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The Mathematics Department is involved in an advisory capacity in the mathematics education courses. These form a component of the Education course which all students take. Here the methodology and content of the primary school mathematics curriculum are covered.

The Mathematics Department has a staff of three:

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Frederick S. Klotz	B.S. Ph.D.	(University of Pittsburgh) (Syracuse University)
Olivia Bree	B.Sc. M.Sc.	(U.C.G.) (U.C.G.)

NUMERICAL METHODS IN DYNAMICAL WEATHER PREDICTION

J.R. Bates

1. Introduction

The scientific problem of forecasting the weather using dynamical methods was first tackled successfully by a group working under the leadership of John von Neumann at the Institute for Advanced Study, Princeton, in the late 1940 s. At that time the first electronic computer, the ENIAC, had just become available. Von Neumann recognised that the new machine was ideally suited to performing the hugh volumes of computations necessary to predict the non-linear development of fluid systems, including the motions of the atmosphere. Using observed initial data derived from balloon ascents over the continental U.S., an integration was performed which succeeded in predicting the main features of the actual evolution of the 500-mb flow for a 24-hour period over the area in question. (Charney, Fjørtoft and Von Neumann, 1950). The integration took 24 hours of computer time, however!

An essential ingredient of the success of the integration, due to von Neumann himself, was the development of a computationally stable numerical scheme for representing the differential equations governing the flow. It had been discovered two decades earlier (Courant, Friedrichs and Lewy, 1928) that not all consistent numerical representations of partial differential equations lead to realistic solutions. The method devised by von Neumann was an explicit leapfrog method based on a grid point representation in space and time.

The vast increase in the speed of computers over the past three decades, coupled with progress in devising more efficient numerical schemes for solving the equations of motion, has made it possible to compute the weather fast enough for the forecasts to be used operationally. The computer forecasts have for some time been more accurate than those which can be produced using traditional methods alone. Even with the most powerful computers available today, however, the spatial truncation errors associated with the numerical representation of the governing differential equations are still a factor limiting the accuracy of forecasts. The search for more efficient and accurate schemes therefore remains a central problem in dynamical weather forecasting.

In this article, some of the main numerical techniques used in this field will be briefly outlined.

2. The Simplified Governing Equations

The large-scale motions of the atmosphere are quasi-horizontal and, to a very high degree of approximation, hydrostatically balanced (i.e. the vertical component of the pressure gradient force equals the force of gravity). They are also shallow, in the sense that ~~the~~ vertical scale of the motions is small by comparison with the earth's radius. These facts allow one to adopt simplified versions of the general equations of fluid dynamics for the purpose of weather prediction. A simplified set of equations, which are capable of describing the flow at the atmosphere's middle level (the 500 mb level, approximately) with reasonable accuracy, are the "shallow water" equations,

$$\frac{du}{dt} = -g\frac{\partial h}{\partial x} + fv \quad \text{.... (1)}$$

$$\frac{dv}{dt} = -g\frac{\partial h}{\partial y} - fu \quad \text{.... (2)}$$

$$\frac{dh}{dt} = h\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] \quad \text{.... (3)}$$

where (x,y) are the eastward and northward coordinates, (u,v) are the corresponding velocity components, h is the height of the 500 mb surface, g is the gravitational acceleration, f $(= f_0 + \beta y)$ is the Coriolis parameter representing the effects of the earth's rotation, and $\frac{d}{dt}$ $(= \frac{\partial}{\partial t} + \underset{\sim}{V}_H.\nabla)$ is the derivative following a fluid particle. In these simplified equations, the effects of friction, thermodynamic forcing and spherical geometry (except for the y-variation of the Coriolis parameter) have all been neglected.

The above equations possess two distinct types of linearized wave solution:

(a) gravity-inertia waves, for which the phase speed in the one-dimensional case (with f regarded as constant) is given by

$$c = \pm\left\{gH + \frac{f^2}{k^2}\right\}^{\frac{1}{2}} \quad \text{.... (4)}$$

Here H is the mean height of the surface and k is the wavenumber.

(b) Rossby waves, for which the phase speed in the one-dimensional case (with no perturbation in the height of the surface) is given by

$$c = -\frac{\beta}{k^2} \quad \text{..... (5)}$$

The gravity-inertia waves are fast (phase speeds of hundreds of metres per second) but have only very small amplitudes in the atmosphere. The Rossby waves are slow (phase speeds of ten metres per second or less) and are very important in the development of weather systems. In the original governing equations used by the Princeton group, the gravity-inertia waves were filtered out by using a modified version of equations (1)-(3). The filtering procedure introduces some inaccuracies however, and the modern practice is not to use it.

In almost all meteorological applications, an Eulerian approach has been used for solving the equations of motion, i.e. the derivatives ($\frac{d}{dt}$) are expressed as $(\frac{\partial}{\partial t} + \underset{\sim}{V}_H.\nabla)$ and the $\underset{\sim}{V}_H.\nabla$ term is brought to the right hand side of the equations. Only partial derivatives in space and time then occur and one forgets about fluid particles. An alternative is the

Lagrangian approach in which one follows the fluid particles, retaining ($\frac{d}{dt}$) in its original form and doing the numerical calculations accordingly. The Irish Meteorological Service is the first to use a Lagrangian method operationally; we have been using this method for our daily forecasts since May 1982 and find it to be more efficient than the Eulerian approach (see Section 4 below).

3. Main Categories of Numerical Methods used for Solving the Equations in Eulerian Form

The methods used for solving the equations in Eulerian form can be classified under two headings:

(i) The Grid Point Method

Here the partial derivatives in the equations of motion are replaced by finite difference approximations at a discrete set of points regularly distributed in space and time. The difference equations are then solved using algebraic methods.

(ii) Galerkin Methods

The Galerkin procedure represents the dependent variables as a sum of functions that have a prescribed spatial structure. The coefficient associated with each function is then a function of time. This procedure transforms a partial differential equation into a set of ordinary differential equations for the coefficients. These equations are usually solved with finite differences in time. Examples of the Galerkin method are (a) the Spectral Method (using orthogonal functions as basis functions), and (b) the Finite Element Method (using functions that are zero except in a limited region where they are low-order polynomials).

The grid point method has been the most widely used method in meteorology, but spectral methods, using surface spherical harmonics as basis functions, are now being used

for hemispheric or global forecasting models at a number of centres, e.g. the European Centre for Medium Range Weather Forecasts. Some numerical experiments have also been carried out using the finite element method, but so far this has not been found to be competitive in efficiency with the other methods.

Some examples will now be given to illustrate the stability properties of numerical schemes using the grid point method. We consider the equation

$$\frac{\partial \psi}{\partial t} = -\bar{u}\frac{\partial \psi}{\partial x} \qquad \text{.... (6)}$$

governing the advection of a scalar ψ by a mean flow $\bar{u}$ (here considered a positive constant). Equation (6) contains a subset of the terms of equations (1) - (3). The analytical solution to (6) is

$$\psi = F(x - \bar{u}t)$$

where $F(x)$ is the initial distribution of ψ.

Consider the following simple difference approximation to equation (6):

$$\frac{\left[\psi_j^{n+1} - \psi_j^n\right]}{\Delta t} = -\bar{u}\frac{\left[\psi_{j+1}^n - \psi_{j-1}^n\right]}{2\Delta x} \qquad \text{.... (7)}$$

where $t = n\Delta t$, $x = j\Delta x$. This is a "forward-in-time, centred-in-space" approximation. We examine the stability of (7) by the von Neumann method, i.e. we assume

$$\psi_j^n = \lambda^n e^{ik(j\Delta x)}\psi^0 \qquad \text{.... (8)}$$

where λ is the amplification factor. Substituting in (7) then gives

$$\lambda = 1 - i\,\bar{u}\frac{\Delta t}{\Delta x}\,\mathrm{Sin}\,k\Delta x$$

Thus $|\lambda|^2 > 1$ for all values of $(\bar{u}\Delta t/\Delta x)$ for general values of k, i.e. unlike the analytical solution, the numerical solution amplifies with time and the difference approximation (7) is unstable.

Next consider the difference approximation

$$\frac{\left[\psi_j^{n+1} - \psi_j^n\right]}{\Delta t} = -\bar{u}\frac{\left[\psi_j^n - \psi_{j-1}^n\right]}{\Delta x} \quad \ldots\ldots (9)$$

i.e. a "forward-in-time, upstream-in-space" approximation. Again assuming a solution of the form (8), we find that

$$\lambda = 1 - u\frac{\Delta t}{\Delta x}\left[1 - \exp(-ik\Delta x)\right]$$

so that

$$|\lambda|^2 = 1 - 2\bar{u}\frac{\Delta t}{\Delta x}(1 - \bar{u}\frac{\Delta t}{\Delta x})(1 - \text{Cos}\,k\Delta x)$$

In this case $|\lambda|^2 \leq 1$, (i.e. stability obtains) provided

$$\bar{u}\frac{\Delta t}{\Delta x} \leq 1 \quad \ldots\ldots (10)$$

The difference approximation (9) is thus conditionally stable, the stability criterion being that the distance covered by a particle in the time interval Δt be less than the spatial grid interval.

The difference equations (7) and (9) are both explicit, in the sense that the right-hand sides of both equations involve known quantities, and the unknown ψ_j^{n+1} is obtained by a simple operation at each grid point.

We now consider an implicit numerical representation of (6):

$$\frac{\left[\psi_j^{n+1} - \psi_j^n\right]}{\Delta t} = -\bar{u}\frac{\left[\psi_{j+1}^{n+1} - \psi_{j-1}^{n+1}\right]}{2\Delta x} \quad \ldots\ldots (11)$$

Here the right-hand side involves the unknown quantities at time level (n+1), and the values of ψ_j^{n+1} can only be obtained by a matrix inversion involving the whole grid. Seeking a

solution to (11) of the form (8) we find

$$\lambda = \frac{1}{1 + i(\bar{u}\frac{\Delta t}{\Delta x})\,\text{Sin}\,k\Delta x}$$

i.e.

$$|\lambda|^2 = \frac{1}{1 + (\bar{u}\frac{\Delta t}{\Delta x})^2\,\text{Sin}^2\,k\Delta x}$$

so that $|\lambda|^2 \leq 1$ for all values of $(\bar{u}\,\Delta t/\Delta x)$. The difference equation (11) is thus unconditionally stable.

When the remaining terms of (1) - (3) are included, a linearized analysis shows that Eulerian explicit difference representations are in all known cases either unstable or conditionally stable, the stability criterion being that $(c\,\Delta t/\Delta x)$ be less than some number of order unity, where c represents the maximum signal velocity in the fluid. This represents a severe limitation on the allowable time step in the meteorological context because of the fast gravity-inertia wave solutions of the equations. These solutions are essentially only "noise", the meteorologically interesting information being carried by the slow Rossby wave and advective terms in the equations of motion. This problem can in theory be circumvented by adopting an implicit differencing scheme for all terms of the equations of motion, which gives unconditional stability and allows one to choose a time step which is realistic for describing the slow motions.

It turns out, however, that the matrix inversions involved in integrations with implicit schemes are computationally so costly that one is no better off than if one had adopted an explicit scheme with a very small time step.

A successful compromise between these two extremes is to use a semi-implicit approach, where the terms governing the fast motions are treated implicitly while the terms governing the slow motions are treated explicitly (Kwizak and Robert, 1971). This leads to conditional stability, but with a stability criterion which is much more lenient than that for fully explicit methods. At the same time, the matrix inversions are much simplified. The semi-implicit approach is now widely used, with both grid points and spectral models.

An alternative efficient method is to adopt the splitting approach pioneered by Soviet mathematicians (Marchuk, 1974). In this approach the equations (1)-(3) are split into the two sets:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial t} &= -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} \\ \frac{\partial h}{\partial t} &= -u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y} \end{aligned}\right\} \quad (12) \qquad \left.\begin{aligned} \frac{\partial u}{\partial t} &= -g\frac{\partial h}{\partial x} + fv \\ \frac{\partial v}{\partial t} &= -g\frac{\partial h}{\partial y} - fu \\ \frac{\partial h}{\partial t} &= -h\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] \end{aligned}\right\} \quad (13)$$

A stability analysis shows that these two sets have independent stability criteria. The set (12) can be stepped forward in time with a long (advective) time step, while the set (13) can be updated successively with a fractional time step. With both sets treated explicitly, a ratio of 3:1 in the respective time steps can be used. This leads to an efficiency comparable to that of the semi-implicit method, with much simpler programming. This method is also widely used by meteorologists.

4. The Semi-Langrangian Method

A fully Lagrangian approach to solving the equations of fluid motion would involve following a fixed set of particles throughout the period of the integration. In atmospheric flow, a set of particles which are initially regularly distributed soon become greatly deformed so it is better to adopt a semi-Lagrangian approach, where a set of particles which arrive at a regular set of grid points are traced backwards over a single time interval to their departure points. The values of the dynamical quantities at the departure points are obtained by interpolation from the surrounding grid points. A new set of particles is then considered at each time step.

A splitting approach can be combined with the Semi-Lagrangian technique by writing (12) in the form

$$\frac{du}{dt} = 0, \quad \frac{dv}{dt} = 0, \quad \frac{dh}{dt} = 0 \qquad \ldots . \; (14)$$

while keeping the remaining terms as in (13). The equations (14) are integrated to give

$$(u,v,h)^{n+1}_{i,j} = (u,v,h)^{n}_{*}$$

where the quantities $(\;)^{n+1}_{i,j}$ are the new values at the grid point (i,j) and the quantities $(\;)^{n}_{*}$ are the old values at the departure point of the particle.

Bates and McDonald (1982) have shown that for linear and quadratic interpolation in the one-dimensional case, and for bilinear and biquadratic interpolation in the two-dimensional case, the above explicit method gives unconditional stability for the advective part of the integration. The use of the semi-Lagrangian method has led to a saving of a third in the computer time required to produce our daily forecasts in the Irish Meteorological Service compared to the Eulerian method previously used, while giving equal accuracy.

5. Conclusions

Only the barest outline has been given above of an extensive field of investigation. Many important questions, such as non-linear computational instability, staggered grids and the maintenance of integral constraints (such as energy and squared vorticity) in numerical integrations have not even been touched on. The governing equations used in practice are much more complicated than the simple shallow water equations which have been discussed here. A comprehensive coverage of numerical methods used in meteorology can be found in

WMO (1976), Chang (1977) and Haltiner and Williams (1980).

Despite the progress that has been made, it appears likely that there is still a long way to go before the ideal numerical method is found which integrates the governing equations and gives clearly maximum accuracy for a given computational cost.

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Meteorological Service,
Glasnevin Hill,
Dublin 9.

RECENT DEVELOPMENTS IN LINEAR PROGRAMMING

Fergus J. Gaines

Let A be an mxn matrix, let $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The basic problem in linear programming is to find, for $x \in \mathbb{R}^n$,

$$\max c^t x, \text{ subject to } Ax \leq b,\ x \geq 0 \qquad (1)$$

For vectors, $x \leq y$ means $x_i \leq y_i$ for all i; $x < y$ means $x_i < y_i$ for all i.)

The standard way of solving this problem is to use the celebrated *simplex method* of G. Dantzig [1]. The idea is to note that the *feasible* solutions of (1), i.e. the $x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$, form a convex polytope K in $\mathbb{R}^n$. The vertices of K are those feasible x with either x = 0 or such that the positive components of x correspond to linearly independent columns of A. The typical step in the simplex algorithm proceeds from vertex $x^{(k)}$ to a vertex $x^{(k+1)}$ so that $c^t x^{(k+1)} \geq c^t x^{(k)}$. Since max $c^t x$ is attained at a vertex of K, the algorithm eventually gives the answer.

This algorithm is arguably the most widely used algorithm of the present day and it is probably safe to say that most of those who use it do not understand it, whereas most of those capable of understanding it never use it. Its popularity is probably the reason for the widespread, if in many cases inaccurate, coverage in the newspapers given to the discovery in 1979 of a new algorithm for solving (1), the work of a Soviet "unknown" L.G. Khachiyan [2]. (One American newspaper reported bitterly (but incorrectly) that a Soviet mathematician had solved the "travelling salesman problem", despite the fact that the U.S.S.R has no travelling salesmen!)

The immediate reason why Khachiyan's algorithm is important is because it is *in theory* more computationally efficient