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The Trace and its Extensions in Operator Algebras

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Abstract. We discuss how mathematicians generalize the usual trace on matrices to various finite and infinite-dimensional algebras. We also examine the existence or lack of (faithful) tracial states in the framework of operator algebras.

1. Introduction and Preliminaries

A natural invariant associated to each linear operator T acting on an n-dimensional vector space V is its characteristic polynomial $p_T(\lambda) = \det(T - \lambda I)$, where I is the identity operator on V and λ is a scalar (for simplicity and because of where we are going, we will assume that the field of scalars is $\mathbb{C}$). This polynomial encodes essential information about T; namely its eigenvalues, which are the roots of $p_T(\lambda)$.

In turn, this gives importance to its coefficients, as invariants of the operator. The most well-known of these coefficients is the constant term, that is the determinant $\det T = p_T(0)$. This is equal to the product of the eigenvalues of T, counting multiplicities. Among the other coefficients of $p_T(\lambda)$, the best known is the coefficient of λ^{n-1}. This coefficient is equal to the sum of the eigenvalues of T, counting multiplicities, and it is usually called the *trace* of T, and denoted by $\mathrm{tr}(T)$. Eigenvalues are crucial in understanding the behavior of linear operators, so the trace and the determinant give quick ways to relate a matrix to its eigenvalues without having to compute them.

Via the Jordan form J_T of T, the number $\mathrm{tr}(T)$ can be seen as the sum of the diagonal entries of J_T. A straightforward computation shows that $\mathrm{tr}(ST) = \mathrm{tr}(TS)$ for any two linear operators S and T acting on V, and hence $\mathrm{tr}(STS^{-1}) = \mathrm{tr}(T)$ for all invertible S and all T. From this one can deduce that $\mathrm{tr}(T) = \sum_{i=1}^n a_{ii}$ for any presentation of T as a matrix $A = [a_{ij}]$ with respect to some basis of V. It is not hard to show that tr is the only linear functional on V with the *tracial property*:

$$\mathrm{tr}(ST) = \mathrm{tr}(TS) \qquad \text{for all } S \text{ and } T, \tag{1}$$

up to normalization by a scalar (see Subsection 2.1.1 for a proof of uniqueness).

The trace is particularly meaningful in the case where our finite-dimensional vector space is a Hilbert space $\mathscr{H}$, but its straightforward extension to the infinite-dimensional case cannot work for all bounded operators on $\mathscr{H}$. For example, for the diagonal operator $\mathrm{diag}(1, 1/2, 1/3, \ldots)$ acting on the Hilbert space ℓ^2 of square summable sequences, the sum of its diagonal entries is not finite. Extensions exist, though, and they appear in many flavours. Discussing those extensions is the main goal of this article.

To fix notation, we let $\mathscr{H}$ denote a Hilbert space over the field $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$. We write $\mathbb{B}(\mathscr{H})$ for the $*$-algebra of all linear bounded operators on $\mathscr{H}$; we denote by I the identity operator on $\mathscr{H}$. The space of all compact operators acting on

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$\mathscr{H}$ is denoted by $\mathbb{K}(\mathscr{H})$, which is a closed two-sided ideal of $\mathbb{B}(\mathscr{H})$. In the case where $\dim \mathscr{H} = n$, we can identify $\mathbb{B}(\mathscr{H}) = \mathbb{K}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all complex $n \times n$ matrices. In this latter case the trace takes the form

$$\operatorname{tr}(T) = \sum_{k=1}^{n} \langle Te_k, e_k \rangle,$$

where $\{e_k\}_{k=1}^{n}$ is any orthonormal basis of $\mathscr{H}$. We denote the normalized trace $\frac{1}{n}\operatorname{tr}$ by $\widehat{\operatorname{tr}}$.

As noted by Albrecht Pietsch [26], the definition of the trace for a square matrix mentioned above has been in use since the 18th century. The term "Spur" for this notion was first introduced by Dedekind [11] within the context of algebraic number theory. In his work on the development of mathematical foundations for quantum mechanics ([32, 33, 34], compiled in [35]), von Neumann defined the trace of a positive operator acting on a Hilbert space and considered the ideal of trace-class operators. Incidentally, von Neumann also defined for the first time the idea of an abstract Hilbert space.

As soon as one tries to extend the notion of trace to the infinite-dimensional setting, issues arise: the only linear functional $\varphi : \mathbb{B}(\mathscr{H}) \to \mathbb{C}$ satisfying the tracial property is the zero functional. This is what led von Neumann to consider the trace-class operators, which form in a sense the largest ideal $\mathbb{T}(\mathscr{H})$ where (1) holds. In fact, $\mathbb{T}(\mathscr{H})$ is the set of all operators $T \in \mathbb{B}(\mathscr{H})$ such that $\|T\|_1 := \sum_{e\in\mathbb{E}} \langle |T|e, e\rangle < \infty$, where $\mathbb{E}$ is any orthonormal basis for $\mathscr{H}$. In addition, we can define the trace of $T \in \mathbb{T}(\mathscr{H})$ as $\operatorname{tr}(T) := \sum_{e\in\mathbb{E}} \langle Te, e\rangle$, and this definition is independent of the choice of basis. The trace in this context appears to be intrinsic, as $\mathbb{T}(\mathscr{H})$ can be seen as the predual of $\mathbb{B}(\mathscr{H})$, in the sense that we have isometric isomorphisms

$$\mathbb{K}(\mathscr{H})^* = \mathbb{T}(\mathscr{H}), \qquad \mathbb{T}(\mathscr{H})^* = \mathbb{B}(\mathscr{H}),$$

where the isomorphisms in both cases are given by the trace; that is, a trace-class operator T is seen as a bounded linear functional on $\mathbb{K}(\mathscr{H})$ via $S \longmapsto \operatorname{tr}(ST)$, and $T \in \mathbb{B}(\mathscr{H})$ is seen as a bounded linear functional on $\mathbb{T}(\mathscr{H})$ via the same duality pairing.

A positive linear functional φ on a C^*-algebra $\mathscr{A}$ is called *tracial* if it satisfies (1) for all $S, T \in \mathscr{A}$. As in [16, Proposition 8.1.1], one can observe that (2) and (3), the latter when $\mathscr{A}$ is unital, are each equivalent to (1):

$$\varphi(X^*X) = \varphi(XX^*), \qquad X \in \mathscr{A}\,. \tag{2}$$

$$\varphi(UXU^*) = \varphi(X), \qquad X \in \mathscr{A} \quad \text{and } U \in \mathscr{A} \text{ a unitary.} \tag{3}$$

There are several papers exploring the characterizations of the tracial functionals on matrices and operator algebras; we mention [3] for further reference.

The study of tracial states, which are tracial positive linear functionals of norm one, is an active area in the theory of operator algebras, particularly in Elliott's Classification Program (see [36] as an initial source of a very large number of references). It is a natural question whether certain classes of C^*-algebras or von Neumann algebras admit a tracial state or not.

Besides the intrinsic interest for operator algebras, such studies have applications in other disciplines. From classifying linear operators to enabling quantum computations and optimizing machine learning models, the trace features both in abstract theory and in real-world applications.

In quantum mechanics, the trace is used in defining the notion of density matrix; such a matrix ρ is a positive semidefinite matrix of trace one. The entropy of a quantum system with density matrix ρ is given by $S = -\operatorname{tr}(\rho \ln \rho)$; see [10, 24]. Moreover, the concept of partial trace in quantum information theory is used to describe subsystems.

The partial trace $\operatorname{tr}_1 : \mathbb{M}_n \otimes \mathbb{M}_m \to \mathbb{M}_m$ is the linear map induced by $\operatorname{tr}_1(A \otimes B) = (\operatorname{tr} A)B$ and the partial trace $\operatorname{tr}_2 : \mathbb{M}_n \otimes \mathbb{M}_m \to \mathbb{M}_n$ is induced by $\operatorname{tr}_2(A \otimes B) = (\operatorname{tr} B)A$. In another setting, the trace is also used in defining the Frobenius inner product on $\mathbb{M}_n$ via $\langle A, B\rangle = \operatorname{tr}(B^*A)$. This inner product is useful in optimization problems over matrices, for example in machine learning where one may minimize some loss function that is expressed using the trace. Another application occurs in random matrix theory, where the trace of random matrices is studied, and results such as the law of large numbers for traces of powers of matrices relate to eigenvalue distributions.

For the readers' convenience we have included a brief summary of the basic theory of C^*-algebras and von Neumann algebras in Appendix A. For any undefined notations or terminologies, readers are referred to [2] for matrix theory and to [16, 22, 31] for the theory of operator algebras.

The main objective of this expository article is to discuss various extensions of the usual trace $\operatorname{tr}\colon \mathbb{M}_n \to \mathbb{C}$ to more general settings in operator algebras. Although the literature contains many interesting and deep results on this topic (see, e.g., [20]), we focus on presenting fundamental facts and some new proofs, and illustrative examples for readers familiar with basic operator algebra theory.

2. Extensions of the usual trace, Uniqueness, and Examples

We aim to explore how to extend the usual trace $\operatorname{tr} : \mathbb{M}_n \to \mathbb{C}$ to positive linear maps satisfying the tracial property by considering changes in the domain $\mathbb{M}_n$, codomain $\mathbb{C}$, or both, to some operator algebras. We also examine the existence or lack of tracial states in the framework of operator algebras.

2.1. **Changing domain.**

2.1.1. *Replacing* $\mathbb{M}_n$ *with a finite-dimensional* C^**-algebra.* As mentioned in the introduction, $\widehat{\operatorname{tr}}$ is the only tracial state on $\mathbb{M}_n$.

Indeed, consider the canonical matrix unit system $\{E_{ij}\} \subseteq \mathbb{M}_n$, which satisfies $E_{ij}E_{kl} = \delta_{jk}E_{il}$. For a tracial state φ, if $i \neq j$ then

$$\varphi(E_{ij}) = \varphi(E_{ij}E_{jj}) = \varphi(E_{jj}E_{ij}) = \varphi(0) = 0;$$

and for any i, j

$$\varphi(E_{ii}) = \varphi(E_{ij}E_{ji}) = \varphi(E_{ji}E_{ij}) = \varphi(E_{jj}).$$

Thus, for $A = [a_{ij}] = \sum_{i,j=1}^n a_{ij}E_{ij} \in \mathbb{M}_n$, we have

$$\begin{aligned}\varphi(A) &= \sum_{i,j=1}^n a_{ij}\,\varphi(E_{ij}) = \sum_{i=1}^n a_{ii}\,\varphi(E_{ii}) = \varphi(E_{11})\sum_{i=1}^n a_{ii} = \varphi(E_{11})\operatorname{tr}(A)\\ &= \frac{1}{n}\operatorname{tr}(A) = \widehat{\operatorname{tr}}(A).\end{aligned}$$

If we replace $\mathbb{M}_n$ with a finite-dimensional C^*-algebra $\mathscr{A} = \bigoplus_{k=1}^m M_{k(m)}$, then there are uncountably many tracial states $\varphi : \mathscr{A} \to \mathbb{C}$, of the form

$$\varphi\Big(\bigoplus_{k=1}^m X_k\Big) = \sum_{k=1}^m t_k\,\widehat{\operatorname{tr}}(X_k),$$

where $t_k \geq 0$ for all k and $\sum_{k=1}^m t_k = 1$.

2.1.2. *Substituting $\mathbb{C}$ in $\mathbb{M}_n(\mathbb{C})$ with an arbitrary C^*-algebra $\mathscr{A}$ having a tracial state.* Given a tracial state φ on $\mathscr{A}$, we can define a tracial state on $\mathbb{M}_n(\mathscr{A})$ by

$$\varphi_n([a_{ij}]) := \frac{1}{n}\sum_{i=1}^{n}\varphi(a_{ii}). \tag{4}$$

And this is the only way to construct tracial states on $\mathbb{M}_n(\mathscr{A})$: if $\gamma : \mathbb{M}_n(\mathscr{A}) \to \mathbb{C}$ is a tracial state, then there exists a unique tracial state φ on A, defined by $\varphi(a) = \gamma(a \otimes E_{11}))$, such that $\gamma = \varphi_n$. So there is a natural bijective correspondence between tracial states on $\mathscr{A}$ and tracial states on $\mathbb{M}_n(\mathscr{A})$.

2.1.3. *Replacing $\mathbb{M}_n(\mathbb{C})$ with a commutative C^*-algebra.* One may replace $\mathbb{M}_n$ with a commutative C^*-algebra $\mathscr{A}$. In this case, every state is tracial. It is known that $\mathscr{A}$ is isometrically $*$-isomorphic to $C_0(\Omega)$ for some locally compact Hausdorff space Ω. Therefore, any (tracial) positive linear functional on $\mathscr{A}$ can be represented as $\varphi(f) = \int_\Omega f d\mu$ for a unique positive Borel measure μ on Ω such that $\mu(\Omega) = \|\varphi\|$, where $\|\varphi\|$ denotes the operator norm of φ. Hence there exist uncountably many tracial states on a commutative C^*-algebra, as long as it is not one-dimensional. If $\mathscr{A}$ is finite-dimensional, then Ω must be a finite set with, say, n elements; see [22, p. 57]. In such case, the state space is parametrized by the simplex $\{(t_1, \dots, t_n) \in \mathbb{R}^n : \ t_j \geq 0 \ \text{ for all } j, \ \sum_j t_j = 1\}$.

2.1.4. *Finite factors have unique tracial states.* It is a seminal result of Murray and von Neumann [23] that a finite factor has a unique tracial state (the original Murray–von Neumann ideas are developed with detail in [30, Section 1.3]). The unique tracial state is always faithful and normal. A finite-dimensional example of a finite factor is $\mathbb{M}_n, \ n \geq 1$ with the usual tracial state $\widehat{\operatorname{tr}}$. An infinite-dimensional example of a finite factor is the *hyperfinite II_1-factor*, which can be seen as the double commutant (that is, the sot-completion) of $\bigcup_{n\in\mathbb{N}} \mathbb{M}_{2^n}$ (with the embeddings $A \longmapsto A \oplus A$) via the GNS representation of the tracial state $\varphi((A_n))$ extending the natural normalized trace on each subalgebra. The hyperfinite II_1-factor also appears as the sot-closure of the image of the group algebra, via the left-regular representation, of any amenable countable discrete group G with infinite conjugacy classes.

2.1.5. *C^*-algebras without any tracial states.* There exist C^*-algebras $\mathscr{A}$ without any tracial states. A separable example is the simple C^*-algebra $\mathbb{K}(\mathscr{H})$ for any infinite-dimensional separable Hilbert space $\mathscr{H}$. Given a fixed orthonormal basis $(e_i)_{i=1}^\infty$ for $\mathscr{H}$, the corresponding *matrix units* are the operators $\{E_{ij}\}$, where E_{ij} is the rank-one operator that sends e_j to e_i. As in the matrix case, they satisfy the relations $E_{rs}E_{ij} = \delta_{si}\, E_{rj}$. In particular $\{E_{ii}\}$ are pairwise orthogonal rank-one projections. As φ is tracial, $\varphi(E_{ij}) = 0$ for any $i \neq j$, and

$$\varphi(E_{jj}) = \varphi(E_{ji}E_{ij}) = \varphi(E_{ij}E_{ji}) = \varphi(E_{ii}).$$

Then, with $P_n = \sum_{i=1}^n E_{ii}$,

$$n\,\varphi(E_{11}) = \varphi(P_n) \leq \|\varphi\|\,\|P_n\| = 1.$$

As n is arbitrary, this implies that $\varphi = 0$, which is not a state since its operator norm is not equal to one.

The argument above also demonstrates that $\mathbb{B}(\mathscr{H})$ is a nonsimple C^*-algebra without any tracial state. It is known that a C^*-algebra $\mathscr{A}$ has no tracial states if and only if its universal enveloping von Neumann algebra $\pi(\mathscr{A})''$ is properly infinite.

Haagerup proved that if $\mathscr{A}$ is a unital C^*-algebra, then $\mathscr{A}$ has no tracial state if and only if there exist $n \geq 2$ and a finite set $\{A_1, \dots, A_n\} \subseteq \mathscr{A}$ such that $\sum_{i=1}^n A_i^* A_i = 1$ and $\|\sum_{i=1}^n A_i A_i^*\| < 1$ [12, Lemma 2.1]. Pop [27] showed that a C^*-algebra $\mathscr{A}$ has no tracial state if and only if there exists $n \geq 2$ such that any element of $\mathscr{A}$ can be expressed as a sum of n commutators $[A_i, B_i] = A_iB_i - B_iA_i,\ 1 \leq i \leq n$. An interesting question posed by Pop [27] is that if $\mathscr{A}$ has no tracial state, what is the smallest n such that each element of $\mathscr{A}$ can be expressed as a sum of n commutators?

2.1.6. *Existence of a unique tracial state on a nonnuclear C^*-subalgebra of a separable simple C^*-algebra possessing no tracial state.* The Choi algebra is the C^*-algebra generated by two unitary operators U and V acting on an infinite-dimensional Hilbert space $\mathscr{H}$ such that $U^2 = V^3 = 1$. For a construction of U and V, Choi [7] used suitable decompositions $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ and $\mathscr{H}_1 = \mathscr{H}_\alpha \oplus \mathscr{H}_\beta$ subject to the conditions $\dim \mathscr{H}_0 = \dim \mathscr{H}_1 = \dim \mathscr{H}_\alpha = \dim \mathscr{H}_\beta$. He then defined U and V by block operator matrices

$$\begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}_0 \oplus \mathscr{H}_1) \quad \text{and} \quad \begin{bmatrix} 0 & 0 & V_1 \\ V_2 & 0 & 0 \\ 0 & V_3 & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}_0 \oplus \mathscr{H}_\alpha \oplus \mathscr{H}_\beta),$$

where $U_1 : \mathscr{H}_1 \to \mathscr{H}_0, U_2 : \mathscr{H}_0 \to \mathscr{H}_1, V_1 : \mathscr{H}_\beta \to \mathscr{H}_0, V_2 : \mathscr{H}_0 \to \mathscr{H}_\alpha$, and $V_3 : \mathscr{H}_\alpha \to \mathscr{H}_\beta$ are unitaries between corresponding Hilbert subspaces of the same dimensions. This C^*-algebra has a unique tracial state, even though it is a C^*-subalgebra of the Cuntz C^*-algebra $\mathcal{O}_2$, which has no tracial state. To prove the latter fact, recall that $\mathcal{O}_2$ is generated by two isometries S_1 and S_2 such that $S_1S_1^* + S_2S_2^* = I$. If φ is a tracial state on $\mathcal{O}_2$, then $1 = \varphi(I) = \varphi\left(S_1S_1^* + S_2S_2^*\right) = \varphi(S_1S_1^*) + \varphi(S_2S_2^*) = \varphi(I) + \varphi(I) = 2$, a contradiction. A nonunital simple separable C^*-algebra with a unique tracial state is the so-called Jacelon–Razak C^*-algebra; see [14]. An example of a unital separable, nuclear projectionless infinite-dimensional C^*-algebra with a unique tracial state is the Jiang–Su algebra [15].

2.1.7. *Kaplansky's problem.* What happens if one assumes that the tracial property $\varphi(AB) = \varphi(BA)$ holds for specific classes of elements $A, B \in \mathscr{A}$ but not necessarily all elements of $\mathscr{A}$? For instance one could require that $\varphi(A^*A) = \varphi(AA^*)$ for all $A \in \mathscr{A}$. The linearity of φ then implies (1). But what if φ is not required to be linear? A function $\varphi : \mathscr{A} \to \mathbb{C}$ is called a *quasitrace* if it satisfies $\varphi(A^*A) = \varphi(AA^*)$ for all $A \in \mathscr{A}$, it satisfies $\varphi(A + \mathrm{i}B) = \varphi(A) + \mathrm{i}\,\varphi(B)$ for all A, B selfadjoint, and it is linear on each abelian subalgebra of $\mathscr{A}$. Kaplansky [17] asked whether every II_1 AW*-factor is a von Neumann algebra. This would be true if one can prove that every quasitrace is a trace. While still an open problem, Haagerup [12] was able to prove in 1991 that each quasitrace on a unital exact C^*-algebra is a trace. This result has had significant applications to the theory of C^*-algebras.

2.1.8. *Approximately tracial state.* In perturbation theory, one considers situations where (1) we have an object that approximately fulfills a property, and we try to prove that it is close to an object that exactly satisfies that property; (2) there exists a problem for which we do not know the exact solution, but we can find an approximate solution for it; (3) there are objects with an approximate property and we seek an object that exactly meets the property. Here we deal with the third situation.

We may consider $(\mathcal{F}, \varepsilon)$-almost traces for any given finite subset $\mathcal{F}$ of the closed unit ball of $\mathscr{A}$ and any $\varepsilon > 0$. This means that there is a state $\varphi_{\mathcal{F},\varepsilon}$ on $\mathscr{A}$ such that $|\,\varphi_{\mathcal{F},\varepsilon}(A^*A - AA^*)| < \varepsilon$ for all $A \in \mathcal{F}$. It is shown in [19, Lemma 5.4] that a C^*-algebra $\mathscr{A}$ has a tracial state φ if and only if it has $(\mathcal{F}, \varepsilon)$-almost traces for all $\mathcal{F}$ and ε. Indeed, φ can be taken to be an accumulation point of the net $(\varphi_{\mathcal{F},\varepsilon})$ in the weak*-compact unit ball of the dual of $\mathscr{A}$.

2.2. **Changing codomain.**

2.2.1. *An extension of the trace with values in a C^*-algebra.* In seeking an extension of the trace, one may substitute the C^*-algebra $\mathbb{C}$ with an arbitrary C^*-algebra $\mathscr{A}$. If $\varphi : \mathbb{M}_n \to \mathscr{A}$ is a tracial positive linear map, then by repeating the argument in subsection 2.1.1 we get

$$\begin{aligned}\varphi(A) &= \sum_{i,j=1}^{n} a_{ij}\,\varphi(E_{ij}) = \sum_{i=1}^{n} a_{ii}\,\varphi(E_{ii}) = \varphi(E_{11})\sum_{i=1}^{n} a_{ii} = \varphi(E_{11})\,\mathrm{tr}(A)\\ &= (n\,\varphi(E_{11}))\frac{1}{n}\,\mathrm{tr}(A) = \varphi(I)\widehat{\mathrm{tr}}(A),\end{aligned}$$

where now $\varphi(I)$ is an element of $\mathscr{A}$.

2.3. **Changing both domain and codomain.**

2.3.1. *A generalization of the trace that implies the commutativity of the underlying C^*-algebra.* One can think of replacing $\mathbb{M}_n$ and $\mathbb{C}$ with $\mathbb{M}_n(\mathscr{A})$ and $\mathscr{A}$, respectively, for some unital C^*-algebra $\mathscr{A}$, and then define $\varphi : \mathbb{M}_n(\mathscr{A}) \to \mathscr{A}$ by $\varphi([A_{ij}]) = \sum_{i=1}^{n} A_{ii}$. Then, if I_n denotes the identity element of $\mathbb{M}_n(\mathscr{A})$ and φ satisfies the tracial property (1), we have

$$AB = \frac{1}{n}\,\varphi(ABI_n) = \frac{1}{n}\,\varphi(AI_nBI_n) = \frac{1}{n}\,\varphi(BI_nAI_n) = \frac{1}{n}\,\varphi(BAI_n) = BA.$$

Therefore, $\mathscr{A}$ has to be commutative, every state is tracial, and $\mathscr{A}$ is of the form $C(\Omega)$ for some compact Hausdorff space Ω.

2.3.2. *Replacing $\mathbb{M}_n$ and $\mathbb{C}$ with an arbitrary C^*-algebra and $\mathbb{B}(\mathscr{H})$, respectively.* A linear map $\Phi : \mathscr{A} \to \mathscr{B}$ is called *tracial and positive* if it takes positive elements of $\mathscr{A}$ to those of $\mathscr{B}$ and fulfills the condition (1). A result due to Choi and Tsui [8, pp. 59-60] states that if $\Phi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ is a tracial and positive linear map, then there exist a commutative C^*-algebra $C(X)$, where X is a compact Hausdorff space, and tracial and positive linear maps $\phi_1 : A \to C(X)$ and $\phi_2 : C(X) \to \mathbb{B}(\mathscr{H})$ such that $\Phi = \phi_2 \circ \phi_1$. In particular, any tracial and positive linear map is completely positive.

2.3.3. *Substituting $\mathbb{M}_n$ and $\mathbb{C}$ with a properly infinite von Neumann algebra $\mathscr{M}$ and a unital C^*-algebra $\mathscr{B}$, respectively.* If $\Phi : \mathscr{M} \to \mathscr{B}$ is a unital tracial positive linear map, then Φ is identically zero. The reason is that we can "halve" projections. In particular, there exists a projection $P \in \mathscr{M}$ such that $P \sim I \sim I - P$ [31, Proposition V.1.36]. Hence, there are partial isometries $U, V \in \mathscr{M}$ such that $U^*U = V^*V = I$, $VV^* = P$, and $UU^* = I - P$. By the tracial property of Φ, we have $\Phi(I) = \Phi(P) = \Phi(I - P)$. Therefore, $\Phi(I) = \Phi(P) + \Phi(I - P) = 2\Phi(I)$, whence $\Phi(I) = 0$. Now given any positive element $A \in \mathscr{A}$, we have $A \le \|A\|\,I$. Therefore, $0 \le \Phi(A) \le \|A\|\,\Phi(I) = 0$, and hence, $\Phi(A) = 0$. As any element in $\mathscr{A}$ is a linear combination of four positive elements, it follows that $\Phi = 0$.

2.4. **C^*-algebras and faithful tracial states.**

2.4.1. *A unital C^*-algebra with a faithful tracial state is finite.* Let φ be a faithful tracial state on a unital C^*-algebra $\mathscr{A}$. We show that if $I \sim P$, then $P = I$. To see this, suppose $U^*U = I$. Then, $\varphi(I - UU^*) = \varphi(U^*U - UU^*) = 0$, and because φ is faithful, we infer that $UU^* = I$. This shows that every isometry is a unitary, and in particular the identity I is finite.

2.4.2. *C^*-algebras and von Neumann algebras admitting a faithful tracial state.* As opposed to the case of von Neumann algebras, it is not entirely clear how to characterize a C^*-algebra as *finite*. The naive way is to use the same definition as for von Neumann algebras. This is done for instance on [28], and it is the definition used in 2.4.1 above. The problem with this is that a C^*-algebra may not have enough projections, or it may even fail to have nonzero projections at all; see [18]. This would make C^*-algebras that "feel" infinite be finite, for example $C_0(\mathbb{R}, \mathcal{O}_2)$. A stronger definition is used in [29], where the requirement for finiteness is that all projections are finite, together with the existence of an approximate unit made entirely of projections. With this definition, combining the results from [4] and [12] it is proven that every unital, stably finite, exact C^*-algebra admits a tracial state. Here *stably finite* means that $\mathscr{A} \otimes \mathbb{K}(\mathscr{H})$ contains no infinite projections. Another notion of *finite* was considered by Cuntz and Pedersen in [9]. They consider, instead of equivalence of projections, equivalence of positive elements, where $x \sim y$ in $\mathscr{A}$ if there exists a sequence $\{z_n\} \subset \mathscr{A}$ such that $x = \sum_m z_n^* z_n$ and $y = \sum_n z_n z_n^*$. They say that $\mathscr{A}$ is finite if $0 \leq y \leq x$ and $y \sim x$ implies $x = y$. With this definition of finite, they prove that a separable C^*-algebra $\mathscr{A}$ is finite if and only if it admits a faithful tracial state.

For von Neumann algebras, the situation is simpler. If $\mathscr{M}$ is a finite von Neumann algebra with separable predual, then it has a faithful tracial state. The von Neumann algebra $\mathscr{M}$ is finite precisely when in the central decomposition of $\mathscr{M}$ there exist only types I_n with $n < \infty$ and II_1.

There is a general form for tracial states on finite von Neumann algebras: if $\mathscr{M}$ is a finite von Neumann algebra equipped with a center-valued tracial map $\mathrm{tr}_c : \mathscr{M} \to \mathcal{Z}(\mathscr{M})$, then each tracial state φ on $\mathscr{M}$ is of the form $\varphi = \rho \circ \mathrm{tr}_c$, where ρ is a state on $\mathcal{Z}(\mathscr{M})$. The tracial state φ is normal on $\mathscr{M}$ if and only if the state ρ is normal on $\mathcal{Z}(\mathscr{M})$, as shown in [16, Theorems 8.2.8 and 8.3.6]; see also [5, Theorem 4.1].

2.4.3. *A normal state on a von Neumann algebra gives a faithful normal tracial state on a reduced von Neumann algebra.* Let's consider a similar construction. Let $\mathscr{M}$ be a von Neumann algebra and let φ be a nonzero normal state on $\mathscr{M}$ with support P. Then $P\mathscr{M}P$ is a von Neumann algebra with a faithful state φ. If τ denotes the unique center-valued trace on $\mathscr{M}$, then $\psi = \varphi \circ \tau$ is a faithful normal tracial state on $P\mathscr{M}P$; see [16, Chapter 8] for more details.

2.4.4. *Invertibility in the presence of a faithful tracial state.* If a C^*-algebra $\mathscr{A}$ has a faithful tracial state φ, then the one-sided invertibility of $A \in \mathscr{A}$ implies the two-sided invertibility of A. Indeed, if $BA = I$, then

$$I = (BA)^* BA = A^* B^* BA \leq \|B\|^2 A^* A.$$

This implies that $A^*A \geq \|B\|^{-2} I$, so A^*A is invertible. Let $V = A(A^*A)^{-1/2}$. Then

$$V^*V = (A^*A)^{-1/2} A^*A (A^*A)^{-1/2} = I.$$

We obtain that $\|VV^*\| = \|V\|^2 = \|V^*V\| = 1$; thus $0 \leq VV^* \leq I$. In addition,

$$0 \leq \varphi(I - VV^*) = 1 - \varphi(VV^*) = 1 - \varphi(V^*V) = 1 - 1 = 0.$$

As φ is faithful, $VV^* = I$, so V is unitary (in particular, it is invertible). Thus, $A = V(A^*A)^{1/2}$ is invertible. An analog computation can be made when A is right-invertible.

2.4.5. *Factors with a faithful tracial state.* It is notable that a faithful tracial state φ on a factor $\mathscr{M}$ has the property

$$P \sim Q \Longleftrightarrow \varphi(P) = \varphi(Q)$$

for all projections $P, Q \in \mathscr{M}$. Indeed, if two projections $P, Q \in \mathscr{M}$ are not equivalent, then by the Comparison Theorem in factors [16, Theorem 6.2.7], we may assume $P \prec Q$ (otherwise, we obtain $Q \prec P$ and we can reason the same). That is, $P \sim Q_1 \leq Q$ for some projection Q_1. Therefore, $\varphi(P) = \varphi(Q_1) \leq \varphi(Q)$. If $\varphi(P) = \varphi(Q)$, then $\varphi(Q - Q_1) = 0$ and faithfulness implies that $Q_1 = Q$; this means $P \sim Q$, a contradiction. Thus, $\varphi(P) = \varphi(Q_1) < \varphi(Q)$, and so $\varphi(P)$ and $\varphi(Q)$ are distinct. The converse is clear by the tracial property of φ.

2.4.6. *Examples of nonfaithful tracial states.* Given a unital C^*-algebra $\mathscr{A}$ with a faithful tracial state φ, the extension $\psi : \mathscr{A} \oplus \mathscr{A} \to \mathbb{C}$ defined by $\psi(A, B) = \varphi(A)$ is a nonfaithful tracial state. Furthermore, the restriction of a tracial state on a C^*-algebra to a C^*-subalgebra may fail to be a state. For example, let $\mathscr{A}$ be a C^*-algebra and consider the tracial state $\varphi : \mathscr{A} \oplus \mathbb{M}_n \to \mathbb{C}$ defined by $\varphi(A, B) = \widehat{\mathrm{tr}}(B)$. Then, the restriction of φ to $\mathscr{A}$ is identically 0, which is not even a state. Another example is to consider a non-factor $\mathscr{M}$ with a faithful tracial state φ. Given a nontrivial projection in the center of $\mathscr{M}$, we have $\phi(P) > 0$ by the faithfulness of φ. Then $\psi(A) = \varphi(AP)$ provides a nonfaithful tracial state, since $\Psi(I - P) = 0$. In this situation we can get different faithful tracial states by weighting, in the following sense: for each $t \in [0, 1]$,

$$\psi_t(A) = \frac{t}{\varphi(P)}\, \varphi(AP) + \frac{(1-t)}{\varphi(P^\perp)}\, \varphi(AP^\perp),$$

where $P^\perp = I - P$, is a faithful tracial state.

2.4.7. *GNS construction for a tracial state.* Let us now describe a situation where one extends a faithful tracial state on a unital C^*-algebra $\mathscr{A}$ to a faithful normal tracial state on a certain von Neumann algebra. We use the notation in the GNS construction (described in Appendix A). As φ is faithful, $N_\varphi = 0$, so $\mathscr{H}_\varphi$ is the completion of $\mathscr{A}$ with respect to the norm $\|A\|_{2,\varphi} = \varphi(A^*A)^{1/2}$ induced by the inner product $\langle a, b\rangle = \varphi(b^*a)$. For instance, if $\mathscr{A} = L^\infty[0, 1]$ and φ is integration with respect to the Lebesgue measure, then $\mathscr{H}_\varphi = L^2[0, 1]$.

In addition, the positive linear functional $\tilde{\varphi} : \pi_\varphi(\mathscr{A})'' \to \mathbb{C}$ defined by $\tilde{\varphi}(T) := \langle T x_\varphi, x_\varphi\rangle$ is a faithful normal tracial state on the von Neumann algebra $\pi_\varphi(\mathscr{A})''$ generated by $\pi_\varphi(\mathscr{A})$, since $\tilde{\varphi}(\pi_\varphi(A)) = \varphi(A)$ for all $A \in \mathscr{A}$ (see (5)) and $\pi_\varphi(\mathscr{A})$ is dense in $\pi_\varphi(\mathscr{A})''$ in the strong operator topology. Therefore, $\pi_\varphi(\mathscr{A})''$ is a finite von Neumann algebra; see [1, Lemma 2.2] for details. Furthermore, if f is a continuous real-valued function on an interval containing the spectrum of $A \in \mathscr{A}$, then $\varphi(f(A)) = \tilde{\varphi}(\pi_\varphi(f(A))) = \tilde{\varphi}(f(\pi_\varphi(A)))$. This property is employed in [25] to establish that if f is a monotone (convex) function, then so is $A \mapsto \varphi(f(A))$.

Since φ is a faithful tracial state, the representation $\pi_\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H}_\varphi)$ is one-to-one, for if $\pi_\varphi(A) = 0$, then (5) implies that $\varphi(A^*A) = 0$, and so $A = 0$.

2.4.8. *Constructing a von Neumann algebra with a faithful normal tracial state from a family of C^*-algebras admitting tracial states.* Let J be an infinite set equipped with a nontrivial ultrafilter α, meaning that α is free and there exists a sequence (J_n) in α such that $\cap_n J_n = \varnothing$. Suppose that for each $i \in J$ there exists a unital C^*-algebra $\mathscr{A}_i$ with a tracial state φ_i. Then, the tracial ultraproduct $\prod_{i\in J}^{\alpha} (\mathscr{A}_i, \varphi_i)$ is defined to be the C^*-product $\prod_{i\in J} \mathscr{A}_i$ modulo the ideal of all (A_i) in $\prod_{i\in J} \mathscr{A}_i$ such that $\lim_{i\to\alpha} \|A_i\|_{2,\varphi_i}^2 = \lim_{i\to\alpha} \varphi_i(A_i^* A_i) = 0$. It is established in [13, Theorem 4.1] that a tracial ultraproduct $\prod\limits_{i\in J}^{\alpha} (\mathscr{A}_i, \varphi_i)$ of C^*-algebras is a von Neumann algebra with the faithful normal tracial state $\psi_a((A_i)) := \lim_{i\to\alpha} \varphi_i(A_i)$.

Appendix A. Basics of C* and von Neumann Algebras

A *C^*-algebra* is a complex Banach $*$-algebra $\mathscr{A}$ with an involution such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathscr{A}$. Every C^*-algebra can be realized as a C^*-subalgebra of $\mathbb{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ (Gel'fand–Naimark–Segal; see [22, Theorem 3.4.1]). On $\mathbb{B}(\mathscr{H})$ we can consider the *operator norm*, defined as

$$\|T\| = \sup\{\|Tx\| : \ x \in \mathscr{H}, \ \|x\| = 1\}.$$

An element $A \in \mathscr{A}$ is *selfadjoint* if $A^* = A$ and *positive* if $A = B^*B$ for some $B \in \mathscr{A}$ (equivalently, if $A = A^*$ and $\sigma(A) \subset [0, \infty)$, where $\sigma(A)$ denotes the spectrum of A). We denote by $\mathscr{A}^+$ and $\mathscr{A}^{\mathrm{sa}}$ the subsets of positive and selfadjoint operators in $\mathscr{A}$, respectively. For two self-adjoint operators (matrices) A and B, we say that $A \le B$ whenever $B - A$ is positive (positive semidefinite). A rank-one projection is an operator of the form $e \otimes e$ for some unit vector $e \in \mathscr{H}$, where $(e \otimes e)(f) := \langle f, e\rangle e$ for all $f \in \mathscr{H}$.

By the *commutant* of a set $\mathcal{X} \subseteq \mathbb{B}(\mathscr{H})$, we mean the set $\mathcal{X}' = \{Y \in \mathbb{B}(\mathscr{H}) : \ XY = YX, \ X \in \mathcal{X}\}$. A non-degenerate $*$-subalgebra $\mathscr{M}$ of the algebra $\mathbb{B}(\mathscr{H})$ is called a *von Neumann algebra* acting in the Hilbert space $\mathscr{H}$ if $\mathscr{M} = \mathscr{M}''$. Von Neumann's *Double Commutant Theorem* states that for a non-degenerate $*$-algebra $\mathscr{M}$ we always have $\mathscr{M}'' = \overline{\mathscr{M}}^{\mathrm{sot}}$, where *sot* ("strong operator topology") denotes pointwise convergence. The commutative von Neumann algebra $\mathcal{Z}(\mathscr{M}) := \mathscr{M} \cap \mathscr{M}'$ is referred to as the center of $\mathscr{M}$, which in turn is always of the form $L^\infty(\Omega, \mu)$ for some measure space (Ω, μ). A *factor* is a von Neumann algebra with trivial center. If $P \in \mathscr{M}$ is a projection (that is, $P^2 = P$ and $P^* = P$), the corresponding *reduced von Neumann algebra* is defined as $\mathscr{M}_P = \{PX|_{P\mathscr{H}} : \ X \in \mathscr{M}\}$.

For projections $P, Q \in \mathscr{M}$, we denote $P \sim Q$ (Murray–von Neumann equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathscr{M}$; intuitively this says that both projections have the same rank, but there is a dependence on the algebra for the existence of the partial isometry U, so the notion of equivalentce is intrinsic to $\mathscr{M}$. A von Neumann algebra $\mathscr{M}$ is said to be *finite* if $P = Q$ for any equivalent projections $P, Q \in \mathscr{M}$ with $P \le Q$. Abelian von Neumann algebras are trivially finite. A non-finite projection is said to be *infinite*, and *properly infinite* if it is nonzero and infinite, and for every nonzero central projection $Q \in \mathscr{M}$, either $QP = 0$ or QP is infinite. A von Neumann algebra is said to be *finite* or *properly infinite* if its identity has the corresponding property. It is known that there exists a unique projection P_0 in the center $\mathcal{Z}(\mathscr{M})$ of $\mathscr{M}$ such that P_0 is finite and $I - P_0$ is properly infinite. Hence, we have the direct sum

$$\mathscr{M} = \mathscr{M}P_0 \oplus \mathscr{M}(I - P_0),$$

where $\mathscr{M}P_0$ is finite and $\mathscr{M}(I - P_0)$ is properly infinite.

We say that a projection in $\mathscr{M}$ is *abelian* if the algebra $P\mathscr{M}P$ is commutative. A von Neumann algebra $\mathscr{M}$ is said to be of *type I* if every projection in $\mathcal{Z}(\mathscr{M})$ majorizes a nonzero abelian projection in $\mathscr{M}$. If there is no nonzero finite projection in $\mathscr{M}$, then it is said to be of *type III.* If $\mathscr{M}$ has no nonzero abelian projection and if each nonzero projection in $\mathcal{Z}(\mathscr{M})$ majorizes a nonzero finite projection in $\mathscr{M}$, then it is said to be of *type II.* If $\mathscr{M}$ is type II and finite, then it is said to be of *type* II_1. If $\mathscr{M}$ is of type II and properly infinite, then it is said to be of *type* II_∞.

Every von Neumann algebra $\mathscr{M}$ has a unique *central decomposition* into a direct sum of subalgebras of type I, type II_1, type II_∞, and type *III* [31, Chapter V, Theorem 1.19]. Thus, $\mathscr{M} = \mathscr{M}_{P_I} \oplus \mathscr{M}_{P_{II_1}} \oplus \mathscr{M}_{P_{\mathrm{II}_\infty}} \oplus \mathscr{M}_{P_{\mathrm{III}}}$, where projections $P_I, P_{II_1}, P_{\mathrm{II}_\infty}$, and P_{III} in $\mathcal{Z}(\mathscr{M})$ are such that $P_I + P_{\mathrm{II}_1} + P_{\mathrm{II}_\infty} + P_{\mathrm{III}} = I$; it is possible for one or more of these to be zero.

A linear functional φ on $\mathscr{A}$ is said to be *positive* if $\varphi(X) \ge 0$ for all positive elements $X \in \mathscr{A}$. It is referred to as a *state* if it is positive and its operator norm $\|\varphi\|$ is equal to one. The positivity-preserving property of a linear functional φ is equivalent to $\|\varphi\| = \varphi(I)$; see [22, Corollary 3.3.5]. It is called *faithful* if it is one to one on $\mathscr{A}^+$. A positive linear functional φ on a von Neumann algebra $\mathscr{M}$ is said to be *normal* if $X_j \nearrow X$ (that is, $\langle X_j z, z\rangle \nearrow \langle Xz, z\rangle$ for all $z \in \mathscr{H}$) with $X_j, X \in \mathscr{M}^{\rm sa}$ implies $\varphi(X) = \sup \varphi(X_i)$.

We briefly introduce the GNS construction corresponding to a given state on a unital C^*-algebra $\mathscr{A}$. Suppose that φ is a state and let $N_\varphi = \{A \in \mathscr{A} : \varphi(A^*A) = 0\}$; this is a norm-closed left ideal of $\mathscr{A}$. An inner product on the quotient space $\mathscr{A}/N_\varphi$ can be defined by

$$\langle A + N_\varphi, B + N_\varphi\rangle := \varphi(B^*A)$$

The completion of this inner product space is denoted by $\mathscr{H}_\varphi$. The linear operator $\pi_\varphi : \mathscr{A}/N_\varphi \to \mathscr{A}/N_\varphi$ defined as $\pi_\varphi(A + N_\varphi)(B + N_\varphi) = AB + N_\varphi$ can be extended to a linear operator on $\mathscr{H}_\varphi$ denoted by the same $\pi_\varphi(A)$. Moreover, $\pi_\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H}_\varphi)$ is a $*$-homomorphism between C^*-algebras; that is, a *representation*. In addition, the unit vector $x_\varphi = I + N_\varphi \in \mathscr{H}_\varphi$ is cyclic (meaning that $\pi_\varphi(\mathscr{A})x_\varphi$ is dense in $\mathscr{H}_\varphi$) and

$$\varphi(A) = \langle \pi_\varphi(A)x_\varphi, x_\varphi\rangle. \tag{5}$$

The triple $(\pi_\varphi, \mathscr{H}_\varphi, x_\varphi)$ is called the *GNS representation* (from Gelfand–Naimark–Segal).

The pair $\{\pi, \mathscr{H}\} = \bigoplus_{\varphi\in S(\mathscr{A})} \{\pi_\varphi, \mathscr{H}_\varphi\}$ is known as the *universal representation* of $\mathscr{A}$. Here, $S(\mathscr{A})$ denotes the set of all states on $\mathscr{A}$. The von Neumann algebra $\mathscr{M} = \pi(\mathscr{A})''$ generated by $\pi(\mathscr{A})$ is said to be the *universal enveloping von Neumann algebra* of the C^*-algebra $\mathscr{A}$ [31, Chap. III, Definition 2.3].

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