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Dissecting Rectangles into Squares

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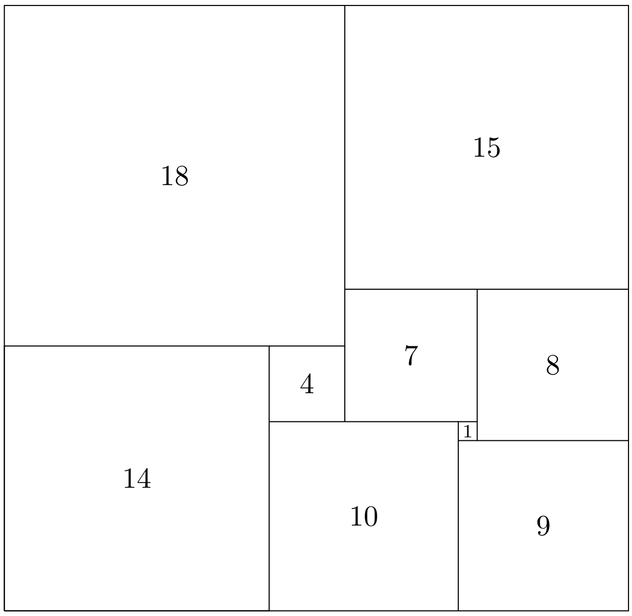
Abstract. Let n be a positive integer less than 100 which can be expressed as the sum of two or more distinct squares of integers. We ask when a rectangle of area n with sides of integer length can be dissected into different squares with just one of the squares cut, and produce several examples. We also present some rectangular dissections where the cut square satisfies the further constraint that the two pieces are rectangular.

1. Introduction

A classical problem in combinatorial geometry asks if it is possible to dissect a non-square rectangle into a finite number of integer-sided squares, no two of which have the same size. This problem was solved by the Polish mathematician Zbigniew Moron [1] in 1925, who gave an example of a $32 \times 33 = 1056$ rectangle which can be dissected into nine squares of sides $\{18, 15, 14, 10, 9, 8, 7, 4, 1\}$ like so:



He also showed that this is the smallest integer example and that, at least, nine squares are necessary.

For smaller integer-sided rectangles and $n < 9$ squares, we ask when a rectangle can be dissected into squares if we allow some of the squares to be cut. Clearly if we can achieve our objective with just one square cut, then this is the best possible result.

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In this note we produce some examples of this situation, e.g. $30 = 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 5$ is the sum of four distinct squares and we achieve a $5 = 4 + 1$ piece dissection of a 6×5 rectangle so that the pieces can be reassembled to form four distinct squares. In some cases the cut square consists of two rectangular pieces – this situation we refer to as a rectangular dissection (R). It involves an extra constraint which is rarely satisfied.

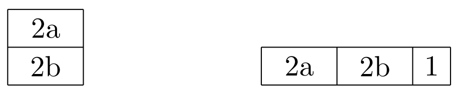
Of course, there are some cases where our objectives cannot be realised. For example, $17 = 4^2 + 1^2$, but a 17×1 rectangle needs at least a five piece dissection to form a 4-square and a 1-square. Also, some integers, for example, 15, are not the sum of distinct squares.

The situation we are looking at for small non-square rectangles appears to differ from that of small squares. See [2]. For example, a dissection of a 5-square to form a 4-square and a 3-square appears to need $4 = 2 + 2$ pieces, based on $5^2 = 4^2 + 3^2$. Intuitively, the unequal length and breadth of a non-square rectangle give more room for manoeuvre. In addition, at least 21 squares are needed to dissect a square into unequal squares.

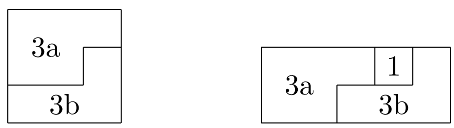
We include the integer equations for which we have failed to find one-cut dissections and where it is not obvious, to us, that no such dissections exist. We would like to hear from readers who have succeeded with some of these. We observe that outside of the one-cut situation, proofs can be extremely difficult and tricky. In this note, we confine ourselves to integer sided rectangles of area less than 100.

2. The Examples

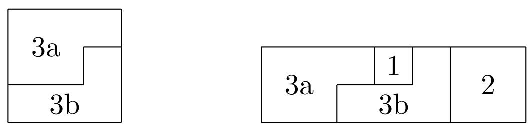
$5 = 2^2 + 1^2 = 5 \times 1$
(R)



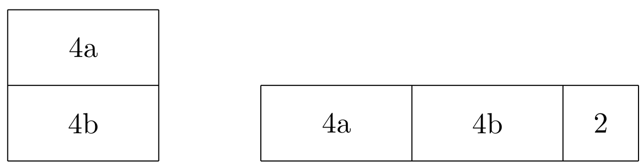
$10 = 3^2 + 1^2 = 5 \times 2$



$14 = 3^2 + 2^2 + 1^2 = 7 \times 2$

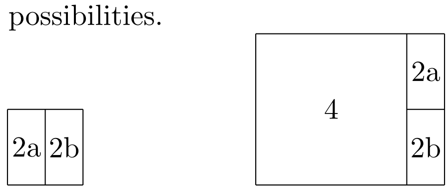


$20 = 4^2 + 2^2 = 10 \times 2$
(R)

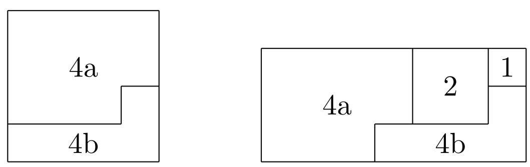


It may be objected that this is merely a ‘blow-up’ of the 5×1 case, but sometimes increasing the scale leads to new possibilities.

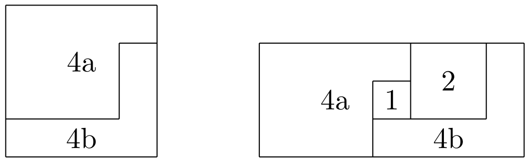
$20 = 4^2 + 2^2 = 5 \times 4$
(R)



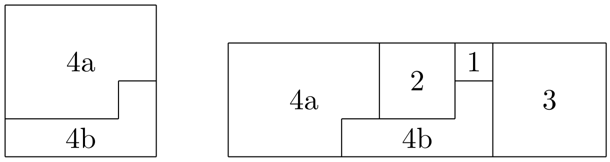
$21 = 4^2 + 2^2 + 1^2 = 7 \times 3$



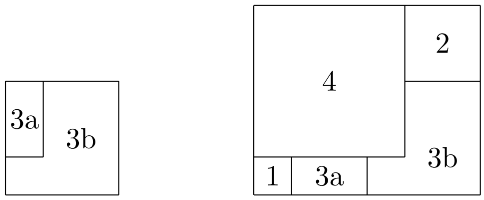
$21 = 4^2 + 2^2 + 1^2 = 7 \times 3$



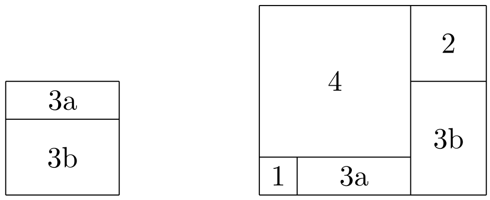
$30 = 4^2 + 3^2 + 2^2 + 1^2$
$= 10 \times 3$



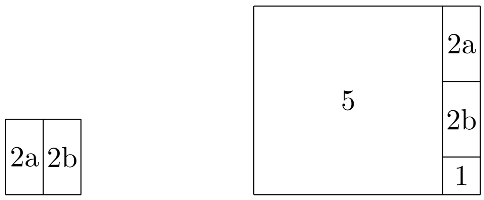
$30 = 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 5$



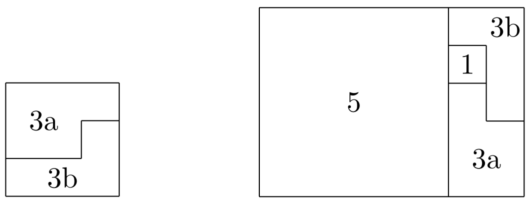
$30 = 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 5$
(R)



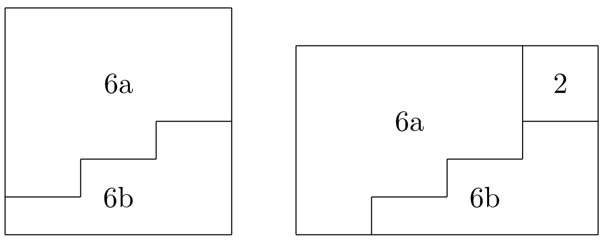
$30 = 5^2 + 2^2 + 1^2 = 6 \times 5$ i
(R)



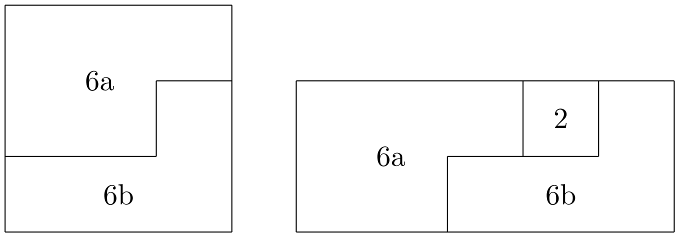
$35 = 5^2 + 3^2 + 1^2 = 7 \times 5$



$40 = 6^2 + 2^2 = 5 \times 8$

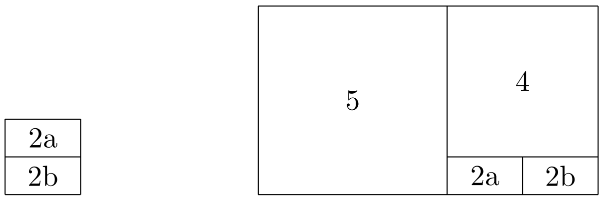


$40 = 6^2 + 2^2 = 10 \times 4$

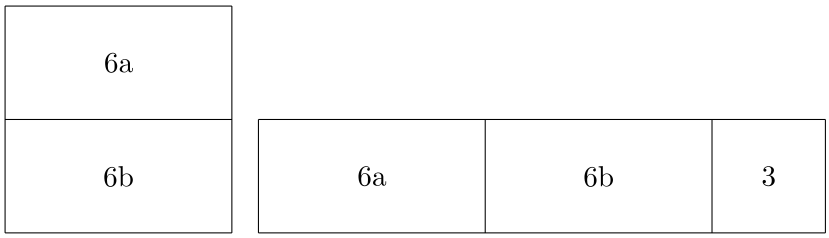


$42 = 5^2 + 4^2 + 1^2 = 6 \times 7$ Not found

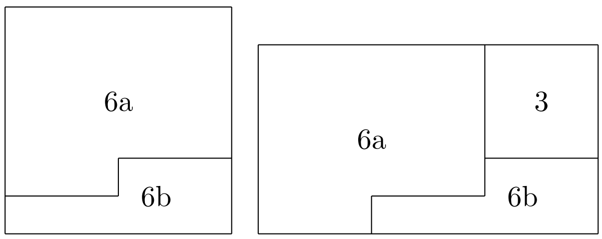
$45 = 5^2 + 4^2 + 2^2 = 5 \times 9$
(R)



$45 = 6^2 + 3^2 = 3 \times 15$
(R)



$45 = 6^2 + 3^2 = 3 \times 15$

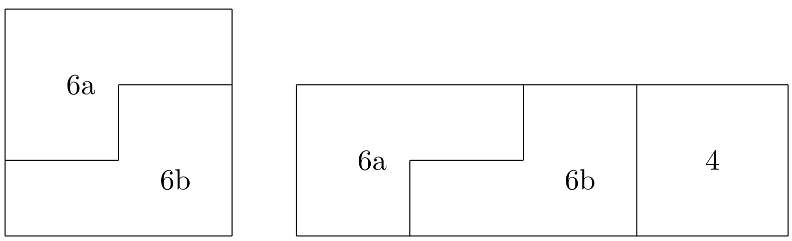


$50 = 7^2 + 1^2 = 5 \times 10$ Not found

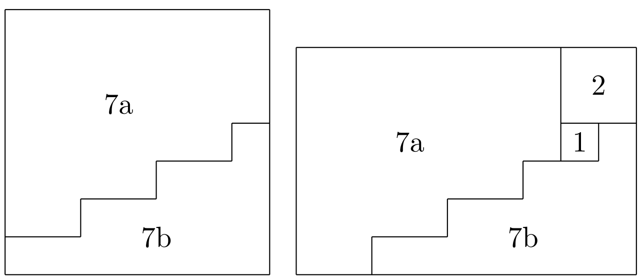
$50 = 6^2 + 3^2 + 2^2 + 1^2 = 5 \times 10$ Not found

$50 = 5^2 + 4^2 + 3^2 = 5 \times 10$ Not found

$52 = 6^2 + 4^2 = 4 \times 13$

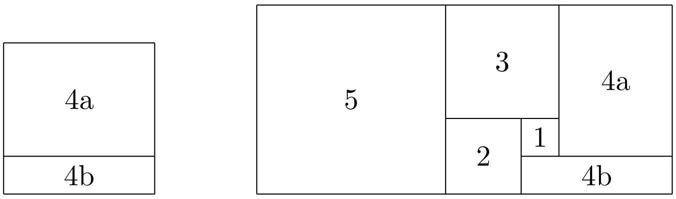


$54 = 7^2 + 2^2 + 1^2 = 6 \times 9$

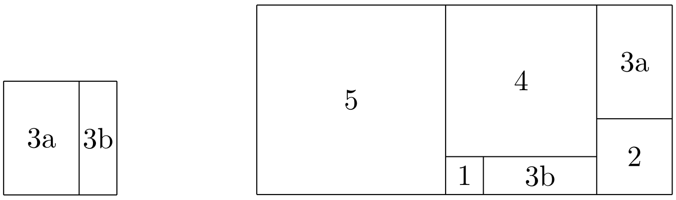


$54 = 5^2 + 4^2 + 3^2 + 2^2 = 6 \times 9$ Not found

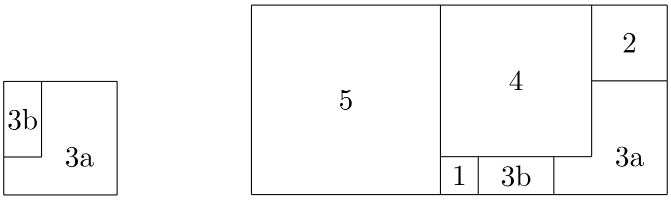
$55 = 5^2{+}4^2{+}3^2{+}2^2{+}1^2$
$= 5 \times 11$ (R)



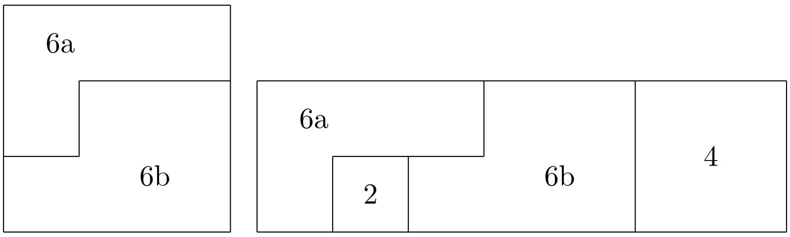
$55 = 5^2{+}4^2{+}3^2{+}2^2{+}1^2$
$= 5 \times 11$ (R)



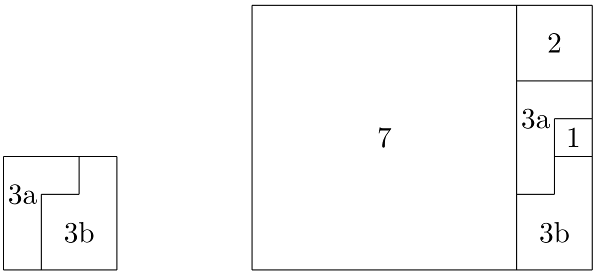
$55 = 5^2{+}4^2{+}3^2{+}2^2{+}1^2$
$= 5 \times 11$



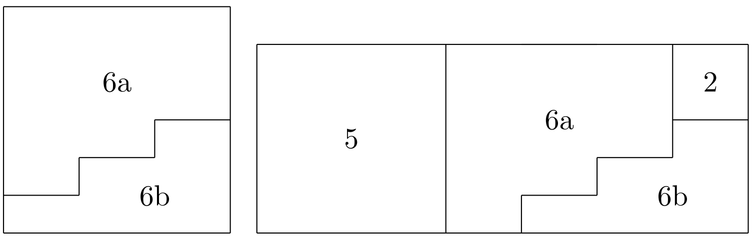
$56 = 6^2 + 4^2 + 2^2$
$= 4 \times 14$



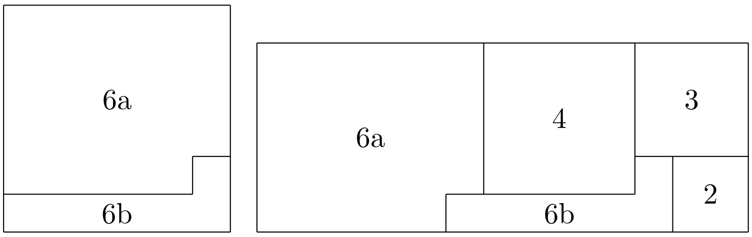
$63 = 7^2 + 3^2 + 2^2 + 1^2 = 7 \times 9$



$65 = 6^2 + 5^2 + 2^2$
$= 5 \times 13$



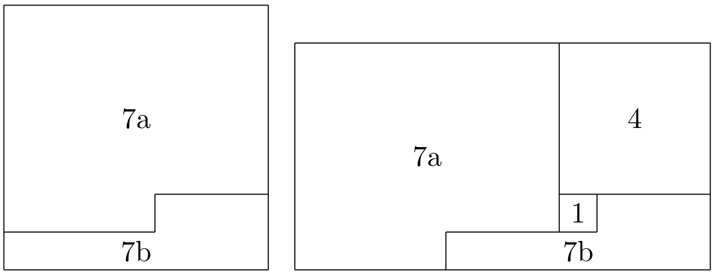
$65 = 6^2 + 4^2 + 3^2 + 2^2$
$= 5 \times 13$



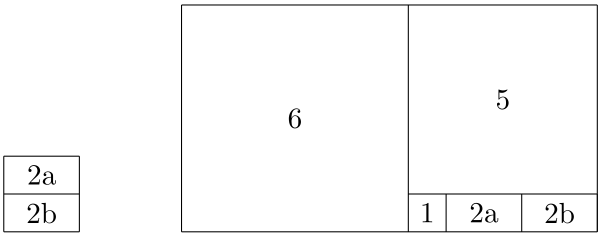
$65 = 8^2 + 1^2 = 5 \times 13$ **Not found**

$65 = 7^2 + 4^2 = 5 \times 13$ **Not found**

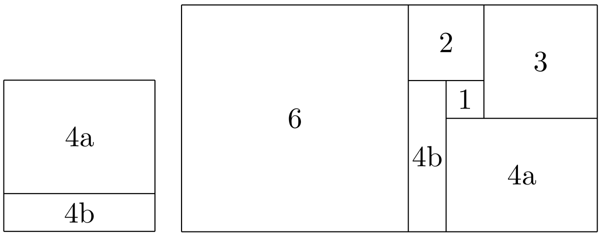
$66 = 7^2 + 4^2 + 1^2 = 6 \times 11$



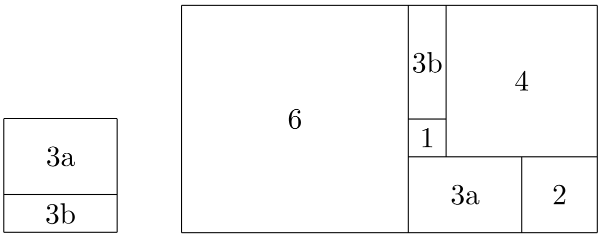
$66 = 6^2 + 5^2 + 2^2 + 1^2 = 6 \times 11$
(R)



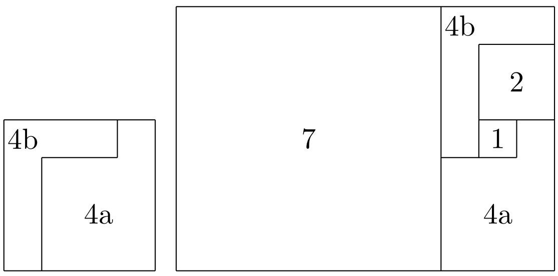
$66 = 6^2 + 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 11$
(R)



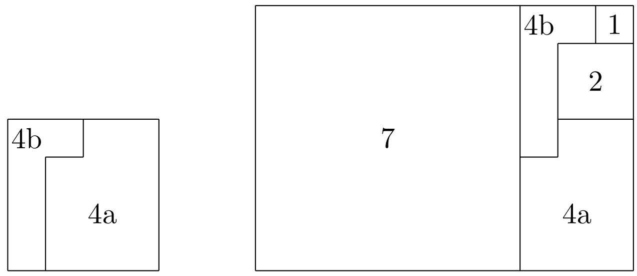
$66 = 6^2 + 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 11$
(R)



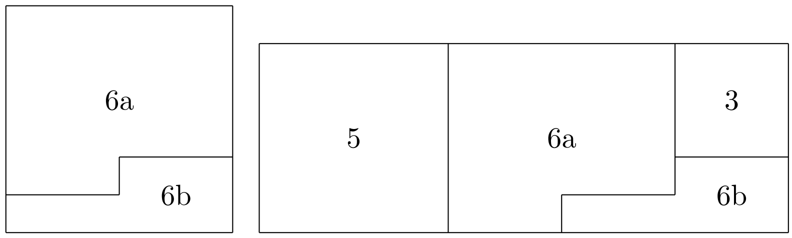
$70 = 7^2 + 4^2 + 2^2 + 1^2 = 7 \times 10$



$70 = 7^2 + 4^2 + 2^2 + 1^2$
$= 7 \times 10$

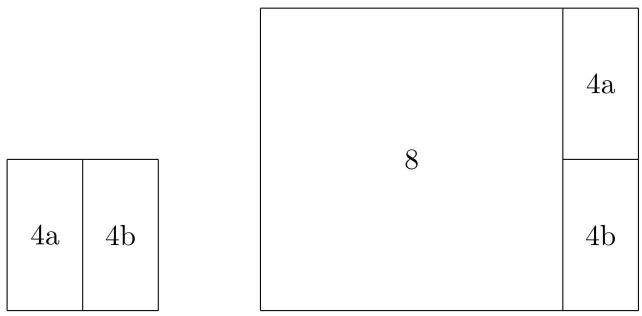


$70 = 6^2 + 5^2 + 3^2$
$= 5 \times 14$

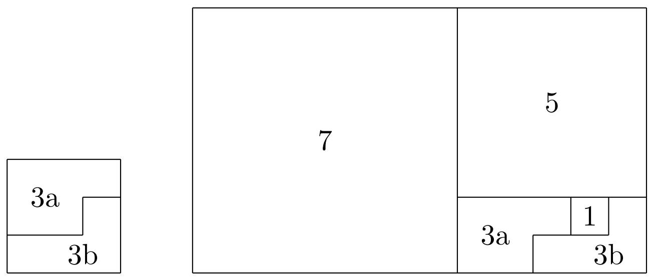


$75 = 7^2 + 5^2 + 1^2 = 5 \times 15$ Not found
$75 = 7^2 + 4^2 + 3^2 + 1^2 = 5 \times 15$ Not found
$75 = 6^2 + 5^2 + 3^2 + 2^2 + 1^2 = 5 \times 15$ Not found
$77 = 8^2 + 3^2 + 2^2 = 7 \times 11$ Not found
$77 = 6^2 + 5^2 + 4^2 = 7 \times 11$ Not found
$78 = 8^2 + 3^2 + 2^2 + 1^2 = 6 \times 13$ Not found
$78 = 7^2 + 5^2 + 2^2 = 6 \times 13$ Not found
$78 = 7^2 + 4^2 + 3^2 + 2^2 = 6 \times 13$ Not found
$78 = 6^2 + 5^2 + 4^2 + 1^2 = 6 \times 13$ Not found

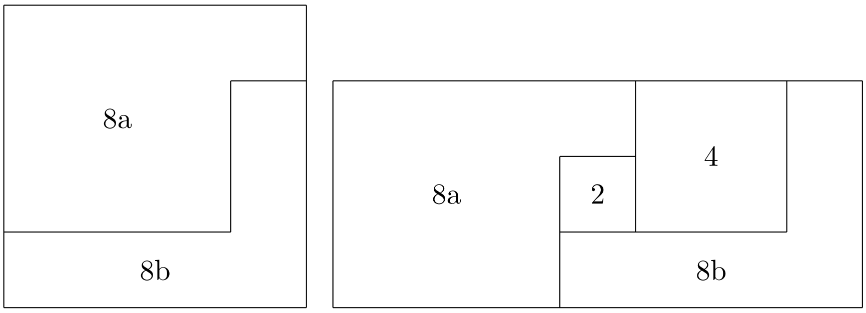
$80 = 8^2 + 4^2 = 8 \times 10$
(R)



$84 = 7^2 + 5^2 + 3^2 + 1^2$
$= 7 \times 12$



$84 = 8^2 + 4^2 + 2^2$
$= 6 \times 14$

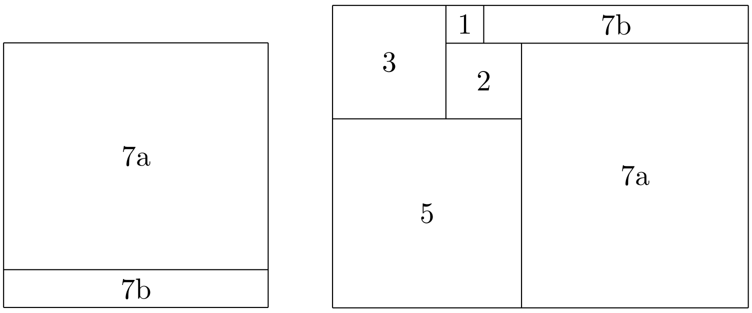


$84 = 7^2 + 5^2 + 3^2 + 1^2 = 6 \times 14$ Not found

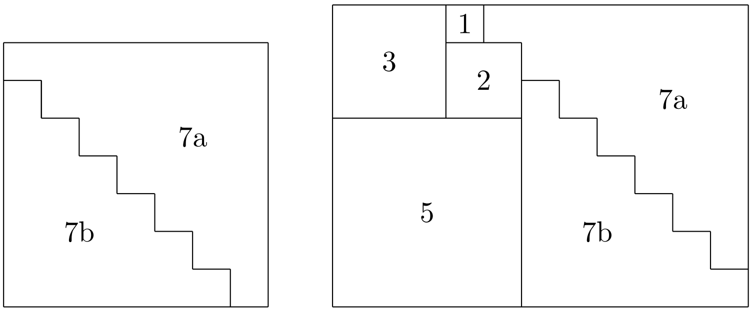
$85 = 9^2 + 2^2 = 5 \times 17$ Not found

$85 = 8^2 + 4^2 +^2 2 + 1^2 = 5 \times 17$ Not found

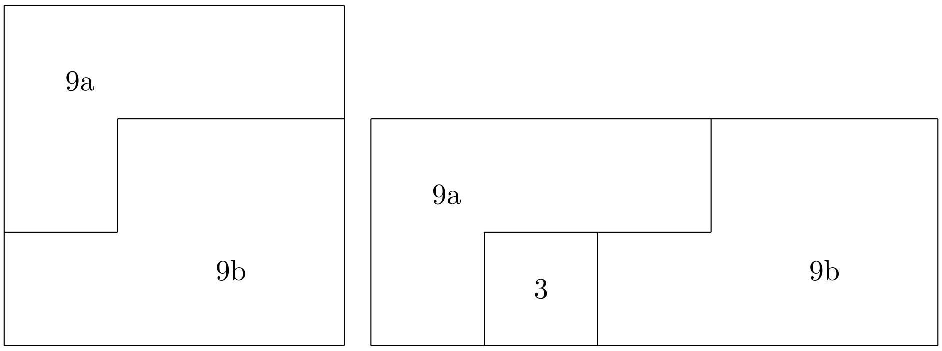
$88 = 7^2 + 5^2 + 3^2 + 2^2 + 1^2$
$= 8 \times 11$ (R)



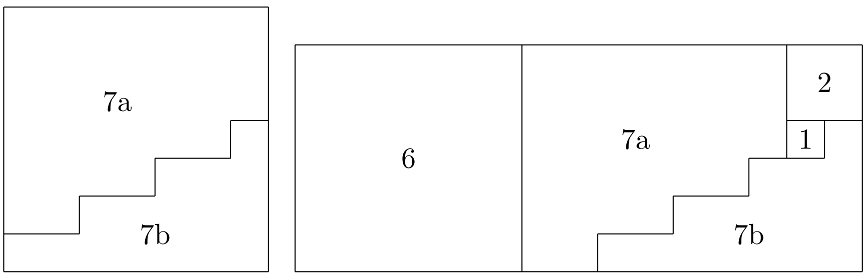
$88 = 7^2 + 5^2 + 3^2 + 2^2 + 1^2$
$= 8 \times 11$



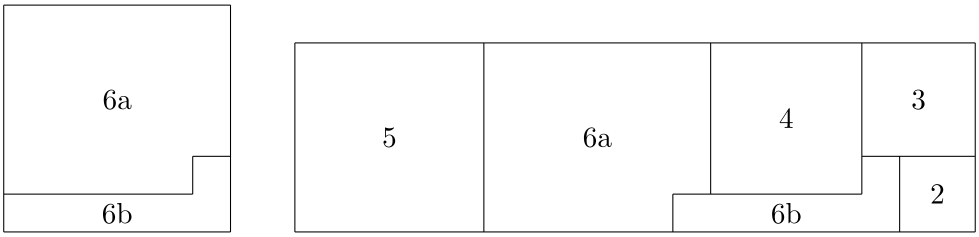
$90 = 9^2 + 3^2$
$= 6 \times 15$



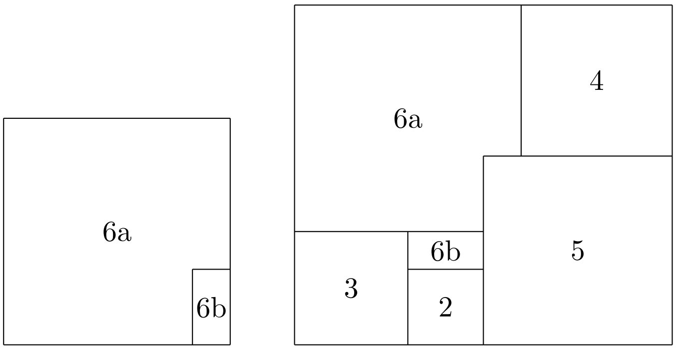
$90 = 7^2 + 6^2 + 2^2 + 1^2$
$= 6 \times 15$



$90 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 = 5 \times 18$



$90 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2$
$= 9 \times 10$



$90 = 8^2 + 5^2 + 1^2 = 5 \times 18$ **Not found**

$90 = 8^2 + 4^2 + 3^2 + 1^2 = 5 \times 18$ **Not found**

$90 = 7^2 + 5^2 + 4^2 = 5 \times 18$ **Not found**

$90 = 8^2 + 5^2 + 1^2 = 6 \times 15$ **Not found**

$90 = 8^2 + 4^2 + 3^2 + 1^2 = 6 \times 15$ **Not found**

$90 = 7^2 + 5^2 + 4^2 = 6 \times 15$ **Not found**

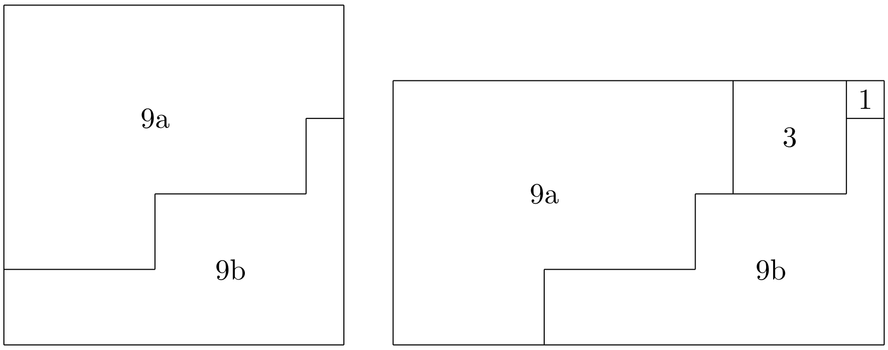
$90 = 8^2 + 5^2 + 1^2 = 9 \times 10$ **Not found**

$90 = 8^2 + 4^2 + 3^2 + 1^2 = 9 \times 10$ **Not found**

$90 = 7^2 + 5^2 + 4^2 = 9 \times 10$ **Not found**

$90 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 = 6 \times 15$ **Not found**

$91 = 9^2 + 3^2 + 1^2$
$= 7 \times 13$



$91 = 7^2 + 5^2 + 4^2 + 1^2 = 7 \times 13$ **Not found**

$91 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2 = 7 \times 13$ **Not found**

$95 = 9^2 + 3^2 + 2^2 + 1^2 = 5 \times 19$ **Not found**

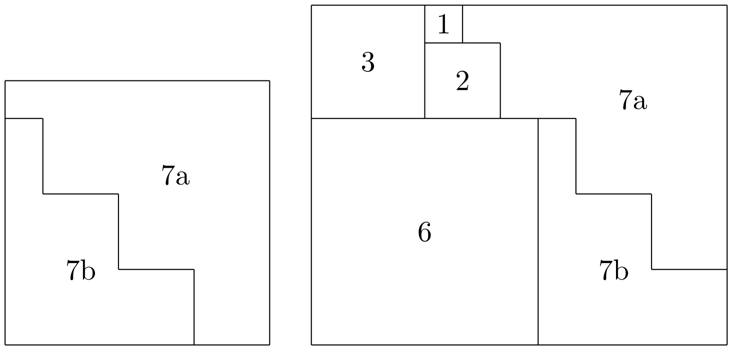
$95 = 7^2 + 5^2 + 4^2 + 2^2 + 1^2 = 5 \times 19$ **Not found**

$98 = 9^2 + 4^2 + 1^2 = 7 \times 14$ **Not found**

$98 = 8^2 + 5^2 + 3^2 = 7 \times 14$ **Not found**

$98 = 7^2 + 6^2 + 3^2 + 2^2 = 7 \times 14$ **Not found**

$99 = 7^2+6^2+3^2+2^2+1^2$
$= 9 \times 11$



$99 = 8^2 + 5^2 + 3^2 + 1^2 = 9 \times 11$ **Not found**

$99 = 7^2 + 5^2 + 4^2 + 3^2 = 9 \times 11$ **Not found**

References

[1] Z. Moron: *On the dissection of rectangles into squares*, Wiadem Mat. (1) 1 (1925), pp. 75—79.
[2] J. Kingston and D. MacHale: *Dissecting Squares*, Mathematical Gazette, Vol 85, Issue 504, November 2001, pp. 403—430.

Joe Kingston taught Mathematics in Hamilton High School, Bandon, for thirty nine years. In the mid-nineties, while studying for a Master's degree at University College, Cork, Des MacHale introduced him to dissections. The introduction should have come with a health warning. Recovery is slow. Other interests include bridge, films, music, puzzles and reading. **Des MacHale** is Emeritus Professor of Mathematics at University College Cork where he taught for forty years. His mathematical interests are in abstract algebra, especially groups and rings, but he has also worked in number theory, Euclidean geometry, combinatorics and the history of mathematics. His other interests include humour, puzzles, words and geology.

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