I feel that students should meet the various constructions separately in a first treatment; the general theorem can follow if there is time - and a desire - for it.

(b) It may be worth mentioning the somewhat surprising fact that the theory of groups acting on (infinite trees is significant in the study of finite groups; see, for example, Goldschmidt's article [3].

(c) For a more topological account of the Bass-Serre Theory see Cohen's notes [2] or the Scott-Wall article [5].

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AN INTUITIVE PROOF OF BROUWER'S FIXED POINT THEOREM IN R^2

Clarence C. Morrison[1] *and Martin Stynes*[2]

Fixed point theorems play a major role in general equilibrium theory. Brouwer's theorem is the most basic of these; it states that any continuous function mapping a closed bounded convex set into itself must contain at least one fixed point (i.e., a point that is its own image).

Elementary discussions invariably give an intuitive proof of the theorem for functions of a single variable, as illustrated in Fig. 1. In R^1 a set is convex if and only if it is an interval; thus a continuous mapping of the closed boundary interval $[x_0, x_1]$ into itself can be represented by a curve f.

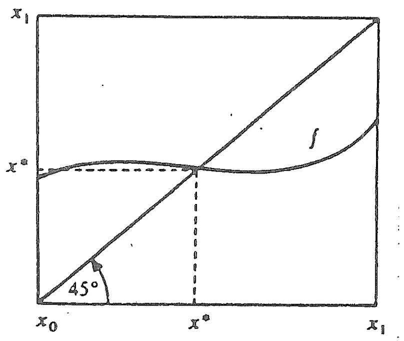

FIGURE 1

Since f connects the left-hand side of the rectangle to the right-hand side of the rectangle, it is intuitively obvious that f must intersect the diagonal of the rectangle at least

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once, and at this point $f(x^*) = x^*$. A bit more formally, if $f(x_0) \neq x_0$ and $f(x_1) \neq x_1$, then $\phi(x_0) = f(x_0) - x_0 > 0$ and $\phi(x_1) = f(x_1) - x_1 < 0$. Since ϕ is continuous on $[x_0, x_1]$, the intermediate value theorem implies that ϕ must assume the val zero somewhere on the open interval (x_0, x_1), which proves the theorem.

An intermediate- or advanced-level student should be a bi street-wise and skeptical of the validity of demonstrations based on two-dimensional diagrams. The purpose of this note is to demonstrate that the intuitive graphic proof generalize to three dimensions (i.e., to functions on R^2) and can be mad rigorous at that level.

To begin, let W be any closed bounded (i.e., compact) co vex set in R^2 and let f be any continuous function mapping W into itself. Since W is bounded it can be contained in a re angle as shown in Fig. 2. We may now extend f to the closed

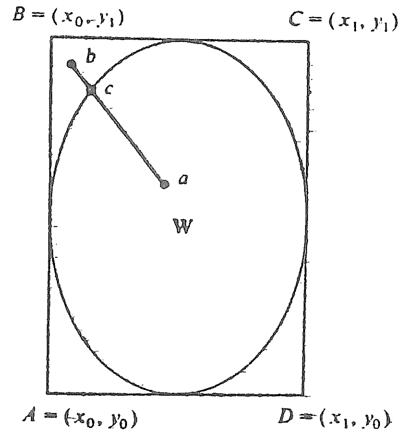

FIGURE 2

rectangle ABCD as follows. Choose an arbitrary interior poi a in W and for each point b in the rectangle but not in W, define f(b) to be the image of the point c at which the line through a and b intersects the boundary of W. The extended

mapping is continuous and maps the closed rectangle ABCD into W. For simplicity we denote the extended mapping by f and represent f by $f(x,y) = (x', y')$ where

$$x' = g(x,y)$$
$$y' = h(x,y) \qquad (1)$$

with $x, x' \in [x_0, x_1]$; $y, y' \in [y_0, y_1]$; and with g and h continuous.

Fig. 3 gives a three-dimensional representation with W and the rectangle ABCD in the horizontal coordinate plane. Represented above the two-dimensional rectangle ABCD is the three-dimensional box EIJFGKLH. The sides EI, FJ, GK and HL as well as EH, IL, JK and FG all correspond to the interval $[x_0, x_1]$. Similarly, the sides EF, IJ, LK and HG all correspond to the interval $[y_0, y_1]$. The graph of g is given by the surface MNOP which is restricted to the closed three-dimensional box since x' is restricted to $[x_0, x_1]$.

Now consider the projection mappings p_x, p_y defined by

$$x = p_x(x,y)$$
$$y = p_y(x,y) \qquad (2)$$

The graph of p_x in Fig. 3 is the diagonal plane EFKL and the intersection of g and p_x is the manifold RQ which projects into the horizontal coordinate plane as TS. Since neither the surface MNOP nor the diagonal plane EFKL have any rips in them, it is intuitively obvious that the intersection of g and p_x must connect the face and back of the three-dimensional box and that the projection TS connects opposite sides of ABCD. Further, TS represents the points (x,y) in ABCD for which $x' = x$. Similarly, the intersection of h and p_y projected to the coordinate plane will connect the left and right sides of ABCD as UV does in Fig. 3. This projection represents the points (x,y) in ABCD for which $y' = y$. Again, intuition tells us that UV must intersect TS (at least once) and any intersection of UV and TS

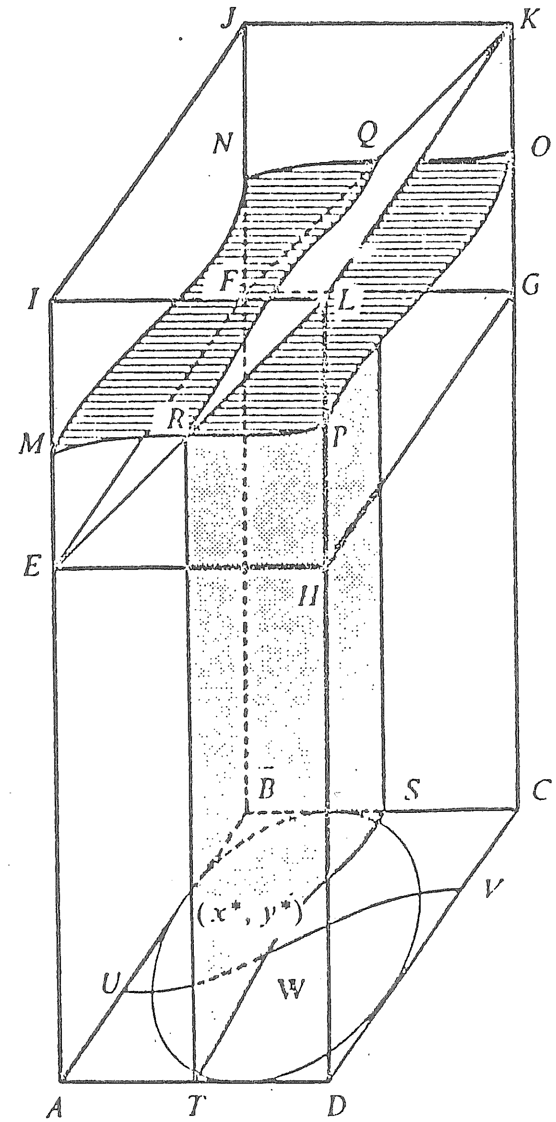

FIGURE 3

is a fixed point of f.

To make this demonstration rigorous, it is necessary to prove that TS (or UV) actually connects opposite sides of ABCD. As a first step we show that if Brouwer's theorem holds for functions which are "very close" to f, then it must hold for f itself. Let $\|q-r\|$ denote the usual Euclidean distance between two points q,r in R^2. For any given $\varepsilon > 0$, suppose that there exists a continuous function f^*:ABCD → ABCD such that $\|f^*(x,y)-f(x,y)\| \le \varepsilon$ for all $(x,y) \in$ ABCD, and such that f^* has a fixed point in ABCD. We claim that this property implies that f has a fixed point in ABCD. Applying the property, we can assume that for each n = 1,2,3,... there exists a continuous function f_n:ABCD → ABCD such that

$$\|f_n(x,y)-f(x,y)\| \le \frac{1}{n}$$

for all $(x,y) \in$ ABCD, and there is a point $Z_n \in$ ABCD such that $f_n(Z_n) = Z_n$. The compactness of ABCD implies that the sequence $\{Z_n\}$ has a limit point, Z^*. We invite the reader to show that Z^* is a fixed point of f.

It is thus sufficient to replace f by another function which closely approximates f, then prove the Brouwer theorem for the replacement function. The Weierstrass approximation theorem (a generalized version is proven in [4, §36]; for the specific R^2 case see [2, p. 187, problem 2]) yields, for a given $\varepsilon > 0$, a function $\hat{f} = (\hat{f}_1, \hat{f}_2)$:ABCD → R^2 such that $\|f(x,y)-\hat{f}(x,y)\| \le \varepsilon$ for all $(x,y) \in$ ABCD, with $\hat{f}_1$ and $\hat{f}_2$ polynomials in x and y. However $\hat{f}$ may give values lying at a distance ε outside of ABCD, so we must shrink its range slightly. To do this, replace $\hat{f}_1(x,y)$ by

$$\frac{x_0 + x_1}{2} + (x_1 - x_0)\left(\frac{\hat{f}_1(x,y) - \frac{x_0 + x_1}{2}}{x_1 - x_0 + 4\varepsilon}\right)$$

and replace $\hat{f}_2$ by a similar expression. A short calculation shows that these new functions (which for simplicity we again

call $\hat{f}_1$ and $\hat{f}_2$) approximate f and give us $\hat{f} = (\hat{f}_1, \hat{f}_2)$:ABCD $\rightarrow$
1 interior of ABCD.

Now define $\hat{g}$ on ABCD by:

$$\hat{g}(x,y) = \hat{f}_1(x,y) - x$$

Then

$$\hat{g} > 0 \text{ on AB and } \hat{g} < 0 \text{ on CD} \qquad (3)$$

We must modify $\hat{g}$ still further so that its partial derivatives satisfy certain conditions, while retaining property (3). First, on AD, $y = y_0$ is constant so on AD $\hat{g}(x,y) = \hat{g}(x,y_0)$ is just a polynomial in x. By altering $\hat{g}(x,y)$ slightly if necessary we can ensure that $\hat{g}(x,y_0)$ has no repeated factors. There are then no points on AD where $\hat{g}(x,y)$ and $\partial\hat{g}(x,y)/\partial x$ vanish simultaneously. A further slight perturbation of $\hat{g}$ will ensure that at least one partial derivative of $\hat{g}$ is non-zero at each point in ABCD where $\hat{g}(x,y) = 0$. This assertion follows from Sard's theorem ([3], [6, Chapter 13, §14]; or for a proof of a special case of this theorem which can easily be adapted to the present situation, see [1, p. 35]).

We can now proceed directly. By the implicit function theorem (see any advanced calculus book) the above condition on $\partial\hat{g}/\partial x$ and $\partial\hat{g}/\partial y$ in ABCD guarantees that $\hat{g}^{-1}(0)$ is a simple one-dimensional curve in a neighbourhood of each point on $\hat{g}^{-1}(0)$. Consequently $\hat{g}^{-1}(0)$ is a collection of simple curves, no two of which intersect. Wherever one of these curves intersects AD, our earlier condition on $\partial\hat{g}/\partial x$ guarantees that the curve is not tangent to AD. By (3) and the $\partial\hat{g}/\partial x$ condition, the number of points in AD $\cap$ $\hat{g}^{-1}(0)$ is odd, since at each such point $\hat{g}(x,y_0)$ changes sign. Curves which originate and terminate on AD account for an even number of these points so there must be a curve that has only one endpoint on AD. The other end of this curve cannot be on AB or CD by (3), so it must join AD and BC and we are free to label the endpoints T and S.

To complete the proof, we define $\hat{h}(x,y) = \hat{f}_2(x,y) - y$. Clearly $\hat{h}$ is continuous. Since $\hat{h}(S) \leq 0 \leq \hat{h}(T)$, the intermediate value theorem assures us of the existence of at least one point (x^*,y^*) on TS such that $\hat{h}(x^*,y^*) = 0$ or (equivalently) $\hat{f}_2(x^*,y^*) = y^*$. Since all points (x,y) on TS satisfy $\hat{g}(x,y) = x$, we have $\hat{g}(x^*,y^*) = x^*$. Thus there exists at least one point (x^*,y^*) in ABCD such that $(x^*,y^*) = \hat{f}(x^*,y^*)$. Since $\hat{f}$:ABCD $\rightarrow$ W we must have $(x^*,y^*) \in W$. Thus (x^*,y^*) is a fixed point of $\hat{f}$.

It is well known that in the class of compact sets, the fixed point property is not restricted only to convex sets [5, p. 9]. It can be shown that if a set has the fixed point property, then any set to which it is homeomorphic also has the fixed point property [5, p. 9]. This theorem can be used to prove that various plane sets with amoeboid shapes have the fixed point property. Our proof given above shows that *any bounded set S in R^2 having an interior point x such that each ray from x has only one intersecting point with the boundary of S has the fixed point property.*

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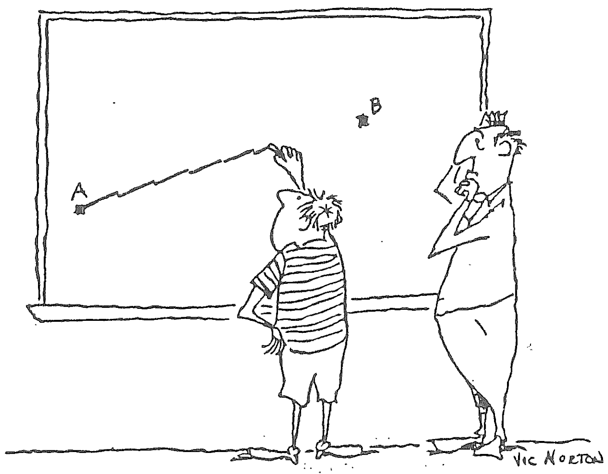
Cartoon without caption:
The computer-age generation



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THE ANALYTICAL REFORM OF IRISH MATHEMATICS

1800 - 1831

N.D. McMillan

The Origin of the Dublin Mathematical School

The mathematical tradition established by the Dublin Philosophical Society of William Molyneux (Fig. 1) had a major influence on the character of mathematics in Ireland [1]. The convergence of interests at the University of Dublin on specific aspects of mathematics, e.g. the theory of equations, optics, potential theory and variational principles [2], and the strong Irish tradition in statistics [3] had their origins in the interests and contributions of the members of the society.

STATISTICS	W. Petty, *Political Arithmetick* (London, 1690). F. Robartes, *An Arithmetical Paradox Concerning the Chances of Lotteries*, Phil. Mag. XVII (1693) pp.677-84.
GEOMETRY	St. George Ashe, *A New and Easy Way of Demonstrating Some Propositions in Euclid*, Phil. Mag. XIV (1684), pp. 672-6.
OPTICS	W. Molyneux, *Solution of a Dioptric Problem*, Bibliotheque Universelle et Historique, III (1686).
ENGINEERING MATHEMATICS	W. Molyneux, *A Demonstration of an Error Committed by Common Surveyors .....*, Phil. Mag., XIX (1677) pp. 625-31.
ASTRONOMY	W. Molyneux, *Concerning the Parallax of Fixed Stars*, Phil. Mag. VXII (1693) pp. 844-9. J. Walley, *Ptolemy's Quadripartite*, (Dublin, 1701).
ACOUSTICS	N. Marsh, *An Introductory Essay to the Doctrine of Sounds*, Phil. Mag. VIX (1684) pp. 472-88.

FIGURE 1: Mathematical Interest of the Dublin Philosophical Society Illustrated by Selection of Works.