SIMPLE GEOMETRIC PROOFS IN LINEAR ALGEBRA

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The orthogonal decomposition of symmetric, skew-symmetric matrices etc. is usually established by computational type proofs. The following geometric approach I feel is easier and more illuminating.

Preliminary ideas: Let $(V,\langle,\rangle)$ be a finite dimensional inner-product space over $\mathbb{R}$. If $A:V\to V$ is linear and we express everything with respect to some orthonormal basis, then $\langle AX,Y\rangle = \langle X,A^tY\rangle$ for all X,Y in V, where A^t is the transpose of the matrix A. By definition the map A is symmetric with respect to $\langle,\rangle$ if and only if $\langle AX,Y\rangle = \langle X,AY\rangle$ for all X,Y in V and is skew-symmetric with respect to $\langle,\rangle$ if and only if $\langle AX,Y\rangle = -\langle X,AY\rangle$ for all X,Y in V.

Thus the notions of (skew)-symmetric maps and (skew)-symmetric matrices are equivalent provided the maps are represented with respect to an orthonormal basis. We note that the same ideas follow through in the Hermitian case with A^t replaced by A^{-t}. The *crucial point* is contained in the following lemma.

Lemma 1: If the linear map $A:V\to V$ is symmetric and leaves the subspace U invariant (i.e. $AU \equiv U$), then it also leaves $U^\perp$, the orthogonal complement of U, invariant.

Proof: If $Y \in U^\perp$, then for all $X \in U$ we have $0 = \langle AX,Y\rangle = \langle X,AY\rangle$ so $AY \in U^\perp$.

Remark: Of course this lemma also holds if A is skew-symmetric and correspondingly in the (skew)-Hermitian case.

The equation $(A-\lambda B)X = 0$: We consider this generalised eigenvalue problem when A and B are symmetric matrices and in addition B is positive definite. (The Hermitian case is identical.) Throughout $\langle,\rangle$ will denote the usual inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$, the context will make clear which is being used. We define a new inner product on $\mathbb{R}^n$ (or $\mathbb{C}^n$) by $(X,Y):=\langle BX,Y\rangle$.

Lemma 2: The eigenvalues of $(A-\lambda B)X = 0$ are all real.

Proof: If the eigenvector $X \in \mathbb{C}^n$ has eigenvalue $\lambda \in \mathbb{C}$, then $\lambda(X,X) = \lambda\langle BX,X\rangle = \langle AX,X\rangle = \langle X,AX\rangle = \langle X,\lambda BX\rangle = \bar{\lambda}(X,X)$. Thus $\lambda = \bar{\lambda}$ so that $\lambda \in \mathbb{R}$ and in particular $X \in \mathbb{R}^n$.

Theorem 3: If A and B are symmetric nxn-matrices with B positive definite, then there exists a basis of eigenvectors $X_1,\ldots,X_n$ of the equation $(A-\lambda B)X = 0$ which are orthonormal with respect to B (i.e. $\langle BX_i,X_j\rangle = \delta_{ij}$).

Proof: We remark that since B is positive definite B^{-1} exists, so we can define $A^1: = B^{-1}A$ and observe

(i) X_1 is an eigenvector of $(A-\lambda B)X = 0$ with eigenvalue λ_1, if and only if it is an eigenvector of $(A - \lambda I)X = 0$ with eigenvalue λ_1.

(ii) A_1 is symmetric with respect to $(\, , \,)$ because $(A^1X,Y) = \langle BA^1X,Y\rangle = \langle AX,Y\rangle = \langle X,AY\rangle = \langle X,BA^1Y\rangle = \langle BX,A^1Y\rangle = (X,A^1Y)$.

(iii) By lemma 2 there exists an eigenvector X_1 of A^1 which we may assume to have unit length with respect to $(\, , \,)$. If $[X_1]$ denotes the subspace spanned by X_1, then $[X_1]$ is invariant under A^1 and therefore (by lemma 1) $[X_1]^\perp$ (its orthogonal complement with respect to $(\, , \,)$) is also invariant under A^1.

(iv) The argument is now completed by induction with A^1 restricted to $[X_1]^\perp$.

<u>Corollary 4:</u> If A and B are as in theorem 3 then there exists a matrix Q such that

$$Q^tAQ = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

and $Q^tBQ = I$.

<u>Proof:</u> Let $Q = [X_1, X_2, \ldots, X_n]$ the matrix whose i^{th} column is X_i the i^{th} eigenvector (as in theorem 3) with eigenvalue λ_i.

<u>Remarks:</u> 1° If B = I, then corollary 4 is the usual statement that every symmetric matrix can be orthogonally diagonalized. That symmetry is necessary here is obvious since $Q^tAQ = D$ where D is diagonal and $Q^tQ = I$ imply $A = QDQ^t = (QDQ^t)^t = A^t$.

2° While symmetry is used to show that the eigenvalues are real it is not the key point. Indeed, $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has real distinct eigenvalues but cannot be orthogonally diagonalized. Of course we see that $\binom{1}{0}$ is an eigenvector whose orthogonal complement, i.e. the line $\{\binom{0}{t}: t \in \mathbb{R}\}$ is not invariant under A.

<u>Theorem 5:</u> If A is an nxn skew-symmetric matrix, then there exists an orthonormal basis for $\mathbb{R}^n$ with respect to which A is tri-diagonal. That is there exists Q satisfying $Q^tQ = I$ and

$$Q^tAQ = \left(\begin{array}{cc|cc} 0 & -b_1 & & \\ b_1 & 0 & & \\ \hline & & 0 & -b_2 \\ & & b_2 & 0 \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{array}\right)$$

<u>Proof:</u> By the argument of lemma 2 one sees that all the eigenvalues of A are pure imaginary. If m(t) denotes the minimal polynomial for A (over $\mathbb{R}$) and if ib, $b \neq 0$, is an eigenvalue of A with eigenvector X, then $0 = m(A)X = m(ib)X$ implies ib is a root of $m(t) = 0$. Accordingly $m(t) = q(t)(t^2+b^2)$. Therefore, since m(t) is minimal, there exists a unit vector $X_1 \in \mathbb{R}^n$ such that $(A^2+b^2I)X_1 = 0$. If we define $X_2 = (AX_1)/b$ then

(i) $AX_1 = bX_2$ and $AX_2 = (A^2X_1)/b = -bX_1$.

(ii) $\langle X_2, X_2\rangle = \langle (AX_1)/b, (AX_1)/b\rangle = -\langle X_1, A^2X_1\rangle/b^2 = \langle X_1, X_1\rangle = 1$.

(iii) $\langle X_1, X_2\rangle = \langle X_1, (AX_1)/b\rangle = -\langle (AX_1)/b, X_1\rangle = -\langle X_2, X_1\rangle$ implies $\langle X_1, X_2\rangle = 0$

(iv) The subspace $[X_1, X_2]$ spanned by X_1 and X_2 is invariant under A and therefore (by lemma 1) so also is its orthogonal complement $[X_1, X_2]^\perp$. We now continue by induction on A restricted to $[X_1, X_2]^\perp$.

(v) The case of zero eigenvalues is easily taken care of and it is clear that the basis produced is the required one with Q being the change of basis matrix.

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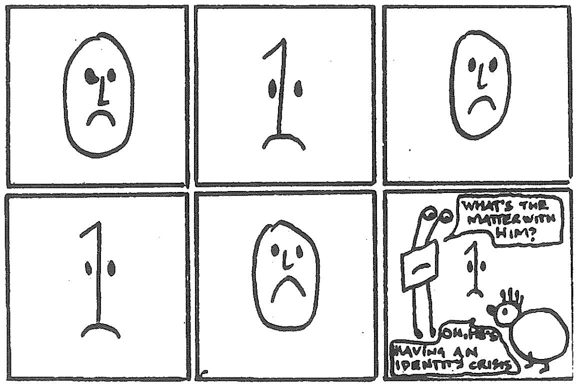

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