORT COURSE

An Introduction to
Text Processing Systems:
Current Problems and Solutions

21 - 22 October 1985

Both events will be held at

BERKELEY COURT HOTEL, DUBLIN, IRELAND

CONFERENCE: 23 - 25 OCTOBER 1985

The conference provides a forum for the discussion of the latest research on computer-aided generalised text processing systems. It is intended that participation will be evenly divided between academia and industry, and an even balance will be maintained between hardware and software topics. Several formal discussions on controversial issues of current interest will be organised. The Proceedings of the Conference will be published in book form within two months of the conference. An idea of the scope of the conference may be obtained from the contents of the PROTEXT I Conference Proceedings.

SHORT COURSE: 21 - 22 OCTOBER 1985

This is a carefully coordinated introduction to computer-aided generalised text processing systems. It is intended both for the beginner, who has little previous knowledge of

the subject, and also for the experienced person who wishes to broaden his knowledge into new areas. The scope of the short course will be the same as that of the Conference, but it will be at an introductory level. The Lecture Notes will be published in book form in advance of the short course.

CALL FOR PAPERS

Abstracts of contributed papers on topics relevant to the Conference should be submitted to the Organising Committee by 1st June 1985. Notification of acceptance will be mailed by 1st July. Abstracts should be at most one A4-page in length. For information, contact PROTEXT II Organising Committee, c/o Boole Press Limited, P.O. Box 5, 51 Sandycove Road, Dun Laoghaire, Co. Dublin.

PSYCHOMETRICS AND CLASSIFICATION

The Fourth European Meeting of the Psychometric Society and the Classification Societies will be held jointly from Tuesday 2nd to Friday 5th July 1985 in Cambridge, England.

Papers on psychometrics and classification are invited for this meeting. Suitable topics include:

Latent trait models, Factor analysis, Structural models, Scaling, Measurement theory, Correspondence analysis, Hierarchical and non-hierarchical classifications, Taxonomy and cladistics, Pattern recognition, Comparison of classifications, Data analysis, etc., etc.

Titles, preferably with abstracts, should be sent together with a completed form to the address below.

Accommodation and meals will be provided at Queen's College, and the sessions will take place in University lecture theatres nearby. The conference will assemble during the afternoon/evening of Tuesday 2nd July, and will close with

lunch on Friday 5th July.

All correspondence should be sent to:

Dr Ian Nimmo-Smith,
MRC Applied Psychology Unit,
15 Chaucer Road,
Cambridge CB2 2EF,
United Kingdom.

GROUPS IN GALWAY

This conference will take place on 10th and 11th May, 1985 at University College, Galway. The speakers will include Dr D. MacHale (University College, Cork), Dr T. Murphy (Trinity College, Dublin), Prof. T. Laffey (University College, Dublin) and Prof. S. Tobin (University College, Galway).

Those who wish to present short communications are asked to get in contact with Dr R. Dark (Mathematics Department, University College, Galway) from whom further details are available.

AN IRISHMAN'S DIARY

Long, long ago, the Murdoch parents gazed into the pram where little Brian lay, his face contorted in concentration. Ah, they said proudly to one another; ah, and then they made those revolting gooey noises that adults are wont to make in the company of small incontinent offspring. And just think, murmured one Murdoch parent to the other: soon the little fellow will be walking. Ah, they repeated together, and little Brian's face wrinkled up even more, so that it resembled an empty prune skin. Gootchy gootchy goo, cried a parent, mistaking the look of concentration; 'oo's a naughty ickle boy, doing number twos in his nappy wappy?

But young Brian was doing no such thing. Young Brian was thinking about walking. Not the physical feat of walking; for a child like Brian, that would have been tiresome, trivial stuff. No no, what interested him were the mathematics involved in random walks; and so while his parents simpered, his brow puckered in a frown of deep mathematical contemplation.

'Oo's a good ickle boy? Gutchy gutchy goo.

Random Walks

The foregoing might, or might not have happened. There is no documentary evidence as to when Brian Murdoch became interested in the mathematics of random walks. It seems, though, that when one takes an interest in the mathematics of random walks, the condition becomes permanent. Because after he did his BA in maths at Trinity, Brian went off to Princeton where he did a PhD in random walks.

Random walks, egad, I murmured; dear heaven above. A PhD on random walks. Whatever next? PhDs on the colour of

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footballers' socks, PhDs on nail varnish tints as used by the teenage natives of Ulan Bator, possibly even the master's on scout camps in Powerscourt? Full of these speculations, I rang Brian up. What, I said with a testy sneer in my voice, the kind of testy sneer which would make Henry Kissinger break out in a sweat and tremble, what can be mathematical about walking? I mean, for heaven's sake, and so on.

Sighed Patiently

Brian sighed patiently. When you've given your life ever since pramhood to the mathematics of random walkings, you get used to being misunderstood. "It is," he conceded, "very hard to explain, even to fellow mathematicians, never mind to lay people."

Poo, I cried, stuff and nonsense; what can be remotely mathematical about having a ramble in the countryside? Eh? Got you there, haven't I? Hey? Go on, admit it. Game set and match, checkmate, a clean sweep in one. If Brian had wept at this point, I would not have blamed him. All his life has been like this, ever since his drooling parents were gutchy gutchy goeing.

"You see," he said slowly - I could sense him shaping his words with awful care, as if he were talking to a very old, very deaf person, which I think is generally considered a fair comparison with me on one of my better days - "what we are dealing with is a question of probabilities. Now, for example, imagine we're talking about a man who's hopelessly drunk"- well he could hardly have chosen a better example when talking to the Diary, drunk every morning by 10 o'clock - "and he doesn't know what direction he's going in and can't remember what direction he last stepped in. He can go forwards, backwards, left or right; that's where the mathematics of probabilities come in."

Harmonic Functions

Ah, I ahhed. Ah. And then assuming daylight had entered my brain, when in fact it had shut tight as an anemone, he started telling me about discrete harmonic functions. Hoy, I interrupted, hoy, and so stilled the flow.

"Yes, well, I did warn you there's no simple way of explaining. And our random walks are not limited to a two-dimensional plane. They can be in three dimensions, or, if you can grasp this, four dimensions, or five." Well, even my simple mind can grasp that speculations concerning four or five dimensions are the kind of thing to give you vertigo. "For example," added Brian cheerfully, "your drunk on a two-dimensional plane is certain to get to his destination over an infinite amount of time. But a drunken spaceman, in three dimensions, is not. And there's no simple way of explaining that either."

If you think that mathematicians engaged in the probabilities of random walks find themselves with no-one to talk to, you're right; so his fellow random walkers regularly keep in touch with their random walks theories.

Even at Trinity, where Brian is professor in charge of the sums department, there are few if any random walkers; his wife is not one, nor are his children. So spare a thought, good reader, on this early day of a brand new, shining 1985, on the random walkers of this world. Theirs is not an easy lot.

Kevin Myers

THE IRISH MATHEMATICAL SOCIETY

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