The result is certainly true if $f(y) = f(x)$. If not, form the function $\phi(t)$ by

$$\phi(t) = \langle f(y)-f(x), f(x+t(y-x))\rangle / |f(y)-f(x)|$$

where $\langle , \rangle$ denotes the inner product in $\mathbb{R}_m$. Then

$$\phi(1) = \langle f(y)-f(x), f(y)\rangle / |f(y)-f(x)|,$$

$$\phi(0) = \langle f(y)-f(x), f(x)\rangle / |f(y)-f(x)|,$$

$$\phi'(t) = \langle f(y)-f(x), f'(x+t(y-x))(y-x)\rangle / |f(y)-f(x)|.$$

In the last line we used the chain rule twice, once because f is differentiable, and once because the inner product is also differentiable. Of course, ϕ itself is well-defined because D is convex.

By the usual mean value theorem,

$$\phi(1)-\phi(0) = \phi'(t) = \langle f(y)-f(x), f(y)-f(x)\rangle / |f(y)-f(x)| = |f(y)-f(x)|,$$

and by the Schwarz inequality,

$$\phi'(t) < |f(y)-f(x)|\,|f'(x+t(y-x))(y-x)| / |f(y)-f(x)|$$

$$< \|f'(x+t(y-x))\|\,|y-x| \leqslant \sup_{0<t<1} \|f'(x+t(y-x))\|\,|y-x|.$$

This proves the theorem.

It is clear that if $\mathbb{R}_n$ is replaced by any Banach space and $\mathbb{R}_m$ is replaced by any real Hilbert space, then the method of the proof remains valid.

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A SIMPLE PROOF OF TAYLOR'S THEOREM

Raymond A. Ryan

This proof uses only the Fundamental Theorem of Calculus in the form:

$$\frac{d}{dx}\int_x^b g(t)\,dt = -g(x)$$

Taylor's Theorem: If f is n+1 times continuously differentiable in an open interval containing the points a and b, then

$$f(b) = f(a) + f'(a)(b-a) + \ldots + \frac{1}{n!} f^{(n)}(a)(b-a)^n + \frac{1}{n!}\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt$$

Proof:

Let $$F(x) = f(x) + f'(x)(b-x) + \ldots + \frac{1}{n!} f^{(n)}(x)(b-x)^n + \frac{1}{n!}\int_x^b f^{(n+1)}(t)(b-t)^n dt$$

Then F is differentiable, and

$$F'(x) = f'(x) - f'(x) + f''(x)(b-x) - \ldots - \frac{1}{(n-1)!} f^{(n)}(x)(b-x)^{n-1} + \frac{1}{n!} f^{(n+1)}(x)(b-x)^n - \frac{1}{n!} f^{(n+1)}(x)(b-x)^n$$

$$= 0$$

Hence $F(x)$ is constant. Furthermore, $F(b) = f(b)$, and so

$$f(b) = F(b) = F(a) = f(a) + \ldots + \frac{1}{n!} f^{(n)}(a)(b-a)^n + \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,dt$$

Q.E.D.

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