ISOTROPIC TENSORS AND SYMMETRIC GROUPS

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Introduction

Students usually meet tensors in courses on Mathematical Physics [1, Chapter 3], and are sometimes told that all isotropic tensors can be expressed as sums of products of Kronecker's δ and the alternating tensor ε (see below). Several years ago, the first author was asked by an inquisitive student how to prove this fact. The author in question did not, at that time, know of a proof in the literature, and (having more energy then than now) he tried to find his own proof. He was led to a question about the group rings of symmetric groups, which he found interesting in its own right, and which he answered to the satisfaction of both himself and the student who had prompted the question. Soon afterwards, he learned that the statement about isotropic tensors is equivalent to a result proved by Weyl [4, page 53, Theorem (2.9.A)] using a different method. More recently, an error was noticed in the calculations in the group ring of the symmetric group, and we give here an exposition of a corrected version of this proof.

Notation

For each positive integer m, put $\underline{m} = \{1,2, \ldots ,m\}$, and let $\underline{n}^{\underline{m}}$ be the set of maps from $\underline{m}$ to $\underline{n}$. An element of $\underline{n}^{\underline{m}}$ can be identified with a *polyindex* $(i) = (i_1,i_2, \ldots ,i_m)$ where $1 \leq i_r \leq n$ $(1 \leq r \leq m)$. With respect to given axes in n-dimensional Euclidean space E^n, a *tensor* u can be defined as a map from $\underline{n}^{\underline{m}}$ to the field of real numbers: for each polyindex $(i) \in \underline{n}^{\underline{m}}$, we have a real coordinate $u(i)$. We shall say that u has *dimension* n and *order* m.

Examples A tensor v of order 1 is the same as a *vector* in E^n with coordinates $v(1)$, $v(2)$, $\ldots$,$v(n)$. Similarly, a tensor T or order 2 can be regarded as an nxn *matrix* with entries

$T(i_1,i_2)$. In particular, we put

$$\delta(i_1,i_2) = \begin{cases} 1 & \text{if } i_1 = i , \\ 0 & \text{if } i_2 \neq i , \end{cases}$$

which corresponds to the identity matrix. Moreover, for each polyindex $(i) = (i_1,i_2, \ldots ,i_n) \in \underline{n}^{\underline{n}}$, we can define

$$\varepsilon(i) = \begin{cases} 1 & \text{if } (i) \text{ is an even permutation of } \underline{n}, \\ -1 & \text{if } (i) \text{ is an odd permutation of } \underline{n}, \\ 0 & \text{if } (i) \text{ is not a permutation;} \end{cases}$$

then ε is a tensor of order n, equal to its dimension. If T is a tensor of order 2, then we can use ε to give a formula for its *determinant*:

$$\det T = \sum_{(i)\in \underline{n}^{\underline{n}}} \varepsilon(i_1,i_2, \ldots ,i_n)T(1,i_1)T(2,i_2)\ldots T(n,i_n).$$

Remark The indices i_1, i_2, $\ldots$,i_m are usually written as subscripts or superscripts, but we prefer to avoid double suffices; we consider only perpendicular axes in E^n, so we do not need to distinguish between cogredient and contragredient indices. Also, we shall not use the Summation Convention [3, page 59]. Examples of physical quantities represented by the concept are the *strain tensor*

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)$$

in the Theory of Elasticity [1, page 99, Section 3.101], and the *metric tensor* g_{ik} of Differential Geometry (as used in the Theory of Relativity) which determines the incremental distance

$$ds = \sqrt{\sum_{i,k} g_{ik}dx^i dx^k} \;;$$

in our case $g_{ik} = \delta(i,k)$.

Definition: When we change the axes in E^n, there is an nxn matrix $T(h,k)$ such that the coordinates of a vector v are altered to

$$v'(h) = \sum_{k=1}^{n} T(h,k)v(k)$$

For u to be a tensor of order m, it is required that its coordinates become

$$u'(i_1, \ldots, i_m) = \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} T(i_1,j_1)\ldots T(i_m,j_m)u(j_1,\ldots,j_m)$$

A tensor is said to be *isotropic* if its coordinates remain unaltered when we change from one set of perpendicular axes to any other such set with the same orientation; this means that $u'(i) = u(i)$ whenever the matrix $T(h,k)$ is orthogonal with determinant 1.

Examples It can be verified that the tensors δ and ε defined above are both isotropic [1, page 87, Section 3.03], and it is known that, in the 3-dimensional case, any isotropic tensor of order 2 or 3 is a multiple of δ or ε respectively. Moreover, it can be shown [1, page 88, Section 3.031] that the isotropic tensors of dimension 3 and order 4 are the linear combinations of the tensors

$$\delta(i_1,i_2)\delta(i_3,i_4),\ \delta(i_1,i_3)\delta(i_2,i_4),\ \delta(i_1,i_4)\delta(i_2,i_3)$$

which are called *outer products* of δ [1, page 115]. These facts can be used to motivate the next definition.

Notation

Taking outer products of δ and ε, we define the following isotropic tensors of dimension n and order m:

$$d(i) = \begin{cases} \delta(i_1,i_2)\delta(i_3,i_4)\ \ldots\ \delta(i_{m-1},i_m) & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

$$e(i) = \begin{cases} \varepsilon(i_1,i_2, \ldots, i_n)d(i_{n+1}, \ldots, i_m) & \text{if } m \geq n, \\ 0 & \text{if } m < n. \end{cases}$$

If α is an element of the symmetric group S_m of all per-

mutations of $\underline{m}$, and if hα denotes the image under α of an element $h \in \underline{m}$, then α acts on the polyindices (i) as a place permutation: $(\alpha i) = (i_{1\alpha}, i_{2\alpha}, \ldots, i_{m\alpha})$. This leads to an action on the tensors of order m given by $(u\alpha)(i) = u(\alpha i)$; we shall call the tensor uα a *conjugate* of u. It is clear that the conjugates of d and e are still isotropic. Our aim is to give a proof of the following result.

Theorem Let D_m be the set of linear combinations of the conjugates of the tensors d and e of order m, and let U_m be the set of isotropic tensors of order m. Then $D_m = U_m$.

Remark We have seen that $D_m \leq U_m$, so we have to show that $U_m \leq D_m$. Our proof uses induction on m, and our main tool is the conjugation action of S_m, which is said to have been first exploited by Schur, and developed by Weyl [4, page 96, Section 6]. We begin by deriving certain relations between the coordinates of an isotropic tensor.

Lemma 1 If u is isotropic, then

(a) $$\sum_{r=1}^{m} \delta(p,i_r)u(i_1, \ldots, i_{r-1}, q, i_{r+1}, \ldots, i_m) = \sum_{r=1}^{m} \delta(q,i_r)u(i_1, \ldots, i_{r-1}, p, i_{r+1}, \ldots, i_m)$$

(b) $$(n-1).u(i_1, \ldots, i_m) + \sum_{r=2}^{m} u(i_r, i_2, \ldots, i_{r-1}, i_1, i_{r+1}, \ldots i_m) = \sum_{r=2}^{m} \delta(i_1,i_r) \sum_{p=1}^{n} u(p, i_2, \ldots, i_{r-1}, p, i_{r+1}, \ldots i_m).$$

Proof The equation (a) is trivial when p = q, so we suppose $p \neq q$. Define

$$T(h,k) = \begin{cases} \cos t & \text{when } h = k = p \text{ or } h = k = q, \\ \sin t & \text{when } h = p \text{ and } k = q, \\ -\sin t & \text{when } h = q \text{ and } k = p, \\ \delta(h,k) & \text{otherwise.} \end{cases}$$

Then $T(h,k)$ represents a rotation of the pq plane by an angle t, and is an orthogonal matrix with determinant 1. Therefore

$$\sum_{j_1=1}^{n} \sum_{j_m=1}^{n} T(i_1,j_1) \ldots T(i_m,j_m)u(j_1, \ldots ,j_m) = u(i_1, \ldots,i_m).$$

We differentiate this equation with respect to t, and then take $t = 0$. Note that, when $t = 0$,

$$T(h,k) = \delta(h,k), \quad \frac{dT}{dt}(h,k) = \delta(p,h)\delta(q,k) - \delta(p,k)\delta(q,h);$$

hence we get

$$\sum_{r=1}^{m} \sum_{j_1=1}^{n} \ldots \sum_{j_m=1}^{n} \delta(i_1,j_1) \ldots \delta(i_{r-1},j_{r-1}).(\delta(p,i_r)\delta(q,j_r)-$$

$$\delta(p,j_r)\delta(q,i_r)).\delta(i_{r+1},j_{r+1}) \ldots \delta(i_m,j_m)u(j_1,\ldots,j_m)=0.$$

Now δ has the substitution property [1, page 59, Section 2.021]

$$\sum_{j_r=1}^{n} \delta(i_r,j_r)u(j_1, \ldots ,j_m) = u(j_1, \ldots ,j_{r-1},i_r,j_{r+1}, \ldots ,j_m)$$

Using this, we can deduce (a) from the last equation.

In (a), take $i_1 = p$, and add the resulting relations for $p = 1,2, \ldots ,n$. This gives

$$n.u(q,i_2, \ldots ,i_m) + \sum_{r=2}^{m} u(i_r,i_2, \ldots ,i_{r-1},q,i_{r+1}, \ldots ,i_m)$$

$$= u(q,i_2, \ldots ,i_m) + \sum_{r=2}^{m} \delta(q,i_r)\sum_{p=1}^{n} u(p,i_2, \ldots ,i_{r-1},p,i_{r+1}, \ldots ,i_m).$$

Replacing q by i_1, we get (b).

Definition The equation (b) suggests the following notation and Lemma. Let R_m be the rational group algebra of the symm-

etric group S_m [3, page 42, Section 2.2]. The elements of R_m are the expressions $\theta = \sum t_\alpha \alpha$, where the coefficients t_α are rational, and where α runs through S_m. If further, u is a tensor of order m, we define $u\theta = \sum t_\alpha (u\alpha)$, where $u\theta$ is given by the conjugation action. As usual, 1 denotes the identity permutation in S_m, and (h,k) is the transposition which interchanges h and k, but fixes the other elements of $\underline{m}$. We write

$$\phi = (n-1).1 + (1,2) + (1,3) + \ldots + (1,m) \in R_m.$$

Lemma 2 (a) If u is isotropic, then

$$u\phi = \sum_{r=2}^{m} \delta(i_1,i_r)\sum_{p=1}^{n} u(p,i_2, \ldots ,i_{r-1},p,i_r, \ldots ,i_m).$$

(b) $\{\theta \in R_m : U_m\theta \leq D_m\}$ is a (2-sided) ideal of R_m.

Proof (a) is a restatement of Lemma 1(b) in terms of the above definition. To prove (b), note that if $\alpha,\beta \in S_m$ then clearly $U_m\alpha = U_m$, $D_m\beta = D_m$. Hence $U_m\alpha\theta\beta \leq D_m\beta = D_m$ as required.

Proof of the Theorem We use induction on m. We interpret a tensor of order 0 as a *scalar* (a real number whose value does not depend on the choice of axes). This means that when $m=0$, then every tensor is a multiple of d, so the Theorem is trivial. When $m=n=1$, then a tensor is again the same as a scalar, and is a multiple of e, and so lies in D_m as required. Next suppose $m=1$ and $n \geq 2$, and let u be isotropic. Taking $p \neq q$ and (i) = (q) in Lemma 1(a), we get $u(p) = 0$ for all p, so the Theorem is again true. We have now proved the result when m=1 or 2, so we may suppose $m > 2$, and assume that the Theorem holds for orders less than m.

The contracted tensor [1, page 87]

$$\sum_{p=1}^{n} u(p,i_2, \ldots ,i_{r-1},p,i_{r+1}, \ldots ,i_m)$$

is clearly isotropic of order m-2, so it lies in D_{m-2} by the inductive hypothesis. Hence Lemma 2(a) implies that $u\phi \in D_m$ whenever u is isotropic, and therefore

(1) $$U_m\phi \leq D_m.$$

The following result will be proved later.

Lemma 3 Suppose $1 \leq m < n$, and let X be the ideal of R_m generated by ϕ. Then $1 \in X$.

It follows from (1) and Lemma 2(b) that $U_mX \leq D_m$. Assuming Lemma 3, we deduce that $U_m \leq U_mX \leq D_m$. This proves the Theorem when $m < n$.

Remark For each subset $\{i_1, i_2, \ldots, i_r\}$ of $\underline{m}$, define

$$\pi(i_1, i_2, \ldots, i_r) = \sum e_\alpha \alpha$$

where α runs through the permutations of $\{i_1, i_2, \ldots, i_r\}$ and where

$$e_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ is even,} \\ -1 & \text{if } \alpha \text{ is odd.} \end{cases}$$

It can be shown that if $m = n$, then $\phi.\pi(1,2, \ldots, m) = 0$, which implies that the conclusion of Lemma 3 no longer holds. Indeed, the tensor ε appears in U_m for the first time in this case, so a different argument is needed.

Returning to the proof of the Theorem, suppose that $m = n > 2$, and that u is isotropic. Taking $p = 2$, $q = 1$, and $(i) = (2,2,3,4, \ldots, m)$ in Lemma 1(a), we get

$$u(1,2, \ldots, m) = -u(2,1,3,4, \ldots, m).$$

Similarly, one can show that, if the polyindex $(i) \in \underline{m}^{\underline{m}}$ is a permutation, then interchanging any 2 indices alters the sign of $u(i)$. Since S_m is generated by transpositions [2, page 136, Theorem 21], it follows that if (i) is a permutation, and

if $u_0 = u(1,2, \ldots, m)$, then $u(i) = u_0\varepsilon(i)$. Writing $v(i) = u(i) - u_0\,\varepsilon(i)$, we deduce that v is an isotropic tensor with

(2) $$v(i) = 0 \text{ when } (i) \text{ is a permutation.}$$

But if (i) is not a permutation, then it must have a repeated entry. Now if $i_1 = i_2$ and $\theta = 1 - (1,2) \in R_m$, and if w is any tensor, then

$$(w\theta)(i) = 0.$$

We note also that if μ is the sum of the even permutations of $\underline{m}$, then $\mu\theta = \pi(1,2, \ldots, m)$ [2, page 137, Proposition 26], whence the last equation implies that $(w.\pi(1,2, \ldots, m))(i) = 0$ when $i_1 = i_2$. Similarly, one can see that if (i) is a polyindex with a repeated entry, and if w is any tensor of order m, then $(w.\pi(1,2, \ldots, m))(i) = 0$. Combining this with (2), we conclude that

(3) $$(v.\pi(1,2, \ldots, m))(i) = 0 \text{ for all } (i) \in \underline{m}^{\underline{m}}.$$

The following result will be proved later.

Lemma 4 Suppose $m = n \geq 1$, and let Y be the ideal of R_m generated by $\pi(1,2, \ldots, M)$ and ϕ. Then $1 \in Y$.

Copying the proof of Lemma 2(b), we see that if V_m is the set of isotropic tensors v which satisfy (3), then $\{\theta \in R_m : V_m\theta \leq D_m\}$ is an ideal of R_m. Using (1) and (3), we deduce that $V_mY \leq D_m$, so Lemma 4 implies that

$$v = v1 \in V_mY \leq D_m.$$

Hence $u = u_0\varepsilon + v \in D_m$ as required.

Finally, suppose $m > n$. Then there must be a repeated index among $i_1, i_2, \ldots, i_{n+1}$, and it follows as before that $u.\pi(1,2, \ldots, n+1) = 0$. In the same way as in the previous case, the Theorem is a consequence of the following result, which will be proved later.

Lemma 5 Suppose $m > n \geq 1$, and let Z be the ideal of R_m generated by $\pi(1,2, \ldots ,n+1)$ and ϕ. Then $1 \in Z$.

It now remains to prove the Lemmata 3, 4 and 5. This will be done with the help of the following calculations. We write

$$\underline{\theta} = n.1 + (1,2) + (1,3) + \ldots + (1,r) \in R_m.$$

Lemma 6 (a) $\pi(1,r+1,r+2, \ldots ,m) = (1,r).\pi(r,r+1, \ldots,m).(1,r)$

(b) $\underline{\theta}.\pi(r+1, r+2, \ldots ,m) = \phi.\pi(r+1,r+2, \ldots ,m) + \pi(1,r+1,r+2, \ldots ,m)$.

Proof (a) follows from the rule for conjugating permutations [2, pages 129-130]. To prove (b), let G and H be the symmetric groups of permutations of the sets $\{1,r+1,r+2, \ldots ,m\}$ and $\{r+1,r+2, \ldots ,m\}$ respectively. Then H is a subgroup of G of index $|G:H| = (m-r+1)!/(m-r)! = m-r+1$. If $i \neq j$ then $(1,i)^{-1}(1,j) = (1,i,j) \notin H$ and therefore the cosets $(1,i)H$ and $(1,j)H$ are distinct [2, page 33, Proposition 5]. It follows [2, page 34] that $\{1,(1,r+1),(1,r+2), \ldots ,(1,m)\}$ is a set of representatives (or *transversal*) for the cosets of H in G, and that

$$G = H \cup (1,r+1)H \cup (1,r+2)H \cup \ldots \cup (1,m)H.$$

Noting that multiplication by a transposition $(1,i)$ changes even to odd permutations and *vice-versa*, we deduce that

$$\pi(1,r+1,r+2, \ldots ,m)$$
$$= (1 - (1,r+1) - (1,r+2) - \ldots - (1,m)).\pi(r+1,r+2,\ldots ,m)$$
$$= (\theta - \phi).\pi(r+1,r+2, \ldots ,m),$$

which is equivalent to the required relation.

Proof of Lemma 3 If $m = 1 < n$, then $\phi = (n-1),1$, whence $1 \in X$. We may therefore suppose $m > 1$, and use induction on m. In particular, the inductive hypothesis allows us to assume that if $1 \leq r < m$, then 1 is in the ideal generated by θ, so there are permutations α_i, β_i of $\underline{r}$ such that

$$(4) \qquad 1 = \sum \alpha_i \theta \beta_i .$$

We shall prove by induction on r that

$$(5) \qquad \pi(r+1,r+2, \ldots ,m) \in X \qquad (0 \leq r < m).$$

We note first that if $1 < i \leq m$, then $(1,i).\pi(1,2, \ldots ,m) = -\pi(1,2, \ldots ,m)$ and hence $\phi.\pi(1,2, \ldots ,m) = (n-m).\pi(1,2, \ldots,m)$. Since $n > m$, it follows that (5) holds when $r = 0$. We may therefore suppose $r > 0$, and assume that $\pi(r,r+1, \ldots ,m) \in X$. Then $\pi(1,r+1,r+2, \ldots ,m) \in X$ by Lemma 6(a). Using Lemma 6(b) we deduce that

$$\theta.\pi(r+1,r+2, \ldots ,m) \in X.$$

Now the permutations β_i in (4) are disjoint from $\pi(r+1,r+2, \ldots ,m)$ and so commute with it. Hence we can multiply (4) by $\pi(r+1,r+2, \ldots ,m)$ to get

$$\pi(r+1,r+2, \ldots ,m) = \sum_i \alpha_i \theta.\pi(r+1,r+2, \ldots ,m)\beta_i \in X.$$

This proves (5). Taking $r = m-1$, we conclude that $1 = \pi(m) \in X$, as required.

Proof of Lemma 4 We shall prove by induction on r that

$$(6) \qquad \pi(r+1,r+2, \ldots ,m) \in Y \qquad (0 \leq r < m).$$

By the definition of Y, (6) holds when $r = 0$, so we may suppose $r > 0$, and assume that $\pi(r,r+1, \ldots ,m) \in Y$. Using Lemma 6, as above, we deduce that $\theta.\pi(r+1,r+2, \ldots ,m) \in Y$. Since $m = n > r$, it follows from Lemma 3 that 1 is in the ideal generated by θ. As before, this enables us to prove (6), and we get the result by taking $r = m-1$.

Proof of Lemma 5 If $m = 2$, then $n = 1$ and $\phi = (1,2)$, so $1 = \phi.(1,2) \in Z$. We may therefore suppose $m > 2$, and use induction on m. In particular, we may assume that if $n+1 < r < m$, then 1 is in the ideal generated by $\pi(1,2, \ldots ,n+2)$ and ϕ. Since

$$\pi(1,2, \ldots ,n+2) = (1 - (1,n+2) - (2,n+2) - \ldots -(n+1,n+2)).\pi(1,2, \ldots ,n+1)$$

it follows that there are permutations α_i, β_i, λ_j, μ_j, such that

$$(7) \qquad 1 = \sum_i \alpha_i \theta \beta_i + \sum_j \lambda_j \pi(1,2, \ldots ,n+1)\mu_j,$$

provided $n+1 < r < m$. If $r = n+1$, then the same result follows from Lemma 4. Moreover, if $r < n+1$, then Lemma 3 implies that 1 is in the ideal generated by θ, so we again get the equation (7), but now with $\lambda_j = \mu_j = 0$. Thus (7) holds whenever $r < m$.

We shall prove by induction on r that

$$(8) \qquad \pi(r+1,r+2, \ldots ,m) \in Z \quad (m-n-1 \leq r < m).$$

If γ is in the m-cycle $(1,2, \ldots ,m) \in S_m$, then

$$\pi(m-n,m-n+1, \ldots ,m) = \gamma^{-m+n+1}\pi(1,2, \ldots ,m)\gamma^{m-n-1} \in Z,$$

so (8) is true when $r = m-n-1$. We may therefore suppose that $r \geq m-n$, and assume that $\pi(r,r+1, \ldots ,m) \in Z$. Using Lemma 6, we deduce that $\theta.\pi(r+1,r+2, \ldots ,m) \in Z$. As before, we can multiply (7) by $\pi(r+1,r+2, \ldots ,m)$ to get

$$\pi(r+1,r+2, \ldots ,m) = \sum_i \alpha_i \pi(r+1,r+2, \ldots ,m)\beta_i + \sum_j \lambda_j \pi(1,2, \ldots ,n+1)\pi(r+1,r+2, \ldots ,m)\mu_j \in Z.$$

This proves (8). Taking $r = m-1$, we obtain the required result.

Remark It can be shown that if $1 \leq m < n$, then 1 is in the right or left ideal generated by ϕ. We do not know whether Y or Z can be replaced by a one-sided ideal in Lemma 4 or 5.

We mentioned in an earlier remark that $\pi(1,2, \ldots ,m)$ is needed in Lemma 4. It can also be shown that if either $m = 2$ and $n = 0$, or $m = 4$ and $n = 1$, or $m = 6$ and $n = 2$, then 1 is not in the ideal of R_m generated by ϕ. However, we do not know for which values of m and n, $\pi(1,2, \ldots ,n+1)$ is needed in Lemma 5.

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