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THE MEAN VALUE THEOREM FOR VECTOR VALUED FUNCTIONS: A SIMPLE PROOF

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It is well known that the mean value theorem in one dimension extends readily to real-valued functions of several variables, but fails for the vector-valued case. For example, let $f(t) = (\cos t, \sin t)$ and suppose there is a point ξ in $(0,2\pi)$ such that $f'(\xi) = 0$. Then $-\sin\xi = \cos\xi = 0$, an impossible situation. A useful and correct generalization is the inequality

$$|f(y) - f(x)| < \sup_{0<t<0} ||f'(x+t(y-x))|| \, |y-x|$$

where $f: D \subset \mathbb{R}_n \to \mathbb{R}_m$ is a differentiable vector-valued function on a convex open set D, f' is the matrix $\partial f_i/\partial x_j$, $i = 1,2 \ldots m$, $j = 1,2 \ldots n$, $|\ |$ is the appropriate norm (in $\mathbb{R}_n$, or in $\mathbb{R}_m$), $||\ ||$ is the usual norm in the set of linear maps from $\mathbb{R}_n$ to $\mathbb{R}_m$, and x,y are arbitrary points in the domain D.

Many undergraduate calculus and analysis texts prove the mean value theorem in the real case but omit the result above. Those that do present this more general form usually give either a "sloppy" proof, using components, or a "slick" proof with the Hahn-Banach Theorem. Here we present a direct approach, requiring only the chain rule and the mean value theorem in $\mathbb{R}$. It is worth noting that f' at each point is a linear map (given by the Jacobian matrix) and that the usual norm for a linear map (matrix) is given by

$$\sup_{|x|=1} |Ax|.$$

However, other norms such as $(\Sigma\alpha_{ij}^2)^{\frac{1}{2}}$ where $A = (\alpha_{ij})$ are frequently used in advanced calculus courses. All we really use is that $|Ax| \leqslant ||A||\,|x|$.

The result is certainly true if $f(y) = f(x)$. If not, form the function $\phi(t)$ by

$$\phi(t) = \langle f(y)-f(x), f(x+t(y-x))\rangle / |f(y)-f(x)|$$

where $\langle , \rangle$ denotes the inner product in $\mathbb{R}_m$. Then

$$\phi(1) = \langle f(y)-f(x), f(y)\rangle / |f(y)-f(x)|,$$

$$\phi(0) = \langle f(y)-f(x), f(x)\rangle / |f(y)-f(x)|,$$

$$\phi'(t) = \langle f(y)-f(x), f'(x+t(y-x))(y-x)\rangle / |f(y)-f(x)|.$$

In the last line we used the chain rule twice, once because f is differentiable, and once because the inner product is also differentiable. Of course, ϕ itself is well-defined because D is convex.

By the usual mean value theorem,

$$\phi(1)-\phi(0) = \phi'(t) = \langle f(y)-f(x), f(y)-f(x)\rangle / |f(y)-f(x)| = |f(y)-f(x)|,$$

and by the Schwarz inequality,

$$\phi'(t) < |f(y)-f(x)|\,|f'(x+t(y-x))(y-x)| / |f(y)-f(x)|$$

$$< \|f'(x+t(y-x))\|\,|y-x| \leqslant \sup_{0<t<1} \|f'(x+t(y-x))\|\,|y-x|.$$

This proves the theorem.

It is clear that if $\mathbb{R}_n$ is replaced by any Banach space and $\mathbb{R}_m$ is replaced by any real Hilbert space, then the method of the proof remains valid.

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A SIMPLE PROOF OF TAYLOR'S THEOREM

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This proof uses only the Fundamental Theorem of Calculus in the form:

$$\frac{d}{dx}\int_x^b g(t)\,dt = -g(x)$$

Taylor's Theorem: If f is n+1 times continuously differentiable in an open interval containing the points a and b, then

$$f(b) = f(a) + f'(a)(b-a) + \ldots + \frac{1}{n!} f^{(n)}(a)(b-a)^n + \frac{1}{n!}\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt$$

Proof:

Let $$F(x) = f(x) + f'(x)(b-x) + \ldots + \frac{1}{n!} f^{(n)}(x)(b-x)^n + \frac{1}{n!}\int_x^b f^{(n+1)}(t)(b-t)^n dt$$

Then F is differentiable, and

$$F'(x) = f'(x) - f'(x) + f''(x)(b-x) - \ldots - \frac{1}{(n-1)!} f^{(n)}(x)(b-x)^{n-1} + \frac{1}{n!} f^{(n+1)}(x)(b-x)^n - \frac{1}{n!} f^{(n+1)}(x)(b-x)^n = 0$$

Hence $F(x)$ is constant. Furthermore, $F(b) = f(b)$, and so

$$f(b) = F(b) = F(a) = f(a) + \ldots + \frac{1}{n!} f^{(n)}(a)(b-a)^n + \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,dt$$

Q.E.D.

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