the Jackknife, the Bootstrap and Other Methods. Biometrika 68, 589-599.

Efron, B. (1982). The Jackknife, the Bootstrap, and Other Resampling Plans. Siam: Philadelphia.

Hampel, F.R. (1974). The Influence Curve and its Role in Robust Estimation. J. Amer. Stat. Assoc. 69, 383-393.

Huber, P.J. (1977). Robust Statistical Procedures. Siam: Philadelphia.

Kendall, M.G. and Stuart, A. (1961). The Advanced Theory of Statistics, Vol. 2. Hafner: New York.

Miller, R., Jr., editor (1980). Biostatistics Casebook. Wiley.

Von Mises, R. (1947). On the Asymptotic Distribution of Differentiable Statistical Functions. Ann. of Math. Statist. 18, 309-348.

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FEIGENBAUM'S NUMBER

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In 1975 Mitchell J. Feigenbaum [2] of the Los Alamos National Laboratory, whose work concerns the transition from periodic to aperiodic bahaviour, discovered a new universal constant which has since been called Feigenbaum's number. He had been using a programmable calculator to examine the iteration of one-parameter families of maps of a finite interval into itself. One map he looked at was $x \to f_B(x) = Bx(1-x)$; another was $x \to B \sin \pi x$, both on the interval $[0,1]$. Feigenbaum observed some common features of the parameter dependence of these maps which he suspects would not have been noticed had the calculations been carried out on a large computer rather than a small calculator. The theory of these maps has been extended by Pierre Collet of Paris, Jean-Pierre Eckmann of Geneva and H. Koch of Harvard. The topic is reviewed in the book by Collet and Eckmann [3] on which this note is based.

For the most part, Collet and Eckmann consider mappings $x \to f(x)$ which are C^1-unimodal. A mapping f of the interval $[-1,1]$ into itself is C^1-unimodal if f is continuous; $f(0)=1$; f is strictly decreasing on $(0,1]$ and strictly increasing on $[-1,0)$; and f is once continuously differentiable with $f'(x) \neq 0$ when $x \neq 0$.

Denoting by f^0 the identity, $f^1 = f$, $f^2 = f \circ f$, $f^n = f \circ f^{n-1}$ the sets of iterates of points $x \in [-1,1]$

$$O_f(x) = \{x, f(x), f^2(x), f^3(x) \ldots\}$$

are called the orbits of f. A point $x \in [-1,1]$ is called a periodic point for f if $O_f(x)$ is a finite set. The cardinality of this set is called the period of x and $O_f(x)$ is called the periodic orbit of x. It is then also the periodic orbit of

$f^n(x)$ for all $n \geq 0$.

It is well known that if $\bar{x}$ is a fixed point of $x \to g(x)$ it will be a stable fixed point provided $|Dg(\bar{x})| < 1$ where Dg means $\frac{dg}{dx}$. Then iterations of the map which start in the neighbourhood of $\bar{x}$ will eventually converge to $\bar{x}$. If P is a periodic orbit of period p for f the orbit is called a stable periodic orbit when, for $x \in P$, $|Df^p(x)| < 1$. By the chain rule, $Df^p(x)$ takes the same value for all $x \in P$, and all the points of P are then fixed points of f^p. Thus in the case of a stable periodic orbit, many starting points give rise to similar behaviour as the number of iterations becomes large. A periodic orbit P is called superstable if $0 \in P$. From the definition of a C^1-unimodal function this means that a periodic orbit is superstable iff $Df^p(x) = 0$ for $x \in P$. Singer [4] has used the concept of negative Schwartzian derivative defined by

$$S f(x) \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

to discuss the existence of stable periodic orbits. The requirement that $S f(x) < 0$ for all $x \in [-1,1]$ is a more precise statement of the shape of functions considered than C^1-unimodality. Ignoring some technical details, Singer concludes that a function f which is C^1-unimodal and for which $S f(x) < 0$ for all $x \in [-1,1]$ has at most one stable periodic orbit plus possibly a stable fixed point in the interval $[-1, f(1))$. If 0 is not attracted to a stable periodic orbit, then f has no stable periodic orbit in $[f(1), 1]$. Further, there exist functions satisfying Singer's conditions which do not have a stable periodic orbit and maps of such functions have ergodic properties.

Collet and Eckmann use numerical results for the mapping

$$x \to f_B(x) = 1 - Bx^2,$$

where B is a parameter in the range $[0,2]$ to illustrate these ideas. For a value of $B \leq B_1 = .75$ iteration of the mapping

converges to a single stable fixed point. For $.75 = B < B \leq B_2 = 1.25$ initial points from the interval $[-1,1]$ are eventually attracted to a stable periodic orbit of period 2. B_1 is called a bifurcation value of the parameter B. At $B_2 = 1.25$ both of the branches bifurcate again and a stable periodic orbit of period $2^2 = 4$ appears. Further bifurcations from period 2^{n-1} to 2^n take place at values B_n which converge to $B_\infty = 1.401 \ldots$. For most values of B in the range $(B_\infty, 2]$ orbits which have very large periods or which are aperiodic appear. For some values of B in this range not only do orbits with period 2^k appear but also orbits of period 6, 5 and 3. Much is known about this type of behaviour but it will not be discussed further here except to say that what happens as the parameter approaches and then passes B_∞ is typical of the transition from regular to chaotic or turbulent behaviour of many systems [3].

The first few values of B_n can be found by using a programmable calculator although the convergence to stable periodic orbits is slow when the parameter is near a bifurcation point. For higher values of B_n multiple-precision arithmetic is required. Collet and Eckmann give the following table.

n	B_n	$\frac{B_n - B_{n-1}}{B_{n+1} \quad B_n}$
1	.75	
2	1.25	
3	1.3680989394	4.233738275
4	1.3940461566	4.551506949
5	1.3996312389	4.645807493
6	1.4008287424	4.663938185
7	1.4010852713	4.668103672
8	1.401140214699	4.668966942
9	1.401151982029	4.669147462
10	1.401154502237	4.669190003
11	1.401155041989	4.669196223

Not only do the B_n converge to a limit B_∞ but the ratios

$$\frac{B_n - B_{n-1}}{B_{n+1} - B_n}$$

also seem to converge to a limit δ.

It has been established ([2], [5]) that, for sufficiently smooth families of maps, the number δ does not in general depend on the family and families similar to the one discussed here produce asymptotically

$$|B_n - B| \sim \text{const. } \delta^{-n},$$

where

$$\delta = 4.6692016 \ldots$$

is Feigenbaum's number. More precisely, the universality of δ is somewhat relative in that its value does depend on the function space in which the mapping functions are assumed to lie.

Feigenbaum discovered the universality of δ experimentally and then proposed an explanation suggested by the renormalization group approach to critical phenomena. Collet, Eckmann and Lanford [5] have proved rigorously, at least in a certain limiting regime, the existence of the scheme outlined by Feigenbaum.

There is another type of scaling which can again be illustrated by the mapping.

$$x \to 1 - Bx^2.$$

For large n it is found that as each bifurcation point is passed the pattern of the orbits exhibits a certain regularity. This may be exemplified by considering the parameter range $(B_n, B_{n+1}]$ where stable orbits of period 2^n occur. It is found that there is always one value of B in this range which gives rise to a superstable orbit containing the point $x = 0$. It is further found that on such a superstable orbit the distance from the point at zero to its nearest neighbour is, for large n, λ^n where λ is another universal constant

$$\lambda = -0.3995 \ldots$$

Indeed if α is defined by

$$\delta^{\alpha} = |\lambda|$$

and a plot is made of the points of the stable orbit for different values of B using a vertical scale $x(B-B_\infty)^{\alpha}$ and a horizontal scale $\log|B-B_\infty|$ then, for large n, a periodic diagram appears. Once again this behaviour is independent of the one parameter family in question.

Finally the value, B_∞, of the parameter at which transition to ergodic behaviour occurs is, in general, different for different maps of the class considered, as are the functions f_B. However the function

$$f(x) = \lim_{n\to\infty} f^{2^n}_{B_\infty}(\lambda^n x)$$

is a universal function up to a change of scale. If scaled to $f(0) = 1$ it has the expansion

$$f(x) = 1 - 1.52763x^2 + 0.104815x^4 - 0.267057 \ldots x^6 + \ldots.$$

By construction f satisfies

$$f \circ f(\lambda x) = \lambda f(x)$$

which evaluated at $x=0$ gives

$$\lambda = f(1)$$

Collet and Eckmann [3], [5] also show that f may be used to characterize δ as the largest eigenvalue of the linear operator (on the function space in question)

$$h(.) \to \frac{1}{\lambda}h(f(\lambda .)) + \frac{1}{\lambda}f'(f(\lambda .))h(\lambda .).$$

Recently Rollins and Hunt [6] have modelled a simple, nonlinear, electronic system which exhibits the period-doubling route to chaos with universal scaling.

References

[1]. R.B. May; Simple Mathematical Models with very Complicated dynamics. Nature 261, 459, (1976).

[2]. M. Feigenbaum; Quantitative universality for a class of nonlinear transformations. J. Stat. Phys. 19, 25 (1978); 21, 669 (1979).

[3]. P. Collet and J.P. Eckmann; Iterated maps on the interval as dynamical systems. Birkhauser PPh progress in physics series (1980).

[4]. D. Singer; Stable orbits and bifurcations of maps of the interval SIAM J. Appl. Math. 35, 260 (1978).

[5]. P. Collet, J.P. Eckmann, O.E. Lanford; Universal Properties of maps on an interval. Comm. Math. Phys. 76, 211, (1980).

[6]. P.W. Rollins and E.R. Hunt; Exactly solvable model of a physical system exhibiting universal chaotic behaviour. Phys. Rev. Lett. 49, 1295 (1982).

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PATHS IN A GRAPH

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In a connected graph any two vertices can be joined by a sequence of edges. This is the definition of connectedness for graphs. However, how do you find a path joining a given pair of vertices, and how do you decide effectively if a graph is connected? These are the questions I shall discuss in this note. The graphs we consider are finite, undirected and have no loops or multiple edges. A path is a sequence $\{v',v_1\} = e_1$, $\{v_1,v_2\} = e_2$, ..., $\{v_{r-1},v''\} = e_r$ of edges without repetition (of edges: vertices *may* occur repeatedly). The vertices v' and v'' are the end vertices of the path.

A popular version of this problem is to find the exit in a maze. We have to distinguish two cases. In the first instance, imagine that we are actually inside a maze without knowing its overall design. Here the only solution seems to be trial and error. A successful route to the exit is very unlikely to be a path according to our definition. In fact, the probability to reach the exit on a path is less than 2^{-c}, where c is the number of intermediate junctions on a path to the exit (provided that there is only one such path in the maze). In other words, it is almost impossible to avoid walking into a cul-de-sac! However, most commonly, maze puzzles are done with paper and pencil, and the design of the maze is right in front of your eyes. In this situation, can you avoid a cul-de-sac? The answer is yes, there is a *construction* for a path to the exit!

From a set P of edges let V(P) be the set of end vertices of edges in P. For a vertex v in the graph, let $d_P(v)$ be the number of edges in P that end at v. A cycle is a path that ends in its initial vertex. Our construction is based upon the following simple observation: