CONFERENCE ANNOUNCEMENTS

LIMERICK MATHEMATICS 1982.

A two-day Algebra Conference will be held at Mary Immaculate College, Limerick on November 12th and 13th, 1982. The main speakers are Prof. J.G. Thompson (Cambridge), Prof. T.J. Laffey (U.C.D.), Prof. M.L. Newell (U.C.G.) and Dr. D. MacHale (U.C.C.). Persons seeking further details or willing to give short talks of up to 20 minutes duration are invited to contact the organiser Dr. G.M. Enright, Mathematics Department, Mary Immaculate College, South Circular Road, Limerick. Phone (061)44588.

NASECODE III.

The Third International Conference on the Numerical Analysis of Semiconductor Devices and Integrated Circuits will be held in Galway from 15th to 17th June 1983, under the auspices of the Numerical Analysis Group and cosponsored by the Electron Devices Society of the IEEE, the Institute for Numerical Computation and Analysis and the Irish Mathematical Society.

Contributed papers are solicited from engineers, physicists and mathematicians on any topic relevant to the numerical analysis, modelling and optimisation of electronic, opto-electronic and quantum electronic semiconductor devices and integrated circuits. The deadline for the receipt of abstracts and preliminary versions of 20-minute contributed papers is 18th February, 1983. All correspondence should be addressed to: NASECODE Conference, 39 Trinity College, Dublin 2. Phone (01) 772941 ext. 1669/1949.

I.C.M.E. - V

The Fifth International Congress on Mathematics Education will be held in Adelaide, South Australia, in August 1984. The chairman of the International Program Committee is Dr. M.F. Newman, Department of Mathematics, Australian National University, Canberra.

INFINITE EXPONENTIALS

P.J. Rippon

A question of the following type appeared during 1981 in a Regional Math. Contest for high school students in the U.S.A. and caused some difficulties for the referees present.

"Find all the real numbers a such that

$$a^{a^{a^{\cdot^{\cdot^{\cdot}}}}} = 8." \qquad (1)$$

The expected answer, presumably, was that

$$a^{a^{a^{\cdot^{\cdot^{\cdot}}}}} = a^{\left(a^{a^{\cdot^{\cdot^{\cdot}}}}\right)} = a^8 = 8,$$

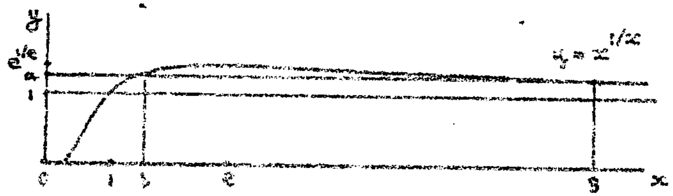
and so $a = 8^{1/8}$. Here (1) has been taken to mean that the sequence $a, a^a, a^{(a^a)}, \ldots$, which we shall denote throughout by

$$a_1 = a, \quad a_{n+1} = a^{a_n}, \qquad (n = 1, 2, \ldots) \qquad (2)$$

converges to 8. Now it is not immediately obvious that the sequence a_n will be convergent when $a = 8^{1/8} \approx 1.3$ and this difficulty occurred to the referees at the Math. Contest. In fact it turns out that the sequence is convergent. Unfortunately however it does not converge to 8 and the question was very nearly scrubbed!

In this article I shall attempt to explain the somewhat surprising facts outlined above and survey a number of the known results about the convergence of such 'infinite exponentials'. Both the real and complex cases will be discussed and some new results given. I am greatly in debt to the survey by Knoebel [4] which appeared coincidentally at about the same time as the Math. Contest and which contains a huge bibliography on this topic. I am also grateful to Leon Greenberg, whose ingenious approach to the convergence of a_n (for real a) first kindled my interest in this problem, and to many colleagues and students at U.C.C. for advice and encouragement, particularly with the computer.

To see why $a = 8^{1/8}$ is not a solution of (1) we consider the graph of $y = x^{1/x}$, $x > 0$.



If b is the unique number less than e satisfying

$$b^{1/b} = a,$$

then $1 < a < b$. Since $a^b = b$ we deduce by induction that the increasing sequence a_n is bounded above by b and so converges to some number c, $1 < c \leq b$. Because $a^c = c$ we must have $c = b$. A little work with a calculator shows that this limit b is approximately 1.46.

It turns out that the behaviour of the sequence a_n for $a > 0$ was studied as long ago as 1777 by Euler [3]. He, it seems, was already aware of the fact (since rediscovered many times) that the sequence is convergent if and only if

$$(1/e)^e \leq a \leq e^{1/e}.$$

The proof of convergence when $1 \leq a \leq e^{1/e}$ uses the argument given above for $8^{1/8}$. On the other hand if a_n converges to b then $a = b^{1/b}$ and so convergence is impossible if $a > e^{1/e}$.

The case $0 < a < 1$ is less straightforward since the sequence a_n is no longer monotonic. In fact

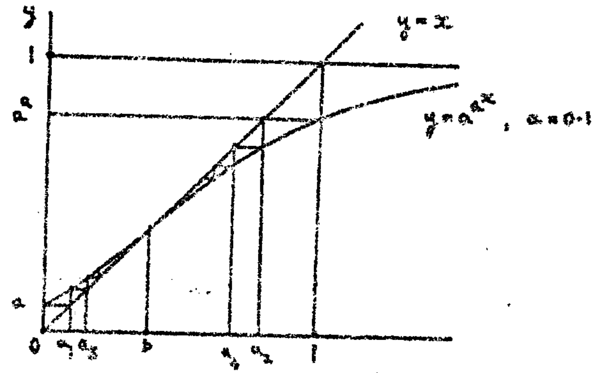
$$0 < a_1 < a_3 < a_5 < \dots < a_6 < a_4 < a_2 < 1,$$

that is, a_{n+1} lies strictly between a_{n-1} and a_n for each $n = 2, 3, \dots$. The proof is by induction, starting from $0 < a < 1$, using repeated exponentiation. It is natural then to study the graph of $y = a^{a^x}$, $x > 0$. As before we let b denote the (unique) solution of $a^x = x$, which will also be a solution of $a^{a^x} = x$. In this case $a < b < 1$.

By elementary calculus,

$$\frac{d}{dx}(a^{a^x}) \leq (-\log a)/e, \qquad (a > 0)$$

with equality only at the single point of inflection of $y = a^{a^x}$, which occurs when $a^x = (-\log a)^{-1}$. If $(1/e)^e \leq a < 1$, therefore, the graph of $y = a^{a^x}$ crosses the graph of $y = x$ exactly once, at $x = b$.

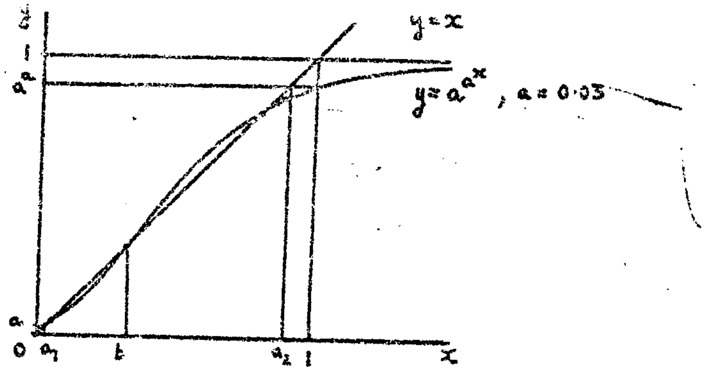


The convergence of the sequence a_n to b is then immediate from the graph.

On the other hand we have

$$\frac{d}{dx}(a^{a^x})\Big|_{x=b} = (\log b)^2,$$

which is strictly greater than 1 if $0 < b < 1/e$, that is if $0 < a < (1/e)^e$. This shows that the graph of $y = a^{a^x}$ crosses the graph of $y = x$ exactly three times if $0 < a < (1/e)^e$.



In this case it is clear from the graph that the sequence a_n does not converge.

The problem becomes much harder if we allow a to assume complex values. In this case we put $a_1 = a$ and

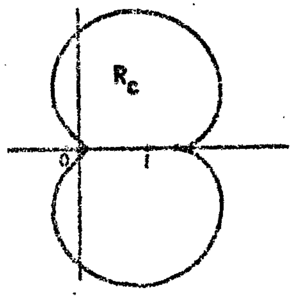
$$a_{n+1} = \exp(a_n \log a) = a^{a_n}, \quad (n = 1, 2, \dots) \qquad (3)$$

where the principal value of log a is taken. This excludes numbers on the negative real axis from consideration, though we note that $a = -1$ is an exceptional special case. Quite a lot is known about the convergence problem for complex a but there are still a number of interesting open questions.

The main problem concerns the set

$$R_C = \{e^{\xi e^{-\xi}} : |\xi| \le 1\},$$

which is illustrated below.



This set meets the real axis precisely in the interval $R_R = [(1/e)^e, e^{1/e}]$ on which the sequence a_n is convergent. Computing evidence suggests that a_n may in fact be convergent throughout R_C and we shall survey a number of partial results towards this.

The set R_C was first identified by Carlsson in his thesis [2] of 1907, where he showed that if $a_n \to w$ as $n \to \infty$ and if $a_n \neq w$, $n = 1, 2, \dots$, then $a \in R_C$. In fact, by (3), we have $w = \exp[w \log a]$ and on putting $\xi = w \log a$ we obtain $w = e^{\xi}$ and $a = e^{\xi e^{-\xi}}$. It only remains to show that $|\xi| \le 1$. To do this we let

$$b_n = a_n w^{-1} - 1, \qquad (n = 1, 2, \dots) \tag{4}$$

so that, by (3),

$$(1 + b_{n+1}) w = \exp[(1 + b_n) w \log a]$$

and hence

$$b_{n+1} = \exp[\xi b_n] - 1. \tag{5}$$

Since $b_n \neq 0$, $n = 1, 2, \dots$, and $b_n \to 0$ as $n \to \infty$ we have

$$\frac{b_{n+1}}{b_n} = \xi + o(1), \quad n \to \infty. \tag{6}$$

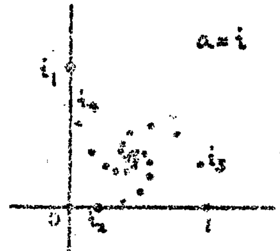
Thus

$$|\xi| = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| \le 1,$$

as required. We also observe from (4) and (6) that if $\arg \xi \neq 0$ then

$$\arg\left(\frac{a_{n+1} - w}{a_n - w}\right) = \arg\left(\frac{b_{n+1}}{b_n}\right) \to \arg \xi, \quad n \to \infty.$$

This shows that the sequence a_n converges to w in a spiral-like manner. As an example we illustrate the convergence of the sequence $i, i^i, i^{(i^i)}, \dots$, which will be proved later.



We shall deduce positive results from the following

Lemma. Let Ω be any domain bounded by a simple closed curve Γ and let $f : \bar{\Omega} \to \bar{\Omega}$ be a continuous function, analytic in Ω, which does not map Ω conformally onto itself. If f has no fixed point on Γ then f has a unique fixed point z_0 in Ω and for any z_1 in Ω the sequence

$$z_{n+1} = f(z_n), \qquad (n = 1, 2, \dots) \tag{7}$$

converges to z_0.

I am grateful to Peter Walker for pointing out how this lemma can be used to prove the convergence of $i, i^i, i^{(i^i)}, \dots$. It also serves to unify and simplify many of the approaches which have been made to the general problem. For other results of this type we refer to Burckel [1].

To prove the lemma first note that f must have at least one fixed point z_0 in Ω by Brouwer's theorem (in any particular application the existence of an interior fixed point can often be demonstrated more directly). We assume, as we may by the Riemann mapping theorem, that Ω is the unit disc and that $z_0 = 0$. By

Schwarz's lemma

$$|f(z)| < |z| , \qquad (0 < |z| < 1) \tag{8}$$

the inequality being strict because f is not a rotation of Ω. Thus $|z_n|$ is decreasing and so

$$r = \lim_{n \to \infty} |z_n|$$

exists. Some subsequence of z_n is convergent, to z say, and by the continuity of f we have

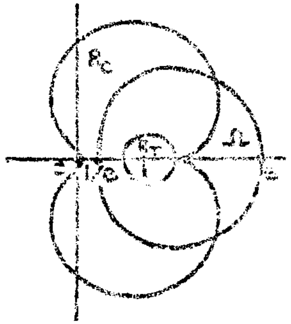
$$|f(z)| = r = |z|.$$

According to (8), therefore, we must have z = 0 and so $z_n \to 0$, as required.

As a first application we show that

$$R_T = \{ a : |\log a| \le e^{-1} \}$$

is a set of convergence for a_n. This is due to Thron [7].



Letting

$$\Omega = \{ z : |\log z| < 1 \} ,$$

we find that

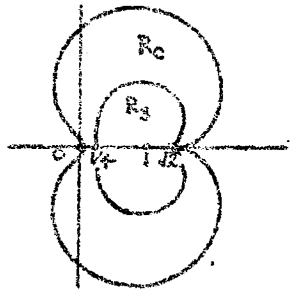
$$|\log(a^z)| = |z \log a| \le 1, \quad (z \in \bar{\Omega}, a \in R_T)$$

since $|z| \le e$ when $z \in \bar{\Omega}$. The inequalities here are strict, except when z = e,

and so a^z maps Ω properly into itself. The only possible fixed point of a^z on $\partial\Omega$ is z = e and this can only occur when $|\log a| = e^{-1}$. This would mean that $a = e^{1/e}$ and we already know that a_n is convergent for this value of a. For any other a in R_T the lemma gives the desired convergence of 1, a, a^a, ...

Another major contribution towards the solution of this problem was made by Shell in his thesis of 1959, part of which was published in [6]. We shall prove here one result of his, that a_n is convergent when a lies in

$$R_S = \{ e^{\xi e^{-\xi}} : |\xi| \le \log 2 \}.$$



For any $a = e^{\xi e^{-\xi}}$, $|\xi| \le \log 2$, consider the disc

$$\Omega_\xi = \{ z : |z - e^\xi| < |e^\xi| \}.$$

If $z \in \bar{\Omega}_\xi$ then $|ze^{-\xi} - 1| \le 1$ and so (†)

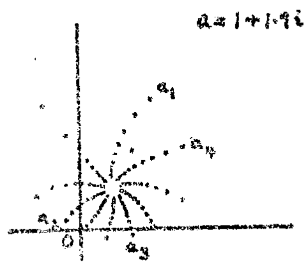
$$\begin{aligned} |a^z - e^\xi| &= |\exp(z \log a) - e^\xi| \\ &= |e^\xi| \; |\exp[\xi(ze^{-\xi} - 1)] - 1| \\ &\le |e^\xi| (\exp[|\xi| \; |ze^{-\xi} - 1|] - 1) \\ &\le |e^\xi| . \end{aligned} \tag{9}$$

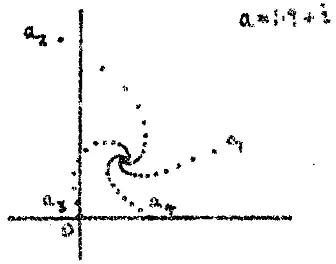
It follows that a^z maps $\bar{\Omega}_\xi$ into itself and evidently has $z = e^\xi$ as a fixed point. This mapping is not a rotation (for instance, its derivative at e^ξ is equal to ξ) and it is easy to check that $z_1 = 1$ lies in Ω_ξ except when $\xi = -\log 2$ (corresponding to $a = \sqrt[4]{}$, where the convergence of a_n is already known). According to the proof of the lemma the sequence 1, a, a^a, ... converges to e^ξ, as required.

(†) The inequality $|e^z - 1| \le e^{|z|} - 1$, which we use here, follows immediately from the Taylor series for e^z.

A more careful study of (9) reveals [6] that if $s = |\xi| < 1$ then there is a number $\rho(s) > 0$ (which depends continuously on s and tends to zero as s tends to 1) such that a^z maps the disc Δ_ξ, with centre e^ξ and radius $\rho(s)$, into itself. Thus if $a_n \in \Delta_\xi$ for any n we must have $a_n \to e^\xi$ as $n \to \infty$. By continuity we deduce that the set of interior points of R_C for which the sequence a_n is convergent forms an open set.

By now the reader will appreciate that the approach being used is to find a 'domain of invariance' for the mapping a^z which is large enough to include the poi[nt] a itself. To give some idea of the types of domains of invariance which might be en[countered] we illustrate the sequence a_n in two particular cases where its convergen[ce] has not been established.





An equivalent way to look at this problem is to make the substitution used earlier in (4), namely,

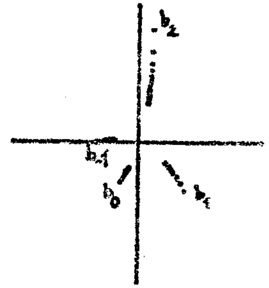
$$b_n = a_n e^{-\xi} - 1, \qquad (n = 1,2, \ldots)$$

where $a = e^{\xi e^{-\xi}}$. We then have $b_1 = ae^{-\xi} - 1 = \exp[\xi(e^{-\xi} - 1)] - 1$, and

$$b_{n+1} = \exp\{\xi b_n\} - 1, \qquad (n = 1,2, \ldots)$$

and it is natural to also put $b_0 = e^{-\xi} - 1$ and even $b_{-1} = -1$. The problem then is to show that $b_n \to 0$ whenever $|\xi| \le 1$. Shell's argument showed that this was true when $|\xi| \le \log 2$ by observing that the unit disc is invariant under the mapping $e^{\xi z} - 1$, for such ξ.

To provide some food for thought we illustrate the spiral-like behaviour of the sequence b_n when $\xi = i$ (which corresponds to $a \approx 1.99 + 1.19i$).



$b_{-1} = -1$,
$b_{n+1} = e^{ib_n} - 1$, $n = -1, 0, 1, \ldots$

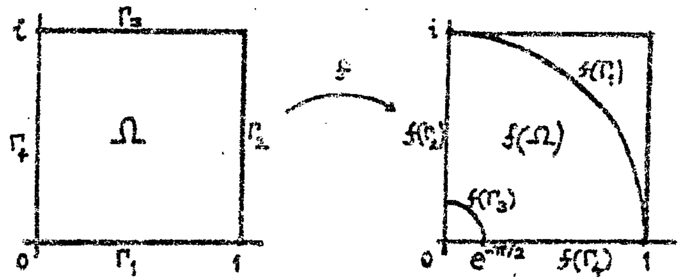
It seems clear that this sequence converges to 0, albeit slowly. For instance, calculations on a computer indicate that when $n = 200{,}000$ the modulus of b_n is about 0.1. In this case however there would be no hope of finding a suitable domain of invariance Ω for $f(z) = e^{iz} - 1$ since $|f'(0)| = 1$, which would force f to map Ω conformally onto itself.

Neither of the positive results proved earlier covers the case $a = i$, corresponding to $\xi \approx -0.6+0.7i$, which is naturally of particular interest. Apparently a proof of the convergence when $a = i$ was given in Shell's thesis, but this does not appear in his subsequent paper. However the following stunningly simple proof was published by Macintyre [5] in 1966. Let

$$\Omega = \{x+iy : 0 < x,y < 1\}.$$

Then $f(z) = i^z$ maps Ω properly into itself, since

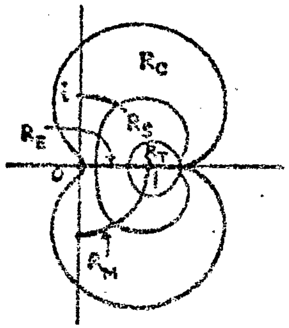
$$i^z = e^{iz\pi/2} = e^{-y\pi/2} e^{ix\pi/2}.$$



By inspection f has no fixed points on $\partial\Omega$ and so we can apply the lemma with $z_1 = i_3 = \exp[i\frac{\pi}{2}e^{-\pi/2}]$, which lies in Ω. The same argument shows that the sequence a_n converges whenever a is of the form $e^{i\theta}$, $0 \le \theta \le \pi/2$. By conjugacy, therefore, we have convergence throughout

$$R_M = \{e^{i\theta} : |\theta| \le \pi/2\}.$$

To get an impression of the results so far we illustrate the sets R_C, R_E, R_T, R_S and R_M together and remind the reader that the set of convergence is open in the interior of R_C.



It is interesting that there is no relationship of containment amongst R_E, R_T, R_S and R_M.

We finish with an intriguing fact that does not seem to have been mentioned in the literature. The sequence a_n is in fact convergent for many numbers lying outside R_C. An obvious example is $a = -1$ but there are also, for instance, unbounded sequences of numbers a in the first and fourth quadrants each having the property that $a^a = a$. For such numbers we clearly have $a_1 = a_2 = a_3 = \dots$.

To demonstrate their existence we show that, for each positive integer k, there is a number z lying in

$$S = \{x+iy : x > 0,\ 0 < y < \pi/2\},$$

such that

$$z(e^z - 1) = 2\pi ki. \qquad (10)$$

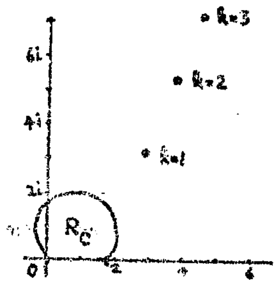
In that case $a = e^z$ lies in the first quadrant and

$$a^a = \exp[a \log a] = \exp[ze^z] = \exp[z + 2\pi ki] = a.$$

Moreover the solutions z of (10) will be unbounded with k. One way to show that these solutions exist is to rewrite (10) as

$$z = \log\left(1 + \frac{2\pi ki}{z}\right).$$

The function on the right-hand side of this equation maps the first quadrant into S and has no fixed points on the boundary. Apart from proving that (10) has a unique solution z in S for each k, the lemma allows us to compute approximate values for these solutions and for the corresponding numbers a. The first few of these are illustrated below and it is clear that they all lie outside R_C, a fact which can also be proved analytically.



Further work along these lines (details of which will appear elsewhere) leads to the existence of other numbers such that $a_2 = a_3$, $a_3 = a_4$, ... We give some examples, which the reader is invited to check-preferably with the help of Fortran IV! Due allowance should be made for round-off errors.

$a_1 = a_2 = \dots$	$a_2 = a_3 = \dots$	$a_3 = a_4 = \dots$	$a_4 = a_5 = \dots$
$2.8629 + 3.2233i$	$2.4293 + 0.55465i$	$1.9813 + 0.16031i$	$1.78285 + 0.082166i$
$3.7273 + 5.3180i$	$2.6921 + 0.58785i$	$2.0599 + 0.16702i$	$1.81660 + 0.071851i$
$4.4332 + 7.1938i$	$2.8513 + 0.60067i$	$2.1024 + 0.15843i$	$1.83403 + 0.066124i$

Finally I should be grateful if someone would disprove (or, better still, prove) the following (rather wild) conjecture which has been bothering me for some time now. Could it be that the set of numbers a such that the sequence a_n converges is actually dense in the complex plane?

References

[1] R.B. Burckel, Iterating analytic self-maps of discs, Amer. Math. Monthly, 88(1981), 396-406.

[2] A. Carlsson, Om Iterader Funktioner, Ph.D., Uppsala (1907).

[3] L. Euler, De formulis exponentialibus replicatus, Opera Omnia, Series Primus XV, 268-297; Acta Academiae Petropolitanae, 1 (1777), 38-60.

[4] R.A. Knoebel, Exponentials reiterated, Amer. Math. Monthly, 88(1981), 235-252.

[5] A.J. Macintyre, Convergence of $i^{i^{\cdots}}$, Proc. Amer. Math. Soc., 17(1966), 67.

[6] D.L. Shell, On the convergence of infinite exponentials, Proc. Amer. Math. Soc., 13(1962), 678-681.

[7] W.J. Thron, Convergence of infinite exponentials with complex elements, Proc. Amer. Math. Soc., 8(1957), 1040-1043.

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Majorization and Schur Functions

Philip J. Boland
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The concepts of majorization and Schur functions lay the basis for a rich and elegant theory in which many classical and applicable inequalities may be viewed. In this expository paper the basic definitions and properties of majorization and Schur functions are presented, together with a variety of applications emphasizing in particular some in reliability theory. For a thorough and recent account of majorization and Schur functions, the interested reader should consult the excellent Inequalities: Theory of Majorization and its Applications by Marshall and Olkin (1979).

1. Majorization

Given a vector $\underline{x} = (x_1,\ldots,x_n) \in R^n$, let $x_{[1]} \geq \ldots \geq x_{[n]}$ denote a decreasing rearrangement of $x_1,\ldots,x_n$.

Definition 1.1 If $\underline{x},\underline{y} \in R^n$, then $\underline{x} \prec \underline{y}$ if

$$\sum_1^j x_{[i]} \leq \sum_1^j y_{[i]} \quad \text{for } j = 1,\ldots,n-1$$

and

$$\sum_1^n x_{[i]} = \sum_1^n y_{[i]}.$$

If $\underline{x} \prec \underline{y}$, we say that $\underline{x}$ is majorized by $\underline{y}$. Note that if $\underline{x} \prec \underline{y}$, then the components of $\underline{y}$ are more "spread out" than those of $\underline{x}$. For example $(\frac{1}{n},\ldots,\frac{1}{n},\frac{1}{n}) \prec (\frac{1}{n-1},\ldots,\frac{1}{n-1},0) \prec (1,0,\ldots,0)$, and $(\bar{x},\bar{x},\ldots,\bar{x}) \prec (x_1,\ldots,x_n)$ where $\bar{x} = \Sigma x_i/n$.