Wilder, R.L. (1973). *Evolution of Mathematical Concepts:An Elementary Study.* London, Transworld Publishers.

The following selections from *The World of Mathematics* are to be treated as part of your reading assignment:

The Axiomatic Method by Wilder, R.L. Vol. 3, pp. 1647-67

The Essence of Mathematics by Peirce, C.S. Vol. 3, pp. 1773-83.

How to Solve it by Polya, G. Vol. 3. pp. 1980-99.

A Mathematician's Apology by Hardy, G.H. Vol. 4. pp. 2027-38.

Mathematical Creation by Poincare, H. Vol. 4. pp. 2041-50.

The Mathematician by von Neumann, J. Vol. 4. pp. 2053-63.

Note: As a future teacher you would be well advised to establish a small personal collection of mathematics books. Why not begin by selecting your favourites from those listed in the readings!

References

1. Wain, G.T. (Editor) (1978). *Mathematical Education*, Van Nostrand Reinhold Ltd., New York.

2. May, K.O. (1972). Teachers should know *about* mathematics. *Int. J. Math. Educ. Sci. Technol.*, Vol. 3, 157-158.

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BOOK REVIEW

"COMBINATORICS ON WORDS" (Encyclopedia of Mathematics and its Applications Volume 17)

By *M. Lothaire*

Published by *Addison-Wesley Publishers Ltd.*, 1983, Stg. £24.70, pp. xix + 238.

ISBN 0-201-13516-7

M. Lothaire is the pseudonym chosen by a group of mathematicians led by Dominique Perrin who have contributed to the writing of this volume - the first devoted wholly to the study of combinatorics on *words* or finite sequences of symbols (letters). Repetitions, decompositions, unavoidable regularities and equations in words are all analysed and connections are established with such classical areas as free groups, Lie algebras, algebras with polynomial identity and coding theory. Combinatorics on words also has significant applications to, and indeed many of its results arise from, the theory of automata, information theory and linguistics. This book attempts to draw together from these diverse areas the principal results on words and to introduce the reader to the essential methods of a new area of mathematics. To quote from the foreword by Roger Lyndon (written in his capacity as Algebra section editor for the series);

> "It is a pleasure to witness such an auspicious official inauguration of a newly recognised mathematical subject, one which carries with it certain promise of continued increasingly broad development and application".

The individual chapters of the book are written by the different co-authors, but they have collaborated to produce a unified text with consistent notation and cross-referencing throughout. The first chapter (by Dominique Perrin) introduces the reader to free monoids ("the natural habitat of words") and their morphisms, submonoids and minimal generating sets (*codes*). As one would expect, words have prefixes and suffixes and the free monoid A^* on an alphabet A also lives quite happily in the (noncommutative) formal power series ring $Z\langle\langle A\rangle\rangle$ and the free associative (polynomial) algebra $Z\langle A\rangle$.

Chapters 2, 3 and 4 (Jean Berstel/Christophe Reutenauer, Jean Eric Pin, and Giuseppe Pirillo respectively) form a block devoted to the study of *unavoidable regularities*, that is, properties shared by all sufficiently long words. Thus it is shown, roughly speaking, that "each sufficiently long word over a finite alphabet behaves locally in a regular fashion". Of course the type of regularity must be specified, a classical example being provided by van der Waerden's theorem: If N is partitioned into k classes, one of the classes contains arbitrarily long arithmetic progressions. Several formulations of this theorem and two proofs, one combinatorial, the other topological, are given in Chapter 3. If A is an alphabet the set of all nonempty words over A is denoted by A^+. A morphism $\phi: A^+ \to S$ from A^+ to a set S is called *repetitive* if each sufficiently long word contains a factor of the type $w_1 w_2 \ldots w_n$ with $\phi(w_1) = \ldots = \phi(w_n)$ and *uniformly repetitive* if all the w_i can be chosen of equal length. In Chapter 4 it is shown that if S is a finite set then ϕ is repetitive and the special case where S is itself a semigroup is investigated. In particular if S is a finite semigroup then ϕ is uniformly repetitive, a result which is shown to be a generalisation of van der Waerden's theorem. The dual problem of *avoidable regularities* is the subject of Chapter 2. These are properties not automatically shared by all long words: for such a property there exist infinitely many words (over a finite alphabet) that do not satisfy it. For example there are infinitely many *square free* words provided that the alphabet has at least three letters, so it is not true that every sufficiently long word contains a square.

Chapters 5, 6 and 7 (Dominique Perrin, Jacques Sakarovitch/Imre Simon, and Christophe Reutenauer respectively) also form a block. These deal with properties of words related to classical noncommutative algebra. In Chapter 5 we find the study of *factorizations* of free monoids which may be thought of as bases, and their relationship to bases of free Lie algebras. The principal tool is a factorization via the so-called *Lyndon words* and among the results analysed are the Witt formula, the Poincaré - Birkhoff-Witt theorem and the Campbell-Baker-Hausdorff formula. Chapter 6 is devoted to *subwords*. It is a simple combinatorial problem to determine the set of subwords of a given word and its cardinality. Of more interest however is the converse problem: under what conditions is a given set of words of a specified kind the set of subwords of a word w? Here one uses the notion of *division* (u divides v if u is a subword of v) and the partial order it induces on A^*, the main property of which is given by a well-known result of Higman: any set of words over a finite alphabet which are pairwise incomparable in the division ordering is finite. Also introduced is the *binomial coefficient* $\binom{u}{v}$ of two words which is intimately related to the Magnus representation of free groups and to Fox's free differential calculus. In Chapter 7 the relationship between words and algebras with polynomial identity is studied. The aim here is to prove the theorem of Shirshov which answers both the Levitski and Kurosch problems for pi-algebras thus: Let *A* be a finitely generated K-algebra (K is a commutative ring with 1) generated by $m_1, \ldots, m_k$ and suppose *A* is a pi-algebra (with polynomial identity) of degree n. If any product of at most n-1 of the m_i is nilpotent (resp. integral over K) then *A* is nilpotent (resp. a finitely generated K-module). What is interesting is that the proof (taken essentially from Shirshov's 1957 paper) is entirely combinatorial and requires no deep knowledge of ring theory.

Each of the last four chapters of the book introduces a new aspect of words and, as indicated by the exercises, each could be considerably extended. Chapter 8 on The Critical Factorization Theorem is written by Marcel Paul Schützenberger (who, incidentally, is acknowledged as the initiator of the systematic study of monoids and combinatorics on words) and deals with *periodic* properties (where the period $\pi(w)$ of a word w is defined as the minimum length of words admitting w as a factor of some power). Chapter 9 (Christian Choffrut) gives an introduction to the vast subject of *equations in words* (here again the name of Lyndon arises frequently in the discussion). In Chapter 10 Dominique Foata describes how *rearrangements* of words can be used in the enumeration of permutations of finite sequences with certain specified properties (such as a given number of descents or a fixed up-down sequence). The final Chapter 11 (Robert Cori) covers the relationship between *plane trees*, *parenthesis systems* and certain families of words. An interesting aspect of this chapter is the use of the combinatorial properties of Lukaciewicz language to give a purely combinatorial proof of the Lagrange inversion formula of complex analysis!

In reading a book of this nature one is of course prepared to accept a certain amount of "unavoidable irregularity" in the writing due to the varied authorship of the different chapters. In fact the style is surprisingly consistent throughout signifying a remarkable degree of cooperation among the (eleven) writers. The index has one or two omissions and I found just one instance of a term (*biprefix code* on p. 27) being used without having been defined (the natural place would have been in Chapter 1). But on the whole the cross-referencing and indexing are adequate to the reader's needs. There is a number of misprints but most of these are textual rather than symbolic and along with several (typically French) non-standard uses of the English language can be forgiven in an otherwise excellent production.

The book is written lucidly and for the most part so as to be accessible to anyone with a standard mathematical background. It contains a wealth of information and many topics not mentioned in this review are included. Very few results are taken for granted and each chapter ends with a good selection of detailed exercises designed to bring out applications and extensions of the theory. Also contained in each chapter are comprehensive bibliographic and historical notes and discussion on a fair number of open problems to whet the appetite for further investigation. This book is sure to become the standard reference work in a new and potentially fruitful area of mathematics.

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"CALCULUS AND ANALYTIC GEOMETRY"

by *Donald W. Trim*

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Academic life is financially secure, but the chances of making a "killing" are few and far between. It is, of course, interesting to speculate how we might get by if we were paid by the theorem; if the mortgage payment next month depended on settling that result you have been trying to prove over the past two years. It might well extend the active mathematical life of many, exposing as a myth the belief that creative mathematics is done by the young. It would certainly make life interesting; it would probably make it shorter too. Most of us are glad that this is not the way things are arranged, and in a society governed by supply and demand we may deduce that we have to be grateful that someone somewhere is giving us the time to prove theorems at all. (Notable exceptions to these observations, indeed most observations, are the U.S.A. and France. The hiring system in the United States has created