third-level education and 'education for life'. Indeed we would argue that the simple mathematical skills we have isolated as being required for elementary physics are also precisely those required for survival in our modern technological society. A serious lack of these skills in an adult population is as serious a problem as illiteracy; those lacking such skills today will inevitably find themselves exploited by society. Many of the students with the problems discussed above would also be unable to determine whether a packet of 6, 10, 16 or 36 fish fingers is the best value. Consider also the level of competence in simple mathematics that is required in order to follow, let alone participate in, a debate on the safety of asbestos or of nuclear power. Similar skills are required if one is to make any judgement about modern economic theories expounded by politicians. The teachers of mathematics at all levels carry a grave responsibility and deserve every support. All those in a position to influence educational policy in mathematics should urgently consider whether present practices are achieving the desired objectives. On the evidence available to us we are forced to conclude that, for a significant number of school leavers, this is not the case.

Physics Department,
University College, Cork.

BOOK REVIEWS

RIESZ AND FREDHOLM THEORY IN BANACH ALGEBRAS

B.A. Barnes, G.J. Murphy, M.R.F. Smyth and T.T. West
Research notes in Mathematics, PITMAN (82) 117 pp.

A linear operator T on a Banach space X is continuous if and only if it is bounded, in the sense of mapping bounded sets into bounded sets, and if it actually maps bounded sets into totally bounded sets then it is called compact. If K is a compact operator and $0 \neq \lambda \in \mathbb{C}$ then $T = K - \lambda I$ is Fredholm, in the sense that its null space $T^{-1}0$ is finite dimensional and its range TX is closed and of finite dodimension. An operator K for which $T = K - \lambda I$ is Fredholm for each $\lambda \neq 0$ is called Riesz. It turns out that if K is Riesz then it need not be compact, and for each $\lambda \neq 0$ the index of T is zero, in the sense that the two finite dimensions coincide, and indeed that T will be invertible except for at worst a sequence of values of λ tending to 0.

As is familiar this is a circle of ideas of crucial importance to those who would use bounded linear operators to study differential equations: differential operators can never be bounded, but the associated integral equations are built upon operators which are not only bounded but actually compact, and it was among integral equations that Fredholm theory was born. Thus Fredholm theory should be tightly bound to the core of the theory of bounded operators. One perhaps rather perverse way of doing this is to try and "algebraicise" the theory - a sort of "Wilson cloud chamber" approach in which the properties of being compact, Fredholm or Riesz are to be expressed not spatially but in terms of all the other bounded linear operators on the space. If one should succeed in doing that, then there is an overwhelming compulsion to go on and study corresponding elements of more general Banach algebras.

For better or worse, this is the subject of the volume under review. It falls into six chapters, indexed by descriptive letters rather than by arabic numerals. The preliminary Chapter O, OPERATOR THEORY, was intended to be a summary of the basic definitions of compact, Fredholm and Riesz operators and an account of the algebraicisation of the second two properties - compactness is rather more elusive. It has however been augmented by a discussion of a new "enlargement process" for spaces and operators which gives a very sharp description of semi-Fredholm and of Fredholm operators; this technique is applied to obtain a new characterization of Riesz operators and to derive range inclusion theorems for them.

Chapter F, FREDHOLM THEORY, is the core of the book, and sets out to characterize the Fredholm elements of a Banach algebra. If A is a semi-simple Banach algebra then the sum of its minimal left ideals is equal to the sum of its minimal right ideals and forms a two-sided ideal called the socle: then an element x ∈ A will be called Fredholm if and only if it is invertible modulo the socle. If A is not semi-simple then this does not work: in that case first remove the radical and make it semi-simple. To see that this Fredholm theory works we must first restrict attention to primitive algebras and then climb back to a general semisimple algebra. If A is primitive, in the sense of having at least one faithful irreducible representation as operators on a linear space, and if the socle is not zero, then it turns out that a Fredholm element x induces an actual Fredholm operator of left multiplication on the quotient A/J by each minimal left ideal J, and further that the index and the two finite dimensions associated with this operator are independent of the particular ideal J. Thus the usual decomposition and perturbation theory for operators transfers back to elements of primitive algebras. If more generally A is semisimple then the authors show that x ∈ A is Fredholm if and only if the coset x+P is Fredholm for each primitive ideal P. The index and the two finite dimensions are now computed by adding the corresponding quantities for each coset x+P, having first shown that they vanish for all but finitely many P; it is then still possible to transfer decomposition and perturbation theory, and the "punctured neighbourhood theorem" to algebra elements.

Chapter R, RIESZ THEORY, turns to the concept of a Riesz element of an algebra A relative to an ideal K, and shows that for the theory to work it is necessary that the ideal K be closed and have the same "hull-kernel" as the socle, or presocle if the algebra is not semisimple. The authors prove a theorem showing that a left or right ideal lies in the hull-kernel of the socle if and only if 0 is not an accumulation point of the spectrum of any of its elements; if the whole algebra satisfies this condition it is called a Riesz algebra. For example the LCC and the RCC algebras of Kaplansky, and the group algebra of a compact group, are Riesz algebras in this sense.

Chapter C*, C*-ALGEBRAS, starts by picking up a "wedge operator" construction used in Chapter O to characterise Riesz operators and uses it

to define the finite rank elements and the compact elements of a C*-algebra: it then turns out that these just form the socle and its closure. The main burden of this chapter is to reproduce the famous West decomposition, and express a Riesz element of C*-algebra as the sum of a normal compact element and a quasinilpotent. Stampfli's generalization of the West decomposition is obtained on the way; the authors then go on to an improved characterization of Riesz algebras, and a version of the classic Gelfand-Naimark-Siegel construction which also preserves compactness and Fredholmness.

Chapter A, APPLICATIONS, looks at operators leaving a fixed subspace of a Banach space invariant, at triangular and quasitriangular operators, and at measures on a compact group. Chapter BA, BANACH ALGEBRAS, is a summary of the background material on minimal ideals and the socle, on primitive ideals and the hull-kernel topology, and on C*-algebras. There is finally a comprehensive bibliography and a valuable index.

"Riesz and Fredholm theory ... " is an exciting book for specialists, and makes available to a wider audience work of Smyth which has hitherto only been in preprints. Roger Smyth also contributes a unique author address to the title page. It is fair to warn the general reader that this is officially a "research note", and that it bears the signs of having been written by four people. This is a pity, because hidden in an exciting research note there is an even more exciting book. In seeking properly to algebraicise Fredholm and Riesz theory, these authors have started to write a new chapter of the theory of Banach algebras. Perhaps some specialist will become sufficiently irritated as he turns back and forth between Chapter BA, Chapter F and Chapter R to follow the advice offered, by another of these four authors, at the bottom of page 264 of the May 1982 issue of the Bulletin of the London Mathematical Society.

Robin Harte,
Mathematics Department,
University College, Cork.