A note on a class of Fourier transforms

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Abstract. We consider functions \( f \in L^2(\mathbb{R}^n) \) for which
\[
\int_{\mathbb{R}^n} |\hat{f}(t)|^2 (1 + \log^+ |t|)^{2\beta} dt < \infty, \quad \beta > 0,
\]
where \( \hat{f} \) is the Fourier transform of \( f \), and we identify a kernel \( K_\beta \) such that \( f \) satisfies this integral condition if, and only if,
\[
f(x) = (K_\beta \ast F)(x) = \int_{\mathbb{R}^n} K_\beta(x-t) F(t) dt
\]
for some function \( F \in L^2(\mathbb{R}^n) \). We also address the question of ‘Fourier inversion’ for this class by showing that certain Bochner-Riesz means of the transforms of \( f = K_\beta \ast F \) converge to \( f \) outside small exceptional sets of points in \( \mathbb{R}^n \) of capacity zero.

1. Introduction

It was conjectured by Lusin in 1915 that the Fourier series of a periodic function \( f \in L^2(-\pi, \pi) \) converges almost everywhere, that is, if \( c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx, k \in \mathbb{Z} \), denote the Fourier coefficients of \( f \), then the partial sums
\[
s_n(f)(x) = \sum_{k=-n}^{k=n} c_k e^{ikx}
\]
converge almost everywhere to \( f(x) \) as \( n \to \infty \). The conjecture remained unproven for several decades and, as doubts began to arise regarding its veracity, some research was directed towards constructing a counterexample. It came as a major surprise therefore when, in a famous and very difficult paper [2], Lennart Carleson proved Lusin’s conjecture in 1966. This result was widely celebrated within mathematics and particularly, perhaps, by those analysts who (like this author) had been nurtured mathematically on Zygmund’s Trigonometric Series! Carleson’s result was extended to \( L^p \) functions, \( p > 1 \), by Hunt [6].

We are concerned with Fourier transforms, and the question that arises in this context is whether Carleson’s result has an analogue in \( \mathbb{R}^n \), specifically whether for a function \( f \in L^2(\mathbb{R}^n) \), the spherical partial integral
\[
S_R f(x) = \int_{|t| \leq R} \hat{f}(t) \exp(2\pi i x \cdot t) dt, \quad R > 0, \quad x \in \mathbb{R}^n, \quad n \geq 2,
\]
converges almost everywhere to \( f(x) \) in \( \mathbb{R}^n \) as \( R \to \infty \), where \( \hat{f} \) is the Fourier transform of \( f \). This question remains open, but by analogy with partial results established for Fourier series prior to Carleson’s paper (see [14, 1.13, p. 163]), it is natural to begin
seeking answers by investigating functions \( f \) which satisfy conditions such as
\[
\int_{\mathbf{R}^n} |\hat{f}(t)|^2 (1 + \log^+ |t|)^{2\beta} \, dt < \infty, \quad \beta > 0, \tag{2}
\]
a stronger requirement than \( f \in L^2(\mathbf{R}^n) \). We note that it has been shown by Carberry and Soria [4] (see also [5]) that if \( f \) satisfies (2) with \( \beta = 1 \) then \( S_R(f) \to f \), as \( R \to \infty \), almost everywhere. We focus in this article on providing a characterisation of functions \( f \) for which (2) holds and, to this end, we define a kernel \( \mathcal{K}_\beta \) which, as we shall prove in section 2, has the property that \( f \) satisfies (2) if, and only if, \( f = \mathcal{K}_\beta \ast F \) for some \( F \in L^2(\mathbf{R}^n) \). A kernel \( K \) is a non-negative, unbounded, and integrable function on \( \mathbf{R}^n \) which is radially symmetric and decreasing, i.e. \( K(x) = K(t) \) if \( |x| = |t| \) and \( K(x) \leq K(t) \) if \( |x| \geq |t| \). We write \( L^2_K(\mathbf{R}^n) \) to denote the class of potentials
\[
(K \ast F)(x) = \int_{\mathbf{R}^n} K(x - t) F(t) \, dt,
\]
where \( K \) is a kernel on \( \mathbf{R}^n \) and \( F \in L^2(\mathbf{R}^n) \), with \( n \geq 2 \). (From here on we shall write \( L^2_K \) for \( L^2_K(\mathbf{R}^n) \), and \( L^2 \) for \( L^2(\mathbf{R}^n) \).) We note [11, p.3] that if \( f \in L^2_K \) then \( f \in L^2 \) and hence has a Fourier transform \( \hat{f} \in L^2 \) by the Plancherel theorem. It follows that \( \hat{f} \) is integrable in \( \{x : |x| \leq R\} \) for every fixed \( R > 0 \), and the integral for the mean \( \overline{S}_Rf \) in (1), and the mean \( \overline{T}_R^\lambda f \) in (3) below, are thus well-defined for \( f \in L^2_K \).

An important alternative summability method to the one based on the mean \( S_Rf \) is Bochner-Riesz summability ([11, pp.170-172], [8], [13]) with
\[
T^\lambda_R f(x) = \int_{|t| \leq R} (1 - \frac{|t|^2}{R^2})^\lambda \hat{f}(t) \exp(2\pi ix \cdot t) \, dt, \quad \lambda > 0, \tag{3}
\]
a more amenable mean than the spherical partial integral. In section 3, using the characterisation \( f = \mathcal{K}_\beta \ast F \), we derive a result on the convergence of \( T^\lambda_R f \) means, outside sets of capacity zero, for functions satisfying (2).

2. THE MAIN THEOREM

We begin with the definition of the kernel \( \mathcal{K}_\beta \). We set
\[
\mathcal{K}_\beta(x) = \int_0^1 \frac{P_s(x)}{s (\log \frac{2}{s})^{\beta+1}} \, ds, \quad x \in \mathbf{R}^n, \quad \beta > 0,
\]
where, for \( n \geq 1 \) and \( s > 0 \),
\[
P_s(x) = \frac{\lambda_n s}{(s^2 + |x|^2)^{(n+1)/2}}, \quad \lambda_n = \Gamma(\frac{n+1}{2})/\pi^{(n+1)/2},
\]
is the Poisson kernel for \( \mathbf{R}^{n+1}_+ \). Since \( \int_{\mathbf{R}^n} P_s(x) \, dx = 1 \) for each \( s > 0 \) [11, p. 9], and \( P_s \) is radially symmetric and decreasing, it follows that \( \mathcal{K}_\beta \) is a kernel.

To prepare for our theorem we present three lemmas, the first two of which provide estimates for \( \mathcal{K}_\beta \) and \( \hat{\mathcal{K}}_\beta \), and the third establishes an equivalence relation for the classes \( L^2_K \) which is central to the proof of the theorem. We will not use Lemma 1 in the proof but, as it answers obvious questions, we include the lemma for the sake of completeness.

**Lemma 2.1.** We have
\[
\frac{c_\beta}{|x|^n (\log \frac{2}{|x|})^{\beta+1}} \leq \mathcal{K}_\beta(x) \leq \frac{c'_\beta}{|x|^n (\log \frac{2}{|x|})^{\beta+1}}, \quad 0 < |x| \leq 1. \tag{1}
\]
We also have \( \mathcal{K}_\beta(x) \leq c_\beta |x|^{-(n+1)} \) for \( |x| > 1 \).
Lemma 2.2. If $2 \mid \cdots \geq |r_1|$, this gives the lefthand inequality in (1). To obtain the second inequality we note that, in Lemmas 2.1 and 2.2, $r_1$ proof. Since $r \geq 1/2$ implies $2/r|x| \leq 4/|x|^2$,

$$I(x) \geq \int_{|x|/2}^{1/|x|} \varphi_x(r)dr \geq 2^{-\gamma} \left( \log \frac{2}{|x|} \right)^{-\gamma} \int_{1/2}^{1} \frac{dr}{(1 + r)^{n+1}} \geq 2^{-(\gamma + n + 2)} \left( \log \frac{2}{|x|} \right)^{-\gamma}.$$ 

This gives the lefthand inequality in (1). To obtain the second inequality we note that $1/|x|^{1/2} \leq 1/|x|$ when $0 < |x| \leq 1$, and write $I(x) = I_1(x) + I_2(x)$, where in $I_1$ we integrate over $(0, 1/|x|^{1/2})$ and in $I_2$ over $(1/|x|^{1/2}, 1/|x|)$. Since $r \leq 1/|x|^{1/2}$ implies $2/r|x| \geq (2/|x|)^{1/2}$,

$$I_1(x) = \int_{0}^{1/|x|^{1/2}} \varphi_x(r)dr \leq 2^{\gamma} \left( \log \frac{2}{|x|} \right)^{-\gamma} \int_{0}^{\infty} \frac{dr}{(1 + r)^{n+1}} \leq 2^{\gamma} \left( \log \frac{2}{|x|} \right)^{-\gamma}.$$ 

We have $r|x| \leq 1$ in $I_2(x)$, so

$$I_2(x) \leq (\log 2)^{-\gamma} \int_{1/|x|^{1/2}}^{1/|x|} \frac{dr}{2^{\gamma}} \leq (\log 2)^{-\gamma} |x|^{1/2} \leq c_\beta \left( \log \frac{2}{|x|} \right)^{-\gamma}$$

since $|x| \leq 1$ implies $(\log 2 |x|)^{\gamma} \leq c_\beta (2/|x|)^{1/2}$, for a big enough constant $c_\beta$. Inequality (1) follows. The estimate for $|x| > 1$ is easily obtained and the Lemma is proved.

Lemma 2.2. If $K_\beta$ is the kernel defined as above for $\beta > 0$, then the Fourier transform $\hat{K}_\beta(x) = \int_{\mathbb{R}^n} K_\beta(t) \exp(-2\pi i x \cdot t) dt$ satisfies

$$\frac{c_\beta}{(1 + \log^+ |x|)^\beta} \leq \hat{K}_\beta(x) \leq \frac{c_\beta'}{(1 + \log^+ |x|)^\beta}, \quad x \in \mathbb{R}^n. \quad (2)$$

Proof of Lemma 2.2. We note to begin with, since $\tilde{P}_s(x) = \exp(-2\pi s |x|)$ [11, p. 5], that, by an interchange of integrals,

$$\hat{K}_\beta(x) = \int_{\mathbb{R}^n} \left( \int_{0}^{1} \frac{P_s(t)}{s (\log \frac{2}{s})^{\beta + 1}} ds \right) \exp(-2\pi i x \cdot t) dt$$

$$= \int_{0}^{1} \frac{\exp(-2\pi s |x|) ds}{s (\log \frac{2}{s})^{\beta + 1}} = \int_{0}^{1/|x|} \frac{\exp(-2\pi r) dr}{r (\log \frac{2|x|}{r})^{\beta + 1}}. \quad (3)$$

Assume that $|x| > 1$. Then

$$\hat{K}_\beta(x) \geq \int_{1/4|x|^2}^{1/2|x|} \frac{\exp(-2\pi r) dr}{r (\log \frac{2|x|}{r})^{\beta + 1}}$$

$$\geq e^{-2\pi} 3^{-\beta - 1} (\log 2|x|)^{-\beta - 1} \int_{1/4|x|^2}^{1/2|x|} \frac{1}{r} dr$$

$$= e^{-2\pi} 3^{-\beta - 1} (\log 2|x|)^{-\beta} \geq c_\beta (1 + \log |x|)^{-\beta} = c_\beta (1 + \log^+ |x|)^{-\beta}. \quad (4)$$
This gives the lower bound in (2). To obtain the upper bound for $|x| > 1$, we note from the second equality in (3) that

$$\beta \hat{K}_\beta(x) = \int_0^1 e^{-2\pi s|x|} d\left(\log \frac{2}{s}\right)^{-\beta} = e^{-2\pi|x| (\log 2)^{-\beta}} + 2\pi|x| J(x),$$

where

$$J(x) = \int_0^1 e^{-2\pi s|x|} (\log \frac{2}{s})^{-\beta} ds.$$

We write $J$ as $J_1 + J_2$ where $J_1 = \int_0^{1/|x|^{1/2}}$, $J_2 = \int_{1/|x|^{1/2}}^1$. Then

$$J_1(x) \leq (\log 2|x|^{1/2})^{-\beta} \int_0^{1/|x|^{1/2}} e^{-2\pi s|x|} ds = \frac{2^{-\beta}(\log 4|x|)^{-\beta}}{2\pi|x|} \int_0^{2\pi|x|^{1/2}} e^{-u} du \leq c_\beta |x|^{-1}(1 + \log^+ |x|)^{-\beta},$$

since $|x| > 1$. Next, by a similar argument,

$$J_2(x) = \int_{1/|x|^{1/2}}^1 \leq (\log 2)^{-\beta} \frac{1}{2\pi|x|} \int_0^{2\pi|x|} e^{-u} du \leq c_\beta (1 + \log^+ |x|)^{-\beta}/|x|,$$

choosing $c_\beta$ large enough. Since $J = J_1 + J_2$, and $e^{-2\pi|x|} \leq c_\beta (1 + \log^+ |x|)^{-\beta}, |x| \geq 1$, the required upper bound follows from (4) when $|x| > 1$. The proof of the inequalities (2) for the case $|x| \leq 1$, i.e. that $c_\beta \leq \hat{K}_\beta(x) \leq c_\beta'$, is easy and is omitted. The proof of Lemma 2.2 is complete.

**Remark** If we apply the result $(\exp(-2\pi s|x|)) = P_s(x)$ [11, p. 6] to the middle integral in (3) we see that $(\hat{K}_\beta)^\wedge = K_\beta$.

**Lemma 2.3.** Let $K$ be a kernel with $\hat{K} > 0$ and suppose that $f \in L^2$. Then

$$\int_{\mathbb{R}^n} \hat{K}(t)^{-2} |\hat{f}(t)|^2 dt < \infty \iff f \in L^2_{\hat{K}}.$$

**Proof.** Note first, by $L^2$ transform theory, that if $g \in L^2$ then the transform $\hat{g} \in L^2$ and $\hat{(g)^\wedge}(-t) = g(t)$.

Assume that $\int_{\mathbb{R}^n} \hat{K}(t)^{-2} |\hat{f}(t)|^2 dt < \infty$ and set $F(t) = \hat{K}(t)^{-1} \hat{f}(t)$, so that $F \in L^2$. Then

$$\hat{f}(t) = \hat{K}(t) F(t) = \hat{K}(t) (\hat{F})^\wedge(-t) = \hat{K}(t) \hat{H}(t) = (K * H)^\wedge(t)$$

where $H(t) = \hat{F}(-t)$, so $H \in L^2$, and we have applied the multiplication formula from [7, theorem 5.8], with $p = 1$ and $q = 2$, for the last equality. Hence

$$(\hat{f})^\wedge = ((K * H)^\wedge)^\wedge, \quad i.e. \quad f(-t) = (K * H)(-t),$$

or $f(t) = (K * H)(t)$, $t \in \mathbb{R}^n$. This proves the first part of the Lemma.

For the second part assume that $f \in L^2_{\hat{K}}$ so that $f = K * Q$ where $Q \in L^2$. Then $\hat{f} = \hat{K} \hat{Q}$ and $\hat{Q}(t) = \hat{K}(t)^{-1} \hat{f}(t)$. Since $\hat{Q} \in L^2$, the converse implication follows and the proof is complete.

Our main theorem now follows immediately from Lemmas 2.2 and 2.3 (with $K = K_\beta$).

**Theorem 2.4.** If $f \in L^2$ then

$$\int_{\mathbb{R}^n} |\hat{f}(t)|^2 (1 + \log^+ |t|)^{2\beta} dt < \infty, \quad \beta > 0,$$

if, and only if, $f \in L^2_{K_\beta}$. 

3. BOchner-Riesz summability

We obtain our final theorem on the convergence of certain Bochner-Riesz means in $L^2_K$ by simply combining two known results. We begin by noting from [11, Corollary 4.16 (b), p.172] that

if $f \in L^2$ has a Lebesgue point at $x \in \mathbb{R}^n$, that is, if

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t) - f(x)| dt = 0,$$

where $m(B(x,r))$ denotes the Lebesgue measure of the ball $B(x,r)$, then the Bochner-Riesz mean $T^R \beta f(x)$ in (1.3) converges to $f(x)$ for all $\lambda > (n-1)/2$ as $R \to \infty$.

Since $L^2_K \subset L^2$, as noted above, and $L^2$ functions have Lebesgue points almost everywhere, it follows that for $f \in L^2_K$ the set of $x \in \mathbb{R}^n$ for which (1.3) fails to converge for $\lambda > (n-1)/2$ has Lebesgue measure zero. We strengthen this by combining the Stein-Weiss result with the following consequence of [12, Theorem 1]:

if $f \in L^2_K$ then $f$ has a Lebesgue point at all points in $\mathbb{R}^n$ except possibly for a set of points of $C_{K,2}$-capacity zero.

The capacity referred to here is the particular case $p = 2$ of the $L^p$-capacities of Meyers ([9], [1, Chapter 2]). For a brief summary of the basic properties of these capacities see [10, pp. 341-2]. If $C_{K,2}(E) = 0$, then $E$ has measure zero, and Meyers’ capacities provide a way of differentiating between sets of measure zero.

Taking $K = K_\beta$ we immediately deduce the following convergence result for functions in $L^2_{K_\beta}$.

**Theorem 3.1.** If $f \in L^2_{K_\beta}$, or equivalently if (2.5) holds, and $\lambda > (n-1)/2$, then

$$\lim_{R \to \infty} T^R \beta f(x) = f(x) \text{ for all } x \in \mathbb{R}^n \text{ outside an exceptional set of } C_{K,2}$-capacity zero.$$

It has been shown in [3, Theorem A] that if $f \in L^2$ then $\lim_{R \to \infty} T^R f(x) = f(x)$ almost everywhere in $\mathbb{R}^n$ for all $\lambda > 0$, and an obvious question here therefore is whether the range of $\lambda$ in Theorem 3.1 can be extended to all positive values. There is also the question of whether the characterisation $f = K_\beta \ast F$ can be used to obtain convergence results in $L^2_{K_\beta}$ for the spherical partial integral $S_R f$. These questions are open.

**References**


Brian Twomey obtained his PhD in Complex Analysis at Imperial College in 1967, and he spent three years in the US (Syracuse University, University of South Florida), after which he returned to Cork to take up a lectureship post. He is now an Emeritus Professor at UCC. Areas of research interest have included univalent functions, boundary behaviour of analytic functions in the unit disc, Poisson integrals in half-spaces, Dirichlet-type spaces, and, latterly, the mysteries of Fourier transforms.

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