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The next goal is to collect money for travel grants for the 1986 International Congress of Mathematicians in Berkeley.

With best thanks for your cooperation,

Yours sincerely,

Ollie Lehto

THE EVOLUTION OF RESONANT OSCILLATIONS IN CLOSED TUBES

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1. INTRODUCTION

This paper discusses the formulation and solution of a nonlinear initial value, boundary value problem that arises from a simple experiment in gas dynamics. A tube which is closed at one end, contains a gas. The gas in the tube is driven by an oscillating piston. It is observed that when the frequency of the piston is near to a natural frequency of the tube the resulting gas motion is periodic and characterised by a shock wave travelling over and back along the tube. The theoretical work to explain the final periodic motion goes back to Betchov [2] and Chester [3]. The reader should consult Seymour and Mortell [9] for more recent work on the problem. However, the problem of the evolution of the periodic motion of the gas from an initial state has not until now [4] been solved.

It is worthwhile noting, at this juncture, that nonlinear effects, such as shocks, can occur without any dramatically large input into the system. For example, in the present case when the piston is operating at the fundamental frequency of the tube, a shock has been observed even though the ratio of piston displacement to tube length is of the order 10^{-2} [8].

Before giving the details of the particular problem, the broader background in which it is set will be sketched. The study of nonlinear waves began with the pioneering work of Stokes [10] and Riemann [7]. Whitham [11] distinguishes two main classes of waves, hyperbolic and dispersive waves. Hyperbolic waves are solutions of a set of hyperbolic partial differential equations and our problem fits into this class. The intersection of characteristics for a nonlinear hyperbolic equation gives rise to the physical phenomenon of a shock.

If the nonlinear wave is travelling in one direction only and into a constant state, the exact solution is called a 'simple wave' and was known to Riemann. Riemann also exposed the fundamental difficulty when nonlinear waves are travelling in opposite directions. It is not, in general, possible to integrate the equations for the characteristics. This corresponds to the fact that nonlinear waves interact and one must know the details of the right-going waves to calculate how the left-going wave will propagate, and *vice-versa*. The fundamental difficulty still remains, and even in such authoritative works as [11] and [6] problems of nonlinear waves travelling through each other in opposite directions receive scant attention.

The problem discussed in this paper involves waves in a tube of finite length and, since shocks appear, automatically involves the propagation of nonlinear waves through each other in a finite space domain. The problem will be approached through a novel use of perturbation methods.

Finally it should be noted that at resonance, aside from the appearance of a shock which is a nonlinear phenomenon, linear theory predicts the evolution to an unbounded motion.

2. FORMULATION

In terms of nondimensional Lagrangian variables the equations expressing conservation of mass and linear momentum for the gas are

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial x} = 0 \qquad (2.1)$$

$$\frac{\partial u}{\partial t} + a^2 \frac{\partial \rho}{\partial x} = 0 \qquad (2.2)$$

where u, ρ, a, denote gas velocity, density, and sound speed at the gas particle x, for time t. The equation of state for the isentropic flow of an ideal gas may be written as:

$$a^2 = \rho^{\gamma-1} \qquad (2.3)$$

where γ is the gas constant.

Equations (2.1), (2.2) and (2.3) are supplemented by the boundary conditions

$$u(0,t) = 0$$

$$u(1,t) = -2\pi\varepsilon\omega \sin 2\pi\omega t, \quad \varepsilon \ll 1 \qquad (2.4)$$

and by the initial conditions

$$\begin{aligned} u(x,t) &= 0 \qquad 0 \leq x \leq 1, \qquad t \leq 0, \\ a(x,t) &= 1 \qquad 0 \leq x \leq 1, \qquad t \leq 0. \end{aligned} \qquad (2.5)$$

The problem is to follow the evolution of the gas motion under the prescribed boundary conditions (2.4) from the initial undisturbed state (2.5) to the final periodic state.

Replacing ρ in (2.1) and (2.2) by using (2.3) the resulting equations can be combined to form the coupled system

$$\begin{aligned} \left[\frac{\partial}{\partial t} + a^{\frac{\gamma+1}{\gamma-1}} \frac{\partial}{\partial x}\right]\left[u + \frac{2}{\gamma-1}a\right] &= 0 \\ \left[\frac{\partial}{\partial t} - a^{\frac{\gamma+1}{\gamma-1}} \frac{\partial}{\partial x}\right]\left[u - \frac{2}{\gamma-1}a\right] &= 0 \end{aligned} \qquad (2.6)$$

The Riemann Invariant $u + \frac{2}{\gamma-1}a$ is constant on the characteristic curves $\alpha(x,t)$ = constant given by

$$\left.\frac{\partial x}{\partial t}\right|_{\alpha} = -a^{\frac{\gamma+1}{\gamma-1}} \qquad (2.7)$$

with a similar statement for the other Riemann Invariant. Equation (2.7) cannot be integrated since $a(x,t)$ is unknown until the solution is found.

The approach adopted to solve the system (2.6) is to assume a regular perturbation expansion for u and a in (2.6) of the form

$$u(x,t) = \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \ldots$$
$$a(x,t) = 1 + \varepsilon a_1(x,t) + \varepsilon^2 a_2(x,t) + \ldots \qquad (2.8)$$

Linear theory results from terms at $O(\varepsilon)$ with a nonlinear correction at $O(\varepsilon^2)$. The linear terms u_1, a_1 satisfy

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right]\left[u_1 + \frac{2}{\gamma-1} a_1\right] = 0$$
$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right]\left[u_1 - \frac{2}{\gamma-1} a_1\right] = 0 \qquad (2.9)$$

with general solution

$$u_1 = f(t+x-1) - g(t-x)$$
$$a_1 = -\frac{(\gamma-1)}{2}[f(t+x-1) + g(t-x)] \qquad (2.10)$$

where f,g are arbitrary functions - the linear Riemann Invariants. It should be noted that the two sets of linear characteristic curves t+x-1 = constant and t-x = constant are parallel straight lines and are independent of the solution u_1, a_1. In other words the linear waves neither distort, nor interact with each other.

The boundary conditions on x=0, x=1 given by equations (2.4) imply that on x=1

$$f(t) - f(t-2) = -2\pi\varepsilon\omega \sin 2\pi\omega t, \quad t > 0 \qquad (2.11)$$

an equation which is augmented by the initial condition

$$f(t) = 0, \quad t \le 0 \qquad (2.12)$$

When $\omega = \frac{n}{2}$, n = 1,2,3 ... equation (2.11) predicts that

$$f(t) \text{ is asymptotic to } -nt \sin(n\pi t), \text{ as } t \to \infty, \qquad (2.13)$$

in other words predicting unbounded growth frequencies equal to the natural frequencies of the gas tube.

On substituting for u_1 and a_1 into (2.6), (2.8) we obtain the particular integral

$$u_2(x,t) = \frac{\gamma+1}{2} x [g(t-x)g'(t-x) + f(t+x-1)f'(t+x-1)]$$
$$+ \frac{\gamma+1}{4} x [f'(t+x-1)G(t-x) - g'(t-x)F(t+x-1)]$$

where

$$G(t) = \int^t g(y)dy \quad \text{and} \quad F(t) = \int^t f(y)dy.$$

We note that the complementary functions associated with u_2 may be absorbed into the representation for u_1.

The novel feature of the approach now outlined is that the boundary conditions are applied not to u_1 and u_2 separately but to the combined approximation $\varepsilon u_1 + \varepsilon^2 u_2$, i.e.

$$\varepsilon u_1(0,t) + \varepsilon^2 u_2(0,t) = 0 \qquad (2.15)$$

and

$$\varepsilon u_1(1,t) + \varepsilon^2 u_2(1,t) = -2\pi\varepsilon\omega \sin 2\pi\omega t.$$

The aim is to formulate in one relationship a mechanism of controlling the linear growth by the nonlinear terms.

The boundary condition on x = 0 implies that

$$f(t-1) = g(t), \quad t \ge 0 \qquad (2.16)$$

After some manipulation the boundary condition on x = 1 implies that f(t), the linear Riemann Invariant, satisfies the nonlinear equation

$$-2\pi\omega \sin 2\pi\omega t = f(t)-f(t-2) + \varepsilon\frac{(\gamma+1)}{2}[f(t)f'(t)+f(t-2)f'(t-2)]$$
$$+ \varepsilon\frac{(\gamma+1)}{4}[-f'(t-2)\int^{t} f(y)dy + f'(t)\int^{t-2} f(y)dy \quad (2.17)$$

with initial condition (2.12).

Equation (2.17) is a nonlinear functional differential equation of neutral type, see [5]. The equation of linear theory is included in (2.17): the nonlinear terms in (2.17) which are in the brackets associated with $\frac{(\gamma+1)}{2}$ represent the effect of amplitude dispersion by which shocks form, while the remaining term represents the nonlinear interaction of opposite travelling waves.

The solution of the nonlinear initial value, boundary value problem on the semi-infinite strip $0 \leq x \leq 1$, $t \geq 0$ and defined by equations (2.1) - (2.5) has now been reduced to a solution of the nonlinear equation (2.17) with the initial conditions (2.12). When the linear Riemann Invariant, f, is known, the particle velocity and sound speed in the tube can be found from the representations (2.10), (2.16).

3. GOVERNING PARTIAL DIFFERENTIAL EQUATION

The functional differential equation (2.17) was derived using only a regular perturbation expansion and retaining the sum of the first two terms as the basic approximation. We now show how equation (2.17) can be simplified to a hyperbolic partial differential equation by the use of a two variable expansion technique. There are two natural time scales in the physical problem under consideration: the time for a signal to travel the length of the tube and the time for a shock to form. The basic assumption underlying the simplification of (2.17) is that the latter time scale is much larger than the former. The fast time scale is $t^+ = t$ and the slow time scale is $\tilde{t} = \varepsilon t$. The function f(t) is then expanded in the form

$$f(t;\varepsilon) = f_1(t^+,\tilde{t}) + \varepsilon f_2(t^+,\tilde{t}) + .. \qquad (3.1)$$

Since the primary motivation is to find solutions near the resonant frequency $\omega = \frac{1}{2}$, we introduce the small detuning parameter

$$\Delta = 2\omega - 1 << 1. \qquad (3.2)$$

We now seek solutions which are periodic in the fast time variable t^+ with the same period as the piston, viz, $1/\omega$ and are slowly modulated on the long time scale.

Therefore we assume that

$$f_i(t^+ - \frac{1}{\omega},\tilde{t}) = f_i(t^+,\tilde{t}) \qquad (3.3)$$

On using (3.1) - (3.3) in the functional differential equation (2.17) we obtain the partial differential equation

$$2\varepsilon\frac{\partial f_1}{\partial \tilde{t}} + \frac{\Delta}{\omega}\frac{\partial f_1}{\partial t^+} + \varepsilon(\gamma+1)f_1\frac{\partial f_1}{\partial t^+} = -2\pi\varepsilon\omega \sin 2\pi\omega t^+ \quad (3.4)$$

where terms involving $O(\Delta^2)$, $O(\varepsilon\Delta)$, $O(\varepsilon^2)$ have been neglected. We note that the integral terms in (2.17) which represent the interaction of oppositely travelling waves are $O(\varepsilon\Delta)$ and thus negligible.

The initial condition corresponding to (2.12) and the state of rest is

$$f_1(t^+,0) = 0 \qquad (3.5)$$

Since the solution of (3.4) is periodic in t^+ with periods $\frac{1}{\omega}$, integration of (3.4) over a time interval of length $\frac{1}{\omega}$, with an appeal to weak shock conditions when necessary (see [11]), yields the mean condition

$$\int_0^{\frac{1}{\omega}} f_1(s,\tilde{t})ds = 0 \qquad (3.6)$$

Thus the mean value of f remains constant on lines of constant $\tilde{t}$ as the signal evolves. In order to put (3.4) in a form more amenable for analysis we define

$$F(\eta,\tau) = (\gamma+1)\varepsilon\omega f_1(t^+,\tilde{t}) + \Delta \qquad (3.7)$$

where

$$\eta = \omega t^+, \quad \tau = \frac{\tilde{t}}{2\varepsilon} \qquad (3.8)$$

Then (3.4) becomes

$$\frac{\partial F}{\partial \tau} + F\frac{\partial F}{\partial \eta} = -A\sin(2\pi\eta) \qquad (3.9)$$

$$A = 2\pi\varepsilon\omega^2(\gamma+1) << 1. \qquad (3.10)$$

The initial condition becomes

$$F(\eta,0) = \Delta \qquad (3.11)$$

The remainder of this paper is concerned with the analysis of (3.9) subject to (3.11). It should be noted that the physical properties of the system are all contained in the similarity parameter A, given by (3.10). Variations in the piston amplitude, ε, and frequency ω, or the gas properties γ, corresponding to different experiments are immaterial to the solution of (3.12) as long as the parameter A remains constant.

4. EXACT SOLUTION

The nonlinear partial differential equation (3.9), which describes the evolution on the boundary x = 1 of the linear Riemann Invariant, is hyperbolic and can thus be studied by the method of characteristics. The transport equation

$$\frac{dF}{d\tau}(\eta,\tau) = -A\sin 2\pi\eta \qquad (4.1)$$

describes the variation of the signal F on the characteristic curves $\alpha(\eta,\tau)$ = constant given by

$$\frac{d\eta}{d\tau} = F(\eta,\tau) \qquad (4.2)$$

The characteristic curves are parameterised by $\alpha(\eta,0) = \eta$. The coupled system (4.1), (4.2) and the initial condition (3.11) are equivalent to the second order equation

$$\frac{d^2\eta}{d\tau^2} = -A\sin 2\pi\eta \qquad (4.3)$$

with initial conditions

$$\eta(0) = \alpha, \quad -\tfrac{1}{2} \le \alpha \le \tfrac{1}{2} \qquad (4.4)$$

and

$$\frac{d\eta}{d\tau}(0) = \Delta.$$

Thus the characteristic paths are given by the nonlinear pendulum equation (4.3) with the signal profile given by (4.1).

On using (4.2), equation (4.3) is written as

$$F\frac{dF}{d\eta} = -A\sin 2\pi\eta \qquad (4.5)$$

Integration of (4.5) yields

$$\left(\frac{d\eta}{d\tau}\right)^2 = F^2 = \frac{\beta^2}{4\pi^2}\{1 - m^2\sin^2(\pi\eta)\} \qquad (4.6)$$

where

$$m^2(\alpha,0) = \frac{1}{\sin^2\pi\alpha + \frac{\pi\Delta^2}{2A}}, \quad \beta^2 = \frac{8\pi A}{m^2} \qquad (4.7)$$

In a standard manner, integration of (4.6) then yields exact solutions for $\eta(\alpha,\tau)$ expressed in terms of elliptic func-

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tions (see [1]). With F given by (4.6) the solution of (3.9) can then be tabulated.

5. SOLUTION CURVES AND DISCUSSION

Fig. 1 corresponds to A = 0.01, Δ = 0.02 and shows the growth of the amplitude of the signal over the initial periods (as predicted by linear theory), but with a simultaneous cumulative distortion of the signal shape until a shock forms in the seventh cycle of the piston. One can see in this output what was anticipated in applying the boundary conditions as in (2.15).

Fig. 2 shows how the signal settles down to the periodic state, containing a shock, after about 30 cycles. Figs 3 and 4 show the evolution of the signal for the case A = 0.01, Δ = 0.06. This is a particularly interesting case since experiments show that the periodic state is continuous and is, in fact, essentially determined by linear theory. The figures show how a shock forms and then decays out of the system so that the eventual steady state is shockless. The solution thus goes through a nonlinear regime but eventually reaches a periodic state which is closely approximated by linear theory.

A shock is a dissipative mechanism so that when the piston is operating at or near resonance the shock dissipation balances the energy buildup due to the phase matching of the input and response to allow a final periodic state containing a shock. Away from resonance, e.g. when Δ = 0.06, there is a sufficient mismatch of phase to obviate an energy buildup and a shock cannot be sustained. Thus a continuous periodic steady state results.

The analysis given here is based on numerical calculations of the exact solution. It is also instructive to consider the phase plane associated with (4.3), (4,4) and the reader will find this in [4].

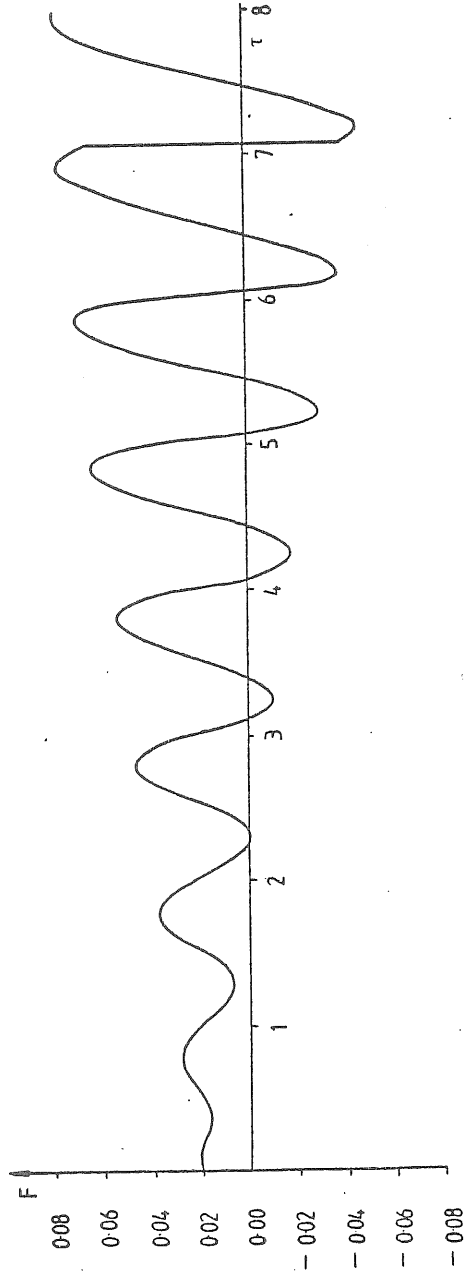

FIGURE 1

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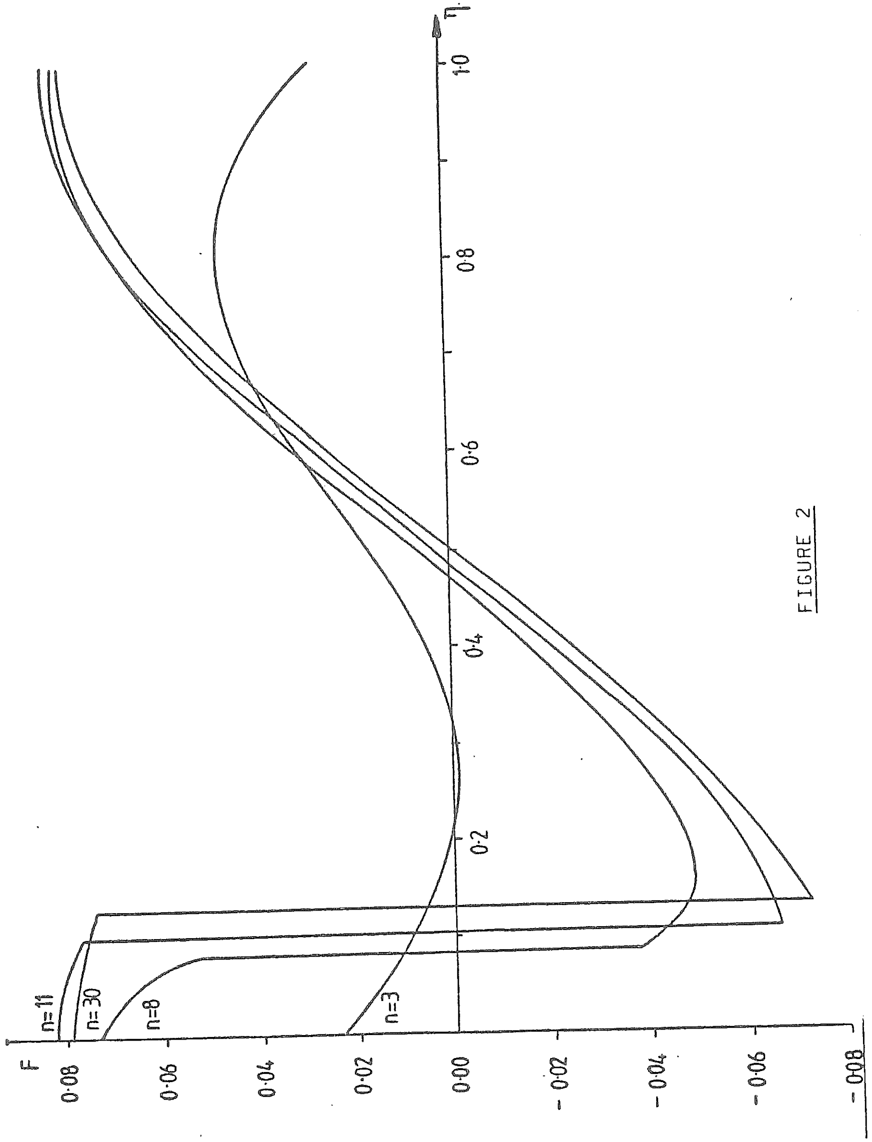

FIGURE 2

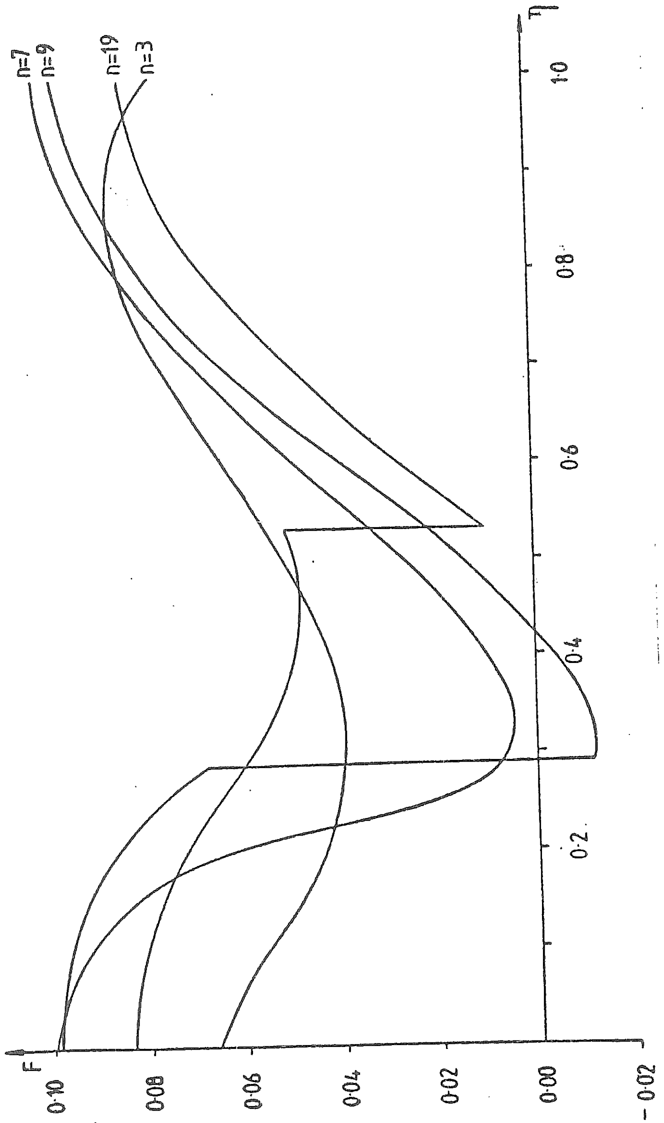

FIGURE 3

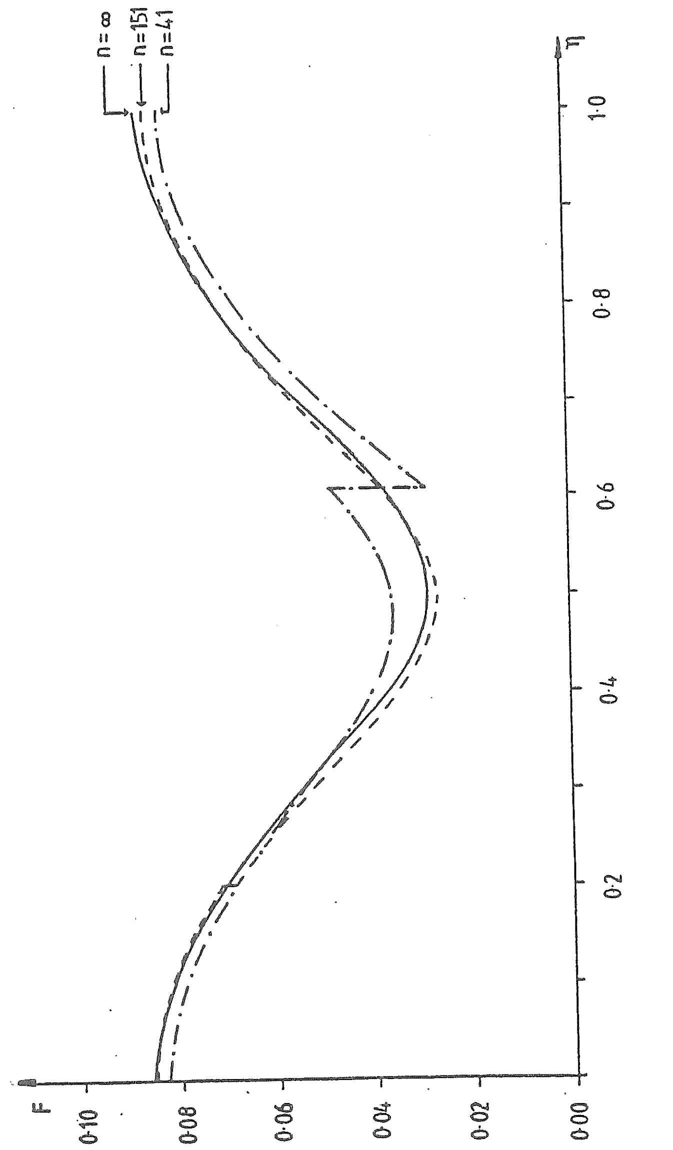

FIGURE 4

REFERENCES

1. M. Abramowitz and I.A. Stegun (Eds.) *Handbook of Mathematical Functions.* National Bureau of Standards, D.C. 1964.

2. W. Betchov, Nonlinear Oscillations of a Column of Gas. *Phys. Fluids 1*, 205-212 (1958).

3. W. Chester, Resonant Oscillations in Closed Tubes. *J. Fluid Mech., 18*, 44-64 (1964).

4. E.A. Cox and M.P. Mortell, The Evolution of Resonant Oscillations in Closed Tubes. *Z. Angew. Math. Phys., 34*, 845-866 (1983).

5. J. Hale, *Theory of Functional Differential Equations.* Springer-Verlag, New York, 1977.

6. J. Lighthill, *Waves in Fluids*, Cambridge University Press, Cambridge 1978.

7. B. Riemann, *Uber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite.* Gottingen Abhandlungen. Vol. 8, p. 43.

8. R.A. Saenger and G.E. Hudson, Periodic Shock Waves in Resonating Gas Columns. *J. Acoustic. Soc. Amer., 32*, 961-970, (1960).

9. B.R. Seymour and M.P. Mortell, Resonant Acoustic Oscillations with Damping: Small Rate Theory. *J. Fluid Mech., 58*, 353-373 (1973).

10. G.G. Stokes, On the Theory of Oscillating Waves. *Camb. Trans., 8*, 441-473 (1847).

11. G.B. Whitham, *Linear and Nonlinear Waves.* Wiley, New York (1974).

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