no place in statistics unless clearly relevant to a statistical problem. He frequently emphasised that a problem generally involved much more than its relevant statistics and resented any charge of materialism on statisticians as much as he resented the use of mathematics for its own sake in the ostensible address of a statistical problem.

He was undoubtedly himself a powerful and energetic mathematician and a magnificently creative statistician, with an unusual emphasis on applicability throughout his work at all times. He had an amazing all-round talent and was a man whose contributions to statistics will not be forgotten.

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PUTTING COORDINATES ON LATTICES

John Hannah

In this article I shall show how the problem of putting coordinates on certain types of lattices leads to the class of von Neumann regular rings, and discuss briefly the resulting connexion between ring and lattice properties. The article is based on a talk I gave at the Group Theory Conference in Galway on May 13th, 1983, and I would like to thank the organizers both for their invitation to speak and for tolerating the presence of a ring theorist.

Recall that a lattice is a partially ordered set in which any pair of elements a,b have a greatest lower bound $a \wedge b$ and a least upper bound $a \vee b$. We shall be considering complemented modular lattices in what follows. A lattice L is said to be ***complemented*** if it has a least element (denoted by 0) and a greatest element (denoted by 1) and if every element $a \in L$ has a complement $a' \in L$; that is, $a \wedge a' = 0$ and $a \vee a' = 1$. Such complements are not usually unique. We say that L is ***modular*** if whenever $a,b,c, \in L$ with $a \leq c$ then $(a \vee b) \wedge c = a \vee (b \wedge c)$.

Example 1: Let V be any vector space (possibly infinite dimensional) and let L be the set of all subspaces of V ordered by inclusion, so that $a \wedge b = a \cap b$ and $a \vee b = a + b$. Then L is a complemented modular lattice.

2. : Von Neumann, studying rings of operators on Hilbert spaces, came across rings whose sets of projections (that is, self-adjoint idempotent operators p, so that $p = p^* = p^2$) formed lattices if $p \leq q$ was taken to mean that $p = qp$ (so that q is a left and right identity for p). Although there was no simple algebraic formula for $p \wedge q$ and $p \vee q$ in this case, von Neumann was able to show that this lattice was complemented and modular.

Consider for a moment the special case of Example 1 where $V = \mathbb{R}^3$ and L is the lattice of subspaces of V. We can view L as the real projective plane $\mathbf{P}$ with the one-dimensional subspaces being the points of $\mathbf{P}$ and the two-dimensional subspaces the lines of $\mathbf{P}$. In this picture of L the operation $\wedge$ is still intersection but $a \vee b$ corresponds now to the line joining the points a and b. The original space $\mathbb{R}^3$ is now the usual homogeneous co-ordinates for $\mathbf{P}$ and von Neumann wondered if such a "co-ordinatization" was available for his lattice of projections too. It would of course be desirable for such co-ordinates to be related to the original ring structure of his examples, and to see what we should expect we shall look more closely at Example 1.

Notice that the lattice of subspaces of $\mathbb{R}^3$ is "the same" as the lattice of right ideals of the ring $M_3(\mathbb{R})$ of 3x3 matrices with real entries. Indeed a subspace U of $\mathbb{R}^3$ corresponds to the right ideal $\hat{U}$ of $M_3(\mathbb{R})$ consisting of all matrices whose columns belong to U. Thus, for example, the subspace U consisting of all vectors of the form

$$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \quad \text{is sent to the set } \hat{U} \text{ of all matrices of the form} \quad \begin{pmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(We could equally use left ideals of $M(\mathbb{R})$: since left multiplication in $M_3(\mathbb{R})$ corresponds to row operations we could make the rows of the matrices come from U in that case.)

To see what sort of right ideals should correspond to the lattice elements in general we need to consider the infinite dimensional case of Example 1. In this case we can represent the elements of V as infinite columns almost all of whose entries are zero. For our ring this time we take the ring of all linear transformations on V, represented as infinite square matrices (having dim V rows and columns) each column of which has only finitely many nonzero entries. This representation allows us to use the same lattice isomorphism as above: a sub-

space U is sent to the set $\hat{U}$ of all matrices whose columns come from U. For example:

$$U = \begin{bmatrix} a \\ a \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \longmapsto \hat{U} = \begin{bmatrix} a & b & c & \cdots \\ a & b & c & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

(Notice that using left ideals here would lead to a different ring since these infinite matrices lack the left-right symmetry which the transpose operation imposes on $M_3(\mathbb{R})$. The choice between left and right here is made when we decide whether to write V as row or column vectors.)

The set $\hat{U}$ is again a right ideal but not all right ideals arise this way (as they do in the case of $M_3(\mathbb{R})$). In fact $\hat{U}$ is a principal right ideal. To find a generator for $\hat{U}$ simply choose a generating set for the subspace U, making sure that the set contains dim V elements, and use the matrix whose columns make up the generating set. The fact that right multiplication corresponds to column operations allows us to generate all of $\hat{U}$ from this one matrix. This example leads us to the definition we have been seeking:

Definition: *Putting co-ordinates on a lattice* L means finding a ring R such that L is lattice-isomorphic to the set $\mathcal{L}(R)$ of principal right ideals of R.

We cannot expect any old ring to be suitable for this purpose: in most rings $\mathcal{L}(R)$ is not even a lattice, let alone complemented and modular. Von Neumann showed that the key property here is being complemented. Indeed let aR be any principal right ideal of R. If aR has a complement in $\mathcal{L}(R)$ then there is some $b \in R$ such that $aR \wedge bR = 0$ and $aR \vee bR = R$.

Regardless of what $\wedge$ and $\vee$ mean in $\mathcal{L}(R)$ (all we have at this stage is the partial order of inclusion) this gives $aR \cap bR = 0$ and $aR + bR = R$. Thus we can write

$$ax + by = 1.$$

Hence

$$axa + bya = a$$

so that

$$bya = a - axa \in bR \cap aR = 0$$

and so

$$a = axa.$$

Thus, in a co-ordinatizing ring for a complemented modular lattice, for any element a there is some x such that $a = axa$. Rings with this property are said to be (*von Neumann*) *regular*, and von Neumann showed that if R is regular then $\mathcal{L}(R)$ is a complemented modular lattice with $A \wedge B = A \cap B$ and $A \vee B = A + B$.

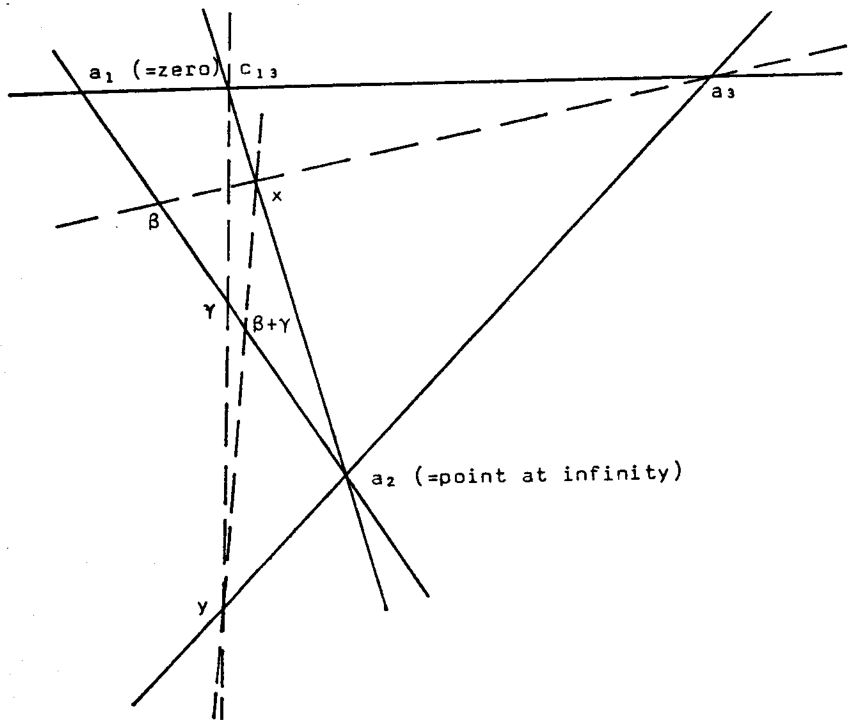
Example 3: If V is any vector space then the ring of all linear transformations of V is a regular ring.

4. It is not hard, using Maschke's Theorem and Example 3, to see that if F is any field and G is any locally finite group having no elements of order equal to the characteristics of F then the group algebra $F[G]$ is a regular ring. Auslander and others have shown that these are the only regular group algebras.

5. The ring of integers Z is not regular.

The original problem, finding what we now know should be a regular ring which co-ordinatizes a given complemented modular lattice, has been solved only in a few special cases. Von Neumann produced a solution which looked after his lattice of projections by imitating the usual construction of co-ordinates in a projective plane. For this method the lattice must have a "homogeneous n-frame" with $n \geq 4$. Such a frame consists of n independent lattice elements $a_1, a_2, \ldots a_n$ (so that $a_i \wedge (a_1 \vee a_2 \vee \ldots a_{i-1} \vee a_{i+1} \ldots \vee a_n) = 0$ for each i) with the properties that $a_1 \vee a_2 \vee \ldots \vee a_n = 1$ and each distinct pair

a_i, a_j are *perspective* (that is, have a common complement c_{ij} so that $a_i \vee c_{ij} = a_j \vee c_{ij} = a_i \vee a_j$ and $a_i \wedge c_{ij} = a_j \wedge c_{ij} = 0$). This frame plays a similar role to the frame in $\mathbb{R}^3$ consisting of the x, y and z axes. Indeed if we look at the lattice L of all subspaces of $\mathbb{R}^3$ we can get a frame by letting a_1, a_2, a_3 be the x, y and z axes (respectively); a common complement c_{13} of a_1 and a_3 would be, for example, the line $x = z$, $y = 0$. Using the projective plane picture of L we can retrieve the usual addition of points on the line $a_1 \vee a_2$ by using just the lattice operations shown in the diagram: the solid lines represent the frame, the broken lines give the construction for the sum of any two "finite" points β and γ on the line $a_1 \vee a_2$.



1. Join β to a_3 to meet the line joining c_{13} and a_2 at x.
2. Join γ to c_{13} to meet the line joining a_2 and a_3 at y.
3. The line joining x and y meets $a_1 \vee a_2$ at β + γ.

Adding points on $a_1 \vee a_2$

A similar diagram can be used to define the product $\beta\gamma$. In terms of the original lattice and its frame we are making a ring from the set of complements of a_2 in $a_1 \vee a_2$: two such complements β and γ are added and multiplied according to the rules

$$\beta+\gamma = ([(\beta \vee a_3) \wedge (c_{13} \vee a_2)] \vee [(\gamma \vee c_{13}) \wedge (a_2 \vee a_3)]) \wedge (a_1 \vee a_2)$$

$$\beta\gamma = \;\; [\beta \vee c_{23}) \wedge (a_1 \vee a_3)] \vee [(\gamma \vee c_{13}) \wedge (a_2 \vee a_3)]) \wedge (a_1 \vee a_2).$$

Von Neumann used these same operations in his more general setting, the extra dimension ($n \geq 4$) being needed to verify that these operations had the desired properties. It can then be shown that the original lattice is the same as the lattice $\mathcal{L}(R)$ where R is the ring of mxn matrices over the ring just constructed. Furthermore this ring is unique up to isomorphism (only a 3-frame, as illustrated in the diagram, is needed for this assertion; the uniqueness breaks down in lower dimensions since, for example, the lattice {0,1} may be co-ordinatized by any division ring). In von Neumann's original setting - where the lattice was the set of projections of a ring of operators - it turns out that there is often an embedding of the original ring in the co-ordinatizing ring, and so the aim of getting a ring compatible with the original structure is achieved in these cases.

In the general setting it is now natural to seek lattice characterizations of the various classes of regular rings and we conclude by considering an example of such a characterization. In a recent paper Munn introduced the class of *bisimple* rings: if the ring R has an identity element, we say it is bisimple if for any pair of nonzero elements a, b there is a third element c such that aR = cR and Rc = Rb. These are just the rings whose multiplicative semigroups are bisimple. Any division ring is bisimple but all other bisimple rings are quite large; for example if S is the ring of all linear transformations on a countable-dimensional vector space, and if I is the ideal of finite-rank transformations then the factor ring

S/I is bisimple. Munn showed that all bisimple rings are simple (in the above notation we have RaR = RcR = RbR) and regular, and so we can ask which complemented modular lattices correspond to these rings.

A hint is given by another of Munn's results: any pair of nonzero principal right ideals of a bisimple ring R must be isomorphic as R-modules (replacing a by the c given by the definition we may assume the right ideals are aR, bR where Ra = Rb; then a = xb and b = ya and left multiplication by y gives an isomorphism from aR to bR). Hence in the lattice $\mathcal{L}(R)$ any two nontrivial intervals [0,A] and [0,B] are isomorphic as lattices. Unfortunately such lattice isomorphisms throw away too much of the ring structure for this property to characterize bisimple rings. However a stronger isomorphism is provided by the notion of perspectivity that we have already met: if lattice elements A, B are perspective they have a common complement C, say, so that

$$A \vee C = B \vee C = A \vee B$$

with

$$A \wedge C = B \wedge C = 0.$$

Hence

$$A \cong \frac{A \vee B}{C} \cong B$$

where the isomorphisms will be module isomorphisms if we are working inside (R). Perspectivity by itself is too strong for our purposes (since if A < B we want $A \cong B$ but clearly cannot have A and B perspective) but a simple splitting trick allows us to get round this problem:

<u>**Result**</u>: Let R be a regular ring with identity and let $\mathcal{L}(R)$ be its lattice of principal right ideals. Then R is bisimple if and only if $\mathcal{L}(R)$ satisfies

(*) ... for any nonzero a, b $\in \mathcal{L}(R)$ there are splittings

$$a = a_1 \vee a_2 \quad \text{where} \quad a_1 \wedge a_2 = 0$$

and

$$b = b_1 \vee b_2 \quad \text{where} \quad b_1 \wedge b_2 = 0$$

such that a_1, b_1 and a_2, b_2 are perspective pairs of elements of $\mathcal{L}(R)$.

The key idea of the proof here is that if A, B are principal right ideals of a bisimple ring R such that $A \cap B = 0$ then A and B are perspective (as before we may assume $A = aR$ and $B = bR$ where $Ra = Rb$; then $c = (a+b)R$ is a common complement of A and B).

A stronger result is also true: any complemented modular lattice satisfying the condition (*) is easily seen to possess a homogeneous 4-frame (or else be the lattice $\{0,1\}$) and so, by von Neumann's result, can be co-ordinatized by a (necessarily bisimple) regular ring.

Some References

1. D. Handleman, Co-ordinatization Applied to Finite Baer * rings, *Trans. Amer. Math. Soc.*, 235 (1978), 1-34.
2. D. Munn, Bisimple Rings, *Quart. J. Math.*, (2), 32 (1981), 181-191.
3. J. von Neumann, On Regular Rings, *Proc. Nat. Acad. Sci. U.S.A.*, 22 (1936), 707-713.
4. J. von Neumann, Continuous Geometry, Princeton University Press, Princeton 1960.

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TEN COUNTEREXAMPLES IN GROUP THEORY

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Introduction

Many major theorems in the theory of finite groups have been proved by the minimum counterexample technique, which works as follows. We assume that the theorem is false and let G be a counterexample of smallest possible order. The assumption that G exists is then used to force a contradiction and the theorem in question is thereby established. In practice, the contradiction frequently arises from the existence of a counterexample of order less than that of the presumed minimum counterexample (m.c.e.). This technique of course is merely a disguised form of induction or the method of infinite descent used in number theory.

However, even when a conjecture about finite groups turns out to be false, it is often of interest to discover an m.c.e., or "least criminal" as it is often called. Note that an m.c.e. need not be unique. Searching for an m.c.e. is a very good method of becoming familiar with the groups of small order and perhaps the size of an m.c.e. is an indication of how plausible the conjecture was in the first place!

In this article we discuss ten "not implausible" conjectures about finite groups and produce an m.c.e. in each case. We outline the arguments used in establishing that a given group is an m.c.e.

The material in this paper is based on the author's M.A. thesis "Minimum Counterexamples in Group Theory", University College, Cork, 1982, prepared under the supervision of Dr. D. MacHale. I wish to thank Dr. MacHale for suggesting this problem and the Mathematics Department of U.C.C. for their co-operation and facilities.