NEWSLETTER

EDITOR

Donal Hurley

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The aim of the *Newsletter* is to inform Society members about the activities of the Society and also about items of general mathematical interest.

The *Newsletter* also seeks articles of mathematical interest written in an expository manner. All parts of mathematics are welcome, pure and applied, old and new.

Manuscripts should be typewritten and double-spaced on A4 paper. Authors should send two copies and keep one copy as protection against possible loss. Prepare illustrations carefully on separate sheets of paper in black ink, the original without lettering and a copy with lettering added.

Correspondence relating to the *Newsletter* should be sent to:

Irish Mathematical Society Newsletter
Department of Mathematics,
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Cork.

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Editorial

At its General Meeting in March 1983, the Irish Mathematical Society decided to seek closer links with the Irish Mathematics Teachers' Association. This move, together with other activities of some of our members, e.g. the organisation of the National Mathematics Contest, contributions to the recent Irish Times *Supplement on Mathematics (September 27, 28), the submission on the Intermediate Certificate Geometry course to the Syllabus Committee etc., show a greater awareness of the necessity of cooperation with those involved in second level teaching of Mathematics.*

The majority of our members are involved in teaching and research at third level institutions. If we wish our students to derive maximum benefit from our activities and encourage some of them to take up careers in Mathematics after graduating, then we should be fully aware of the mathematical experiences students have at second level schools. Any curriculum changes in our courses designed to take account of the changing role and nature of Mathematics must be explained to those who are preparing students to embark on that curriculum. Furthermore, we should examine the syllabi and books used in schools and, where we feel necessary, make effective, cogent and constructive criticisms. School teachers often feel victimised by poor advice from third level mathematicians, while the latter plead that their ideas were never well understood or properly explained.

The links between the two professional organisations should be strengthened and utilized to further the common aim of both groups; the teaching of Mathematics.

Donal Hurley

IRISH MATHEMATICAL SOCIETY

ORDINARY MEETING

22 DECEMBER 1983

12:15 at DIAS

Agenda

1. Minutes of Ordinary Meeting of 31st March, 1983
2. Matters Arising
3. Reciprocity with I.M.T.A.
4. Aer Lingus Young Scientist Exhibition
5. Elections of Secretary, Treasurer, Four Committee Members (all for two years) and One Committee Member for one year
6. Any Other Business

PERSONAL ITEMS

Dr. Richard Aron has left the Mathematics Department at T.C.D. to take up an Associate Professorship at Kent State University in Ohio.

Dr. Don Barry has been appointed College Lecturer in the Statistics Department, U.C.C. He did his Ph.D. studies at Yale University and works in Nonparametric Regression and Bayesian Analysis.

Dr. Peter J. Birch has been appointed to a temporary position in the Mathematics Department, U.C.C. He did his Ph.D. studies at Teesside Polytech. and works in Near Ring Theory.

Dr. J.W. (Bill) Bruce has left the Mathematics Department, U.C.C. to take up a Lectureship at the University of Newcastle. He works in Singularity Theory.

Dr. Emmanuel Buffet, Postdoctoral Fellow at the Mathematical Physics Department, U.C.D., has been appointed to a position as Scholar at the School of Theoretical Physics, DIAS.

Dr. Edward Cox has been appointed to a position at the Mathematical Physics Department, U.C.D. He did his Ph.D. studies in U.C.C. and works in the Theory of Nonlinear Waves.

Dr. Murray Golden of An Foras Forbartha is visiting Simon Frazer University in British Columbia this year.

Dr. Maciej Klimek who was a Postdoctoral Fellow at the Mathematics Department, T.C.D., has been appointed to a temporary position at the Mathematics Department, U.C.D. His research interests are in analysis.

Dr. Paul McGill of the New University of Ulster has been appointed to a position in the Mathematics Department, St. Patrick's College, Maynooth. He is on leave of absence in France this year.

Dr. Denis O'Brien who spent the session 1982-83 at the Mathematical Physics Department, U.C.D., has left to take up a position in the Max Planck Institute for Physics in Munich.

Dr. Donal O'Donovan of the Mathematics Department, T.C.D., spent six months at the University of California at Berkeley.

Dr. Patrick O'Leary of the Mathematical Physics Department, U.C.G., is on leave of absence at the University of Colorado, Boulder.

Dr. Niall O'Murchadha of the Experimental Physics Department, U.C.C., recently spent four weeks in September visiting the Institute of Theoretical Physics, University of Vienna, supported by a Royal Irish Academy/Austrian Academy of Sciences travel grant.

Dr. Andrew N. Prensley of Oxford University has been appointed to a Lectureship in Pure Mathematics at T.C.D. His field of interest is Lie Groups.

Dr. David Reynolds has been appointed to a permanent position at the School of Mathematics, N.I.H.E (D).

Professor William Ruckle of Clemson University, South Carolina, is visiting the School of Mathematics, T.C.D., as a Fulbright Fellow this session. His interests are in Functional Analysis.

Dr. Benedict Seifert has resigned from his position at the Mathematics Department, U.C.D.

Dr. Johannes Siemons who spent the session 1982-83 at the Mathematics Department, U.C.D., has left to take up a position at the University of Milan.

Professor C.J. Van Rijsbergen of the Department of Computer Science, U.C.D., is on leave of absence at Cambridge University for the session 1983-84.

Dr. Colin Walter of the Mathematics Department, U.C.D., is planning to spend the second semester of 1983-84 at the Department of Computing Studies of the University of East Anglia. He is to do research on graph theory.

Dr. James Ward has been appointed to a temporary position at the Mathematics Department, U.C.C. He did his Ph.D. studies at the University of Freiburg and works in Group Theory and Ring Theory.

Finally, the following promotions at the Department of Physical and Quantitative Sciences, R.T.C., Waterford.

Mr. D. O'Maidin (Head of Department) to Senior Lecturer 1.

and the following to Lecturer 2

Mr. P. Barry, Mr. P. Fallon, Mr. T. Power, Dr. J. Stynes and *Dr. M. Stynes.*

SIMPLE GEOMETRIC PROOFS IN LINEAR ALGEBRA

Michael Clancy

The orthogonal decomposition of symmetric, skew-symmetric matrices etc. is usually established by computational type proofs. The following geometric approach I feel is easier and more illuminating.

Preliminary ideas: Let $(V, <,>)$ be a finite dimensional inner-product space over $\mathbb{R}$. If $A: V \to V$ is linear and we express everything with respect to some orthonormal basis, then $<AX,Y> = <X,A^tY>$ for all X,Y in V, where A^t is the transpose of the matrix A. By definition the map A is symmetric with respect to $<,>$ if and only if $<AX,Y> = <X,AY>$ for all X,Y in V and is skew-symmetric with respect to $<,>$ if and only if $<AX,Y> = -<X,AY>$ for all X,Y in V.

Thus the notions of (skew)-symmetric maps and (skew)-symmetric matrices are equivalent provided the maps are represented with respect to an orthonormal basis. We note that the same ideas follow through in the Hermitian case with A^t replaced by A^{-t}. The *crucial point* is contained in the following lemma.

Lemma 1: If the linear map $A: V \to V$ is symmetric and leaves the subspace U invariant (i.e. $AU \equiv U$), then it also leaves $U^\perp$, the orthogonal complement of U, invariant.

Proof: If $Y \in U^\perp$, then for all $X \in U$ we have $0 = <AX,Y> = <X,AY>$ so $AY \in U^\perp$.

Remark: Of course this lemma also holds if A is skew-symmetric and correspondingly in the (skew)-Hermitian case.

The equation $(A-\lambda B)X = 0$: We consider this generalised eigenvalue problem when A and B are symmetric matrices and in addition B is positive definite. (The Hermitian case is identical.) Throughout $<,>$ will denote the usual inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$, the context will make clear which is being used. We define a new inner product on $\mathbb{R}^n$ (or $\mathbb{C}^n$) by $(X,Y) := <BX,Y>$.

Lemma 2: The eigenvalues of $(A-\lambda B)X = 0$ are all real.

Proof: If the eigenvector $X \in \mathbb{C}^n$ has eigenvalue $\lambda \in \mathbb{C}$, then $\lambda(X,X) = \lambda<BX,X> = <AX,X> = <X,AX> = <X,\lambda BX> = \bar{\lambda}(X,X)$. Thus $\lambda = \bar{\lambda}$ so that $\lambda \in \mathbb{R}$ and in particular $X \in \mathbb{R}^n$.

Theorem 3: If A and B are symmetric nxn-matrices with B positive definite, then there exists a basis of eigenvectors $X_1, \ldots, X_n$ of the equation $(A-\lambda B)X = 0$ which are orthonormal with respect to B (i.e. $<BX_i,X_j> = \delta_{ij}$).

Proof: We remark that since B is positive definite B^{-1} exists, so we can define $A^1 := B^{-1}A$ and observe

(i) X_1 is an eigenvector of $(A-\lambda B)X = 0$ with eigenvalue λ_1, if and only if it is an eigenvector of $(A - \lambda I)X = 0$ with eigenvalue λ_1.

(ii) A_1 is symmetric with respect to (,) because $(A^1X,Y) = <BA^1X,Y> = <AX,Y> = <X,AY> = <X,BA^1Y> = <BX,A^1Y> = (X,A^1Y)$.

(iii) By lemma 2 there exists an eigenvector X_1 of A^1 which we may assume to have unit length with respect to (,). If $[X_1]$ denotes the subspace spanned by X_1, then $[X_1]$ is invariant under A^1 and therefore (by lemma 1) $[X_1]^\perp$ (its orthogonal complement with respect to (,)) is also invariant under A^1.

(iv) The argument is now completed by induction with A^1 restricted to $[X_1]^\perp$.

Corollary 4: If A and B are as in theorem 3 then there exists a matrix Q such that

$$Q^tAQ = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

and $Q^tBQ = I$.

Proof: Let $Q = [X_1, X_2, ..., X_n]$ the matrix whose i^{th} column is X_i the i^{th} eigenvector (as in theorem 3) with eigenvalue λ_i.

Remarks: 1° If $B = I$, then corollary 4 is the usual statement that every symmetric matrix can be orthogonally diagonalized. That symmetry is necessary here is obvious since $Q^tAQ = D$ where D is diagonal and $Q^tQ = I$ imply $A = QDQ^t = (QDQ^t)^t = A^t$.

2° While symmetry is used to show that the eigenvalues are real it is not the key point. Indeed, $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has real distinct eigenvalues but cannot be orthogonally diagonalized. Of course we see that $\binom{1}{0}$ is an eigenvector whose orthogonal complement, i.e. the line $\{\binom{0}{t} : t \in \mathbb{R}\}$ is not invariant under A.

Theorem 5: If A is an nxn skew-symmetric matrix, then there exists an orthonormal basis for $\mathbb{R}^n$ with respect to which A is tri-diagonal. That is there exists Q satisfying $Q^tQ = I$ and

$$Q^tAQ = \begin{pmatrix} \begin{array}{cc|cc} 0 & -b_1 & & \\ b_1 & 0 & & \\ \hline & & 0 & -b_2 \\ & & b_2 & 0 \end{array} & & \\ & \ddots & \\ & & 0 \\ & & & \ddots \\ & & & & 0 \end{pmatrix}$$

Proof: By the argument of lemma 2 one sees that all the eigenvalues of A are pure imaginary. If $m(t)$ denotes the minimal polynomial for A (over $\mathbb{R}$) and if ib, $b \neq 0$, is an eigenvalue of A with eigenvector X, then $0 = m(A)X = m(ib)X$ implies ib is a root of $m(t) = 0$. Accordingly $m(t) = q(t)(t^2+b^2)$. Therefore, since $m(t)$ is minimal, there exists a unit vector $X_1 \in \mathbb{R}^n$ such that $(A^2+b^2I)X_1 = 0$. If we define $X_2 = (AX_1)/b$ then

(i) $AX_1 = bX_2$ and $AX_2 = (A^2X_1)/b = -bX_1$.

(ii) $<X_2, X_2> = <(AX_1)/b, (AX_1)/b> = -<X_1, A^2X_1>/b^2 = <X_1, X_1> = 1$.

(iii) $<X_1, X_2> = <X_1, (AX_1)/b> = -<(AX_1)/b, X_1> = -<X_2, X_1>$ implies $<X_1, X_2> = 0$

(iv) The subspace $[X_1, X_2]$ spanned by X_1 and X_2 is invariant under A and therefore (by lemma 1) so also is its orthogonal complement $[X_1, X_2]^\perp$. We now continue by induction on A restricted to $[X_1, X_2]^\perp$.

(v) The case of zero eigenvalues is easily taken care of and it is clear that the basis produced is the required one with Q being the change of basis matrix.

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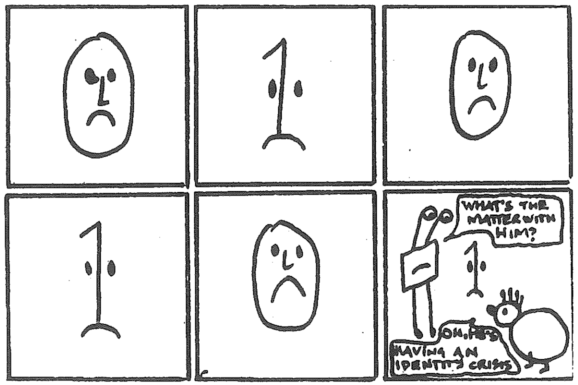


From 2-*Manifold*, No. 1

POPULATION BIOLOGY OF INFECTIOUS DISEASES

R.G. Flood

The Population biology of infectious diseases breaks roughly into two main classes - those diseases causing immunity and those not causing immunity. These classes correspond (again roughly) to microparasitic infections and macroparasitic infections.

I don't intend to become too technical in this discussion but will aim to show how some simple mathematical models can be very powerful. In this paper I will discuss mainly microparasitic infections and in particular the construction and effects of vaccination programmes. However to start I want to describe briefly the other major category.

MACROPARASITES

Examples of diseases: Hookworm, Schistosomiasis

These are widespread and serious diseases. Approximately 200 million people suffer from schistosomiasis.

In general macroparasites have quite long generation times, and direct multiplication within the host is either absent or occurs at quite a low rate. The immune response elicited generally depends on the number of parasites present in a given host and tends to be of relatively short duration. Macroparasitic infections therefore tend to be of a persistent nature with hosts being continually reinfected. The pathogenicity of the infection is related to the worm burden. Typically, theoretical work had taken the worm burden to be Poisson distributed. However recent field trials have indicated that a much more realistic model for some infections, e.g. Hookworm, is to take the worm burden to be distributed according to the

negative Binomial distribution.

This distribution is more highly exaggerated, for example less than 10% of the population harbours 80% of the parasites.

Distribution	Probability Generating Function
Poisson	$\pi(z) = e^{-\mu(1-z)}$
Negative Binomial	$\pi(z) = [1 + \frac{\mu}{k}(1-z)]^{-k}$

This has major implications for control strategy since an obvious thing to do is to try and identify those people with a high worm burden and treat them. This is being attempted at the moment in two villages in Burma for Hookworm.

An interesting question concerns the reason for this aggregation of the worm burden. One speculation is that there is a genetic predisposition to high worm density.

MICROPARASITIC INFECTIONS

These are caused by most viruses, most bacteria and many protozoans.

They are characterized by small size, short generation times, extremely high rates of direct reproduction within the host and a tendency to induce immunity to reinfection in those hosts that survive the initial onslaught. The duration of infection is typically short in relation to the lifespan of the host and therefore is of a transient nature. However there are many exceptions.

Some examples are given in the Table on the following page:

	Incubation Period (days)	Duration of Infectiousness (days)	Pathogenicity
Measles	9-12	5-7	Low-High
Smallpox	12-14	10	High
Rubella	17-20	14	Low
Mumps	10-20	7	Low
Whooping Cough	7-10	14+	Medium
Polio	5-20	Long	Medium
Herpex Simplex Virus	5-8	Long	Very Low

All these examples induce lifelong immunity. However this need not be the case - typhoid is an example.

The reason why the pathogenicity of measles varies from low to high is that in developing countries measles can kill 30% of those who obtain it. Smallpox has been eradicated but we will see that it is an interesting example to consider when studying the present controversy regarding vaccination programmes for Whooping Cough.

Rubella (German measles) and Mumps will illustrate another aspect of the effects of a vaccination programme, which is, that vaccination increases the average age at which the infection is obtained. This is of particular concern for these two infections. Mumps in adolescent and adult males can cause intense discomfort. In women, rubella, which is normally a mild infectionaccompanied by a fever may cause serious diseases in offspring if the infection is acquired during the first three months of pregnancy; infants born with congenital rubella syndrome may suffer deafness and neurological and other disorders.

The first result I want to obtain is a connection between the force of infection λ and the average age A at which infection is obtained. The result is:

$$\lambda = \frac{1}{A}$$

where

λ = force of infection - percapita rate of acquiring infection; the probability to acquire infection in unit time.

A = average age of infection.

To obtain this we use a compartment model. Divide the host population into discrete classes, at age a and at time t, let

$X(a,t)$ = number susceptible,
$Y(a,t)$ = number infectious,
$Z(a,t)$ = number recovered and immune, at age a at time t.

The basic partial differential equations for this system are

$$\frac{\partial X}{\partial t} + \frac{\partial X}{\partial a} = -[\lambda(t) + \mu(a)]X(a,t)$$

$$\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial a} = \lambda X - [\alpha(a) + \mu(a) + \nu]Y(a,t)$$

$$\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial a} = \nu Y - \mu(a)X(a,t)$$

with initial and boundary conditions

$t=0$ specify $Z(a,0)$, $Y(a,0)$, $Z(a,0)$

$a=0$ specify $X(0,t) = B$, $Y(0,t) = X(0,t) = 0$

$\lambda(t)$ = force of infection

$\mu(a)$ = age specific death rate

$\alpha(a)$ = disease induced death rate

ν = recovery rate (constant).

<u>Observations:</u> This compartment model can be modified in many ways. Some of these are as follows:

(1) A latent class (infected but not yet infectious) may be added.

(2) Maternal antibodies may protect for first three to nine months so infants are born into new protected class and lose immunity in the first year.

(3) Immunity may be lost, not lifelong, as above.

(4) We have assumed a constant recovery rate v but recovery may be after some defined interval or some more general statistical recovery.

(5) The assumption of homogeneous mixing assumes that we can average out all local details - school, family etc. This allows us to write

$$\lambda(t) = \beta\int_0^\infty Y(a,t)da.$$

These, or similar differential equations are often the starting point for the analysis. However I now want to restrict attention to the equilibrium situation with the assumptions that

(a) Births and deaths exactly balance - justified usually by stating that population densities remain roughly constant on the time scale appropriate to the pathology of most diseases. This is clearly not a reasonable assumption for many countries.

(b) $\alpha = 0$; infection does not cause significant number of deaths. Again this is not a reasonable assumption for some diseases in the developing countries.

Under equilibrium the partial differential equations reduce to

$$\frac{dX}{da} = -(\lambda + \mu(a)X(a))$$

$$\frac{dX}{da} = \lambda x - (v + \mu(a)Y(a))$$

$$\frac{dZ}{da} = vY - \mu(a)Z(a)$$

$$N(a) = X(a) + Y(a) + Z(a)$$

$$X(0) = N(0);\ Y(0) = Z(0) = 0.$$

Then

$$X(a) = N(0)\phi(a)\exp(-\lambda a)$$

$$N(a) = N(0)\phi(a),$$

where

$$\phi(a) = \exp(-\int_0^a \mu(s)ds).$$

The fraction of people of age a who are susceptible is

$$x(a) = \frac{X(a)}{N(a)} = \exp(-\lambda)$$

Therefore the average age of infection

$$\equiv \frac{\int_0^\infty a\lambda x(a)da}{\int_0^\infty \lambda x(a)da} = \frac{1}{\lambda}$$

So

$$A = \frac{1}{\lambda}.$$

This relates the 'observable' A with the more abstract λ - provided we treat λ as independent of age. This is frequently done in mathematical work but usually is not true. From $\lambda A = 1$ we conclude that the lower the force of infection the greater the age at which infection is obtained. Therefore weakening this force of infection, e.g. by vaccination, increases this average age which is of concern for infections such as rubella.

We now need two further concepts. The basic reproduction rate R_0 is the number of secondary cases produced, on average, when everyone is susceptible. R_0 combines the biology of the infection with social and behavioural factors influencing contact rates.

The effective reproductive rate, R, when X out of N are susceptible is, assuming homogeneous mixing, given by $R=R_0X/N$.

It is the number of secondary cases produced on average when X out of N are susceptible.

However at equilibrium we must have R = 1, therefore

$$(R_0 \frac{X}{N})_{eq.} = 1$$

which implies

$$R_0 = (\frac{N}{X})_{eq.}$$

Now

$$N = \int_0^\infty N(a)da, \qquad X = \int_0^\infty X(a)da$$

To find N,X we need to specify $\mu(a)$ the age specific death rate. I will take this as all who have lived to age L when they die.

There are obvious alternatives, e.g. taking $\mu(a)$ to be a constant. Then from the expressions previously obtained for N(a), X(a) we obtain

$$R_0 = \frac{L}{A} \text{ if L is much greater than A.}$$

Note: if $\mu(a)$ is taken as a constant then

$$R_0 = 1 + \frac{L}{A}.$$

Examples: Before immunization began in the U.S. and U.K., children typically caught measles and whooping cough around 4 - 5 years. If we take L = 70 then

R_0 = 13 - 15 for measles and whooping cough.

For rubella A is 9 - 10 years giving

R_0 = 7 - 8 for rubella.

Effects of Vaccination

Mass vaccination as a means of controlling diseases has two main effects. Most obviously there is the direct effect that those effectively immunized are protected against infection. The second and indirect effect which is less obvious

arises because a susceptible individual has less chance of acquiring the infection in a partially vaccinated community than in an unvaccinated population; there are fewer people around him to give him the disease, thus it is not necessary to immunize everyone to eradicate the infection. The crucial factor is the effective reproductive rate R of the disease. If $R \geq 1$, i.e. if each infected individual infects one or more persons before he shakes off the disease then the infection will persist. But if $R < 1$ the disease will die out even if there are susceptible people in the community.

Let us suppose a proportion p are vaccinated at age b. Then we can find the new equilibrium for the previous system of differential equations. From this we can calculate the new force of infection, which will depend on p. Eradication of the disease corresponds to the force of infection going to zero. To achieve this it is then seen that the critical proportion requiring to be vaccinated is given by

$$P \text{ critical} > 1 - \frac{1 - \frac{b}{A}}{R_0 - \frac{b}{A}}$$

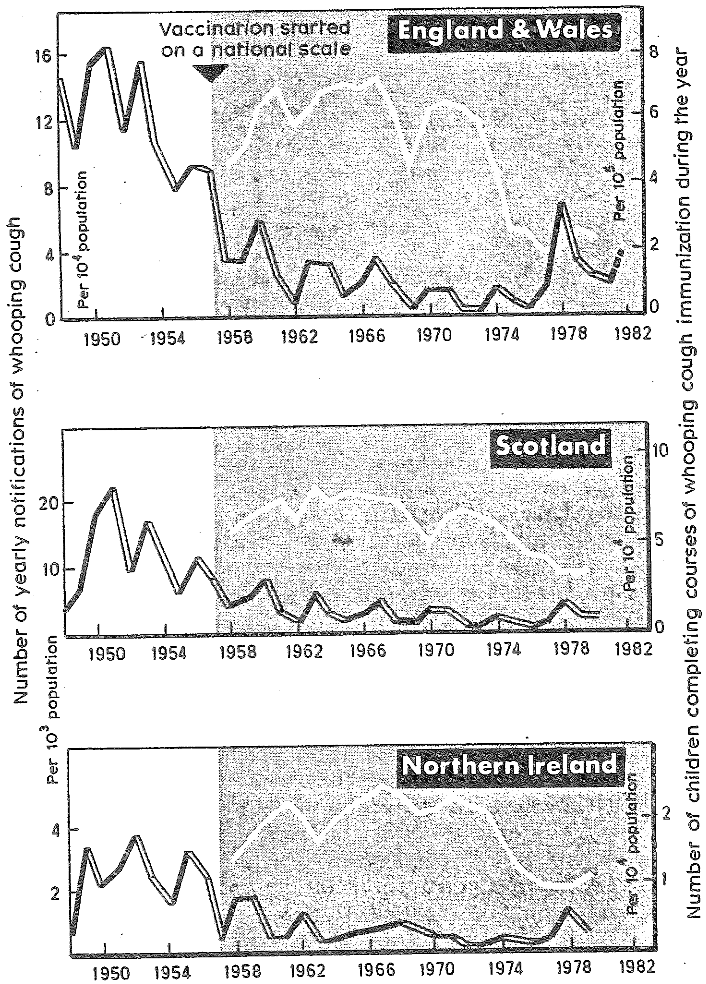
If b = 0, i.e. vaccination occurs at birth then

$$P \text{ critical} > 1 - \frac{1}{R_0}$$

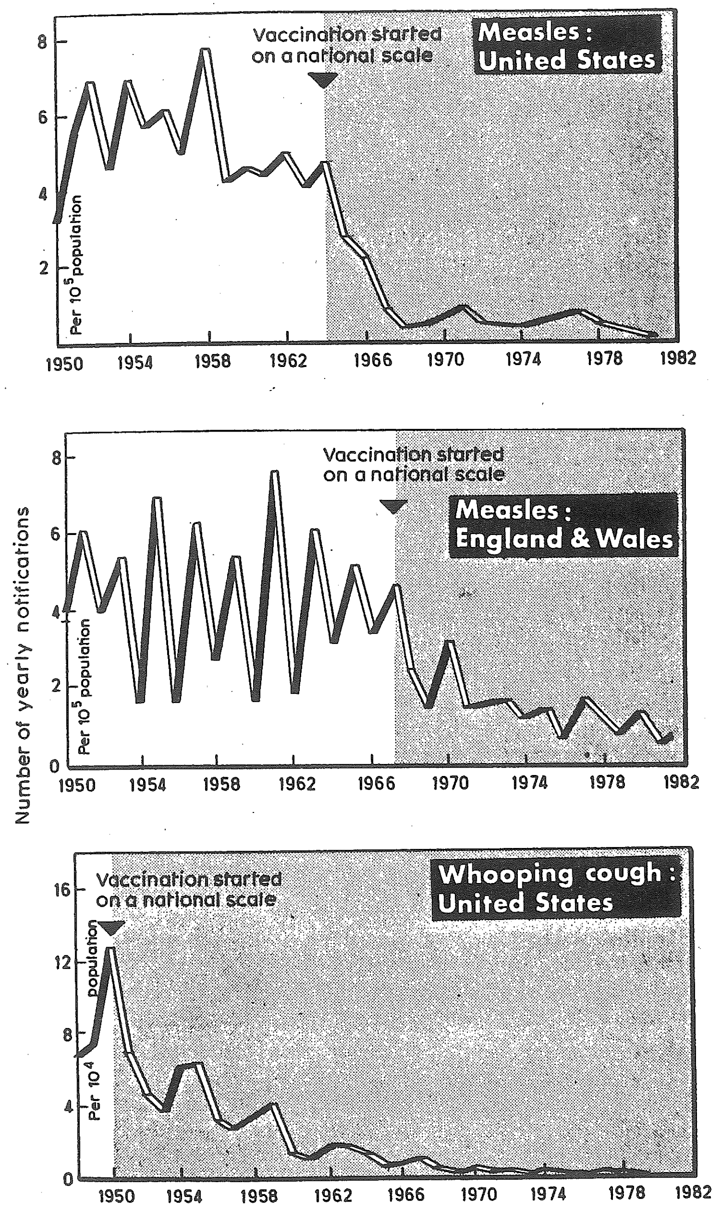
For measles and whooping cough we need 93% approximately. For polio and diptheria the situation is a little better since we require 80% and 85% respectively. If vaccination occurs at age 2 then we need approximately 95% coverage to eradicate measles and whooping cough.

These figures are depressingly high, but in the U.S. where indigenous measles has virtually disappeared more than 95% of children are currently vaccinated before reaching school age. This is the result of the childhood immunization initiative which began in 1977 (although widespread vaccination began much earlier). The goal is supported by laws which require docum-

Examples of trends in the numbers of reported cases of whooping cough (pertussis) and measles in Britain and the United States. The three graphs below show the trends in the United Kingdom of reported pertussis cases and the numbers of children vaccinated annually. The graphs to the right denote the reported cases of measles in the United States and England and Wales, plus pertussis in the United States



From *New Scientist*, 18 November, 1982



From *New Scientist*, 18 November, 1982

entary proof that a child has been vaccinated by the time it enters first grade; all 50 states had enforced this law by January 1982. Currently more than 95% of children in the U.S. are vaccinated against measles and pertussis (Whooping cough) before they enter school.

In the U.K. immunization is not enforced by law and high levels of vaccine coverage have proved difficult to achieve. Over the past decade levels of vaccination of children against diptheria, polio and measles have remained approximately constant - around 80% for diptheria and polio and around 50% for measles. In the case of whooping cough, as a result of the much publicised debate on the safety of the vaccine during the mid-1970s, vaccination fell from 80% in 1970 to less than 40% in 1981.

The controversy continues today and the current whooping cough epidemic is a direct result of the low levels of vaccination. It is interesting to compare this situation with that of smallpox.

In the U.K. from 1951 to 1970 there were roughly 100 deaths from smallpox vaccination and approximately 37 from smallpox itself, and in the U.S. the centre for disease control estimated that it would require 15 importations per year to produce the same mortality currently associated with smallpox vaccinations in the U.S. As a result vaccination against smallpox was discontinued both in the U.S. and U.K. in 1971.

The risk benefit analysis for whooping cough vaccine is considerably more complex than it was for smallpox. For one thing whooping cough has never been close to eradication in developing countries the way smallpox was in the 1960s. For another the range of neurological illnesses associated with whooping cough vaccination is clinically indistinguishable from that occurring in children who have not been immunized. In smallpox, by contrast, any disease produced by the vaccine can be clearly distinguished from the natural disease. Estimation

of the risk of vaccination must therefore come from large and statistically well designed studies that aim to distinguish neurological damage caused by the vaccine from the background of similar cases that arise in infancy and early childhood caused by bacterial and viral infections, congenital diseases and other processes that appear to be poorly understood.

The three year National Childhood encephalopathy study in the U.K. examined the records of every child below three years of age admitted to hospitals in Britain with neurological illness between June 1976 and July 1979. The study concluded that both pertussis and measles vaccines can indeed cause acute neurological reactions but that in both cases these are rare events. The study estimated the risk of persistent neurological damage one year after vaccination to be 1 in 310,000 immunizations.

The 1977-79 epidemic in Britain was responsible for 36 deaths and 17 cases of brain damage. Others suffer from continuing illness. The lower level illness associated with the disease should not be underestimated. The illness is protracted and debilitating, usually lasting 10 to 12 weeks and led to 5000 hospital admissions.

Based on the National Childhood encephalopathy study and a birth rate of 1.2%, i.e. 600,000 births, we can estimate that 6 cases of brain damage might be expected each year if every child completes the full course of 3 injections.

The consequences of the 1977-1979 epidemic were much worse.

Inter-Epidemic Period

From the graph one can see variation in the patterns of incidence. The models I have described also exhibit this behaviour. The incidence varies both from season to season and over longer periods. The seasonal trend is in part determined by patterns of social behaviour such as timing of school holidays. The best known examples of longer term fluctuations, taking place on time scales greater than one year, are the two to three year cycles in measles and the three to four year cycles in whooping cough. Indeed many directly transmitted viral infections such as measles, and bacterial infections such as whooping cough, typically follow such a pattern of recurrent epidemic largely because the susceptible population varies. First the number of susceptible people decreases as immunity is acquired by recovering from infection and then the number of susceptibles increases slowly as children are born.

This can be explained by the model we have proposed. In the partial differential equations we integrate out the age variable a to obtain a system of ordinary differential equations in which we will take the death rate to be constant. We obtain

$$\frac{dX}{dt} = \mu N - (\lambda + \mu)X(t)$$

$$\frac{dY}{dt} = \lambda X - (\nu + \mu)Y(t)$$

$$\frac{dZ}{dt} = \nu Y - \mu Z(t)$$

$$\frac{dN}{dt} = 0 \qquad N = X(t) + Y(t) + Z(t)$$

Assuming birth rate = death rate.

Then we can analyse the equilibrium and stability behaviour of the solutions, to obtain damped periodic solutions with period

$$T = 2\pi\sqrt{A\tau}$$

where A = average age on infection

τ = average interval between an individual acquiring infection and passing it on to the next infectee.

Examples:

	A years	years	T years
Measles	4-5	1/25	2-3
Whooping Cough	4-5	1/14	3-4
Rubella	9-10	1/17	5

The tendency for the incidence of disease to oscillate in a regular manner raises a further problem in assessing the benefits of mass immunization. If vaccination coverage is high the non-seasonal epidemics will be small and it will be difficult to distinguish epidemic from non-epidemic years.

Under low to moderate levels of vaccination however, there may still be more cases in an epidemic year than there were in non-epidemic years before vaccination.

Thus in any assessment of the risk of exposure to infection we must base our calculations on the average risk over the inter-epidemic period covering years of low and high risk.

Average Age of Infection

I wish now to discuss the effect of a vaccination programme to increase the average of infection. I will illustrate this with respect to rubella, which as I have mentioned, is normally a mild infection accompanied by a fever but can cause serious damage to offspring if acquired during the first three months of pregnancy.

In the U.S. boys and girls are vaccinated against rubella around the age of two years with the aim of creating sufficient levels of herd immunity to virtually eliminate rubella from the population. Currently more than 90% of children are vaccinated before entering school and the incidence of rubella has dropped to very low levels.

In the U.K. the aim is to facilitate the natural circulation of the virus in the population so that most girls have contracted rubella before they reach child-bearing age. Currently the vaccination coverage is 60-80%. The average age (before vaccination programmes began) at which children caught rubella was 9-10 years. If they are still susceptible in early childhood then girls and only girls are immunized at around 12 years of age. This policy has had little impact on the incidence of rubella as such but has reduced the number of cases of congenital rubella syndrome.

Recent theoretical research has yielded the satisfactory conclusion that Britain's policy is best for Britain (since high levels of vaccination cannot be achieved) while the U.S. policy is best for its circumstances.

The important thing in the U.S. policy is that the level of vaccination does not fall below 50% to 55% otherwise more cases of congenital rubella syndrome will be obtained - due to increased average age of infection.

Define

D_a = number of people acquiring infection between ages a_1 and a_2 at equilibrium after vaccination

D_b = number of people acquiring infection between ages a_1 and a_2 at equilibrium before vaccination.

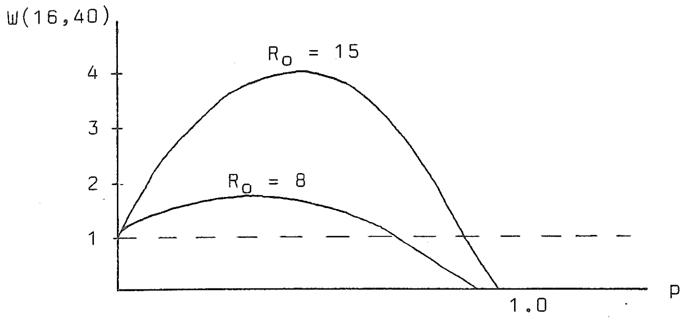
$W(a_1,a_2) = D_a/D_b$

We wish to ensure that the ratio $W(a_1,a_2)$ is less than 1. From the theory we have developed it can be shown that

$$W(a_1,a_2) = (1 - p)\,\frac{e^{-\lambda_1 a_1} - e^{-\lambda_1 a_2}}{e^{-\lambda a_1} - e^{-\lambda a_2}}$$

where p = proportion vaccinated
λ_1 = force of infection at equilibrium after vaccination
λ = force of infection at equilibrium before vaccination.

We sketch below the graph of $W(16,40)$ for rubella and measles. The graphs illustrate that the number getting sick at older ages can increase if the coverage p is not approaching 1 whenever R_0 is large.



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VAN DER WAERDEN'S CONJECTURE ON PERMANENTS AND ITS RESOLUTION

Thomas J. Laffey

Let $A = (a_{ij})$ be an nxn matrix. The *permanent*, per A, of A is given by the formula

$$\text{per } A = \sum_{\sigma \in S_n} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

where the sum is over the symmetric group S_n. Thus per A is obtained from det A by formally replacing the factors sign(σ) in the expansion of det A by +1. Let a_i be the i^{th} column of A. Then it is clear that per A = per$(a_1,\ldots,a_n)$ is multilinear. Also if A(i,j) is the (n-1)x(n-1) submatrix obtained from A by deleting row i and column j, we have the Laplace-like expansions

$$\text{per } A = \sum_{i=1}^{n} a_{ij} \text{per } A(i,j) \qquad (j=1,2,..n)$$
$$= \sum_{j=1}^{n} a_{ij} \text{per } A(i,j) \qquad (i=1,2,..n).$$

However, per A does not have the alternating properties of det A and it is not in general multiplicative, so it is not a similarity invariant. However, it is clear that per $P^T AP$ = per A for all permutation matrices P,Q. This last property enables one to replace A by a matrix equivalent to A by permutation matrices in carrying out calculations and it is used many times without explicit mention in this article.

A real nxn matrix is called *doubly-stochastic* if its entries are non-negative and

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad (i=1,2,\ldots,n)$$
$$\sum_{i=1}^{n} a_{ij} = 1 \qquad (i=1,2,\ldots,n)$$

Let DS(n) be the set of nxn doubly-stochastic matrices. Then DS(n) is a compact subset of $\mathbb{R}^{n^2}$. Let

$$f(n) = \inf\{\text{per } A \mid A \in DS(n)\}.$$

By compactness, there exist elements $A \in DS(n)$ with per A = f(n). Such a matrix A is called a ***minimizing matrix***. Thus A is a minimizing matrix if A is an (nxn) doubly-stochastic matrix such that its permanent achieves the ***absolute minimum*** of the permanent on the set of all doubly-stochastic matrices.

The famous van der Waerden conjecture (1926) states

Van der Waerden Conjecture

(1) $f(n) = n!/n^n$

(2) there is exactly one minimizing matrix, namely the matrix J_n that has all its entries equal to 1/n.

This conjecture was resolved in the affirmative by G.P. Egorychev of Krasnoyarsk in the U.S.S.R. in 1980. Independently D.I. Falikman, also from the U.S.S.R., proved part (1) of the conjecture in a paper submitted in 1979. Various special cases of the conjecture had been resolved earlier by various authors. Of particular beauty was the verification of the conjecture for the class of positive semi-definite symmetric doubly-stochastic matrices by Marcus and Newman (1962), later improved by Minc (1963), and the work of Friedland in the 1970s who showed in particular that per A > 1/n! Of particular relevance to subsequent interest in the problem as well as to its solution was the verification by Marcus and Newman (1959) of the conjecture for matrices that have all their entries positive. While the verification of the van der Waerden conjecture for n=2 is an elementary exercise, the problem quickly increases in difficulty as n increases and it was not until 1968 that Eberlein and Mudholkar settled the case n=4 and 1969 that Eberlein settled the case n=5.

In this expository article we present an account of Egorychev's work and describe the necessary background results. As well as Egorychev's own account [2] which appeared in English in Advances in Mathematics, an account of his work has been published by van Lint [10] and a detailed account with the background filled in has been given by Knuth in the American Mathematical Monthly [5]. The presentation here has been greatly influenced by the accounts of van Lint and Knuth. In the final section we describe a few more recent results.

A full and authoritative account of the properties and importance of permanents has been given by Minc in his enjoyable book [7]. The problem of computing permanents is described by Nijenhuis and Wilf in Chapter 23 of [8]. Permanents arise in many combinatorial problems and the "permanental polynomial" per $(xI-A)$ is sometimes referred to as one of the isomorphism invariants of a graph with incidence matrix A.

1. Preliminaries

Let $A = (a_{ij})$ be an nxn matrix. The (directed) graph $G(A)$ is the graph with vertices $1,2,3,..,n$ and such that for $i \neq j$, ij is a (directed) edge of $G(A)$ if and only if $a_{ij} \neq 0$. $G(A)$ is connected if for all $i \neq j$, there exists $s \geq 1$ and a sequence $i_0 = i, i_2, ..., i_s = j$ such that $i_0 i_1, i_1 i_2, .., i_{s-1} i_s$ are edges of $G(A)$. Equivalently, A is irreducible under permutation similarity, i.e. there is no permutation matrix P such that

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is an rxr, A_{22} an $(n-r)x(n-r)$ matrix, some $1 \leq r < n$. (A special case of) the Perron-Frobenius theorem states that if A is a permutation irreducible non-negative real matrix, then A has a real eigenvalue r with $r \geq |\lambda|$ for all eigenvalues λ of A and r is a simple eigenvalue.

A theorem of Birkhoff states that the set $DS(n)$ of doubly-stochastic matrices is precisely the set of convex combinations of the permutation matrices, i.e. $A \in DS(n)$ if and only if there exist non-negative real numbers $a(\sigma)$ with $\sum a(\sigma) = 1$ such that

$$A = \sum_{\sigma \in S_n} a(\sigma)P(\sigma)$$

where $P(\sigma)$ is the permutation matrix corresponding to σ. Note that this result in particular implies that $f(n) > 0$.

2. Minimizing Matrices

Throughout this section $A = (a_{ij}) \in DS(n)$ is such that per $A = f(n)$.

Lemma 2.1 A is irreducible under permutation similarity.

Proof Suppose not. Since per $(P^T AP)$ = per A for P a permutation matrix, we may assume

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is rxr, A_{22} $(n-r)x(n-r)$, some $1 \leq r < n$. Since $A \in DS(n)$, looking at the sum of all the entries in A_{11}, we see that $A_{11} \in DS(r)$, that $A_{12} = 0$ and thus that $A_{22} \in DS(n-r)$. Note that per A = per A_{11} per A_{22}.

Now a simple induction yields that per $A_{11} > 0$ and that per $A_{22} > 0$. We may assume that $a_{ii} > 0$, $(i=1,2,...,n)$. Let $A(e)$ be the matrix obtained from A by replacing a_{11} by $a_{11}-e$, $a_{1,r+1}$ by $a_{1,r+1}+e$, $a_{r+1,1}$ by $a_{r+1,1}+e$, $a_{r+1,r+1}$ by $a_{r+1,r+1}-e$. Then for sufficiently small $e > 0$, $A(e) \in DS(n)$. But a simple calculation yields per $A(e) <$ per A for all sufficiently small $e > 0$. This is a contradiction.

The next result, due to Marcus and Newman, is crucial to the discussion.

Theorem 2.2 For all i,j for which $a_{ij} > 0$, we have $\text{per } A(i,j) = \text{per } A$.

Proof Let $Z = \{B \in DS(n) \mid b_{ij} = 0 \text{ if } a_{ij} = 0\}$.

Using Lemma 2.1, we see that A is an interior element of Z and hence it must satisfy the analytic criteria for a local minimum. A matrix $X = (x_{ij}) \in Z$ if the following conditions hold

$$x_{ij} \geqq 0 \quad (\text{all } i,j)$$

$$x_{ij} = 0 \quad \text{if } a_{ij} = 0$$

$$\sum_{j=1}^{n} x_{ij} - 1 = 0 \quad (i=1,..,n)$$

$$\sum_{i=1}^{n} x_{ij} - 1 = 0 \quad (j=1,..,n)$$

Introducing Lagrange multipliers, we consider the function

$$F(X) = \text{per }(X) - \sum_{i=1}^{n} \lambda_i \Big(\sum_{j=1}^{n} x_{ij} - 1\Big) - \sum_{j=1}^{n} \mu_j \Big(\sum_{i=1}^{n} x_{ij} - 1\Big).$$

If $x_{ij} \neq 0$, the partial derivative

$$\frac{\partial F}{\partial x_{ij}} = \text{per } X(i,j) - \lambda_i - \mu_j$$

so

$$(*) \qquad \text{per } A(i,j) = \lambda_i + \mu_j \quad \text{if } a_{ij} \neq 0.$$

Now the expansions

$$\text{per } A = \sum_{i=1}^{n} a_{ij} \text{ per } A(i,j)$$

$$= \sum_{j=1}^{n} a_{ij} \text{ per } A(i,j)$$

yield

$$(1) \qquad \text{per } A = \lambda_i + \sum_{j=1}^{n} a_{ij}\mu_j$$

$$(2) \qquad \text{per } A = \sum_{i=1}^{n} a_{ij}\lambda_i + \mu_j.$$

Let $e = (1,\ldots,1)^T$, $\lambda = (\lambda_1,\ldots,\lambda_n)^T$, $\mu = (\mu_1,\ldots,\mu_n)^T$. The equations become

$$(\text{per } A)e = \lambda + A\mu$$

$$(\text{per } A)e = A^T\lambda + \mu.$$

Since $A \in DS(n)$, $Ae = A^Te = e$. Thus we obtain

$$A^T\lambda + A^TA\mu = A^T\lambda + \mu$$

$$\lambda + A\mu = AA^T\lambda + A\mu.$$

Thus $A^TA\mu = \mu$, $AA^T\mu = \lambda$. But A and therefore AA^T, A^TA are irreducible with maximum eigenvalue 1 and corresponding eigenvector e. By the Perron-Frobenius theorem, 1 is a simple eigenvalue, so

$$\lambda_1 = \ldots = \lambda_n = a, \text{ say}$$

$$\mu_1 = \ldots = \mu_n = b, \text{ say}.$$

But then $\text{per } A = \sum_{i=1}^{n} a_{ij} \text{ per } A(i,j) = a + b$ and the result follows.

Remark We note that Knuth [5] gives a purely combinatorial argument to establish (*).

We note also that Marcus and Newman were able to obtain a proof that if $a_{ij} > 0$ for all i,j, then $A = J_n$ easily from (2.2). This is not used in Egorychev's work, so we omit it. Details are given in Minc [7], page 79.

The following partial extension of (2.2) to the case where $a_{ij} = 0$ is due to London (1971) ([7], page 85).

Theorem 2.3 For all i,j, $\text{per } A(i,j) \geqq \text{per } A$.

Proof In proving the result for a pair i,j we may assume $a_{ij} = 0$. Using the remark on permutation equivalence in the introduction and (2.1) we may assume i=1, j=1 and further that $a_{kk} \neq 0$ (k=2,...,n).

Note that for sufficiently small $e > 0$, $(1-e)A + eI \in DS(n)$ and using the fact that for $C = (c_{ij})$, $D = (d_{ij})$,

$$p(C + eD) = \text{per } C + e \sum_{i,j=1}^{n} d_{ij} \text{ per } C(i,j) + O(e^2)$$

and (2.2) we obtain

$$\text{per}((1-e)A + eI) = \text{per } A + e(\text{per } A(1,1) - \text{per } A) + O(e^2).$$

Since A is minimizing, we obtain per $A(1,1) \geqq$ per A, as required.

3. Aleksandrov's Inequality

The next ingredient in Egorychev's solution is (a special case of) an inequality of Aleksandrov (1938) [1]. This arose in the context of computing the volumes of convex sets.

Suppose $a_1,\ldots a_{n-2}$ are (column)-vectors in $\mathbb{R}^n$. We can define an inner product by

$$x.y = \text{per }(a_1,\ldots a_{n-2},x,y)$$

for $x,y \in \mathbb{R}^n$. (Of course this is not a positive definite inner product.) We may write $x.y = x^TQy$ for a symmetric matrix Q.

The result of Aleksandrov we require is

Theorem 3.1 Let $a_1,\ldots a_{n-1}$ be elements of $\mathbb{R}^n$ with all their entries positive. Then (using the notation above) for $x \in \mathbb{R}^n$

(*) $$(x.a_{n-1})^2 \geqq (x.x)(a_{n-1}.a_{n-1})$$

with equality if and only if $x = ba_{n-1}$ for some real b. (Note that (*) is the reverse of the Cauchy-Schwarz inequality valid for positive definite inner products.)

We show that Theorem (3.1) follows from

Theorem 3.2 Let $a_1,\ldots a_{n-2} \in \mathbb{R}^n$ have positive entries. Then (in the notation above) Q is non-singular and has exactly one positive eigenvalue.

For suppose (3.2) holds. Suppose x and a_{n-1} are independent. Then on the two dimensional space span(x,a_{n-1}), there exists an element $x + ha_{n-1}$ such that $(x + ha_{n-1}).(x + ha_{n-1}) < 0$. Thus

$$h^2a_{n-1}.a_{n-1} + 2hx.a_{n-1} + x.x = 0.$$

Since $a_{n-1}.a_{n-1} > 0$ (as all the a_i have positive entries) the discriminant of the polynomial

$$\lambda^2a_{n-1}.a_{n-1} + 2\lambda x.a_{n-1} + x.x$$

is positive, proving (*).

We now prove (3.2) by induction on n.

If n=2, $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the result is trivial. Suppose $n > 2$ and that the result holds for n-1. We first show Q is non-singular. For suppose $Qx = 0$. Since $Q = (q_{ij})$ where

$$q_{ij} = \text{per}(a_1,\ldots,a_{n-2},e_i,e_j)$$

(where $e_1,\ldots e_n$ is the standard basis of $\mathbb{R}^n$) we see that

(†) $$\text{per}(a_1,\ldots a_{n-2},x,e_j) = 0 \quad (j=1,2,..,n).$$

This equation is the same as

$$\text{per}((a_1,\ldots,a_{n-2},x,e_j)(i,n)) = 0.$$

Applying the induction hypothesis and hence (*) to the (n-1)x(n-1) matrix

$$(a_1,\ldots,a_{n-2},x,e_j)(j,n)$$

and the fact that $(a_1,\ldots,a_{n-2},a_{n-2},e_j)(j,n)$ has a positive permanent, we obtain

$$\text{per}((a_1,\ldots,a_{n-3},x,x,e_j)(j,n)) \leq 0$$

with equality if and only if $x - ca_{n-2}$ is zero at all positions except possibly the j^{th} for some real c. But in the case of equality we must have $c = 0$ since a_{n-2} has positive entries.

Hence

(††) $$\text{per}(a_1,\ldots a_{n-3},x,x,e_j) \leq 0$$

with equality if and only if x has all its entries except possibly the j^{th} zero.

But by (†)

$$\text{per}(a_1,\ldots,a_{n-2},x,x) = 0$$

and since a_{n-2} has positive entries, this with (††) gives

$$\text{per}(a_1,\ldots,a_{n-3},e_j,x,x) = 0$$

for all j and hence x has all its entries zero.

Thus Q is non-singular. Let $Q(\lambda)$ be defined by replacing a_i by $\lambda e + (1-\lambda)a_i$ where $e = (1,1,\ldots,1)^T$. Applying the above argument to $Q(\lambda)$ we conclude that $Q(\lambda)$ is non-singular for $0 \leq \lambda \leq 1$. Hence by continuity, the number of positive eigenvalues of $Q(0) = Q$ is the same as that of $Q(1)$. But $Q(1) = (n-1)!\,(E-I)$ where E is the nxn matrix that has all its entries 1 so the eigen values of $Q(1)$ are

$$(n-1)!\,(n-1),\ -(n-1)!,\ldots,\ -(n-1)!$$

So (3.2) holds.

By continuity we obtain from (3.1)

<u>Corollary 3.3</u> If $a_1,\ldots a_{n-1} \in \mathbb{R}^n$ have non-negative entries, then for $x \in \mathbb{R}^n$

$$(x.a_{n-1})^2 \geq (x.x)(a_{n-1}.a_{n-1}).$$

4. <u>Egorychev's Resolution</u>

Suppose $A \in DS(n)$, $n \geq 3$ with per $A = f(n)$.

We first show that for all i,j

(†) $$\text{per } A(i,j) = \text{per } A.$$

This is true by (2.2) if $a_{ij} > 0$ and by (2.3), per $A(i,j) \geq$ per A if $a_{ij} = 0$. Suppose that for some i,j, per $A(i,j) >$ per A. Now for some t, $a_{it} > 0$. By Corollary 3.3,

$$\text{per}(a_1,\ldots,a_i,\ldots,a_t,\ldots,a_n)^2 \geq$$

$$\text{per}(a_1,\ldots,a_i,\ldots,a_i,\ldots a_n)\text{per}(a_1,\ldots,a_t,\ldots,a_t,\ldots,a_n).$$

Using the fact that per $A(u,v)$ = per A for $a_{uv} \neq 0$ and a_{it} per $A(i,j) > a_{it}$ per A, we see, by expanding the terms along the t^{th} column, that the right-hand side is greater than (per A)(per A). This is a contradiction.

Next, note that using (†)

$$\text{per}(a_1,a_2,\ldots a_n) = \text{per}(\tfrac{1}{2}(a_1+a_2),\ \tfrac{1}{2}(a_1+a_2),a_3,\ldots a_n)$$

and since the matrix on the right is also in DS(n) and hence minimizing, we may repeat this process to find a minimizing matrix

$$(b_1,b_2,\ldots,b_{n-1},a_n)$$

in which $b_1,b_2,\ldots b_{n-1}$ have positive entries. But now using (†) again

$$\text{per}(b_1,\ldots,b_{n-1},a_n)^2 =$$

$$\text{per}(b_1,\ldots b_{n-1},b_{n-1})\text{per}(b_1,\ldots,b_{n-2},a_n,a_n)$$

so by Aleksandrov's result (3.1), $b_{n-1} = c_{n-1}a_n$ for some real c_{n-1}. Expanding

$$\text{per}(b_1,\ldots b_{n-1},a_n)$$

by its $(n-1)^{st}$ and n^{th} columns and using (†) gives $c_{n-1} = 1$. Thus $b_{n-1} = a_n$. Similarly $b_{n-2} = a_n,\ldots,b_1 = a_n$. Hence since $A \in DS(n)$, $a_n = e/n$ where $e = (1,1,\ldots,1)^T$. Similarly $a_1 = a_2 = \ldots = a_{n-1} = e/n$. Thus $A = J_n$ and the conjecture is proved.

5. <u>More Recent Developments</u>

With the solution of the van der Waerden conjecture, the interest in permanents has increased rather than waned. Many conjectures related to the van der Waerden conjecture had been formulated and while several were special cases of the conjecture, some were more general. A detailed account is given in Minc [7] Chapter 8. We refer briefly to some recent work on a few of these conjectures.

Let $A \in DS(n)$ and let $\sigma_k(A)$ be the sum of the permanents of all the kxk submatrices of A. (Thus for example $\sigma_1(A) = n$, $\sigma_n(A) = \text{per } A$.)

The *Tverberg conjecture* (1963) states that if $A \in DS(n)$ and $2 \leq k \leq n$, then

$$\sigma_k(A) \geq \sigma_k(J_n)$$

with equality only if $A = J_n$. (The case k=n is the van der Waerden conjecture.) In a beautiful paper [4], Friedland has proved this conjecture. He first expresses $\sigma_k(A)$ as a permanent of a (2n-k)x(2n-k) doubly-stochastic matrix having an (n-k)x(n-k) block of zeros. Modifying Egorychev's methods and using many ingenious arguments, he then solves the more general problem of finding min(per A) taken over all $B \in DS(m)$ having a given rxr block of zeros.

Another conjecture more general than the van der Waerden conjecture is due to Djokovic (1967). In the notation of the last paragraph, Djokovic conjectures that for $k = 2,\ldots,n$, $A \in DS(n)$

$$\sigma_k((1-\theta)J_n + \theta A)$$

is strictly increasing for $0 \leq \theta \leq 1$. Many special cases of this have been settled. Friedland and Minc [7] proved it for $k = n$, $A = J_n$ or $(nJ_n - I_n)/(n-1)$. London [6] has proved it for $k = n$, $A = \alpha I_n + \beta P_n$, $\alpha \geq 0, \beta \geq 0, \alpha+\beta = 1$, where P_n denotes the permutation matrix corresponding to the n-cycle (1 2 3 ... n) and for $A = (nJ_n - I_n - P_n)/(n-2)$ $(n > 2)$.

An important advance on this problem has been reported by Egorychev in his review of London's paper (MR 83g 15005). He asserts that if $f_0,f_1,f_{m+1},\ldots,f_n$ are column n-vectors with positive entries and $f_\lambda = \lambda f_0 + (1-\lambda)f_1$, $0 \leq \lambda \leq 1$, then the function $\text{per}^{1/m}(B)$ where

$$B = (f_\lambda,f_\lambda,\ldots,f_\lambda,f_{m+1},\ldots f_n)$$

is concave (convex upwards).

Finally we describe a conjecture of Schrijver and Valiant [9] which in his review of their paper, Minc (MR 82a 15004) suggests is a worthy successor to the van der Waerden conjecture.

Let Λ_n^k be the set of all nxn matrices with non-negative *integer* entries such that each row sum and each column sum equals k. Let

$$\lambda(n) = \min\{\text{per}(a) \mid A \in \Lambda_n^k\}$$

$$\theta_k = \lim_{n\to\infty} \lambda_k(n)^{1/n}$$

In their paper Schrijver and Valiant show that

(1) $\lambda_k(n) \leqq k^{2n}/\binom{nk}{n}$

(2) $\theta_k \leqq (k-1)^{k-1}/k^{k-2}$

and their conjecture states that (2) is an equality for all k. (The positive solution of the van der Waerden conjecture yields $\theta_k \geqq k/e$ here.)

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SOME APPLICATIONS OF THE CLASSIFICATION OF FINITE SIMPLE GROUPS TO PERMUTATION GROUP THEORY [1]

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The classification of finite simple groups has made it possible to prove many new and striking results in the theory of finite permutation groups. We survey some of these results and describe some of the methods used in proving them. We also present a theorem on maximal subgroups of finite classical groups which is of use in extending the techniques.

(A) The Classification Theorem This states that any finite simple group is isomorphic to one of the following groups:

cyclic	Z_p	
alternating	A_n $(n \geqq 5)$	
groups of Lie type	classical:	$PSL(n,q)$, $PSp(2m,q)$, $PSU(n,q)$, $P\Omega^{\pm}(n,q)$
groups of Lie type	Chevalley:	$G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$
	twisted:	${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$, ${}^3D_4(q)$, ${}^2E_6(q)$
26 sporadic groups		

See [5] for descriptions of the groups of Lie type.

(B) Some Recent Applications to Permutation Groups. As explained, for example, in Sections 2 and 3 of [2], at the heart of

[1] This is the text of a talk given by the author to the Group Theory Conference held in Galway, 13-14 May, 1983, under the auspices of the I.M.S.

the theory of permutation groups lies the study of primitive permutation groups. Many old problems in this field have been solved using the classification theorem. Here are some examples.

THEOREM 1 ([2], Section 5). All finite 2-transitive groups are known. Any 6-transitive permutation group of finite degree n must be A_n or S_n.

THEOREM 2 ([3]). For almost all positive integers n the only primitive groups of degree n are A_n and S_n. More precisely, if $e(x) = |\{n \leq x|$ there is a primitive group G of degree n with $A_n \nleq G\}|$ then $e(x) \sim 2x/\log x$.

THEOREM 3 (Sims' Conjecture:[4]). There is a function $f:\mathbb{N} \to \mathbb{N}$ such that if G is primitive on a finite set Ω and for $\alpha \in \Omega$, G_α has an orbit $(\neq\{\alpha\})$ of size d, then $|G_\alpha| < f(d)$.

THEOREM 4 ([9]). All primitive groups of degree kp, with p prime and $k < p$, are known.

THEOREM 5 ([8], Example 5). All primitive groups G of degree p^r with p prime, and G having no elementary abelian regular normal subgroup, can be classified.

THEOREM 6 ([2], Theorem 6.1). Let G be primitive of degree n. Then one of the following holds:

(i) G has an elementary abelian regular normal subgroup;

(ii) G is a known group;

(iii) $|G| < n^{10\log\log n}$.

(C) Methods of Reduction. The basic tool for reducing problems about general primitive groups to problems about primitive *simple* groups is the following theorem of O'Nan and Scott (see Theorem 4.1 of [2]; note that there is an error in the statement

IN [2] - possibility (ii) below is omitted).

O'Nan-Scott THEOREM 7. Let G be primitive of finite degree n on Ω and let $N = \mathrm{soc}\, G$, the product of the minimal normal subgroups of G. Then $N = T_1 \times \ldots \times T_r$ with all $T_i \simeq T$, a fixed simple group, and one of the following holds:

(i) $T \simeq Z_p$, $N \simeq (Z_p)^r$ and $G \leq AGL(r,p)$;

(ii) T is nonabelian, $N_\alpha = 1$ $(\alpha \in \Omega)$ and $n = |T|^r$;

(iii) T is nonabelian and either

(a) *wreath action:* $T = \mathrm{soc}\, G_0$ for some primitive group G_0 of degree n_0 and $G \leq G_0 \mathrm{\ wr\ } S_r$, with $n = n_0^r$, or

(b) *diagonal action:* $N_\alpha = D_1 \times \ldots \times D_m$ where $r = km$ for some $k > 1$, D_i is a diagonal subgroup of $T_{(i-1)k+1} \times \ldots \times T_{ik}$ and $n = |T|^{r-m}$.

As an example of the use of the O'Nan-Scott Theorem we consider Theorem 4 above, and prove

PROPOSITION 8. If G is primitive of degree kp with p prime and $k < p$ then $T = \mathrm{soc}\, G$ is simple (so that $T \lhd G \leq \mathrm{Aut}\, T$).

Proof. Let $N = \mathrm{soc}\, G = T_1 \times \ldots \times T_r$ as above. If case (i) of the O'Nan-Scott Theorem occurs then $k = 1$ and $G \leq AGL(1,p)$. If (ii) holds then $kp = |T|^r$ with T a nonabelian simple group, which is impossible. Similarly case (iii)(b) cannot hold. Finally, in case (iii)(a) we must have $r = 1$ and $n_0 = kp$, so that $T = \mathrm{soc}\, G$ is simple.

The O'Nan-Scott Theorem thus focuses attention of primitive permutation representations of finite simple groups. Such representations are determined by the conjugacy classes of maximal subgroups.

(D) Maximal Subgroups. Let T be a finite simple group and G a group with $T \triangleleft G \leq \mathrm{Aut}\, T$ (i.e. T = soc G). We describe some recent progress in determining the maximal subgroups of such groups G.

(1) *T sporadic* It seems certain that the maximal subgroups of G can be determined explicitly in this case; in fact this has already been accomplished for 14 of the 26 groups.

(2) *T exceptional of Lie type* Here the determination of the maximal subgroups of G requires special techniques for each type of group. It seems possible that a complete determination will eventually be achieved; this has been done for several of the cases of low rank.

(3) *T alternating* In this case much can be said. Let $T = A_c$ and suppose that $c > 6$, so that G is A_c or S_c. Let G act naturally on $C = \{1,...,c\}$ and let H be a maximal subgroup of G. Then one of the following occurs:

(i) H is intransitive on C: then $H = (S_k \times S_{n-k}) \cap G$ for some k with $1 \leq k \leq n-1$;

(ii) H is transitive and imprimitive on C: then H permutes b blocks of size a, where ab = c and a > 1, b > 1, so $H = (S_a \text{ wr } S_b) \cap G$;

(iii) H is primitive on C: in this case our results on primitive groups (such as O'Nan-Scott or Theorem 6, for example) apply to H.

Finally we discuss the case

(4) *T classical* I have recently obtained (in [10]) the following result on the orders of maximal subgroups of G in this case:

THEOREM 9 ([10]. Let G_0 be a simple classical group with natural projective module V of dimension n over GF(q) (for example $G_0 = \mathrm{PSL}(n,q)$, etc.), and let G be a group such that

$G_0 \triangleleft G \leq \mathrm{Aut}\, G_0$. If H is a maximal subgroup of G then either

(I) H is a known group, and $H \cap G_0$ has well-described (projective) action on V, or

(II) $|H| < q^{3n}$.

Note. $|G|$ is roughly q^{n^2} (if $G_0 = \mathrm{PSL}(n,q)$) or $q^{\frac{1}{2}n^2}$(otherwise), so for large n the maximal subgroups H under (II) are of very small order. Theorem 9 improves substantially the results of [6] and [7] (note, however, that these were obtained without the use of the classification theorem).

The known groups under (I) comprise the following:

(a) stabilisers of

(i) certain subspaces of V,

(ii) certain decompositions of V as a direct sum or tensor product of subspaces,

(iii) fields $F_1 \subset GF(q)$ or $F_0 \supset GF(q)$, of prime index.

(b) classical groups of dimension n over GF(q) contained in G;

(c) A_c or S_c in a representation of smallest degree over GF(q) (n = c-1 or c-2).

The proof of Theorem 9 uses the following fundamental structure theorem of Aschbacher:

THEOREM 10 ([11]). Let G_0, G and H be as in the statement of Theorem 9. Then either

(A) H is known, or

(B) there is a nonabelian finite simple group S with $S \triangleleft H \leq \mathrm{Aut}\, S$, and the representation of the covering group $\overline{S}$ of S on V is absolutely irreducible.

The groups under (A) are as in (a) and (b) above, plus a few extra subgroups. So to prove Theorem 9 we must essentially show that if H satisfies (B) of Theorem 10 then either

$|H| < q^{3n}$ or H is A_c or S_c as in (c) above. This is achieved by obtaining lower bounds for degrees of absolutely irreducible modular representations of groups H satisfying (B) of Theorem 10.

(E) Some Deductions. As a corollary to Theorem 9 we obviously have:

COROLLARY 11. If G is a classical group of dimension n over GF(q) and G acts primitively on a set Ω then either

(I) G^{Ω} is 'known' (i.e. it is the action of G on the cosets of a known subgroup), or

(II) $|\Omega| > |G|/q^{3n}$.

In order to demonstrate an application of this we return to our consideration of Theorem 4. Let G be primitive on of degree kp with p prime and k < p. By Proposition 8, T = soc G is simple and, excluding the case $G \leq AGL(1,p)$ as well known, T is also nonabelian. Clearly $|G:G_{\alpha}| < r^2$ where r is the largest prime divisor of $|T|$ $(\alpha \in \Omega)$. When T is alternating, exceptional or sporadic it is possible to determine the possibilities for G_{α} and hence for G^{Ω}. And if T is classical then Corollary 11 gives the possibilities for G^{Ω}; for example, if T = PSL(n,q) then $|\Omega| = |G:G_{\alpha}| < (q^n-1)^2$ and so, provided $n \geq 6$, G^{Ω} must fall under (I) of Corollary 11.

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ASYMPTOTIC BEHAVIOUR OF GRAVITATIONAL INSTANTONS ON $\mathbb{R}^4$

Niall Ó Murchadha

Instantons are objects which play an important rôle in the transition from a classical to a quantum model for many physical theories. An instanton is a solution of the 'euclidean' classical field equations, satisfying appropriate boundary conditions. "Euclidean" means that the signature of the spacetime is changed from (-, +, +, +) to (+, +, +, +) but the form of the field equations is left unchanged. In particular this means that the D'Alembertian

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

gets changed into a four-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Thus there is an interest in gravitational instantons, asymptotically flat solutions to the Einstein equations, on Riemannian rather than pseudo-riemannian manifolds. Edward Witten [1], in a recent paper, has shown that no gravitational instantons exist on $\mathbb{R}^4$. His proof is very simple but assumes that the (non-existent) solution falls off like r^{-4} (where $r^2 = w^2 + x^2 + y^2 + z^2$). Witten argues that this is a reasonable assumption on the basis that the monopole in four dimensions falls off like r^{-2}, the dipole like r^{-3} and the quadrupole like r^{-4}. Since gravity is a quadrupole theory, then the solutions should fall off like r^{-4}. This argument cannot be trusted, because ordinary general relativity, in addition to the quadrupole term also contains a monopole term, the Newtonian gravitational potential.

Therefore, it would be more reasonable to assume that the gravitational instantons fall off like r^{-2}, rather than r^{-4}. In this article, I will show that if I assume that the instanton

vanishes at infinity, I can prove that it must fall off like r^{-4} and thus complete the Witten proof. This result will depend only on very simple properties of the four-dimensional Laplacian.

Near infinity, the gravitational field is weak and I can legitimately ignore the higher order non-linear terms by comparison with the linear terms. Thus the leading part of the gravitational field must satisfy the (euclidean) linear theory of gravity equations.

The structure of linearized gravity bears a strong resemblance to electromagnetism. In Maxwell's equations the field variable is the vector potential A_μ. The field equations are not hyperbolic, due to the gauge freedom (many different A_μs give the same physical effects). When, however, we reduce the gauge freedom by imposing the Lorentz gauge condition

$$\eta^{\mu\nu} A_{\mu,\nu} = 0 \qquad (1)$$

[η is the Minkowski metric diagram (-1, +1, +1, +1)]

then Maxwell's equations take the nice hyperbolic form

$$\Box A_\mu = 0 \qquad (2)$$

The Lorentz condition does not completely eliminate the gauge freedom. If we have a scalar ϕ satisfying

$$\Box\phi = 0 \qquad (3)$$

and a vector A_μ satisfying (1) and (2), then

$$A'_\mu = A_\mu + \phi_{,\mu} \qquad (4)$$

also satisfies (1) and (2).

The linear theory of gravity looks just like Maxwell's equations except that the field variable is a symmetric tensor $h_{\mu\nu}$ rather than a vector. Again, the field equations are not hyperbolic until we impose the gauge condition

$$\eta^{\alpha\beta}h_{\mu\alpha,\beta} = 0 \qquad (5)$$

The field equations now become

$$\Box h_{\mu\nu} = 0 \qquad (6)$$

Gauge condition (5) does not entirely eliminate the gauge freedom. If we have an $h_{\mu\nu}$ which satisfies (5) and (6), and a vector λ_μ which satisfies

$$\Box\lambda_\mu = 0 \qquad (7)$$

then

$$h'_{\mu\nu} = h_{\mu\nu} + \lambda_{\mu,\nu} + \lambda_{\nu,\mu} \qquad (8)$$

also satisfies (5) and (6).

The euclidean linear gravity equations look just like (5), (6), (7) and (8) except that the Minkowski metric is replaced by the euclidean metric $\delta_{\alpha\beta}$. The field equations are

$$\nabla^2 h_{\mu\nu} = 0 \qquad (9)$$

The gauge condition is

$$\delta^{\alpha\beta}h_{\mu\alpha,\beta} = 0 \qquad (10)$$

and the residual gauge freedom is represented by a vector satisfying

$$\nabla^2\lambda_\mu = 0 \qquad (11)$$

If $h_{\mu\nu}$ falls off at infinity, one can always impose gauge condition (10), just as we can always use the Lorentz gauge in electromagnetism. Therefore we have from (9) that each component of $h_{\mu\nu}$ is a harmonic function of the four dimensional Laplacian. The leading (monopole) term of the Laplacian is $1/r^2$. There are four dipole terms, w/r^4, x/r^4, y/r^4, z/r^4, nine $1/r^4$ harmonic functions, sixteen of order $1/r^5$ and so on. These can be found by taking repeated derivatives of $1/r^2$, because of course any derivative of a harmonic function is a harmonic function.

If we have a solution to (9) which vanishes at infinity it must fall off at least as fast as r^{-2}. If it has an r^{-2} term it must be of the form

$$h_{\mu\nu} = \begin{Bmatrix} A/r^2, & B/r^2, & C/r^2, & D/r^2 \\ B/r^2, & E/r^2, & F/r^2, & G/r^2 \\ C/r^2, & F/r^2, & H/r^2, & K/r^2 \\ D/r^2, & G/r^2, & K/r^2, & L/r^2 \end{Bmatrix}$$

where (A,B,...,L) are ten arbitrary constants. But $h_{\mu\nu}$ must also satisfy the four divergence conditions of (10) as well. This means that each row of $h_{\mu\nu}$ must be divergence-free. Looking just at the first row we get

$$-2A\frac{w}{r^4} - 2B\frac{x}{r^4} - 2C\frac{y}{r^4} - 2D\frac{z}{r^4} = 0 \qquad (12)$$

The only solution to this is A = B = C = D = 0. The other three divergence equations require that all the others of the ten "arbitrary" constants must be zero. Thus there is no r^{-2} solution to (9) and (10).

A counting argument may be illuminating at this point. There are ten components of $h_{\mu\nu}$ and one harmonic function which gives ten arbitrary constant coefficients. The gauge condition (10) involves first derivatives of $h_{\mu\nu}$, so we get the four independent $1/r^3$ harmonic functions (see (12)). We have four divergence conditions and therefore sixteen conditions on the ten coefficients. The only solution is that they all vanish.

If $h_{\mu\nu}$ does not fall off like $1/r^2$, the next possible fall-off is $1/r^3$. In this case we can have that each of the ten components of $h_{\mu\nu}$ is a sum of the four $1/r^3$ harmonic functions. This gives us forty arbitrary constant coefficients. The first derivatives of the $1/r^3$ harmonic functions will give us the nine (linearly independent) $1/r^4$ harmonic functions. Thus the four divergence conditions will give us thirty-six conditions on the forty coefficients. This means that we have four linearly independent $1/r^3$ solutions to (9) and (10).

Of course, this needs that the thirty-six conditions be linearly independent. They are.

This is not the end of the road, however. We still have to account for the residual gauge freedom (11). Since $h_{\mu\nu}$ falls off like $1/r^3$, we seek solutions to (11) which fall off like $1/r^2$ (see (8)). The general $1/r^2$ solution to (11) is of the form

$$\lambda_\mu = (\alpha/r^2, \beta/r^2, \gamma/r^2, \delta/r^2)$$

with four arbitrary constants $(\alpha,\beta,\gamma,\delta)$. Thus we have four linearly independent pure gauge $1/r^3$ solutions to (9) and (10). These are the only solutions that are left after imposing the thirty-six conditions on the forty coefficients. They do not correspond to real solutions because they can be totally eliminated by a gauge transformation.

The next term to consider is $1/r^4$. Now each component of $h_{\mu\nu}$ can be a sum of the nine $1/r^4$ harmonic functions of the Laplacian giving ninety constant coefficients. Since there are sixteen $1/r^5$ harmonic functions, each of the four divergence conditions will give sixteen conditions, sixty-four in all. Therefore we have twenty-six independent solutions falling off like $1/r^4$ of equations (9) and (10). The residual gauge freedom is represented by a vector that falls off like $1/r^3$. Each component of the vector is a sum of the four $1/r^3$ harmonic functions. Thus sixteen of the twenty-six solutions of (9) and (10) are pure gauge, leaving ten independent solutions which cannot be eliminated by a gauge choice.

We see then that if the instanton falls off at infinity, it must fall off like r^{-4}, and so the Witten assumption is not only reasonable but correct, and his proof of the absence of any gravitational instanton goes through.

It is important to notice that the proof in this article is entirely local (although "local at infinity"). I do not

assume that the Einstein equations are satisfied everywhere, I only need that they are satisfied near infinity. Thus, the result here covers cases where one looks for solutions to the euclidean Einstein equations with sources; such solutions will exist, but they must fall off at infinity like $1/r^4$. The ten independent $1/r^4$ solutions that I find must have some interpretation as moments (quadrupole ?) of the sources.

I would like to stress that this proof of the non-existence of gravitational instantons holds only for instantons on manifolds with the topology of $\mathbb{R}^4$. There do exist instantons with $\mathbb{R}^3 \times S^1$ topology [2]. The absence of the $\mathbb{R}^4$ instanton is interpreted as meaning that cold flat space is stable, whereas the existence of $\mathbb{R}^3 \times S^1$ instantons shows that hot flat space is unstable (for example, via the spontaneous formation of a small black hole).

Notes for the Initiated:

(i) Function spaces: To make this whole scheme work I have to assume that $h_{\mu\nu}$ belongs to some weighted space, i.e. that if $h_{\mu\nu}$ falls off like $r^{-\alpha}$ then $h_{\mu\nu;}$ falls off like $r^{-(\alpha+1)}$ and so on. Weighted Sobolev spaces or weighted Holder spaces will do. This is used in several places, particularly when I ignore the non-linear terms in favour of the linear terms, and again when I claim that I can make the gauge choice (10). This involves solving an elliptic equation.

(ii) Gauge freedom: This arises from the fact that the exact theory is geometrical; thus I can make coordinate transformations at will. The linear theory inherits this, and the gauge transformation (8) is nothing more than the Lie derivative of the metric along a coordinate transformation. This is why all solutions to (9) and (10) which can be written in the form $\lambda_{\alpha,\beta} + \lambda_{\beta,\alpha}$ can be eliminated.

Acknowledgements

The whole idea behind this calculation is not my own; it was suggested to me by James York. The actual calculations described here were done in collaboration with James Davis, we have produced explicit expressions for the ten independent non-gauge r^{-4} solutions to (9) and (10). I would also like to thank F.A. Deeney for his very helpful comments on various versions of this article.

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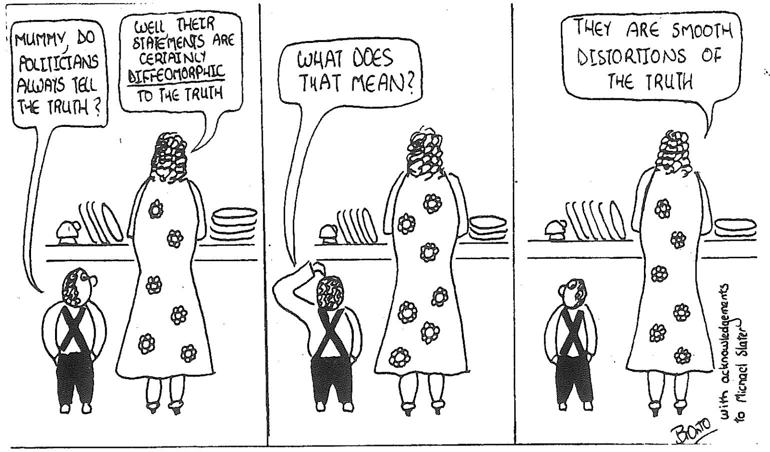

From 2-*Manifold*, No. 3

KNOWING 'ABOUT' MATHEMATICS: A FOCUS ON TEACHING

John O'Donoghue

"Mathematical Education" may be seen then as an operational activity based on a number of areas of study with the analysis of the communication of mathematics as its objective.

(G.T. Wain)

INTRODUCTION

Since all of us have a good intuitive idea of what is meant by mathematical education it is acceptable to start by presenting a definition. The above definition may not suit everyone's tastes but then definitions rarely find universal acceptance. It is not my intention to argue a case for mathematical education as a discipline but rather to focus attention on some important aspects of mathematical education as an activity. This particular definition serves to focus attention on the *communication* of mathematics. All of us at some time or another have been concerned with this aspect of mathematics teaching as students, teachers, lecturers or professors. Many of us have resolved to improve matters given the opportunity. My particular concern has been to improve teacher preparation so that better mathematics teaching results in secondary schools.

The purpose of this paper is to draw attention to a neglected aspect of mathematics teaching at third level which is vitally important for future teachers of mathematics. A case is made for better treatment of this aspect, and finally an outline of an experimental course is given.

A BASIC REQUIREMENT

Traditionally teacher educators have rightly insisted that the fundamental requirement for teachers of mathematics is to know mathematics. In other words subject competence is more important than methodology. One cannot teach mathematics if one does not know mathematics. While this establishes priorities it does little else. It is not prescriptive in any sense. For example, what does it mean to know mathematics? It is important to clarify what is at stake here. We demand of our teachers a certain competence in mathematics. That is to say, it is taken for granted that teachers of mathematics should be trained in the theory of mathematics, its methods and techniques. But mathematics teachers like any other teachers must be concerned to maximise their contribution in schools. Thus as members of the teaching profession they will find it necessary to address four questions:

1. Why teach mathematics?
2. What mathematics to teach?
3. When to teach mathematics?
4. How to teach it?

The nature and quality of teachers' responses to these questions will, in large measure, determine their effectiveness as teachers of mathematics. Success depends upon knowledge and experience of a special kind. It requires of teachers perspective, insight and knowledge adequate for the presentation of mathematics *and* its role in modern culture. In short teachers must know mathematics and know *about* mathematics. I believe that all good mathematics teachers manage somehow to combine these two attributes in their teaching. The matter generates concern because whereas these aims are not mutually exclusive, the attainment of one is no guarantee that the other has been achieved. Some sort of intervention is required. Mathematics teachers can be helped to help themselves in this regard. One way is to provide them with opportunities for talking *about* mathematics and for finding out *about* mathematics by reading as well as doing.

A PROBLEM WITH EXISTING PROGRAMMES

The competing demands on a student's time in existing undergraduate teacher training programmes in mathematics guarantee insufficient time for subject specialists. Inevitably therefore, the main effort is directed towards developing students' subject competence in the allocated time. Programmes as a result are so crowded and demanding that little or no time is available to develop students' perspective or to cultivate an overview of mathematics - important but neglected aspects of mathematical competence. That is to say that little attention is devoted to these aspects *explicitly* in any mathematical programme. The accepted view seems to be that specific attention is unnecessary because it happens anyway or in any case if it does not happen during the undergraduate phase it must surely happen later during study for higher degrees in mathematics. This state of affairs in unsatisfactory for teacher educators for two reasons: (1) many student teachers fail to develop a reasonable overview; (2) the vast majority of mathematics teachers never proceed to higher degrees in mathematics. In practice, therefore, most mathematics teachers forfeit any benefits which would accrue from this activity.

KNOWING 'ABOUT' MATHEMATICS

Who can deny that knowing about mathematics is a legitimate mathematical pursuit? Is the explicit treatment of problem solving and mathematical modelling outside the domain of mathematics? Does the nature of proof and proof techniques constitute appropriate study? Is it not imperative given the nature of school mathematics that students confront the concept of mathematical structure and deal with it comprehensively? Will not a straightforward treatment of mathematical processes such as consolidation, generalization, abstraction etc., contribute to a better mathematical experience. The list can be extended to include history of mathematics and foundations.

No one would deny that any of this activity is valid mathematics but many teacher educators afford it a low priority in practice in their undergraduate programmes. In effect this means that intervention by way of direct teaching is the exception rather than the rule. Rarely are undergraduate students in mathematics confronted by appropriate courses, materials and experiences. As a result few are able to talk about mathematics in an interesting and informed manner.

EXPECTED BENEFITS

Perhaps you feel that there really is 'much ado about nothing' here. I consider this issue to be a matter of some considerable importance for teacher educators in mathematics. I feel strongly that teacher effectiveness is considerably impaired by the absence of these competencies. Further, I attribute some observed shortcomings in practice to this deficiency namely the inability of many mathematics teachers to go beyond the text book, to make mathematics relevant or to instill confidence in doing mathematics.

Teaching mathematics is not simply a matter of showing children how to do mathematics. Pupils have to be motivated and kept interested. Appropriate topics and sequencing have to be used in context. Teachers have to cope even in a single class group with an incredible variation in ability and motivation. Pupils learn in different ways. Appropriate learning experiences and practice have to be devised and so on.....
A teacher must be able to cope with such complexity. It is more likely that he will cope effectively if he can present topics in different and interesting ways, evaluate different approaches and methods, identify significant concepts etc. Teachers cannot be expected to do this unless they have a sound grasp of mathematics, can see connections and interrelations, know something of its history and foundations - in short know about mathematics.

There are other benefits. Many teachers having completed their initial training will never return for further formal education in mathematics. This means that the education and training they receive as undergraduates has to serve for their entire professional lives. It is inconceivable in modern times that teachers could live through their working lives without updating their subject knowledge. If this is not done formally then it must be done outside the system, i.e. by independent study. In any case success is more likely if the endeavour is built on a solid foundation of mathematics. Independent study is more likely to succeed if the teacher is confident in his knowledge of mathematics, knows his way 'around the subject' and can articulate effectively.

CONCLUSION

In this paper I have attempted to highlight an aspect of mathematical education which, I believe, is especially significant for teacher educators and future mathematics teachers. This has been done in a way which separates (perhaps artificially) certain aspects of mathematics. Whether one agrees with this particular approach due to May [2] is not important. As long as the difficulty is recognised the means of describing it may be considered of secondary importance. My attempts to deal with the problem have been based on explicit teaching and directed independent study in a sequence of three courses, namely: History of Mathematics, Foundations of Mathematics and Mathematics Seminar. I leave it to the readers to judge the merit of such an exercise and in particular the use of the mathematics seminar which is outlined below. It is appropriate to raise such issues here in this forum since many of the readers are involved directly or indirectly in teacher education in university colleges and colleges of education. I should point out that I do not consider the list of selected readings to be a definitive list since choice was limited by what was immediately available. Perhaps others would want to substitute their own preferences!

APPENDIX

EXPERIMENTAL COURSE

COURSE : Mathematics Seminar
TUTOR : Dr. J. O'Donoghue
YEAR : Final Year Mathematics Students
DURATION : One Academic Year (30 hours)

1. Introduction

My concern, among other things, has been to ensure that student teachers completing their initial training know mathematics and know *about* mathematics. Obviously these aims are not mutually exclusive but the attainment of one is no guarantee that the other has been achieved. *I believe that all good mathematics teachers manage somehow to combine these two attributes in their teaching.*

The aim of this course is to set you thinking about your mathematics in a way which will benefit you in your profession now and in the future. You will be encouraged in a variety of ways to develop your perspective, insight, intuition and knowledge regarding mathematics. You will be challenged to develop your skills in analysis and synthesis by practising on issues in the nature of mathematics, its concepts and structures, its methodologies, and by examining such processes as abstraction, generalization, unification, consolidation, idealization, modelling as they pertain to mathematics.

The hope is that you will learn to penetrate deeper the mass of detail and apparently disparate areas of mathematics and develop a perception which allows you to achieve a worthwhile synthesis of the mathematics you command. It is my earnest desire that some of you, at least, will advance further and use these ideas *purposefully* at each stage of your mathematical development and thus equip yourself with a powerful methodology for learning to learn about mathematics.

2. Objectives

- To encourage the student teacher to develop a wider perspective and deeper insight into mathematics.
- To promote in the student teacher an attitude of inquiry into mathematics requiring analysis and synthesis.
- To cultivate in the student teacher a worthy sense of the meaning and significance of important mathematical ideas.
- To encourage the student teacher to develop a methodology for learning to learn about mathematics.

3. Course Organization and Content

Themes: The following themes have been selected in an attempt to add structure to the endeavour:

(i) Problem Solving
(ii) Mathematical Modelling
(iii) Mathematical Structure
(iv) Mathematical Knowledge
(v) Mathematical Proof and proof techniques

Readings: Various readings have been assigned. Readings dealing with specific themes have been grouped together. There will be some overlap between readings and groups of readings.

Lectures: The course tutor will deliver a series of occasional lectures (5). Lecture topics will relate to the aforementioned themes. Topic, venue and time will be posted on the mathematics department notice board.

Discussions: The course tutor will be available to deal with individuals as required. Opportunities will be provided occasionally to meet as a group to discuss particular readings. Watch your notice board for information.

Assessment: Assessment is based on the following course elements:

(i) attendance at occasional lectures
(ii) summaries of assigned readings
(iii) short (750 word) essay which brings the totality of your mathematical experience to date to bear on the following topic:

Identify major themes running through your mathematics programme and use them to effect a unification of the programme as a whole.

Notes on Procedure

A. Duration of Course: One full academic year beginning in first term (30 hour equivalent).

B. Timetable: See notice board.

C. (a) Readings: Readings are organised into files as follows:

File 1 - Mathematical Knowledge
File 2 - Mathematical Structure
File 3 - Problem Solving and Mathematical Modelling
File 4 - Mathematical Proof and Proof Techniques.
File 5 - General Reading

(b) Availability: Three copies of each file will be available at the Restricted Loan Counter in the College Library from the beginning of term.

(c) Content: A full list of readings is appended to this outline.

D. Each student is responsible for reading each reading on the list.

E. Each student is responsible for maintaining article summaries in a file which must be available for scrutiny by the tutor.

F. Assessment: Essay must be submitted two weeks prior to the end of last term.

5. Study Notes

A. A number of essays should not be read at one sitting. Time has been provided for a leisurely but measured pace spreading the work over the year.

B. The readings/essays vary in style, difficulty and point of view. Some are short, others are long. However, they do have something in common - each reading from a particular group relates to the theme for that group.

C. You have been asked to summarise each essay in one half page. Why demand such a short summary even for long readings? You will be surprised how many readings really only contain one or two or three fundamental ideas. What about analysis and synthesis?

D. Read essays for impression then for detail but do not devote excessive time to detail.

E. Themes are useful to focus your attention on specific important issues but boundaries between themes/topics are never sharp since themes merge easily or envelop each other. But this is only as it should be!

6. Readings

File 1 - Mathematical Knowledge

Aleksandrov, A.D. *et al* (Editors) (1962). *Mathematics: its Content, Methods and Meaning.* Cambridge, M.I.T. Press, pp. 1-7.

Hogben, L. (1967). *Mathematics for the Millions.* London, Pan Books, pp. 75-117.

Kapur, J.N. (1976). Proposal for a Course on the nature of Mathematical Thinking. *International Journal of Math Education in Science and Technology, 1*, 287-296.

Kasner, E., Newman, J. (1979). *Mathematics and the Imagination*, U.K., Penguin Books, pp. 17-35.

Kline, M. (1964). *Mathematics for Liberal Arts.* Reading, Addison-Wesley, pp. 30-55.

Rees, M. (1962). The Nature of Mathematics. *Mathematics Teacher*, October, pp. 434-440.

Sawyer, W.W. (1943). *Mathematician's Delight*. U.K., Penguin Books, pp. 26-34.

File 2 - Mathematical Structure

Bell, A.W. (1966). *Algebraic Structures*. London, Allen and Unwin, Chapters 1,5 and 6.

Gowar, N. and Flegg, H.G. (1974). *Basic Mathematical Structures 2*. London, Transworld Publisher. Chapter 4.

Jeger, M. (1966). *Transformation Geometry*. London, Allen and Unwin, Chapters 1,5 and 6.

Mansfield, D.E. and Bruckheimer, M. (1965). *Background to Set and Group Theory*. London, Chatto and Windus. Chapters 1, 6 and 8.

Piaget, J. (1972). Mathematical Structures and the Operational Structures of the Intellect. In Lamon, W.E. (Editor). *Learning and the Nature of Mathematics*. Chicago, SRA, pp. 117-136.

Sawyer, W.W. (1955). *Prelude to Mathematics*. U.K., Pelican, Chapters 4 and 5.

File 3 - Problem Solving and Mathematical Modelling

Bajpai, A.C. *et al* (1974). *Engineering Mathematics*. London, John Wiley, Chapters 0 and 1.

Bell, M. (1979). Teaching Mathematics as a Tool for Problem Solving. *Prospects*, IX, 311-320.

Jackson, K.F. (1975). *The Art of Solving Problems*. London, Heinemann, Chapters 1, 2 and 6.

Kac, M. (1969). Some Mathematical Models in Science. *Science*, 166, 695-699.

Kac, M. and Ulam, S. (1971). *Mathematics and Logic*. U.K., Pelican Books, Chapter 3.

Molkevitch, J. and Meyer, W. (1974). *Graphs, Models and Finite Mathematics*. New Jersey, Prentice-Hall, Chapters 1 and 2.

Ormell, C.P. (1972). Mathematics, Science of Possibility. *International Journal of Math. Education in Science and Technology*, 3, 329-341.

Therauf, R.J. and Klekamp, R.C. (1975). *Decision Making through Operations Research*. (2nd. Ed.) New York, John Wiley. pp. 16-24.

File 4 - Mathematical Proof and Proof Techniques

Bell, A.W. (1966). *Algebraic Structures*. London, Allen and Unwin, Chapter 1.

Course Team (1977). *Polymaths Book A: Number Systems*. Cheltenham, Stanley Thornes. pp. 1-15.

Griffiths, H.B. and Hilton, P.J. (1970). *Classical Mathematics*. New York, Van Nostrand. pp. 1-2 and 241-243.

Kline, M. (1962). *Mathematics for Liberal Arts*. Reading, Addison-Wesley, Chapter 3.

Scaaf, W.L. (1969). *Basic Concepts of Elementary Mathematics*. New York, John Wiley, pp. 109-113.

File 5 - General Reading

Committee on Support of Research in the Mathematical Sciences, National Academy of Sciences (1971). "The Mathematical Sciences: A Report Section II. The State of the Mathematical Sciences". *International Journal of Math. Education in Science and Technology*, 2, 345-390.

Lighthill, J. (Editor) (1978). *Newer Uses of Mathematics*. U.K., Penguin Books.

Newman, J.R. (1956). *The World of Mathematics*. 4 Vols. London, Allen and Unwin.

Stewart, I. (1981). *Concepts of Modern Mathematics*. U.K. Pelican Books.

Wilder, R.L. (1973). *Evolution of Mathematical Concepts: An Elementary Study.* London, Transworld Publishers.

The following selections from *The World of Mathematics* are to be treated as part of your reading assignment:

The Axiomatic Method by Wilder, R.L. Vol. 3, pp. 1647-67

The Essence of Mathematics by Peirce, C.S. Vol. 3, pp. 1773-83.

How to Solve it by Polya, G. Vol. 3. pp. 1980-99.

A Mathematician's Apology by Hardy, G.H. Vol. 4. pp. 2027-38.

Mathematical Creation by Poincare, H. Vol. 4. pp. 2041-50.

The Mathematician by von Neumann, J. Vol. 4. pp. 2053-63.

Note: As a future teacher you would be well advised to establish a small personal collection of mathematics books. Why not begin by selecting your favourites from those listed in the readings!

References

1. Wain, G.T. (Editor) (1978). *Mathematical Education*, Van Nostrand Reinhold Ltd., New York.

2. May, K.O. (1972). Teachers should know *about* mathematics. *Int. J. Math. Educ. Sci. Technol.*, Vol. 3, 157-158.

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BOOK REVIEW

"COMBINATORICS ON WORDS" (Encyclopedia of Mathematics and its Applications Volume 17)

By *M. Lothaire*

Published by *Addison-Wesley Publishers Ltd.*, 1983, Stg. £24.70, pp. xix + 238.

ISBN 0-201-13516-7

M. Lothaire is the pseudonym chosen by a group of mathematicians led by Dominique Perrin who have contributed to the writing of this volume - the first devoted wholly to the study of combinatorics on *words* or finite sequences of symbols (letters). Repetitions, decompositions, unavoidable regularities and equations in words are all analysed and connections are established with such classical areas as free groups, Lie algebras, algebras with polynomial identity and coding theory. Combinatorics on words also has significant applications to, and indeed many of its results arise from, the theory of automata, information theory and linguistics. This book attempts to draw together from these diverse areas the principal results on words and to introduce the reader to the essential methods of a new area of mathematics. To quote from the foreword by Roger Lyndon (written in his capacity as Algebra section editor for the series);

> "It is a pleasure to witness such an auspicious official inauguration of a newly recognised mathematical subject, one which carries with it certain promise of continued increasingly broad development and application".

The individual chapters of the book are written by the different co-authors, but they have collaborated to produce a unified text with consistent notation and cross-referencing throughout. The first chapter (by Dominique Perrin) introduces the reader to free monoids ("the natural habitat of words") and their morphisms, submonoids and minimal generating sets (*codes*). As one would expect, words have prefixes and suffixes and the free monoid A^* on an alphabet A also lives quite happily in the (noncommutative) formal power series ring Z<<A>> and the free associative (polynomial) algebra Z<A>.

Chapters 2, 3 and 4 (Jean Berstel/Christophe Reutenauer, Jean Eric Pin, and Giuseppe Pirillo respectively) form a block devoted to the study of *unavoidable regularities*, that is, properties shared by all sufficiently long words. Thus it is shown, roughly speaking, that "each sufficiently long word over a finite alphabet behaves locally in a regular fashion". Of course the type of regularity must be specified, a classical example being provided by van der Waerden's theorem: If N is partitioned into k classes, one of the classes contains arbitrarily long arithmetic progressions. Several formulations of this theorem and two proofs, one combinatorial, the other topological, are given in Chapter 3. If A is an alphabet the set of all nonempty words over A is denoted by A^+. A morphism $\phi: A^+ \to S$ from A^+ to a set S is called *repetitive* if each sufficiently long word contains a factor of the type $w_1 w_2 \ldots w_n$ with $\phi(w_1) = \ldots = \phi(w_n)$ and *uniformly repetitive* if all the w_i can be chosen of equal length. In Chapter 4 it is shown that if S is a finite set then ϕ is repetitive and the special case where S is itself a semigroup is investigated. In particular if S is a finite semigroup then ϕ is uniformly repetitive, a result which is shown to be a generalisation of van der Waerden's theorem. The dual problem of *avoidable regularities* is the subject of Chapter 2. These are properties not automatically shared by all long words: for such a property there exist infinitely many words (over a finite alphabet) that do not satisfy it. For example there are infinitely many *square free* words provided that the alphabet has at least three letters, so it is not true that every sufficiently long word contains a square.

Chapters 5, 6 and 7 (Dominique Perrin, Jacques Sakarovitch/Imre Simon, and Christophe Reutenauer respectively) also form a block. These deal with properties of words related to classical noncommutative algebra. In Chapter 5 we find the study of *factorizations* of free monoids which may be thought of as bases, and their relationship to bases of free Lie algebras. The principal tool is a factorization via the so-called *Lyndon words* and among the results analysed are the Witt formula, the Poincaré - Birkhoff-Witt theorem and the Campbell-Baker-Hausdorff formula. Chapter 6 is devoted to *subwords*. It is a simple combinatorial problem to determine the set of subwords of a given word and its cardinality. Of more interest however is the converse problem: under what conditions is a given set of words of a specified kind the set of subwords of a word w? Here one uses the notion of *division* (u divides v if u is a subword of v) and the partial order it induces on A^*, the main property of which is given by a well-known result of Higman: any set of words over a finite alphabet which are pairwise incomparable in the division ordering is finite. Also introduced is the *binomial coefficient* $\binom{u}{v}$ of two words which is intimately related to the Magnus representation of free groups and to Fox's free differential calculus. In Chapter 7 the relationship between words and algebras with polynomial identity is studied. The aim here is to prove the theorem of Shirshov which answers both the Levitski and Kurosch problems for pi-algebras thus: Let A be a finitely generated K-algebra (K is a commutative ring with 1) generated by $m_1, \ldots, m_k$ and suppose A is a pi-algebra (with polynomial identity) of degree n. If any product of at most n-1 of the m_i is nilpotent (resp. integral over K) then A is nilpotent (resp. a finitely generated K-module). What is interesting is that the proof (taken essentially from Shirshov's 1957 paper) is entirely combinatorial and requires no deep knowledge of ring theory.

Each of the last four chapters of the book introduces a new aspect of words and, as indicated by the exercises, each could be considerably extended. Chapter 8 on The Critical Factorization Theorem is written by Marcel Paul Schützenberger (who, incidentally, is acknowledged as the initiator of the systematic study of monoids and combinatorics on words) and deals with *periodic* properties (where the period $\pi(w)$ of a word w is defined as the minimum length of words admitting w as a factor of some power). Chapter 9 (Christian Choffrut) gives an introduction to the vast subject of *equations in words* (here again the name of Lyndon arises frequently in the discussion). In Chapter 10 Dominique Foata describes how *rearrangements* of words can be used in the enumeration of permutations of finite sequences with certain specified properties (such as a given number of descents or a fixed up-down sequence). The final Chapter 11 (Robert Cori) covers the relationship between *plane trees*, *parenthesis systems* and certain families of words. An interesting aspect of this chapter is the use of the combinatorial properties of Lukaciewicz language to give a purely combinatorial proof of the Lagrange inversion formula of complex analysis!

In reading a book of this nature one is of course prepared to accept a certain amount of "unavoidable irregularity" in the writing due to the varied authorship of the different chapters. In fact the style is surprisingly consistent throughout signifying a remarkable degree of cooperation among the (eleven) writers. The index has one or two omissions and I found just one instance of a term (*biprefix code* on p. 27) being used without having been defined (the natural place would have been in Chapter 1). But on the whole the cross-referencing and indexing are adequate to the reader's needs. There is a number of misprints but most of these are textual rather than symbolic and along with several (typically French) non-standard uses of the English language can be forgiven in an otherwise excellent production.

The book is written lucidly and for the most part so as to be accessible to anyone with a standard mathematical background. It contains a wealth of information and many topics not mentioned in this review are included. Very few results are taken for granted and each chapter ends with a good selection of detailed exercises designed to bring out applications and extensions of the theory. Also contained in each chapter are comprehensive bibliographic and historical notes and discussion on a fair number of open problems to whet the appetite for further investigation. This book is sure to become the standard reference work in a new and potentially fruitful area of mathematics.

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"CALCULUS AND ANALYTIC GEOMETRY"

by *Donald W. Trim*

Published by *Addison-Wesley Publishing Company, Inc.* 1983.
Stg £25.45.

ISBN 0-201-16270-9

Academic life is financially secure, but the chances of making a "killing" are few and far between. It is, of course, interesting to speculate how we might get by if we were paid by the theorem; if the mortgage payment next month depended on settling that result you have been trying to prove over the past two years. It might well extend the active mathematical life of many, exposing as a myth the belief that creative mathematics is done by the young. It would certainly make life interesting; it would probably make it shorter too. Most of us are glad that this is not the way things are arranged, and in a society governed by supply and demand we may deduce that we have to be grateful that someone somewhere is giving us the time to prove theorems at all. (Notable exceptions to these observations, indeed most observations, are the U.S.A. and France. The hiring system in the United States has created star and superstar status for certain mathematicians, with corresponding salaries; in France people *are* paid money for proving good theorems!)

One killing however beckons us all: write the perfect calculus text and get rich! But perhaps I am being a little mercenary. With calculus such a stumbling block for so many perhaps the quest for the perfect calculus text is the mathematician's analogue of the quest for the holy grail. Donald W. Trim lays claim to the Siege Perilous. Is it his? Before pronouncing judgement (how much easier it is to write reviews than to write books) let me describe the text.

The first thing that strikes one on picking up the book is its weight! There are over 900 pages and the range of material covered is impressively complete. I will list the chapter headings, the subheadings can be determined by analytic continuation. Chapter 1: Plane analytic geometry and functions; Chapter 2: Limits and continuity; Chapter 3: Differentiation; Chapter 4: Applications of differentiation; Chapter 5: The indefinite integral or antiderivative; Chapter 6: The definite integral; Chapter 7: Applications of the definite integral; Chapter 8: Transcendental functions and their derivatives; Chapter 9: Techniques of integration; Chapter 10: Conic sections, polar co-ordinates, and parametric equations; Chapter 11: Infinite sequences and series; Chapter 12: Vectors and three-dimensional analytic geometry; Chapter 13: Differential calculus of multivariable functions; Chapter 14: Multiple integrals; Chapter 15: Vector calculus; Chapter 16: Differential equations.

The book is very attractively produced, as one might expect from Addison-Wesley, with many useful diagrams, and an extra wide margin down the lefthand side of the pages. (Would the world be a wiser place if this had been the case in Fermat's time?) There are over 4,400 problems, some of which require the use of an electronic calculator, with answers to the even numbered ones in the back. Other available supplements

include a student's manual containing detailed solutions to even numbered exercises, an instructor's manual containing answers and selected solutions to odd numbered exercises (hopefully the hard ones) and a set of transparencies for the more complicated figures in the text. (Thankfully no inflatable lecturers - at least not yet!).

All of this is no doubt much as you might expect and indeed this is true of the text as a whole. It seems to be quite well written, but I did spot some errors and points of contention. On page 42 we learn that a function $f(x)$ has limit L as x approaches a if $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a. Although this "definition" is not meant to be precise (there is a "mathematical definition" of limit on page 57) it really is completely misleading, it is "sufficiently close" that one chooses and not x. The other surprising error I spotted occurs on page 444, in exercise 35. "Prove the following result: If $\sum_{n=1}^{\infty} c_n$ converges, then its terms can be grouped in any manner, and the resulting series will be convergent with the same sum as the original series." Presumably the author had something fairly restrictive in mind when he wrote "grouped in any manner", but given the standard results on conditionally convergent series one might have hoped for something a little more precise (or perhaps, even better, nothing at all).

Two other complaints: first why do so many calculus texts discuss differentials? In this book we have a definition "An increment Δx in the independent variable x is denoted by

$$dx = \Delta x$$

and when written as dx is called the differential of x." I am unsure what students make of such stuff, it certainly has me puzzled and moreover undermines any other definition appearing in the text. This approach, to a fairly straightforward topic - linear approximation - is almost certain to cause confusion. Also in the preface to the student the author states that it is surprising that neither Leibniz nor Newton formulated the idea of limit. This seems to me to be an alarming confusion of the logical and psychological, which thankfully is not continued in the text.

How does this book compare with its rivals? There has been no fundamental change in the selection or treatment of material used in calculus texts over the past 20 years, and competitors vary by and large very little. My present favourite of books of this type is Fraleigh's *Calculus with* (rather than *and*) *Analytic Geometry*, also published by Addison-Wesley, and I prefer it to the text under review. It covers almost exactly the same material as the book by Trim, but is more direct and considerably shorter (in content, not pages; the print in Fraleigh's book is larger than that in Trim's). It is to be noted that since neither book deals with complex variables or Fourier series they are really not suitable as recommended texts for engineering students, but more of that later.

And my conclusion? Well I have little doubt that such a book would make a useful addition to any university library. What of the holy grail? Well we all know that *that* quest is part of Arthurian *legend* (which in turn appears to be the English attempt to compensate for the unmistakeable fact that God was a foreigner). Similarly, the perfect calculus text is a fiction; especially so on this side of the Atlantic. For in contrast with the situation in the U.S. one rarely has a number of classes being taught the same material simultaneously, and consequently there is not the same need for some unifying influence. Where possible we all usually prefer to give our own treatment, perhaps gleaned from a number of books, or courses we have attended. Perhaps more importantly (back to money again) one certainly could not recommend a text at this price to a class of students here or in the U.K.

There is one other point, which I think even (or especially) dedicated writers and publishers might lose sight of when considering this as a students' textbook. That is the very completeness of such texts is off-putting. This volume

may well be the lifetime's work of its author; it has the look of the same for any prospective reader. Moreover, many of the interesting applications tend to be, if anything, too interesting and distracting, and the exercises suffer a little from the same symptoms, with too few trivial ones. The author says in his preface to the student that "The key word in our approach to calculus is *think*." I hope he does not claim any originality here. But nonetheless thinking can be a rather elusive and overestimated quantity in the learning process. A selected and condensed core of material, to be learnt by rote, and a number of mechanically (and hopefully quickly boring) examples may be great aids to understanding. Indeed, here we may have the main reason why calculus causes so many problems, and why calculus texts are only ever a minor aid to its understanding. With calculus, as with any other worthwhile topic, one has to be willing to soldier on in a fog for a considerable time before (hopefully) sunshine filters in, and the promise of the joys of understanding is usually not enough for (at least normal) students. So how do we do it? How do we get them to soldier on? At school by intimidation, but later? I'm not quite sure, but people really are more interesting than mathematics, so don't lose too many nights sleep worrying about those inflatable lecturers.

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"THE CONVOLUTION PRODUCT AND SOME APPLICATIONS"

By W. Kecs

Published by *D. Reidel*, US $69.50. xvii + 332 pp.

The problem of multiplying distributions has occupied mathematicians for decades. In many ways, this is an excellent problem, leading to some intriguing mathematics which often has genuine use in other areas. Probably, the most successful multiplication is the convolution product, and properties of this product and its uses in engineering are the subject of Kecs' book.

Given two L^1 (i.e. integrable) functions on R^n, f and g, the convolution $f*g$ is defined by $f*g(x) = \int f(y)g(x-y)dy$. An application of Fubini's theorem shows that $f*g = g*f$ is again in L^1. Now, every L^1 function h defines a continuous linear form T_h or distribution, on the space of infinitely differentiable functions on R^n with compact support, via the mapping $T_h(\phi) = \int \phi(x)h(x)dx$. (Of course, a definition of a topology is needed to entitle us to say that T_h is continuous, but we will omit this.) Applying this to $h = f*g$, we can see that $T_{f*g}(\phi) = T_f(\check{g}*\phi)$ where $\check{g}(x) = g(-x)$. In more suggestive notation, we can write $T_{f*g}(\phi) = T_{f_x}(S_{g_y}(\phi(x+y)))$. This, hopefully, motivates the following definition.

For distributions S, T, let $S*T$ be the distribution defined by $S*T(\phi) = S_x(T_y(\phi(x+y)))$. (In fact, $S*T$ *cannot* be defined for all pairs (S,T), just as $f*g$ cannot be defined for all pairs of functions (f,g), and it is of much interest to find when this convolution does make sense. However, it is defined for "very many" pairs (S,T), and we will only consider these.) The most useful, and straightforward, properties of convolutions are that $S*T = T*S$ and that $\delta*T = T$ for all T, where δ is the Dirac distribution, which takes a function ϕ to $\phi(0)$. Also, if P(D) is a differential polynomial, we can define P(D)T to be the distribution given by $P(D)T(\phi) = T(P(-D)\phi)$. Then,

we can verify that $P(D)S*T = (P(D)S)*T$. A distribution E is called a fundamental solution for the differential polynomial P(D) if $P(D)E = \delta$. Fundamental solutions are major building blocks in the theory of differential equations, because if T is a distribution, then $P(D)(E*T) = (P(D)E)*T = \delta*T = T$, and so $E*T$ is a solution to the problem $P(D)X = T$ (always assuming that all convolution products make sense). This observation is, in a general way, the motivating force behind the interest in convolution products

The book under review is the second in the Eastern Europe Reidel series, Mathematics and its Applications. As the editor of the series states, it is "hoped to contribute something towards better communications among the practitioners in diversified fields", by making available to western audiences monographs emanating from the Soviet Union, Eastern Europe and Japan. It is unfortunate that this worthwhile objective has been thwarted by Reidel which has priced this volume ($69.50, for a 330 page book, printed in Romania) well beyond the reach of much of its intended audience. This is a pity, since Kecs' book deserves a larger audience than it will receive.

Put briefly, the book introduces distributions and operations on distributions in the first three chapters, with the aim (Chapters 4 and 5) of describing applications of convolution equations in engineering. Chapter 1 is an introduction to distributions, together with basic underlying definitions from functional analysis. Chapters 2 and 3 deal with convolution products and Fourier and Laplace transforms. Much of the material here is completely standard, with the usual presentation of the basic properties of convolutions and transforms, and the relations between them. Several interesting features of these chapters do stand out, such as an exposition of the author's work on the partial convolution product and a discussion of Mikusinski's operational calculus with several good examples. The idea for this operational calculus is as follows. We consider C(R+), the space of continuous complex valued functions on R+, defining a product in the following manner:

for f,g C(R+), $f \otimes g$ is the function in C(R+) given by

$$f \otimes g(x) = \int_0^x f(t)g(x-t)dt.$$

By a theorem of Titchmarsh, C(R+) is an integral domain with this product, and Mikusinski was led to the quotient algebra Q(R+). It turns out that Q(R+) contains the usual differential and integral operators, as well as many distributions. In particular, the Dirac δ and the distribution s are in Q(R+), where $s(f) = f'-f'(0)$ for C^1 functions f. As a consequence, one can apply Laplace transform techniques, using s, to solve differential equations with constant coefficients, integral equations, etc.

The main body of the book is the last two chapters. Chapter 4 deals with convolution equations in spaces of distributions. It is here that the problem of finding fundamental solutions for various operators is addressed, with Kecs examining the role of special spaces of distributions. The Cauchy initial value problem is considered, in a number of settings, and applications are made to the wave equation, heat equation, etc. Finally, in Chapter 5, the author applies the methods of the previous chapter to solve differential equations arising in electrical and mechanical engineering and in viscoelasticity.

The text is readable, although the English is not always idiomatic. It is evident that the translator has little mathematical experience. Thus, for example, we find ourselves considering the "body of real or complex numbers" and the open unit "bubble" of a normed space. A more substantive criticism can be made of the author's approach, from the mathematician's point of view. Routine results, such as properties of the convolution, are usually proved in full detail. On the other hand, the discussion is often incomplete in terms of (mathematically) more interesting results. For example, no attempt is made to discuss topological properties of the space of distributions, beyond some mention of the weak-* topology. No mention is made of such beautiful results as the Titchmarsh-Lions

theorem on the support of convolutions, the Paley-Wiener theorems, etc. Indeed, the relation of analytic function theory to this subject seems to have been largely ignored. Unlike Schwartz's "Mathematics for the Physical Sciences", this book (which might be considered as "convolution equations for the engineering sciences") has no exercises.

These doubts having been raised, it must in fairness be mentioned that it seems remarkable that, as Kecs shows, one can get many, apparently non-trivial, results in engineering mathematics using only the material developed in this volume. Thus, it may well be that the book serves the very useful purpose of introducing engineers to this fruitful area of mathematics.

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"MATHEMATICAL SNAPSHOTS"

By *H. Steinhaus*

Published by *Oxford University Press,* (311 pp.). Stg £5.95.

PREFACE TO THE GALAXY EDITION

By *Morris Kline, Professor Emeritus of Mathematics at the Courant Institute of Mathematical Sciences, New York University.*

This reprinting of the third, enlarged edition of Steinhaus' *Mathematical Snapshots* is more than welcome.

The book must be distinguished from numerous books on riddles, puzzles and paradoxes. Such books may be amusing but in almost all cases the mathematical content is minor if not trivial. For example, many present false proofs and the reader is challenged to find the fallacies.

Professor Steinhaus is not concerned with such amusements. His snapshots deal with straightforward excerpts culled from various parts of elementary mathematics. The excerpts involve themes of sound mathematics which are not commonly found in texts or popular books. Many have application to real problems, and Steinhaus presents these applications. The great merit of his topics is that they are astonishing, intriguing and delightful. The variety of themes is large. Included are unusual constructions, games which involve significant mathematics, clever reasoning about triangles, squares, polyhedra, and circles, and other very novel topics. All of these are independent so that one can concentrate on those that attract one most. All are interesting and even engrossing.

Professor Steinhaus explains the mathematics and his fine figures and excellent photographs are immensely helpful in understanding what he has presented. He does raise some questions the answers to which may be within the scope of most readers but the reader is warned that some answers have thus far eluded the efforts of the greatest mathematicians. Mathematical proof demands more than intuition, inference based on special cases, or visual evidence.

This book should be and can be read by laymen interested in the surprises and challenges basic mathematics has to offer. Professor Steinhaus is mathematically distinguished, and, as evidenced by the very fact that he has undertaken to present unusual, though elementary, features, is seriously concerned with the spread of mathematical knowledge. The careful reader will derive pleasure from the material and at the same time learn some sound mathematics, which is as relevant today as when the original Polish edition was published in 1938.

PROBLEMS

First the solutions to some previous problems.

1. A car park has spaces numbered 1,2,....,n. Any driver arriving with a ticket for space k parks at space k unless it is occupied, in which case he chooses the first vacant space from k+1, k+2,....,n. If these are occupied he leaves in disgust.

If n drivers arrive in turn, each with a ticket bearing a randomly chosen integer between 1 and n, prove that they can all park with probability $(n+1)^{n-1}/n^n$.

This problem appeared in Vol. 1, No. 1, of the Mathematical Intelligencer and the solution appeared in the next issue. Briefly, the idea is to consider a modified problem in which the tickets bear randomly chosen integers between 1 and n+1, and in which the car park has n+1 spaces and is circular. The n drivers are able to park (since they can go round again) and there is always one space left at the end. The answer to the original problem is then clear because:

(i) a successful outcome in the original problem corresponds to an outcome in the modified problem in which the space n+1 is left vacant, and

(ii) in the modified problem there are $(n+1)^n$ sample points, exactly the same number of which leave any given space vacant (why?).

2. Ship A is moving due east at constant speed and, at a certain moment, ship B is moving due north at the same speed towards A. If B maintains this speed but continuously alters course towards A how closely can B approach A?

Let both ships have speed v and begin at a distance of d miles. Make the construction indicated in Fig. 1 overleaf.

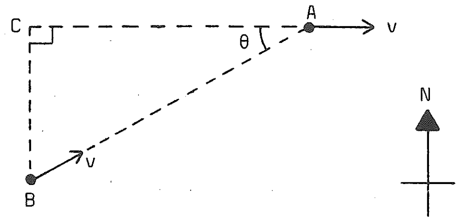


FIGURE 1: Typical Position

Then the distance AB is increasing with speed $v\cos\theta - v$ (so it is decreasing) and the distance AC is increasing with speed $v - v\cos\theta$. Thus the distance AB+AC is constant, and so the distance AB tends to $\frac{1}{2}d$ in the limit.

Now for a problem which can be solved very elegantly by thinking laterally (quite literally).

1. The Plank Problem. Does there exist a positive integer n such that a closed disc of diameter 1 can be covered by fewer than n planks of width $\frac{1}{n}$?

A plank is defined to be a parallel strip which is closed and of infinite length.

2. The Planet Problem. A finite number of equal spherical planets are in outer space. A region on the surface of one of the planets is called hidden if it is invisible from any of the other planets. Find the total area of the hidden regions.

This problem came from a Russian Olympiad.

Phil Rippon,
Mathematics Faculty,
The Open University,
Milton Keynes.

CONFERENCE REPORTS

IRISH MECHANICS GROUP CONFERENCE ON DEVELOPMENTS IN MECHANICS

Several years ago, those working in Mechanics in Ireland felt the need for an informal Association which could provide more definite contact through periodic meetings. From this need, the Irish Mechanics Group was initiated with the objective of organising short, usually one-day, meetings once or twice each year. The general format of such meetings aimed at enabling Mechanicians to present brief talks (usually of thirty minutes duration) on their current areas of research as well as affording them an opportunity to meet and exchange views informally but on a regular basis. On occasion, some more formal meetings, having specific themes and areas of research have been organised. In order to maintain the desirable informality of the meetings, Proceedings are not published.

The meetings/conferences are held in different locations usually shortly before or soon after the end of University or Technical College term. Attendance, which tends to number around thirty, usually includes personnel from the Institutes of Higher Education, Universities, Colleges of Technology, and various research institutes including representation from the Meteorological Office, An Foras Forbartha etc.

For a number of reasons meetings of the Irish Mechanics Group (I.M.G.) had not been held for a few years up to June of this year. A two-day I.M.G. conference on "Developments in Mechanics", sponsored by the Mathematical Physics Department was held in University College Cork on 2/3 June last. The attendance of some thirty Mechanicians included representation from the N.U.I. Colleges, Trinity College, Queen's University, N.I.H.E. Limerick, N.I.H.E. Dublin and some of the R.T.Cs.

The Conference Chairman - Professor P.M. Quinlan, U.C.C., in his opening address stressed both the 'healthiness' of mechanics today and the international reputation which Irish applied mathematicians have held in this field. Three 'invited' lectures were well received by the participants. Professor Michael Hayes, U.C.D., spoke on "Elastic and Viscoelastic Waves", Dr. Michael Quinlan, U.C.C., spoke on "Internal Rupture of Materials", and Professor Matt McCarthy, U.C.G., spoke on "Scattering of Elastic Waves". In addition to these talks, there were nine other presentations on a wide diversity of current research interests. Topics discussed were "Viscoelastic Rayleigh Waves in Low-Loss Material", "Asymptotics of Force-Displacement Relations for a Bonded Elastic Cylinder", "Resonant Oscillation in Water Waves", "Asymptotic Partition of Energy in Linear Viscoelastic Materials", "Free Vibration of Thin Elastic Plates", "Higher Order Equations in Mechanics", "Wave Forces on a Submerged Cylinder", "Stoke's Waves, Body Waves and Rayleigh Pressure Problem", "Cracks, Cavities and Stresses in Two-Dimensional Bodies".

A very pleasant and relaxing reception was provided on the first evening of the Conference and it provided further opportunity for the participants to fruitfully and informally discuss their work.

At a business meeting of the Irish Mechanics Group held during the Conference it was decided to set up a new committee consisting of Professor P.M. Quinlan, U.C.C., (Chairman); Dr. M.J.A. O'Callaghan, U.C.C., (Secretary); Dr. F. Hodnett, N.I.H.E. Limerick; Professor M.A. Hayes, U.C.D.; Dr. A. Wood, N.I.H.E Dublin; Professor M.F. McCarthy, U.C.G.: Dr. J. Fitzpatrick, T.C.D., and Dr. P.J. Donohue, Q.U.B. The committee will draw up a brief constitution, discuss the possibility of membership fees, explore possible relationships with other groups of compatible interests and discuss future meeting schedules. The committee will communicate informally in the interim before it meets at Christmas in conjunction with the Mathematical Symposium in Dublin.

Michael J.A. O'Callaghan, Mathematical Physics Dept., U.C.C.

SUMMER SCHOOL ON COMBINATORIAL OPTIMISATION, N.I.H.E., DUBLIN

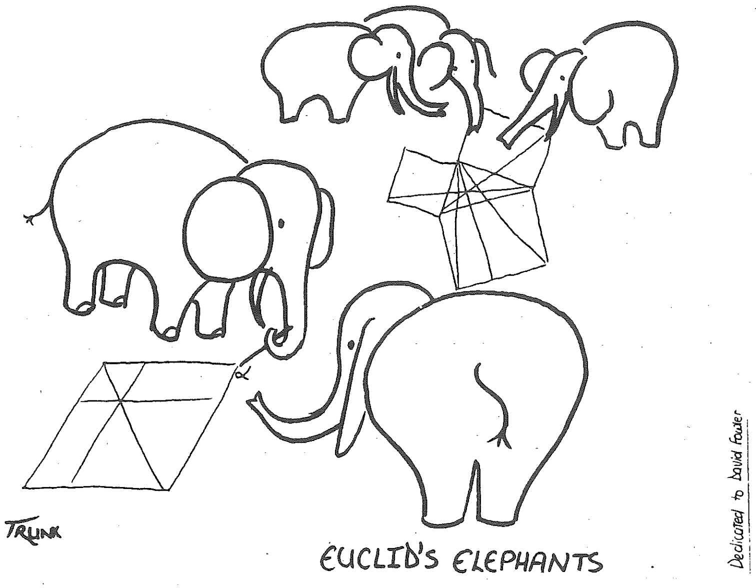
The 1983 conference on Combinatorial Optimisation was held between 4th and 15th July and hosted by N.I.H.E. Dublin, which provided the financial backing to enable many well-known mathematicians to be invited. Considerable effort by the organiser Michael O'hEigeartaigh was amply rewarded by an excellent conference in which the main speakers were N. Christofides, M. Grotschel, R.M. Karp, E.L. Lawler, J.K. Lenstra, G.L. Nemhauser, M.W. Padberg, C.H. Papadimitrou, A.H.G. Rinnooy Kan, and L.E. Trotter, Jr. Each of these gave two instructional talks of a general nature and a lecture on an aspect of recent research.

The most well-known problems in Optimisation are the Travelling Salesman Problem (TSP) and the Vehicle Routing Problem (VRP). In the former, a salesman is required to visit each of n cities and to minimise the distance he has to drive to accomplish this. In the latter, a vehicle of fixed capacity must deliver varying quantities of goods from a depot to each of n customers, again with the restriction of minimising the distance or cost of driving to each customer. In the VRP several journeys may be required from the depot because of the limit on goods which the vehicle can carry. The TSP is really the same problem, except that the salesman has sufficient space in his car to provide an encyclopedia to everyone.

Both of these are "integer programming" problems and the solution of them requires such a considerable amount of computing when n is large that usually only approximate solutions are sought. Much of the conference time was spent considering how one might change these to "linear programming" problems. For which the simplex algorithm almost invariably finds an optimal solution extremely quickly.

The first week of the summer school took place at the Drumcondra site, and it moved out to the Glasnevin campus for the second week. Nevertheless one still had the impression that this was about half way between the Netherlands and the States!

Colin Walter,
Mathematics Department,
University College,
Dublin 4.



From *2-Manifold*, No. 4

CONFERENCE ANNOUNCEMENTS

FIRST ANNOUNCEMENT AND CALL FOR PAPERS

BAIL III

The Third International Conference on
Boundary and Interior Layers -
Computational and Asymptotic Methods
20th to 22nd June, 1984 in Trinity College, Dublin, Ireland
under the auspices of the Numerical Analysis Group
and co-sponsored by the
American Institute of Aeronautics and Astronautics
American Meteorological Society
Irish Mathematical Society

and

Advances in Computational Methods for
Boundary and Interior Layers
An International Short Course held in association with the
BAIL III Conference
18th and 19th June, 1984 in Trinity College, Dublin, Ireland

Aims and Scope

Boundary and interior layers are of great practical importance. They arise in many problems in the aerospace industry, biological fluid flow, chemical engineering, combustion, meteorology, microstructured materials, nuclear engineering, petroleum reservoir modelling and semiconductor device simulation. In BAIL III particular emphasis will be placed on computational methods for solving these problems.

It is important to bring together engineers and scientists who encounter such problems, in order to avoid wasteful duplication of research effort. This is because the technical difficulties are frequently the same although the application areas are quite different. This becomes apparent when researchers, who are not normally in contact, have an opportunity to exchange information

In order to preserve the intimate and informal atmosphere of the previous BAIL conferences, attendance at BAIL III will be limited to a maximum of 120 delegates

Call for Papers

Abstracts of papers on topics in the above or related areas are invited by 1st February 1984. Notification of acceptance will be sent by 1st March 1984. Abstracts should be at most one page in length.

For further information please contact

BAIL III Organising Committee,
P.O. Box 5,
51 Sandycove Road,
Dun Laoghaire,
Co. Dublin,
Ireland.

BRITISH MATHEMATICAL COLLOQUIUM

The 36th British Mathematical Colloquium will be held at the University of Bristol on 9th - 13th April 1984. The principal speakers will be J.P. Serre (Paris), M.O. Rabin (Harvard and Jerusalem) and H. Funstenberg (Jerusalem); fifteen morning speakers have also agreed to speak. There will also be an educational forum on the use of computers in university mathematics teaching.

The registration fee will be £12.00 rising to £18.00 after 31st January 1984. The cost of accommodation for the full period will be £61.50, both are payable in advance. Application forms and further information are available from the colloquium secretary, H.E. Rose, School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW.

APPLIED STATISTICS IN IRELAND

The fourth annual conference of Applied Statistics in Ireland will be held on March 29-30, 1984, at Kilkea Castle in Castledermot, Co. Kildare.

For further information, contact Dr. P.J. Boland, Department of Mathematics, University College, Belfield, Dublin 4, or Dr. F. Murtagh, Department of Computer Science, University College, Belfield, Dublin 4.

THE IRISH MATHEMATICAL SOCIETY

Instructions to Authors

The Irish Mathematical Society seeks articles of mathematical interest for inclusion in the *Newsletter*. All parts of mathematics are welcome, pure and applied, old and new. Articles of an expository nature are preferred.

In order to facilitate the editorial staff in the compilation of the *Newsletter*, authors are requested to comply with the following instructions when preparing their manuscripts.

1. Manuscripts should be typed on A4 paper and double-spaced.
2. Pages of the manuscript should be numbered.
3. Commencement of paragraphs should be clearly indicated, preferably by indenting the first line.
4. Words or phrases to be printed in capitals should be doubly underlined, e.g.

   Print this word in capitals → Print THIS WORD in capitals

5. Words or phrases to be italicized should be singly underlined, e.g.

   Print this word in italics → Print *this word* in italics

6. Words or phrases to be scripted should be indicated by a wavy underline, e.g.

   Print this word in script → Print *this word* in script

7. Diagrams should be prepared on separate sheets of paper (A4) in black ink, the original without lettering and a copy with lettering attached.
8. Authors should send two copies of their manuscript and keep one copy as protection against possible loss.

If the above instructions are not adhered to, correct reproduction of a manuscript cannot be guaranteed.

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Cork.