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On a certain double integral representation of Catalan's constant and other interesting integration formulae

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Abstract. In this note, we discuss an almost certainly known but unfamiliar double integral representation for Catalan's constant, based on a classical trigonometric integral formula. From this foundation, we also derive some interesting integral identities involving a combination of logarithmic and inverse tangent functions.

1. Catalan's constant

Catalan's constant, often denoted by G, is the alternating sum

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots .$$

It is named after the Belgian mathematician Eugène Catalan (1814-1894), who undertook a comprehensive study of it in 1865. There are many representations of Catalan's constant, both as series and integrals; see Bradley [3]. Many other formulae can be found in classical references such as Gradshteyn and Ryzhik [4] and the three-volume collection by Berndt [2].

The simplest integral representation of G seems to be that coming from the arctangent power series

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

In fact, if we divide by x and integrate from 0 to 1 then we obtain

$$G = \int_0^1 \frac{\arctan x}{x}\, dx. \tag{1}$$

Arguably, the easiest way of justifying interchanging summation and integration above is by writing

$$\frac{\arctan x}{x} = \sum_{n=0}^{N} \frac{(-1)^n x^{2n}}{2n+1} + r_N(x)$$

and noting that, since the series is alternating with terms decreasing in magnitude, we have $|r_N(x)| \leqslant x^{2N+2}/(2N+3)$, so that

$$\lim_{N\to\infty} \int_0^1 r_N(x)\, dx = 0.$$

By substituting $x = \tan\varphi$ into (1) and subsequently setting $\theta = 2\varphi$, we obtain

$$G = \int_0^{\pi/4} \frac{\varphi}{\sin\varphi\cos\varphi}\, d\varphi = \frac{1}{2}\int_0^{\pi/2} \frac{\theta}{\sin\theta}\, d\theta. \tag{2}$$

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Moreover, by noticing that

$$\frac{1}{\sin\varphi\cos\varphi} = \frac{\sec^2\varphi}{\tan\varphi} = \frac{d}{d\varphi}\ln(\tan\varphi),$$

integration by parts in the middle expression in (2) yields

$$G = -\int_0^{\pi/4} \ln(\tan\varphi)\, d\varphi. \tag{3}$$

More generally, the following holds.

Lemma 1.1. *For all* $p = 0, 1, 2, \ldots,$

$$\int_0^{\pi/2} \frac{\theta^{p+1}}{\sin\theta}\, d\theta = -2^{p+1}(p+1)\int_0^{\pi/4} \varphi^p \ln(\tan\varphi)\, d\varphi.$$

Proof. By starting with the integral on the right, perform integration by parts (with $u = \varphi^p \ln(\tan\varphi)$ and $dv = d\varphi$), using the derivative

$$\frac{d}{d\varphi}[\varphi^p \ln(\tan\varphi)] = p\,\varphi^{p-1}\ln(\tan\varphi) + \frac{\varphi^p}{\sin\varphi\cos\varphi}.$$

To conclude, make the change of variables $\theta = 2\varphi$. □

For $p = 0$, Lemma 1.1 is just the equality between the right-hand integrals in (2) and (3). For $p = 1, 2$, it well known that

$$\int_0^{\pi/2} \frac{\theta^2}{\sin\theta}\, d\theta = 2\pi G - \frac{7}{2}\zeta(3) \tag{4}$$

and

$$\int_0^{\pi/2} \frac{\theta^3}{\sin\theta}\, d\theta = \frac{3\pi^2}{2}G - 12\beta(4), \tag{5}$$

where $\zeta(3)$ is *Apéry's constant*, namely, the value for $s = 3$ of the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots,$$

and $\beta(4)$ is the value for $s = 4$ of the *Dirichlet beta function*

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots. \tag{6}$$

Note that $\beta(2) = G$. The standard way of deriving formulae (4) and (5) is by using the Fourier series of $\ln(\tan\varphi) = \ln(\sin\theta) - \ln(\cos\varphi)$, see Tolstov [7, Sect. 3.14]. A more general description of the corresponding indefinite integrals in Lemma 1.1 as certain Fourier series can be found in Berndt [2, Part I: p. 261, Entry 14]. By using the Laurent expansion of the co-secant function, one can also express the integrals in Lemma 1.1 as a series involving powers of π and the Bernoulli numbers; see e.g. Sofo and Nimbran [6, Lemma 2.2].

2. An interesting double integral representation of G

There are also some representations of G as double integrals, the most basic being arguably

$$G = \int_0^1\int_0^1 \frac{dx\,dy}{1 + x^2y^2}.$$

This representation can be established directly from (1); see Bradley [3, Formula (40)]. Here, we will prove that

$$G = \int_0^{\pi/2} \int_0^1 \frac{d\theta\, dx}{1 + 2x\cos\theta + x^2}. \tag{7}$$

The proof of (7) will be based on the following classical formula.

Proposition 2.1. *For all* $0 \leqslant x < 1$,

$$\int_0^{\pi/2} \frac{d\theta}{1 + 2x\cos\theta + x^2} = \frac{2}{1 - x^2} \arctan\frac{1 - x}{1 + x}. \tag{8}$$

Proof. Using the rational parametrization $\cos\theta = (1 - t^2)/(1 + t^2)$, the integral on the left in (8) equals

$$\begin{aligned}
\int_0^1 \frac{1}{1 + 2x \cdot \dfrac{1 - t^2}{1 + t^2} + x^2} \frac{2\,dt}{1 + t^2} &= \int_0^1 \frac{2\,dt}{(1 + x)^2 + (1 - x)^2 t^2} \\
&= \frac{2}{(1 - x)^2} \int_0^1 \frac{dt}{\left(\dfrac{1 + x}{1 - x}\right)^2 + t^2} \\
&= \frac{2}{1 - x^2} \arctan\frac{1 - x}{1 + x},
\end{aligned}$$

where in the last equality we have used

$$\int \frac{dt}{a^2 + t^2} = \frac{1}{a} \arctan\frac{t}{a} + C. \qquad \square$$

Now, to prove (7), we integrate (8) over x, from 0 to 1, which yields

$$\begin{aligned}
\int_0^1 \left[\int_0^{\pi/2} \frac{1}{1 + 2x\cos\theta + x^2}\, d\theta\right] dx &= \int_0^1 \frac{2}{1 - x^2} \arctan\frac{1 - x}{1 + x}\, dx \\
&= \int_0^1 \frac{\arctan y}{y}\, dy \\
&= G,
\end{aligned}$$

where the second identity follows by the change of variables $y = (1 - x)/(1 + x)$ and the third follows by (1).

We might ask what happens if we interchange the order of integration in the iterated integral above. The conclusion, in brief, is that nothing particularly interesting arises. In fact,

$$\begin{aligned}
\int_0^1 \frac{dx}{1 + 2x\cos\theta + x^2} &= \int_0^1 \frac{dx}{\sin^2\theta + (x + \cos\theta)^2} \\
&= \frac{1}{\sin\theta} \arctan\frac{x + \cos\theta}{\sin\theta}\Big|_{x=0}^{x=1} \\
&= \frac{1}{\sin\theta}\left[\arctan\frac{1 + \cos\theta}{\sin\theta} - \arctan\frac{\cos\theta}{\sin\theta}\right] \\
&= \frac{1}{\sin\theta} \arctan\frac{\sin\theta}{1 + \cos\theta} \\
&= \frac{\theta}{2\sin\theta},
\end{aligned}$$

and a further integration over θ, from 0 to $\pi/2$, simply yields (2).

Next, we explore identity (8) in other directions.

3. An elegant integration formula

Consider the following classical formulae, both valid for $0 < \theta < \pi$,

$$\int_0^\infty \frac{\ln x}{1 + 2x\cos\theta + x^2}\, dx = 0, \tag{9}$$

and

$$\int_0^1 \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\, dx = \frac{\theta(\pi^2 - \theta^2)}{6\sin\theta}. \tag{10}$$

Formula (9) appears in Gradshteyn and Ryzhik [4, (4.233-5)] and can be easily verified by changing x to $1/x$, which makes the integral equal to its negative, implying its value is 0. Formula (10) appears in Gradshteyn and Ryzhik [4, (4.261-1)], without proof but with a reference to the 1867 publication *Nouvelles tables d'intégrales définies*, by Bierens de Haan, which in turn refers to an even earlier publication.

We will not try to prove (10) here, but we may notice that changing x to $1/x$ yields

$$\int_0^1 \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\, dx = \int_1^\infty \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\, dx,$$

which implies

$$\int_0^\infty \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\, dx = \frac{\theta(\pi^2 - \theta^2)}{3\sin\theta} \quad (0 < \theta < \pi). \tag{11}$$

If we multiply (8) by $\ln x$, then integrate over x, from 0 to $+\infty$, and interchange the order of integration on the left side, we obtain (using (9)),

$$\int_0^\infty \frac{\ln x}{1 - x^2} \arctan\frac{1-x}{1+x}\, dx = \frac{1}{2}\int_0^{\pi/2} \Big[\int_0^\infty \frac{\ln x}{1 + 2x\cos\theta + x^2}\, dx\Big]\, d\theta = 0.$$

This is also derived by simply changing x to $1/x$ in the integral on the left, with no need of formula (9). The same procedure, this time multiplying (8) by $\ln^2 x$, integrating from 0 to 1, and using (10), yields

$$\begin{aligned}\int_0^1 \frac{\ln^2 x}{1 - x^2} \arctan\frac{1-x}{1+x}\, dx &= \frac{1}{2}\int_0^{\pi/2} \Big[\int_0^1 \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\, dx\Big]\, d\theta \\ &= \frac{1}{12}\Big[\pi^2 \int_0^{\pi/2} \frac{\theta}{\sin\theta}\, d\theta - \int_0^{\pi/2} \frac{\theta^3}{\sin\theta}\, d\theta\Big].\end{aligned}$$

On the right, the first integral inside brackets equals $2G$, by (2). Combining this with (4) we obtain the interesting formula

$$\int_0^1 \frac{\ln^2 x}{1 - x^2} \arctan\frac{1-x}{1+x}\, dx = \frac{\pi^2 G}{24} + \beta(4). \tag{12}$$

Note that, in light of (11), the corresponding integral from 0 to $+\infty$ is twice that in (12). Moreover, changing $(1-x)/(1+x)$ to x yields the equally interesting

$$\int_0^1 \ln^2\Big(\frac{1-x}{1+x}\Big) \frac{\arctan x}{x}\, dx = \frac{\pi^2 G}{12} + 2\beta(4). \tag{13}$$

4. Final thoughts: the Basel problem

It is in all likelihood an overstatement to assert that the identity (7) is new and has never been highlighted before. It must be observed, however, that it does not appear, for instance, in Bradley's comprehensive list [3], and despite our best efforts, we were unable to find any record of it in the literature. On the other hand, the computation following the proof of (8), which shows that (7) is essentially (2), renders this double integral representation of G quite natural.

The same goes with formulae (12) and (13). There are some close relatives, for instance, in Vălean's books [8, 1.20, 1.21, 1.24, 1.26] and [9, 1.36, 1.37, 1.38, 1.57, 1.58]. By 'close relative' we mean any integral formula involving logarithms multiplied by inverse tangents divided by polynomials. In the event that those identities are already known, we believe and hope that, at least, the evaluations presented here may be a novel and interesting contribution.

The integral in (8) is more often considered over the intervals $[0, \pi]$ or $[0, 2\pi]$. There are many such formulas in various sections of Gradshteyn and Ryzhik [4]. In particular, Gradshteyn and Ryzhik [4, (3.792-1)] is essentially

$$\int_0^\pi \frac{d\theta}{1 + 2x\cos\theta + x^2} = \frac{\pi}{1 - x^2}, \tag{14}$$

valid for $-1 < x < 1$. As with (8), this is easily obtained using rational parametrization. Now, if we multiply (14) by $\ln x$, then integrate over x, from 0 to 1, and interchange the order of integration on the left side, we obtain

$$\int_0^\pi \Big[\int_0^1 \frac{\ln x}{1 + 2x\cos\theta + x^2}\,dx\Big]\,d\theta = \pi\int_0^1 \frac{\ln x}{1 - x^2}\,dx. \tag{15}$$

As it is well-known, the integral on the right-hand side is related to the so called *Basel problem*, namely, the problem of numerically evaluating the series

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots .$$

This was first solved by Euler in 1734, who showed $\zeta(2) = \pi^2/6$. The connection between (15) and the Basel problem is

$$\int_0^1 \frac{\ln x}{x^2 - 1}\,dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4}\zeta(2). \tag{16}$$

In fact, the first identity above can be established by expanding $(1 - x^2)^{-1}$ in geometric series and using the formula (obtained with integration by parts)

$$\int_0^1 x^{2n} \ln x\,dx = -\frac{1}{(2n+1)^2},$$

after interchanging integration with summation; the second identity comes from splitting $\sum n^{-2}$ into odd and even indices. The leftmost integral in (16) is known to be $\pi^2/8$, a result obtainable independently of the Basel problem. For more details, see e.g. Abreu [1].

Unfortunately, despite (9), the innermost integral on the left side in (15) is not known as a function of θ in terms of elementary functions; in fact,

$$\int_0^1 \frac{\ln x}{1 + 2x\cos\theta + x^2}\,dx = -\frac{\mathrm{Cl}_2(\pi - \theta)}{\sin\theta} \quad (0 < \theta < \pi),$$

where Cl_2 denotes the *Clausen function of order two*, see Moll and Posey [5]. Thus, the Basel problem is equivalent to

$$\int_0^\pi \frac{\mathrm{Cl}_2(\theta)}{\sin\theta}\,d\theta = \frac{\pi^3}{8}.$$

This connection, despite being possibly familiar to the experts in the field (Clausen functions, polylogarithms, etc.), does not seem to be widely known. For those not familiar with these special functions, or not wishing to delve deeper into these matters, it suffices to say that any elementary evaluation of the double integral in (15), yielding the value $-\pi^3/8$, would constitute a genuinely new solution to the Basel problem.

Finally, by an analogous reasoning, we have

$$\int_0^1 \frac{\ln^2 x}{x^2-1}\,dx = -2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = -\frac{7}{4}\zeta(3), \tag{17}$$

this time using the formula

$$\int_0^1 x^{2n}\ln^2 x\,dx = \frac{2}{(2n+1)^3}.$$

Then, using the elementary identity $\arctan(1/u) = \pi/2 - \arctan(u)$ (valid for $u > 0$) in (12), combined with (17), yields the integration formula

$$\int_0^1 \frac{\ln^2 x}{1-x^2}\arctan\frac{1+x}{1-x}\,dx = \frac{7\pi}{8}\zeta(3) - \frac{\pi^2 G}{24} - \beta(4).$$

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References

[1] J. Abreu: *An Elementary Solution of the Basel Problem*, Amer. Math. Monthly (132) 8 (2025), 813-817.

[2] B. C. Berndt: *Ramanujan's Notebooks: Parts I, II and III*, Springer-Verlag, New York (NY), 1985 (Part I), 1989 (Part II), 1991 (Part III).

[3] D. M. Bradley: *Representations of Catalan's Constant*, 2001. Available at `www.researchgate.net/publication/2325473`.

[4] I. S. Gradshteyn and I. M Ryzhik: *Table of Integrals, Series, and Products*, Academic Press, 8th edition, 2015.

[5] V. H. Moll and R. A. Posey: *The integrals in Gradshteyn and Ryzhik. Part 12: Some logarithmic integrals*, Scientia (18) (2009), 77-84.

[6] A. Sofo and A. S. Nimbran: *Integrals with powers of the arctan function via Euler sums*, Mem. Fac. Sci. Eng. Shimane Univ. (56) (2023), 1–17.

[7] G. P. Tolstov: *Fourier Series*, Dover Publications, New York (NY), 1976.

[8] C. I. Vălean: *(Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, Springer, Cham (Switzerland), 2019.

[9] C. I. Vălean: *More (Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, Springer, Cham (Switzerland), 2023.

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