DINEEN ON HOLOMORPHY - A REVIEW

COMPLEX ANALYSIS IN LOCALLY CONVEX SPACES, by Sean Dineen, North-Holland, 1981, 492 pp.

"The main purpose of this book ..... was to provide an introduction to modern infinite dimensional complex analysis, or infinite dimensional holomorphy as it is commonly called, for the graduate student and research mathematician. Since we were more interested in communicating the nature rather than the scope of infinite dimensional complex analysis ..... we chose to develop a single theme which has made much progress in recent years and which exemplifies the intrinsic nature of the subject, namely the study of locally convex topologies on spaces of holomorphic functions in infinitely many variables." Thus begins the author's foreword. There follows a comprehensive introduction to infinite dimensional holomorphy from the topological viewpoint, complete with exercises for the reader, a helpful historical commentary and an extensive bibliography.

Infinite dimensional holomorphy can trace its origins at least as far back as Hilbert, but in the last 18 years a great explosion of research has taken place, and most of the material of the book has come from this period. The unifying theme is the problem of how best to topologise the space of holomorphic functions. Consider the case of one complex variable. $H(U)$ will denote the set of holomorphic functions on the open subset U of the complex plane. Having formed this set, one's immediate instinct is to equip it with some structures. For example, $H(U)$ is a complex vector space. To see how naturally the question of topology arises, consider the convergence of the Taylor series.

Suppose for simplicity that U is a disc with centre a, so that for every f $H(U)$, the Taylor series at a converges to f at every point of U. Let s_n be the n-th partial sum of this series. In what precise sense does the sequence s_n converge to f in the space $H(U)$? If U is not a disc, the Taylor series at one point will not, in general, represent f throughout U. However, Runge's Theorem tells us that it may be possible to approximate f by polynomials, or rational functions; in other words, these functions form a dense subset of $H(U)$ for a certain topology. The "right" topology in this case is the compact open topology, τ_o - a sequence f_n in $(H(U),\tau_o)$ converges to a function f if $f_n(z)$ converges to $f(z)$ uniformly on each compact subset of U. This topology arises naturally in many settings, and has many useful properties; for example, it is compatible with the vector space structure of $H(U)$, it is metrizable and is complete. Thus $(H(U),\tau_o)$ is a Fréchet space. On a deeper level, $(H(U),\tau_o)$ is also Nuclear. These properties open the doors of an armoury of weapons from Functional Analysis which are essential elements in the proofs of many of the classical theorems of complex analysis in one and several variables.

If U is now a domain in an infinite dimensional space, the situation becomes much more complicated. τ_o is defined on $H(U)$ in the same way, but in many important cases, one finds that those properties which made it so useful in finite dimensions, such as metrizability and nuclearity, no longer apply. There appears a galaxy of different topologies on $H(U)$, each with its own justification, sometimes agreeing with one another, more often not. The exploration of these topologies has been central to the development of infinite dimensional holomorphy in recent years, and very many of the great advances which have been made bear Dineen's name.

This book is not simply an account of the topology of H(U), rather is it a comprehensive introduction to infinite dimensional holomorphy, the inspiration for the development coming from these fundamental topological problems. The prerequisites for the reader would include, of course, complex variables, but, more importantly, a reasonable knowledge of functional analysis, including the elements of locally convex spaces. A useful appendix provides a summary of definitions and results from several complex variables and functional analysis.

Chapter 1, Polynomials On Locally Convex Topological Vector Spaces, introduces the building blocks of the Taylor series, the homogeneous polynomials. Several types of polynomials, such as continuous, hypocontinuous and nuclear, are met, and various topologies on the spaces of polynomials are studied. The duality theory of polynomials is here, together with the special features of polynomials on nuclear spaces.

Chapter 2, Holomorphic Mappings Between Locally Convex Spaces, introduces the reader to holomorphic mappings on open sets, and holomorphic germs on compact sets, and their elementary properties. The three most important topologies on H(U), τ_o, τ_ω and τ_δ are introduced.

Chapter 3, Holomorphic Functions On Balanced Sets : The balanced set in infinite dimensions replaces the disc in the complex plane - it has the crucial property that the Taylor series at the centre of the set represents the function throughout the set. Thus H(U) is, in some sense, the direct sum of the subspaces of homogeneous polynomials. One can then hope that topological properties of the spaces of polynomials can be pieced together to give results about H(U). This

idea is exploited here, the main tools being Schauder decompositions and associated topologies.

Chapter 4, Holomorphic Functions On Banach Spaces, and Chapter 5, Holomorphic Functions On Nuclear Spaces With A Basis continue the study of holomorphic functions on two contrasting types of domains. There are no infinite dimensional spaces which are at the same time Banach and nuclear, and the theories for these two types of spaces develop in different ways. For Banach spaces, the emphasis is on the interplay between the geometry of the space and the holomorphic functions. The Maximum Modulus Theorem, Schwarz's Lemma and their applications are here, together with bounding sets, and the equality of the topologies τ_ω and τ_δ on Banach spaces with unconditional bases. In nuclear spaces with bases, we have first a coordinate system, and we find that suitable nuclearity conditions on the space allow us to write the Taylor series using monomials, which are simply products of the coordinates. Again, using the basis, one can construct polydiscs and Reinhardt domains. This leads to a very satisfactory duality theory, and the resolution of many of the topological problems.

Chapter 6, Germs, Surjective Limits, ε-Products and Power Series Spaces. The chapter opens with a further study of spaces of holomorphic germs on compact sets, and their relationship to the study of H(U). Surjective limits provide a method for constructing, by a projective process, spaces with good holomorphic properties. The ε-product, which can be viewed as a generalised tensor product, relates the theory of vector-valued functions to that of scalar values. The chapter closes with some recent results on representations of spaces of holomorphic functions on certain sequence spaces.

Each chapter is accompanied by a set of exercises. Some of these are easy, some challenging, and some, in the author's own words, "quite difficult". They should at least be read, as many indicate further areas of research, and introduce topics not covered in the text. Appendix III, Notes on Some Exercises, has hints and explanations, and references to the literature for the interested problem-solver. Appendix II, already mentioned, consists of definitions and results from functional analysis, complex variables and topology. Appendix I, Further Developments in Infinite Dimensional Holomorphy, is a survey of current research which emphasises areas not treated in the book.

At the end of each chapter is a section entitled Notes and Remarks, comprising a fascinating historical account of the subject matter of the chapter, together with many illuminating insights, suggestions for further research, and a guide to the literature. The Bibliography is enormous, containing some 725 entries in all, ranging from papers by Volterra and Von Koch in the 1880's up to the present. This is the first complete listing of papers in holomorphy and will be of great value to workers in the field, and indeed, to interested spectators.

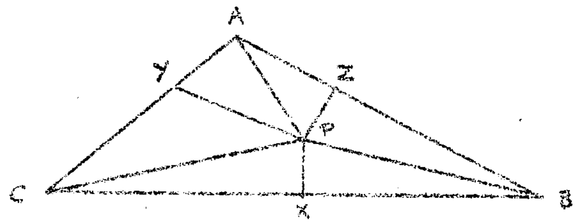
It is the reviewer's opinion that this book is a major contribution to infinite dimensional holomorphy. It succeeds admirably in its stated aims, and while giving a complete account of the theory from the topological viewpoint, is in no way closed or static - one is always led on to think of the next step, the right generalisation, the open problem. This will surely be the holomorphist's bible for many years to come. May it gain many converts!

R. Ryan

Problems Page

New or old, solved or unsolved, published or unpublished, this page will discuss any problem which has that certain something. Please send problems, solutions and references to the Editor.

1. Let P be an arbitrary point in a scalene triangle ABC, and let PX, PY, PZ be the internal bisectors of ∢BPC, ∢CPA, ∢APB respectively.



Prove that

$$|PA| + |PB| + |PC| \geq 2(|PX| + |PY| + |PZ|).$$

As far as I know this is Barrow's inequality, but I have no reference. The weaker inequality, in which PX, PY, PZ are perpendicular to BC, CA, AB respectively, is due to Erdős.

2. This problem came from Tom Laffey. Is

$$\inf \{|n \sin n| : n \in \mathbb{N}\} = 0?$$

Finbarr Holland, working with his ZX81, found the approximation $\pi \approx 355/113$ which gives $355 \sin 355 \approx 0.0107$. Later Jo Manning checked (on the IBM) that this is the smallest value of $|n \sin n|$ for $1 \leq n \leq 10^6$. The problem would be answered in the affirmative if π were approximable by rationals to order $2 + \varepsilon, \varepsilon > 0$, and so presumably this is also open. Bill Bruce found an article by Chudnovsky (Springer Lecture Notes in Math, 751) which gives negative results on the approximation of π by rationals.