Acknowledgements

The whole idea behind this calculation is not my own; it was suggested to me by James York. The actual calculations described here were done in collaboration with James Davis, we have produced explicit expressions for the ten independent non-gauge r^{-4} solutions to (9) and (10). I would also like to thank F.A. Deeney for his very helpful comments on various versions of this article.

References

1. E. Witten, "A New Proof of the Positive Energy Theorem", *Comm. Math. Phys.*, *80*, 381-402 (1981).
2. See for example, D.J. Gross, M. Perry and L.G. Yaffe, "Instability of Flat Space at Finite Temperature", *Phys. Rev.*, *D 25*, 330-355 (1982).

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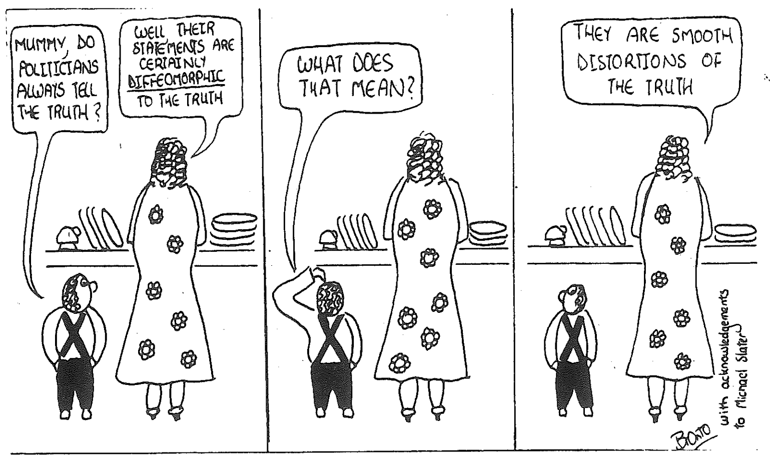

From 2-*Manifold*, No. 3

KNOWING 'ABOUT' MATHEMATICS: A FOCUS ON TEACHING

John O'Donoghue

"Mathematical Education" may be seen then as an operational activity based on a number of areas of study with the analysis of the communication of mathematics as its objective.

(G.T. Wain)

INTRODUCTION

Since all of us have a good intuitive idea of what is meant by mathematical education it is acceptable to start by presenting a definition. The above definition may not suit everyone's tastes but then definitions rarely find universal acceptance. It is not my intention to argue a case for mathematical education as a discipline but rather to focus attention on some important aspects of mathematical education as an activity. This particular definition serves to focus attention on the *communication* of mathematics. All of us at some time or another have been concerned with this aspect of mathematics teaching as students, teachers, lecturers or professors. Many of us have resolved to improve matters given the opportunity. My particular concern has been to improve teacher preparation so that better mathematics teaching results in secondary schools.

The purpose of this paper is to draw attention to a neglected aspect of mathematics teaching at third level which is vitally important for future teachers of mathematics. A case is made for better treatment of this aspect, and finally an outline of an experimental course is given.

A BASIC REQUIREMENT

Traditionally teacher educators have rightly insisted that the fundamental requirement for teachers of mathematics is to know mathematics. In other words subject competence is more important than methodology. One cannot teach mathematics if one does not know mathematics. While this establishes priorities it does little else. It is not prescriptive in any sense. For example, what does it mean to know mathematics? It is important to clarify what is at stake here. We demand of our teachers a certain competence in mathematics. That is to say, it is taken for granted that teachers of mathematics should be trained in the theory of mathematics, its methods and techniques. But mathematics teachers like any other teachers must be concerned to maximise their contribution in schools. Thus as members of the teaching profession they will find it necessary to address four questions:

1. Why teach mathematics?
2. What mathematics to teach?
3. When to teach mathematics?
4. How to teach it?

The nature and quality of teachers' responses to these questions will, in large measure, determine their effectiveness as teachers of mathematics. Success depends upon knowledge and experience of a special kind. It requires of teachers perspective, insight and knowledge adequate for the presentation of mathematics *and* its role in modern culture. In short teachers must know mathematics and know *about* mathematics. I believe that all good mathematics teachers manage somehow to combine these two attributes in their teaching. The matter generates concern because whereas these aims are not mutually exclusive, the attainment of one is no guarantee that the other has been achieved. Some sort of intervention is required. Mathematics teachers can be helped to help themselves in this regard. One way is to provide them with opportunities for talking *about* mathematics and for finding out *about* mathematics by reading as well as doing.

A PROBLEM WITH EXISTING PROGRAMMES

The competing demands on a student's time in existing undergraduate teacher training programmes in mathematics guarantee insufficient time for subject specialists. Inevitably therefore, the main effort is directed towards developing students' subject competence in the allocated time. Programmes as a result are so crowded and demanding that little or no time is available to develop students' perspective or to cultivate an overview of mathematics - important but neglected aspects of mathematical competence. That is to say that little attention is devoted to these aspects *explicitly* in any mathematical programme. The accepted view seems to be that specific attention is unnecessary because it happens anyway or in any case if it does not happen during the undergraduate phase it must surely happen later during study for higher degrees in mathematics. This state of affairs in unsatisfactory for teacher educators for two reasons: (1) many student teachers fail to develop a reasonable overview; (2) the vast majority of mathematics teachers never proceed to higher degrees in mathematics. In practice, therefore, most mathematics teachers forfeit any benefits which would accrue from this activity.

KNOWING 'ABOUT' MATHEMATICS

Who can deny that knowing about mathematics is a legitimate mathematical pursuit? Is the explicit treatment of problem solving and mathematical modelling outside the domain of mathematics? Does the nature of proof and proof techniques constitute appropriate study? Is it not imperative given the nature of school mathematics that students confront the concept of mathematical structure and deal with it comprehensively? Will not a straightforward treatment of mathematical processes such as consolidation, generalization, abstraction etc., contribute to a better mathematical experience. The list can be extended to include history of mathematics and foundations.

No one would deny that any of this activity is valid mathematics but many teacher educators afford it a low priority in practice in their undergraduate programmes. In effect this means that intervention by way of direct teaching is the exception rather than the rule. Rarely are undergraduate students in mathematics confronted by appropriate courses, materials and experiences. As a result few are able to talk about mathematics in an interesting and informed manner.

EXPECTED BENEFITS

Perhaps you feel that there really is 'much ado about nothing' here. I consider this issue to be a matter of some considerable importance for teacher educators in mathematics. I feel strongly that teacher effectiveness is considerably impaired by the absence of these competencies. Further, I attribute some observed shortcomings in practice to this deficiency namely the inability of many mathematics teachers to go beyond the text book, to make mathematics relevant or to instill confidence in doing mathematics.

Teaching mathematics is not simply a matter of showing children how to do mathematics. Pupils have to be motivated and kept interested. Appropriate topics and sequencing have to be used in context. Teachers have to cope even in a single class group with an incredible variation in ability and motivation. Pupils learn in different ways. Appropriate learning experiences and practice have to be devised and so on..... A teacher must be able to cope with such complexity. It is more likely that he will cope effectively if he can present topics in different and interesting ways, evaluate different approaches and methods, identify significant concepts etc. Teachers cannot be expected to do this unless they have a sound grasp of mathematics, can see connections and interrelations, know something of its history and foundations - in short know about mathematics.

There are other benefits. Many teachers having completed their initial training will never return for further formal education in mathematics. This means that the education and training they receive as undergraduates has to serve for their entire professional lives. It is inconceivable in modern times that teachers could live through their working lives without updating their subject knowledge. If this is not done formally then it must be done outside the system, i.e. by independent study. In any case success is more likely if the endeavour is built on a solid foundation of mathematics. Independent study is more likely to succeed if the teacher is confident in his knowledge of mathematics, knows his way 'around the subject' and can articulate effectively.

CONCLUSION

In this paper I have attempted to highlight an aspect of mathematical education which, I believe, is especially significant for teacher educators and future mathematics teachers. This has been done in a way which separates (perhaps artificially) certain aspects of mathematics. Whether one agrees with this particular approach due to May [2] is not important. As long as the difficulty is recognised the means of describing it may be considered of secondary importance. My attempts to deal with the problem have been based on explicit teaching and directed independent study in a sequence of three courses, namely: History of Mathematics, Foundations of Mathematics and Mathematics Seminar. I leave it to the readers to judge the merit of such an exercise and in particular the use of the mathematics seminar which is outlined below. It is appropriate to raise such issues here in this forum since many of the readers are involved directly or indirectly in teacher education in university colleges and colleges of education. I should point out that I do not consider the list of selected readings to be a definitive list since choice was limited by what was immediately available. Perhaps others would want to substitute their own preferences!

APPENDIX

EXPERIMENTAL COURSE

COURSE : Mathematics Seminar
TUTOR : Dr. J. O'Donoghue
YEAR : Final Year Mathematics Students
DURATION : One Academic Year (30 hours)

1. Introduction

My concern, among other things, has been to ensure that student teachers completing their initial training know mathematics and know *about* mathematics. Obviously these aims are not mutually exclusive but the attainment of one is no guarantee that the other has been achieved. *I believe that all good mathematics teachers manage somehow to combine these two attributes in their teaching.*

The aim of this course is to set you thinking about your mathematics in a way which will benefit you in your profession now and in the future. You will be encouraged in a variety of ways to develop your perspective, insight, intuition and knowledge regarding mathematics. You will be challenged to develop your skills in analysis and synthesis by practising on issues in the nature of mathematics, its concepts and structures, its methodologies, and by examining such processes as abstraction, generalization, unification, consolidation, idealization, modelling as they pertain to mathematics.

The hope is that you will learn to penetrate deeper the mass of detail and apparently disparate areas of mathematics and develop a perception which allows you to achieve a worthwhile synthesis of the mathematics you command. It is my earnest desire that some of you, at least, will advance further and use these ideas *purposefully* at each stage of your mathematical development and thus equip yourself with a powerful methodology for learning to learn about mathematics.

2. Objectives

- To encourage the student teacher to develop a wider perspective and deeper insight into mathematics.
- To promote in the student teacher an attitude of inquiry into mathematics requiring analysis and synthesis.
- To cultivate in the student teacher a worthy sense of the meaning and significance of important mathematical ideas.
- To encourage the student teacher to develop a methodology for learning to learn about mathematics.

3. Course Organization and Content

Themes: The following themes have been selected in an attempt to add structure to the endeavour:

(i) Problem Solving
(ii) Mathematical Modelling
(iii) Mathematical Structure
(iv) Mathematical Knowledge
(v) Mathematical Proof and proof techniques

Readings: Various readings have been assigned. Readings dealing with specific themes have been grouped together. There will be some overlap between readings and groups of readings.

Lectures: The course tutor will deliver a series of occasional lectures (5). Lecture topics will relate to the aforementioned themes. Topic, venue and time will be posted on the mathematics department notice board.

Discussions: The course tutor will be available to deal with individuals as required. Opportunities will be provided occasionally to meet as a group to discuss particular readings. Watch your notice board for information.

Assessment: Assessment is based on the following course elements:

(i) attendance at occasional lectures
(ii) summaries of assigned readings
(iii) short (750 word) essay which brings the totality of your mathematical experience to date to bear on the following topic:

Identify major themes running through your mathematics programme and use them to effect a unification of the programme as a whole.

Notes on Procedure

A. Duration of Course: One full academic year beginning in first term (30 hour equivalent).

B. Timetable: See notice board.

C. (a) Readings: Readings are organised into files as follows:

File 1 - Mathematical Knowledge
File 2 - Mathematical Structure
File 3 - Problem Solving and Mathematical Modelling
File 4 - Mathematical Proof and Proof Techniques.
File 5 - General Reading

(b) Availability: Three copies of each file will be available at the Restricted Loan Counter in the College Library from the beginning of term.

(c) Content: A full list of readings is appended to this outline.

D. Each student is responsible for reading each reading on the list.

E. Each student is responsible for maintaining article summaries in a file which must be available for scrutiny by the tutor.

F. Assessment: Essay must be submitted two weeks prior to the end of last term.

5. Study Notes

A. A number of essays should not be read at one sitting. Time has been provided for a leisurely but measured pace spreading the work over the year.

B. The readings/essays vary in style, difficulty and point of view. Some are short, others are long. However, they do have something in common - each reading from a particular group relates to the theme for that group.

C. You have been asked to summarise each essay in one half page. Why demand such a short summary even for long readings? You will be surprised how many readings really only contain one or two or three fundamental ideas. What about analysis and synthesis?

D. Read essays for impression then for detail but do not devote excessive time to detail.

E. Themes are useful to focus your attention on specific important issues but boundaries between themes/topics are never sharp since themes merge easily or envelop each other. But this is only as it should be!

6. Readings

File 1 - Mathematical Knowledge

Aleksandrov, A.D. *et al* (Editors) (1962). *Mathematics: its Content, Methods and Meaning.* Cambridge, M.I.T. Press, pp. 1-7.

Hogben, L. (1967). *Mathematics for the Millions.* London, Pan Books, pp. 75-117.

Kapur, J.N. (1976). Proposal for a Course on the nature of Mathematical Thinking. *International Journal of Math Education in Science and Technology, 1*, 287-296.

Kasner, E., Newman, J. (1979). *Mathematics and the Imagination*, U.K., Penguin Books, pp. 17-35.

Kline, M. (1964). *Mathematics for Liberal Arts.* Reading, Addison-Wesley, pp. 30-55.

Rees, M. (1962). The Nature of Mathematics. *Mathematics Teacher*, October, pp. 434-440.

Sawyer, W.W. (1943). *Mathematician's Delight*. U.K., Penguin Books, pp. 26-34.

File 2 - Mathematical Structure

Bell, A.W. (1966). *Algebraic Structures*. London, Allen and Unwin, Chapters 1,5 and 6.

Gowar, N. and Flegg, H.G. (1974). *Basic Mathematical Structures 2*. London, Transworld Publisher. Chapter 4.

Jeger, M. (1966). *Transformation Geometry*. London, Allen and Unwin, Chapters 1,5 and 6.

Mansfield, D.E. and Bruckheimer, M. (1965). *Background to Set and Group Theory*. London, Chatto and Windus. Chapters 1, 6 and 8.

Piaget, J. (1972). Mathematical Structures and the Operational Structures of the Intellect. In Lamon, W.E. (Editor). *Learning and the Nature of Mathematics*. Chicago, SRA, pp. 117-136.

Sawyer, W.W. (1955). *Prelude to Mathematics*. U.K., Pelican, Chapters 4 and 5.

File 3 - Problem Solving and Mathematical Modelling

Bajpai, A.C. *et al* (1974). *Engineering Mathematics*. London, John Wiley, Chapters 0 and 1.

Bell, M. (1979). Teaching Mathematics as a Tool for Problem Solving. *Prospects*, IX, 311-320.

Jackson, K.F. (1975). *The Art of Solving Problems*. London, Heinemann, Chapters 1, 2 and 6.

Kac, M. (1969). Some Mathematical Models in Science. *Science*, 166, 695-699.

Kac, M. and Ulam, S. (1971). *Mathematics and Logic*. U.K., Pelican Books, Chapter 3.

Molkevitch, J. and Meyer, W. (1974). *Graphs, Models and Finite Mathematics*. New Jersey, Prentice-Hall, Chapters 1 and 2.

Ormell, C.P. (1972). Mathematics, Science of Possibility. *International Journal of Math. Education in Science and Technology*, 3, 329-341.

Therauf, R.J. and Klekamp, R.C. (1975). *Decision Making through Operations Research*. (2nd. Ed.) New York, John Wiley. pp. 16-24.

File 4 - Mathematical Proof and Proof Techniques

Bell, A.W. (1966). *Algebraic Structures*. London, Allen and Unwin, Chapter 1.

Course Team (1977). *Polymaths Book A: Number Systems*. Cheltenham, Stanley Thornes. pp. 1-15.

Griffiths, H.B. and Hilton, P.J. (1970). *Classical Mathematics*. New York, Van Nostrand. pp. 1-2 and 241-243.

Kline, M. (1962). *Mathematics for Liberal Arts*. Reading, Addison-Wesley, Chapter 3.

Scaaf, W.L. (1969). *Basic Concepts of Elementary Mathematics*. New York, John Wiley, pp. 109-113.

File 5 - General Reading

Committee on Support of Research in the Mathematical Sciences, National Academy of Sciences (1971). "The Mathematical Sciences: A Report Section II. The State of the Mathematical Sciences". *International Journal of Math. Education in Science and Technology*, 2, 345-390.

Lighthill, J. (Editor) (1978). *Newer Uses of Mathematics*. U.K., Penguin Books.

Newman, J.R. (1956). *The World of Mathematics*. 4 Vols. London, Allen and Unwin.

Stewart, I. (1981). *Concepts of Modern Mathematics*. U.K. Pelican Books.

Wilder, R.L. (1973). *Evolution of Mathematical Concepts: An Elementary Study.* London, Transworld Publishers.

The following selections from *The World of Mathematics* are to be treated as part of your reading assignment:

The Axiomatic Method by Wilder, R.L. Vol. 3, pp. 1647-67

The Essence of Mathematics by Peirce, C.S. Vol. 3, pp. 1773-83.

How to Solve it by Polya, G. Vol. 3. pp. 1980-99.

A Mathematician's Apology by Hardy, G.H. Vol. 4. pp. 2027-38.

Mathematical Creation by Poincare, H. Vol. 4. pp. 2041-50.

The Mathematician by von Neumann, J. Vol. 4. pp. 2053-63.

Note: As a future teacher you would be well advised to establish a small personal collection of mathematics books. Why not begin by selecting your favourites from those listed in the readings!

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1. Wain, G.T. (Editor) (1978). *Mathematical Education*, Van Nostrand Reinhold Ltd., New York.

2. May, K.O. (1972). Teachers should know *about* mathematics. *Int. J. Math. Educ. Sci. Technol.*, Vol. 3, 157-158.

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BOOK REVIEW

"COMBINATORICS ON WORDS" (Encyclopedia of Mathematics and its Applications Volume 17)

By *M. Lothaire*

Published by *Addison-Wesley Publishers Ltd.*, 1983, Stg. £24.70, pp. xix + 238.

ISBN 0-201-13516-7

M. Lothaire is the pseudonym chosen by a group of mathematicians led by Dominique Perrin who have contributed to the writing of this volume - the first devoted wholly to the study of combinatorics on *words* or finite sequences of symbols (letters). Repetitions, decompositions, unavoidable regularities and equations in words are all analysed and connections are established with such classical areas as free groups, Lie algebras, algebras with polynomial identity and coding theory. Combinatorics on words also has significant applications to, and indeed many of its results arise from, the theory of automata, information theory and linguistics. This book attempts to draw together from these diverse areas the principal results on words and to introduce the reader to the essential methods of a new area of mathematics. To quote from the foreword by Roger Lyndon (written in his capacity as Algebra section editor for the series);

> "It is a pleasure to witness such an auspicious official inauguration of a newly recognised mathematical subject, one which carries with it certain promise of continued increasingly broad development and application".

The individual chapters of the book are written by the different co-authors, but they have collaborated to produce a unified text with consistent notation and cross-referencing