Irish Mathematical Society
Cumann Matamaitice na hÉireann

Bulletin

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Irish Mathematical Society
Bulletin

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

Correspondence concerning books for review in the *Bulletin* should be directed to

`mailto://reviews.ims@gmail.com`

All other correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

`mailto://ims.bulletin@gmail.com`

and only if not possible in electronic form to the address

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Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

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Irish Mathematical Society Bulletin 96, Winter 2025

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EDITORIAL

The year 2026 marks the 50$^{\text{th}}$ anniversary of the founding of the Irish Mathematical Society on 14$^{\text{th}}$ April 1976 in Trinity College Dublin. To mark the occasion, the Summer 2026 issue of this Bulletin will be a special one - see the email from Rachel Quinlan, President, to the membership dated 21$^{\text{st}}$ August last. Quoting from her email: 'Papers are invited on all topics relevant to the Irish mathematical landscape in the last 50 years and into the future, including (but not limited to)

- research articles
- survey articles with an Irish context
- history of mathematics in Ireland (and of the IMS)
- interviews and biographies
- mathematics education (including student experiences)
- mathematical outreach and community engagement
- recreational mathematics
- student events (e.g. mathematical olympiads).

Submissions intended for this special issue should mention "Summer 2026 special issue" in the subject line.' A deadline of 20$^{\text{th}}$ February 2026 is mentioned, though I can work around this within reason (with the cooperation of reviewers). The submission process can be found on the Bulletin webpage. Should you wish to discuss informally any ideas you may have for this special issue, feel free to email me at `ims.bulletin@gmail.com`.

This editor is always keen to receive articles 'written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community'. It is natural to think of the Mathematical Proceedings of the Royal Irish Academy (MPRIA) as a sister publication to this Bulletin: indeed, these two are the only regular mathematics journals published in Ireland. Publication in MPRIA has recently become more attractive. Over and above being listed on Scopus, all articles published in MPRIA during 2026 will be published open access (without author fees), regardless of author affiliation. This development is made possible by the RIA's participation in Project MUSE's Subscribe to Open (S2O) initiative. I would encourage members to consider if this Bulletin or MPRIA might be a suitable forum in which to publicise their work.

Finbarr Holland (UCC) informed me of the sad news that Jim Chadwick, a former UCC student from Tralee who would have been known to many of the more established readers of this Bulletin, passed away on 23$^{\text{rd}}$ November 2025. He was an outstanding student who wrote textbook-style answers to exam questions. He won the Travelling Studentship in 1968 and wrote his Ph.D. at the Australian National University under Ronald William Cross (according to the Mathematics Genealogy Project). He taught in UCC and TCD for brief periods before emigrating to South Africa where he headed a Computer Science Department in Grahamstown. Jim's regular visits home were a summer highlight for his many old friends: he was, by all accounts, a great conversationalist with a witty sense of humour and fun (rip.ie).

The variety of contributions to this issue is noteworthy. We have two particularly interesting reviews, one by Zhenwei Lyu of a book on AI by Susskind and one by Peter Lynch on a bird's eye view of mathematics by Thomas Waters. The latter author was born and grew up in Dublin, attended DCU where he earned his PhD under Brien Nolan and, inter alia, lectured at the University of Galway for three years. He has been a lecturer at the University of Portsmouth since 2010. Tony O'Farrell takes us through many diverse approaches to, and perspectives on, the Fundamental Theorem of Algebra. In a similar vein, Argerami and Moslehian take us on a tour of the trace operator in a variety of contexts. Des MacHale, who is a loyal long-term supporter of both this Bulletin and MPRIA, is co-author on two articles: one with Joe Kingston on dissections of rectangles into squares and a second with Michael Kinyon on providing equational proofs of commutativity theorems in rings. Dospra describes an algorithm for finding small solutions of bivariate linear congruences while Abreu presents some new representations of Catalan's Constant $G = 1 - 1/3^2 + 1/5^2 - 1/7^2 + \ldots$. Every issue of the Bulletin closes with the Problem Page, ably curated by J.P. McCarthy, who is always on the lookout for interesting problems.

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LINKS FOR POSTGRADUATE STUDY

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: mailto://maths@dcu.ie
TUD: mailto://chris.hills@tudublin.ie
ATU: mailto://leo.creedon@atu.ie
MTU: http://mathematics.mtu.ie/datascience
UG: mailto://james.cruickshank@universityofgalway.ie
MU: mailto://mathsstatspg@mu.ie
QUB:
https://www.qub.ac.uk/schools/SchoolofMathematicsandPhysics/Research/culture-environment/PostgraduateResearch/
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: https://www.ucc.ie/en/matsci/study-maths/postgraduate/#d.en.1274864
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://macsi@ul.ie

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor.

E-mail address: ims.bulletin@gmail.com

NOTICES FROM THE SOCIETY

Officers and Committee Members 2025

President	Dr Rachel Quinlan	UG
Vice-President	Prof. David Malone	MU
Secretary	Dr Derek Kitson	MIC
Treasurer	Dr Cónall Kelly	UCC

Assoc. Prof. C. Boyd, Dr R. Flatley, Dr R. Gaburro, Dr T. Huettemann, Dr P. Ó Catháin, Prof. A. O'Shea, Assoc. Prof. H. Šmigoc, Dr N. Snigireva.

Officers and Committee Members 2026

President	Dr Rachel Quinlan	UG
Vice-President	Prof. David Malone	MU
Secretary	Dr Derek Kitson	MIC
Treasurer	Dr Dana Mackey	TUD

Assoc. Prof. C. Boyd, Dr S. Dendrinos, Dr R. Gaburro, Dr T. Huettemann, Dr A. Krishnan, Dr P. Ó Catháin, Prof. A. O'Shea, Prof. K. Wendland.

Local Representatives

Belfast	QUB	Prof. M. Mathieu
Carlow	SETU	Dr D. Ó Sé
Cork	MTU	Dr J. P. McCarthy
	UCC	Dr S. Wills
Dublin	DIAS	Prof. T. Dorlas
	TUD, City	Dr D. Mackey
	TUD, Tallaght	Dr C. Stack
	DCU	Prof. B. Nolan
	TCD	Prof. K. Soodhalter
	UCD	Dr R. Levene
Dundalk	DKIT	Mr S. Bellew
Galway	UG	Dr J. Cruickshank
Limerick	MIC	Dr B. Kreussler
	UL	Dr R. Gaburro
Maynooth	MU	Prof. S. Buckley
Sligo	ATU	Dr L. Creedon
Tralee	MTU	Prof. B. Guilfoyle
Waterford	SETU	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers' Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€250
Ordinary member	€40
Lifetime member	€400
Student member	€20
DMV, IMTA, NZMS, MMS or RSME reciprocity member	€20
AMS reciprocity member	$25
LMS reciprocity member (paying in Euro)	€20
LMS reciprocity member (paying in Sterling)	£20

(3) The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$40.
If paid in sterling then the subscription is £30.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$40.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate.

(6) Those members who have reached 75 years of age, and who have been members in good financial standing with the Society for the previous 15 years, are entitled upon notification to the Treasurer to have their subscription rate reduced to €0.

(7) Subscriptions normally fall due on 1 February each year.

(8) Cheques should be made payable to the Irish Mathematical Society.

(9) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets three times each year.

(10) Please send the completed application form, available at
`https://www.irishmathsoc.org/business/imsapplicn_2024.pdf`
with one year's subscription, either by post or by email, to:

Dr Dana Mackey
School of Mathematics and Statistics
Technological University of Dublin
Central Quad, Grangegorman
Dublin DO7 ADY7
subscriptions.ims@gmail.com

Honorary Members

The Irish Mathematical Society is pleased to announce that it has the following new Honorary Member:

Professor David Conlon (California Institute of Technology).

PRESIDENT'S REPORT 2025

The Irish Mathematical Society is now in its 50$^{\text{th}}$ year, having been founded on 14$^{\text{th}}$ April 1976 in Trinity College Dublin. The society exists and continues to flourish because of the goodwill, energy and commitment of its members, who have maintained it and grown it as a community where people gather to do mathematics, and as a visible locus of mathematical scholarship in the island of Ireland.

A focus of attention for the Society at present is the celebration of its 50$^{\text{th}}$ birthday in 2026. This occasion will bring opportunities to reflect on the vision and initiative of the founding members, and of everyone who has built over the years on those first steps. It also brings a chance to look to the future, at a time when higher education faces opportunities and challenges that were unimaginable in 1976. The argument for preserving and valuing and communicating the human experience of mathematical discovery is more urgent and more compelling than ever. The task of making that argument loudly and clearly is in the hands of communities like ours. I was three when the IMS was founded. There are little children around the world now who will benefit from and cherish the work of the IMS in 2076 and in the years until then. We have a moment in 2026 to look to their future and at our evolution to date, and to be proud of all that has been achieved through collective effort.

The year 2025 has been another eventful one for the IMS. Our 15 new (ordinary, lifetime, student and reciprocity) members are warmly welcome. The Society supported seven conferences and workshops across a wide range of mathematical activities. These included the meeting of the European Mathematical Society Committee for Developing Countries, which was hosted in April at University College Cork. Members are encouraged to respond to the biannual calls for applications to the IMS Conference Support Fund. The level of support on offer is modest but it can make a difference, and the Committee tries to be as flexible and inclusive as possible in supporting diverse actitvies, in terms of location and subject matter.

The RIA-ICEDIM Women in the Mathematical Sciences Day took place at the University of Galway on May 12$^{\text{th}}$, which is International Women in Mathematics Day. The event was co-organised by the Irish Committee for Equality, Diversity and Inclusion in Mathematics (an IMS committee) and the Royal Irish Academy. We welcome this partnership and thank the RIA for their financial support of the meeting, which had over 50 participants and a fascinating and diverse programme of talks including a public lecture. Thanks and congratulations to the local organisers Niall Madden and Nina Snigireva, and to all the members of ICEDIM and the organising team.

The annual meeting of the IMS took place over two days at the end of August, organised by Steve Buckley, Galatia Cleanthous, Christian Ketterer and Ollie Mason. It was a scientifically and socially stimulating event, with an extensive programme of excellent talks across a wide range of specialisms. A huge thank you to the organisers of both the Women in the Mathematical Sciences Day and and the IMS annual meeting, and to everyone who contributed to both events by giving talks or presenting posters, or by taking the time and making the effort to attend and participate.

The IMS Christmas Lecture was delivered on December 15 by Robert Osburn, who was recently appointed to the George Boole Chair of Mathematics at University College Cork. The lecture, on *Unimodal sequences and mixed modularity*, was warmly appreciated by the audience. Thanks to Robert and to all who attended.

The nomination of David Conlon, Professor of Mathematics at the California Institute of Technology, for honorary membership was approved at the Annual General Meeting. David is a graduate of Trinity College Dublin and the University of Cambridge, and a world leading expert in combinatorics, particularly in Ramsey theory. He has maintained his mathematical links to Ireland throughout his international career. We welcome David to the IMS. Members are reminded that nominations for honorary membership are welcome at all times. A nomination requires a proposal to the IMS Committee from three members.

The Fergus Gaines Cup was presented to Justin Ang Yang Li for the highest performace in the Irish Mathematical Olympiad (IrMO) in 2025. The IMS congratulates Justin and all participants in the IrMO.

The IMS has established a new award for Distinguished Service, as a mechanism for recognizing and celebrating extraordinary contributions. The award will be presented once every three years, for the first time in 2026. The nomination process will open on the IMS website in January 2026, with a deadline of 1st March. The awardee will be announced at the 2026 annual meeting.

The IMS has enthusiastically endorsed the Glasgow ICM2030 bid to hold the 2030 International Congress of Mathematicians (ICM) in Glasgow and to hold the General Assembly (GA) of the International Mathematical Union in Dublin a few days before the Congress. The bid was formally submitted to the IMU in November 2025. The IMS commits to supporting the bidding process and the organisational effort (should the bid be successful) in all feasible ways. The decision on the venue for the 2030 ICM and GA will be made at the 2026 GA in New York City. The Glasgow/Dublin bid team is led by Professor Michael Wemyss of the University of Glasgow, and the Irish element is led by Professor David Wraith of Maynooth University.

The Summer 2026 issue of the IMS Bulletin will be a special one, dedicated to the 50th anniversary. Submissions are welcome on all topics relevant to the mathematical landscape in Ireland. These include articles on current research, history and biography, interviews, mathematical education, public engagement, student activities and the cultural role of mathematics. All members are strongly encouraged to submit articles on their interests to the special issue. The hope is that the issue will represent the diversity of mathematical activity in Ireland and showcase a scientifically and culturally engaged community that is open to interactions and active in all forms of mathematical expression.

The 2026 annual meeting will be hosted at Trinity College Dublin from 26th to 28th August. It will be the main focal point for the 50th anniversary celebration. Many thanks to local organisers Tommaso Cremaschi, Marvin Anas Hahn, Nicolas Maschot, Tristan McLoughlin and Katrin Wendland for their commitment to this highly anticipated event. Please save the dates! It will be great to see as many members as possible on some or all of the three days of the meeting.

Finally, a note of appreciation to everyone who has contributed to the activities of the IMS in 2025. Thanks to all the committee members, especially to Ronan Flatley, Helena Šmigoc and Nina Snigireva whose terms conclude in December this year. Particular thanks to Cónall Kelly who leaves the committee at the end of 2025 after six years of outstanding service in the role of treasurer. Thanks to Tom Carroll and all the editorial team of the IMS Bulletin. Thanks to Romina Gaburro and Ann O'Shea for chairing the

ICEDIM and ICME respectively, and to all the members of those committees. Thanks to Michael Mackey for his dedication and initiative in maintaining the IMS website. Welcome to the new committee members, and to Ronan Flatley as the new chairperson of ICME from January 2026.

Rachel Quinlan
December 2025

E-mail address: president@irishmathsoc.org and rachel.quinlan@universityofgalway.ie

Draft minutes of the Irish Mathematical Society Annual General Meeting held on 29th August 2025 at Maynooth University

Present: A. Baykalov, C. Boyd, P. Browne, S. Buckley, L. Creedon, J. Dillon, R. Flatley, R. Gaburro, F. Hegarty, M. Ibrahim, C. Kelly, D. Kitson, B. Kreussler, A. Krishnan J. Lansdown, P. Lynch, D. Mackey, M. Mackey, D. Malone, M. Manolaki, M. Mathieu, P. Mellon, P. Ó Catháin, A. O'Farrell, G. Pfeiffer, K. Pfeiffer, R. Quinlan, H. Šmigoc, N. Snigireva, W. Tang, S. Wills.

Apologies: T. Huettemann, J.P. McCarthy, A. O'Shea.

(1) **Agenda / Conflicts of interest**
The agenda was accepted and no conflicts of interest were declared.

(2) **Minutes**
The minutes of the AGM held on 30th August 2024 at Queen's University Belfast were accepted.

(3) **Matters Arising**
None.

(4) **Correspondence**
- The International Mathematical Union (IMU) is seeking nominations to three committees: the IMU Executive Committee, the Commission for Developing Countries (CDC) and the International Commission on the History of Mathematics. The deadline for nominations is 15th November. Interested members should contact the IMS committee.
- Registration for the International Congress of Mathematicians (ICM) 2026 is now open. The ICM will take place in Philadelphia, USA, from 23-30 July.
- The IMU has circulated a statement from the International Science Council (ISC) titled "International scientific collaboration: Vital yet vulnerable". ISC statements are available at council.science/statements/.

(5) **President's Report**
R. Quinlan highlighted several activities of the Society during the year including the Women in the Mathematical Sciences Day in May, the Annual Meeting in August and the seven conferences which received support from the Society. A full report will appear in the Bulletin. Romina Gaburro and the local organisers of the RIA-ICEDIM Women in the Mathematical Sciences Day, Nina Snigireva and Niall Madden, were thanked. As were the local organisers of the Annual Meeting in Maynooth: Stephen Buckley, Galatia Cleanthous, Christian Ketterer and Oliver Mason. Next year's annual meeting will take place at Trinity College Dublin with thanks to Katrin Wendland and Tristan McLoughlin. Committee members Helena Šmigoc, Ronan Flatley and Nina Snigireva and Treasurer Cónall Kelly have reached their final year on the committee and were thanked for their excellent service to the Society. The Editor of the Bulletin, Tom Carroll, and webmaster Michael Mackey were thanked for their continued service.

(6) **New members**
Nine membership applications were approved since the last AGM. The new members are: Hassan Alkhayuon, David Barnes, Jason Curran, Indranil Ghosh, Jesse Lansdown, Tristan McLoughlan, Jack McNicholl, Donald Laurence McQuillan, Aisling Twohill.

(7) **Nomination for honorary membership**
Professor David Conlon (California Institute of Technology) was nominated for honorary membership of the Society by D. Wilkins, A. O'Farrell and D. Kitson.

D. Malone outlined Professor Conlon's mathematical career: Professor Conlon is an Irish mathematician working in combinatorics, particularly Ramsey theory. He was an undergraduate in Trinity College Dublin and his Ph.D. at the University of Cambridge in 2009 was supervised by Tim Gowers. Following a Junior Research Fellowship at Cambridge, he moved to the University of Oxford and became a Professor in 2016. Since 2019, he has been a Professor at the California Institute of Technology. Prof Conlon represented Ireland at the International Mathematical Olympiad in 1998 and 1999. He has won a range of prizes, including the European Prize in Combinatorics (2011) and the LMS Whitehead Prize (2019). In 2014 he was a sectional speaker at the International Congress of Mathematicians. He is currently a Simons Visiting Professor at the Simons Laufer Mathematical Sciences Institute (formerly MSRI). Prof Conlon is a regular visitor to Ireland, delivering seminars, colloquia and Maths Week talks. The nomination was approved by the meeting.

(8) **Treasurer's Report**
Accounts for 2024 were presented. The amount of funding allocated to conferences has increased this year. Savings certificates have been purchased as a rainy day fund. Membership fees for 2025 have been paid to the EMS and IMU. The report was approved and the Treasurer, C. Kelly, was thanked.

(9) **Conference support fund**
The following workshops were supported this year:

- Irish SIAM student chapter conference (UL), 24$^{\text{th}}$ January 2025.
- EMS-CDC meeting (UCC), 10-12 April 2025.
- Layer Phenomena 2025 (Galway), 24-25 April 2025.
- 4th Irish Linear Algebra and Matrix Theory Meeting (MIC), 29$^{\text{th}}$ April 2025.
- IMS Women in Maths (Galway), 12$^{\text{th}}$ May 2025.
- Groups in Galway, 15-16 May 2025.
- LMS Harmonic Analysis and PDE's (UCC), 15$^{\text{th}}$ May 2025.

Applications to the conference support fund are encouraged.

(10) **Bulletin**
Issue 95 of the Bulletin is now available. Members are encouraged to submit articles to the Bulletin, particularly items with a connection to the Society. A special issue of the Bulletin will be published to mark the 50th anniversary of the Society. Members can contact the Editor, T. Carroll, with any questions.

(11) **Report from Irish Committee for Mathematics Education (ICME)**
R. Flatley gave a report on behalf of A. O'Shea (Chair). A full report will be published on the IMS website.

M. Hanly, R. Flatley and J. Grannell are finalising reports on second-level mathematics textbook quality and have made preliminary contact with a publisher. The Chief Examiner has not yet responded to a report prepared by R. Quinlan, J. Crowley and K. Pfeiffer on the 2023 Higher Level Leaving Cert exam paper. The IMS President wrote to the NCCA in September 2024 to ask that the society have representation on the NCCA committee which is reviewing the Leaving Certificate Mathematics Curriculum. The NCCA responded that they were not able to facilitate this. An ICME survey on the proposed curriculum changes has been submitted to the NCCA and circulated to members of the Mathematics Development Group.

J. Crowley organised a webinar on 13th May where the article "A Scoping Survey of ChatGPT in Mathematics Education" was discussed.

The ICME plans to write to the Minister of Education to raise concerns about the secondary school examinations, syllabi and textbooks.

Feedback and suggestions from members are welcome. Contact the ICME secretary K. Pfeiffer.

(12) **Report from Irish Committee for Equality, Diversity and Inclusion in Mathematics (ICEDIM)**
R. Gaburro (Chair) reported on ICEDIM activities.

The committee currently has nine members. In Autumn the committee ran an online seminar series. For next year the committee is considering holding an in-person event in Autumn. The RIA-ICEDIM Women in the Mathematical Sciences Day 2025 took place on 12th May at the University of Galway as part of the international May12 celebrations. Nina Snigireva and Niall Madden were thanked for organising the meeting. The RIA Physical, Chemical and Mathematical Sciences Committee was also thanked for supporting the meeting.

Organisers are being sought for future meetings. Interested members should contact the ICEDIM secretary N. Madden.

(13) **Elections**
The current terms of the following committee members come to an end this year: Cónall Kelly; Derek Kitson; Christopher Boyd; Ronan Flatley; Thomas Huettemann; Helena Šmigoc; Nina Snigireva.

C. Kelly, R. Flatley, H. Šmigoc and N. Snigireva have reached the end of three consecutive terms and are consequently not eligible for re-election to the committee. The remaining committee members are eligible for re-election.

The following nominations were received and election to these positions was approved by the meeting:

Candidate	**Role**	**Nominated by**	**Seconded by**
Dana Mackey	Treasurer	Cónall Kelly	Helena Šmigoc
Derek Kitson	Secretary	Nina Snigireva	Romina Gaburro
Christopher Boyd	Member	Peter Lynch	Nina Snigireva
Spyridon Dendrinos	Member	Romina Gaburro	Derek Kitson
Thomas Huettemann	Member	Nina Snigireva	Romina Gaburro
Arundhathi Krishnan	Member	Ronan Flatley	Derek Kitson
Katrin Wendland	Member	Rachel Quinlan	Cónall Kelly

(14) **Proposal for a statement of solidarity with scholars in Gaza**
A proposal from a member that the Society issue a statement in relation to Gaza was presented for discussion. It was expressed that a statement should aim to highlight matters specific to mathematics and call for academic activity to continue unhindered. The practicality of issuing statements was raised and a general ethics statement was suggested. It was agreed that the committee would seek further views and suggestions from the membership.

(15) **AOB**
- M. Mathieu announced that the Mathematical Proceedings of the Royal Irish Academy is adopting an open access model and submissions are encouraged.
- E. Gill encouraged members to get involved in events for Maths Week and to register events on the Maths Week website.

Derek Kitson (MIC)
derek.kitson@mic.ul.ie

IMS Annual Scientific Meeting 2025
Maynooth University
28 – 29 August, 2025

The 38th *Annual Scientific Meeting of the Irish Mathematical Society* took place at Maynooth University on Thursday 28th and Friday 29th August 2025 in the Rye Hall Lecture Theatre. The local organising team in 2025 consisted of Stephen Buckley, Galatia Cleanthous, Christian Ketterer, and Ollie Mason.

We would like to gratefully acknowledge the financial support received from the Irish Mathematical Society, the Department of Mathematics and Statistics at Maynooth University, as well as the sponsorship of the UKIE section of SIAM for the poster competition.

The meeting had a mixture of 45 minute talks given by invited speakers, shorter contributed talks, and a poster session. The nine invited talks covered a diverse range of topics across pure and applied mathematics, statistics, and mathematics education. Details on the titles and speakers for these are given below.

- David Barnes (Queen's University Belfast):
  Global dimension of incomplete Mackey Functors and incidence algebras.
- Niamh Cahill (Maynooth University):
  A Bayesian hierarchical spatio-temporal model for extreme sea-level prediction in Ireland.
- Stephen Coombes (University of Nottingham):
  Mathematical Neuroscience: Large-scale brain modelling.
- Aoife Hennessy (South East Technological University):
  A Riordan array framework for enumerating and transforming lattice paths.
- Elise Lockwood (Oregon State University):
  Integrating Computing into Mathematics Education: A Case of Python Programming in Combinatorial Contexts.
- Götz Pfeiffer (University of Galway):
  Reflection Groups in the Light of Formal Concept Analysis.
- Melanie Rupflin (University of Oxford):
  Quantitative estimates for geometric variational problems: Does almost solving a problem almost give you a solution?
- Ian Short (Open University):
  Integer tilings and hypertilings.
- Stephen Wills (University College Cork):
  Construction of quantum Markov processes.

The Society's AGM was held during the lunch break on the 29th of August. There was also a conference dinner for participants held on the previous evening at a restaurant in Maynooth.

In addition to the invited talks, there were five shorter, 25-minute contributed talks, covering topics in algebra, dynamical systems, and the history of mathematics. The list of shorter contributed talks is given below.

- Mariam Al-Hawaj (Trinity College Dublin):
  Generalized pseudo-Anosov maps and Hubbard trees.
- Anton Baykalov (University of Galway):
  Computing zeta functions of groups and algebras.
- Patrick Browne (Technological University of the Shannon)
  Chord Diagrams and Weight systems.
- Ted Hurley (University of Galway):
  Units and zero-divisors: Building blocks for required communications' systems.

- Siobhán McGarry and Ciarán Mac an Bhaird (Maynooth University):
  Euclid's Elements as Gaeilge – Beginnings.

Nearly all of the participants at IMS2025 pictured outside the Iontas building at Maynooth University.

There was also a poster session which ran during coffee breaks on the 28th and 29th of August. During these breaks, participants had an opportunity to mingle and discuss the posters with their presenters. The UKIE Section of SIAM sponsored a prize of €100 for the best poster, which was awarded to Michael Joyce Maher (University of Galway). The poster titles and their presenters are listed below.

- David Cormican (University of Galway):
  Ask Zeta Functions of Unitary Lie Algebras.
- Conor Curtin (Technological University Dublin):
  Hamiltonian & Lagrangian Models for Waves and Currents.
- Joseph Dillon:
  Properties of the square of the modulus of the xi function along the real line.
- Niamh Fennelly (University College Dublin):
  Synaptic Plasticity and Spatial Patterning in the Next-Generation Neural Field Model.
- Ramen Ghosh (Atlantic Technological University):
  Learning Criticality: Statistical Limits of Predicting Phase Transitions in Random Networks.
- Maniru Ibrahim (University of Limerick):
  Modeling Drug Release from Drug-Eluting Devices with Finite Dissolution Rates.
- Michael Joyce Maher (University of Galway):
  Odd inversion sets and their associated Turán graphs.
- David Malone (Maynooth University):
  Pollard's Rho Method.
- Brian Skelly (University College Dublin):
  A biophysical model of AMPA receptor Dynamics.

Generalized pseudo-Anosov maps and Hubbard trees

Mariam Al-Hawaj

Trinity College Dublin

In this talk, I will present a result from my PhD thesis where I develop a new connection between the dynamics of quadratic polynomials on the complex plane and the dynamics of homeomorphisms of surfaces. In particular, given a quadratic polynomial, we show that one can construct an extension of it which is a generalized pseudo-Anosov homeomorphism. Generalized pseudo-Anosov means the foliations have infinite singularities that accumulate on finitely many points. We determine for which quadratic polynomials such an extension exists. My construction is related to the dynamics on the Hubbard tree, which is a forward invariant subset of the filled Julia set that contains the critical orbit.

Computing zeta functions of groups and algebras

Anton Baykalov

University of Galway

In this talk, I will report on ongoing work on explicit computations of zeta functions associated with various types of counting problems attached to groups, algebras, and related algebraic structures. The goal of this project is to combine systematic methods (which can be very computationally involved and limited in scope) and ad hoc approaches driven by human insight intuition.

Global dimension of incomplete Mackey Functors and incidence algebras

David Barnes

Queen's University Belfast

The representation ring R(G) of a finite group G encodes rich structural information about G. To gain deeper insight, one can consider the collection R(H) for each subgroup H of G, along with the natural operations of restriction and induction between them. This leads to the framework of Mackey functors, with further examples such as the Burnside Mackey functor (based on finite H-sets) and the stable equivariant homotopy groups of a topological space with a continuous G-action.

Recent developments in equivariant stable homotopy theory have motivated a generalisation: incomplete Mackey functors, where only a subset of the induction maps is available. These arise naturally in computations and constructions within the field, making it important to understand the algebraic complexity of their categories. One such measure is global dimension, a generalisation of the notion of global dimension for rings, where dimension 0 corresponds to semi-simple rings and dimension 1 to hereditary rings.

In this talk, I will present a somewhat unexpected connection between this modern question (in the case of rational coefficients) and classical work from the 1990s on incidence algebras of partially ordered sets. These algebras, a type of path (or quiver) algebra that received significant attention in the 1970s and 1980s, have well-understood global dimensions. This connection provides insight on the algebraic complexity of categories of rational incomplete Mackey functors.

Chord Diagrams and Weight systems

Patrick Browne
Technological University of the Shannon

In this talk, we explore weight systems in knot theory, i.e. linear functionals on chord diagrams. Chord diagrams, while motivated by singular knots, can be viewed as purely combinatorial objects with rich mathematical structure. The significance of weight systems stems from the fundamental result that every Vassiliev knot invariant determines and is determined by a weight system. Moreover, Lie algebras provide a powerful framework for constructing these weight systems.

This presentation will introduce the connection between chord diagrams, weight systems, and Lie theory. We'll explore this interplay as preliminary research that may reveal new insights into both knot theory and combinatorial structures. The talk will be accessible to those without specialized background in knot theory or Lie algebras, focusing on the connections between these objects.

A Bayesian hierarchical spatio-temporal model for extreme sea-level prediction in Ireland

Niamh Cahill
Maynooth University

Rising sea levels increase the risk of flooding, coastal erosion, and extreme sea-level events. Coastal communities in Ireland are particularly vulnerable due to a combination of long, varied shorelines, low-lying urban areas, and exposure to both Atlantic storm systems and surges propagating from the Irish Sea. Accurate risk assessment depends on understanding the drivers of extreme sea levels, especially storm surges. A Bayesian hierarchical spatio-temporal model is developed to estimate extreme sea-level surges at both gauged and ungauged locations, drawing on tide-gauge records from Ireland and the west coast of Great Britain in the Global Extreme Sea Level Analysis (GESLA) database. Data from Great Britain are incorporated to compensate for the relatively short record lengths at most Irish tide gauges. Annual maxima of sea-level surges are modelled using the Generalised Extreme Value (GEV) distribution, incorporating both spatial and temporal dependencies. A barrier model captures complex spatial correlations along irregular coastlines.

Model evaluation shows that combining spatial and temporal components improves predictive skill. This is particularly valuable for Ireland, where short records limit site-specific analysis; the model's ability to share information across locations enhances estimates for data-sparse areas. The analysis reveals key patterns in extreme sea-level variability and detects an upward trend in surge annual maxima, with the east coast emerging as a higher-risk region. By explicitly integrating spatio-temporal dependencies, the framework offers a flexible, data-driven approach to representing extreme sea-level behaviour, supporting risk management and coastal planning in Ireland and similar coastal settings.

Mathematical Neuroscience: Large-scale brain modelling

Stephen Coombes
University of Nottingham

Neural mass models have been actively used since the 1970s to model the coarse-grained activity of large populations of neurons and synapses. They have proven especially fruitful for understanding brain rhythms. Although inspired by neurobiological principles, these models are largely phenomenological and often fall short of reproducing the complex dynamical repertoire observed in real neural tissue. In this talk I will discuss a simple integrate-and-fire spiking neuron network model that has recently been shown to admit to an exact mean-field description for synaptic interactions. This has many of the features of a neural mass model coupled to an additional dynamical equation that describes the evolution of population synchrony. I will show that this next generation neural mass model is ideally suited to understanding the patterns of brain activity that are ubiquitously seen in whole brain non-invasive neuroimaging recordings. Additionally, I will outline key mathematical challenges in linking structural and functional brain connectivity and discuss how phase-amplitude reduction techniques may provide a path forward. Time permitting, I will also describe the Haken model – a spiking network that can be analysed without mean-field approximations – highlighting its relevance in the era of high-resolution neural recordings from hundreds to thousands of simultaneously monitored neurons.

A Riordan array framework for enumerating and transforming lattice paths

Aoife Hennessy

South East Technological University

This talk explores how Riordan arrays can be used to enumerate and transform families of lattice paths. We introduce a promotion framework that takes classical Dyck paths to more general Motzkin and Schröder paths via two key transformations: the Binomial and Chebyshev transforms. The framework is further extended to study grand paths, which are not restricted by the x-axis. By uncovering patterns within this framework, we construct explicit bijections linking different path families. The Riordan transform approach provides new combinatorial insights and a fresh perspective on lattice paths.

Units and zero-divisors: Building blocks for required communications' systems

Ted Hurley

University of Galway

The talk is about how units and zero-divisors in abstract algebra are used, and can be used, in building required types of structures for communications' systems, such as for Coding Theory, Cryptography, Filter Banks and others.

Integrating Computing into Mathematics Education: A Case of Python Programming in Combinatorial Contexts

Elise Lockwood

Oregon State University

Computational activity, and programming in particular, comprise an increasingly essential aspect of scientific activity, and engaging in computing is as accessible as it ever has been. In mathematics education, there is a need to investigate the ways in which students' computational activity can support their reasoning about mathematical concepts. In this talk, I will present results from a study in which undergraduate students engaged with Python programming tasks designed to support combinatorial thinking. I highlight noteworthy aspects of students' experiences with computing in this mathematical context, including benefits and drawbacks of working in a computational environment. I suggest that even for students with little programming experience, the computational environment supported their combinatorial reasoning in valuable ways. Overall, I seek to frame these specific findings about Python programming in combinatorics as an instance of a broader phenomenon, namely highlighting the ways in which computing may be leveraged to support students' engagement with mathematical concepts and practices.

Euclid's Elements as Gaeilge – Beginnings

Siobhán McGarry & Ciarán Mac an Bhaird
Maynooth University

The title 'Additional Irish ms 2a' in UCD Special Collections reveals nothing of its remarkable mathematical content. The first 16 pages of this manuscript are a translation in Irish of the start of Euclid's Elements from around 1850 by the famous Irish language scholar John O'Donovan (1806 –1861). In this talk, we will provide some background on O'Donovan, including his work for the Ordnance Survey. We will present evidence that suggests that O'Donovan's original source was Robert Simson's Elements, and that O'Donovan may have been aware of the controversy around the parallel postulate. Regarding the translation itself, the terminology that O'Donovan employed is particularly interesting. It attracted a commentary from the leading Irish language expert Eoin MacNeill (1867–1945) and included the repurposing and combination of existing Irish words, and references to original Greek terms. We will close with a brief overview of how the manuscript ended up at UCD and mention other partial Irish language translations of the Elements that have thus far been uncovered.

Reflection Groups in the Light of Formal Concept Analysis

Götz Pfeiffer
University of Galway

Formal Concept Analysis (FCA) is a branch of applied lattice theory, concerned with the study of concept hierarchies derived from collections of objects and their attributes. Introduced by R. Wille in the 1980s, FCA now has found applications in machine learning and related fields. An application of FCA to hyperplane arrangements yields a new Galois connection on the (conjugacy classes of) parabolic subgroups of a finite reflection group. Combined with methods from Serre's recent work on involution centralizers, we obtain a refinement of Howlett's description of the normalizers of parabolic subgroups of a finite Coxeter group. This is joint work G. Roehrle and J.M. Douglass.

Quantitative estimates for geometric variational problems: Does almost solving a problem almost give you a solution?

Melanie Rupflin
University of Oxford

Many interesting geometric objects are characterised as minimisers or critical points of natural geometric quantities such as the length of a curve, the area of a surface or the energy of a map.

For the corresponding variational problems it is important to not only understand the properties of potential minimisers, but to obtain a more general understanding of the energy landscape.

It is in particular natural to ask whether an object with almost minimal energy must essentially "look like" a minimiser, and if so whether this holds in a quantitative sense, i.e. whether one can bound the distance to a minimiser in terms of the energy defect. In this talk we will discuss this and related questions concerning the behaviour of almost critical points and the convergence of gradient flows for some classical geometric problems, including the Dirichlet energy of maps between spheres whose minimisers correspond to meromorphic functions.

Integer tilings and hypertilings

Ian Short
Open University

We begin by discussing frieze patterns, which are periodic arrays of integers introduced by Coxeter in the 1970s. Conway and Coxeter discovered an elegant way of classifying frieze patterns of positive integers using triangulated polygons. Frieze patterns are closely related to certain integer tilings of the plane known as n-tilings. Motivated by Conway and Coxeter's triangulated polygons, we describe geometric models in the hyperbolic plane for n-tilings and their three-dimensional counterparts. These models allow us to construct all rigid integer tilings and hypertilings explicitly. This is joint work with Karpenkov, Van Son, and Zabolotskii.

Construction of quantum Markov processes

Stephen Wills
University College Cork

After giving a brief introduction to the idea and uses of quantum probability spaces and noncommutative random variables, I will discuss the various methods for the construction of continuous-time quantum Markov processes, in particular considering these as dilations of an underlying quantum Markov semigroup. My aim will be to give a flavour of what goes on, explaining some of the challenges that come when working in noncommutative analysis, but without with getting bogged down in technical detail.

Report by Stephen Buckley, Galatia Cleanthous, Christian Ketterer, Oliver Mason
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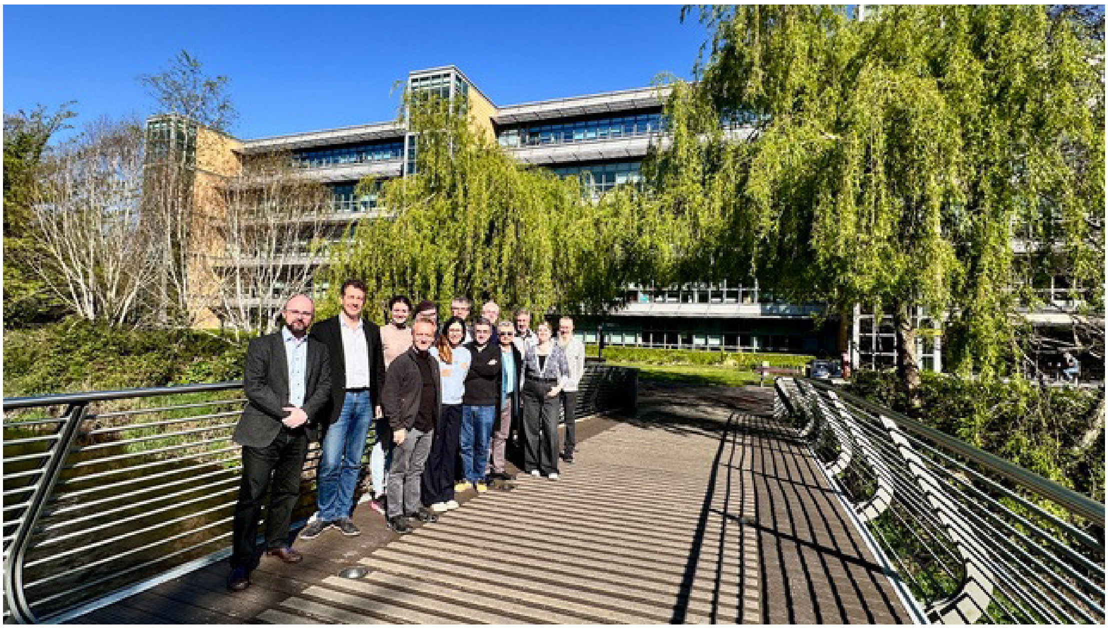
Reports of Sponsored Meetings

Spring 2025 Meeting of the EMS Committee for Developing Countries
10–12 April 2025, UCC

The Spring 2025 meeting of the European Mathematical Society Committee for Developing Countries (EMS-CDC) was hosted by University College Cork (UCC) over 10-12 April 2025. This event had a hybrid format, with 14 participants on-site and 20 more online. The local organisers were Cónall Kelly (currently vice-chair of the EMS-CDC) and Tom Carroll.



The EMS-CDC has a mandate to assist countries in the Global South in ways that include the development of mathematics curricula, libraries, and financial support. Recent major projects of the committee include the Emerging Regional Centres of Excellence (ERCE) programme, which recognises and supports centres of mathematics in the Global South that have achieved a substantial level of research activity and that play a key role in training students in their region.

The IMS President, Dr. Rachel Quinlan, gave welcoming remarks to open the meeting, which started with a session on academic publication practices and their implications for researchers in the Global South. The agenda also included reports from associate members of the committee representing organizations such as Centre International de Mathématiques Pures et Appliquées (CIMPA), the International Centre for Mathematical Sciences (ICMS), and the African Institute for Mathematical Sciences (AIMS).

Please see the committee's webpage for more about the work of the EMS-CDC.

The meeting received funding from the IMS; UCC College of Science, Engineering, and Food Science; and UCC School of Mathematical Sciences.

Report by Cónall Kelly, University College Cork
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Groups in Galway 2025
15–16 May 2025, University of Galway

Groups in Galway 2025 took place at the University of Galway during 15–16 May 2025. The meeting was organised by Joshua Maglione and Rachel Quinlan. It was supported by the de Brún Centre for Mathematics, the Irish Mathematical Society, and Research Ireland. There were 7 invited speakers and nearly 40 participants. We also held a poster session with about 10 posters on display next to the University of Galway's *Women in Mathematics* poster exhibit.

The conference featured a total of seven invited talks covering a wide range of topics in contemporary group theory and related fields. The speakers were:

Anton Baykalov (University of Galway):
Imprimitive partial linear spaces and groups of rank 3
Abstract: A partial linear space (PLS) is a point-line incidence structure such that each line is incident with at least two points and each pair of points is incident with at most one line. We say that a PLS is proper if there exists at least one non-collinear point

pair, and at least one line incident with more than two points. The highest degree of symmetry for a proper PLS occurs when the automorphism group G is transitive on ordered pairs of collinear points, and on ordered pairs of non-collinear points. In this case, G is a transitive rank 3 group on the points. While the primitive rank 3 PLSs are essentially classified, we present the first substantial classification of a family of imprimitive rank 3 examples. We classify all imprimitive rank 3 proper partial linear spaces such that the rank 3 group is innately transitive (including quasiprimitive cases) or semiprimitive and induces an almost simple group on the unique nontrivial system of imprimitivity. We construct several infinite families of examples and ten individual examples. The examples admit a rank 3 action of a linear or unitary group, and to our knowledge most of our examples have not appeared before in the literature. This is a joint work with Alice Devillers and Cheryl Praeger.

Iker de las Heras (University of the Basque Country):
Strong conciseness and equationally Noetherian groups
Abstract: The notion of strong conciseness of a group-word extends the classical concept of conciseness from abstract groups to the profinite setting. A word w is said to be strongly concise in a class $\mathcal{C}$ of profinite groups if, for any $G \in \mathcal{G}$, the cardinality of the set of values taken by w in G being strictly smaller than $2^{\aleph_0}$ implies that the verbal subgroup of G is finite. In this talk we will study the relation between this notion and the notion of equationally Noetherian groups. These groups arise from the theory of algebraic geometry over groups, which we will develop throughout the talk. As a consequence, we will see that every word is strongly concise in the class of profinite linear groups, as well as in the class of profinite completions of virtually abelian-by-polycyclic groups. This is joint work with Andoni Zozaya.

Brita Nucinkis (Royal Holloway, University of London):
Cohomological finiteness conditions for topological groups
Abstract: I will give a quick introduction into the classical finiteness conditions FP_n and F_n for a discrete group and then explain how to extend these to certain topological groups. The search for discrete groups that are of type FP_n but not of type FP_{n+1} has a very interesting and rich history. In this talk will present a new family of discrete and topological groups with this property. This is joint work with I. Castellano, B. Marchionna, and Y. Santos-Rego.

Götz Pfeiffer (University of Galway):
Reflection groups in the light of formal concept analysis
Abstract: Formal Concept Analysis (FCA) is a branch of applied lattice theory, concerned with the study of concept hierarchies derived from collections of objects and their attributes. Introduced by R. Wille in the 1980s, FCA now has found applications in machine learning and related fields. An application of FCA to hyperplane arrangements yields a new Galois connection on the (conjugacy classes of) parabolic subgroups of a finite reflection group. Combined with methods from Serre's recent work on involution centralizers, we obtain a refinement of Howlett's description of the normalizers of parabolic subgroups of a finite Coxeter group. This is joint work G. Roehrle and J.M. Douglass.

Margherita Piccolo (University of Hagen):
Representation zeta functions of subgroups and split extensions of $\mathrm{SL}_2^m(\mathcal{O})$
Abstract: The representation growth of a group G measures the asymptotic distribution of its irreducible representations. Whenever the growth is polynomial, a suitable vehicle for studying it is a Dirichlet generating series called the representation zeta function of G. One of the key invariants in this context is the abscissa of convergence of the

representation zeta function. The spectrum of all abscissae arising across a given class of groups is of considerable interest and has been studied in some cases. In the realm of p-adic analytic groups (with perfect Lie algebra), the abscissae of convergence are explicitly known only for groups of small dimensions. But there are interesting asymptotic results for "simple" p-adic analytic groups of increasing dimension. In this talk, I will give an overview of the main tools and ingredients in this area and I will report on recent work joint with Moritz Petschick to enlarge the class of groups.

Anitha Thillaisundaram (Lund University):
Normal subgroups of non-torsion multi-EGS groups
Abstract: The family of multi-EGS groups form a natural generalisation of the Grigorchuk-Gupta-Sidki groups, which in turn are well-studied groups acting on rooted trees. Groups acting on rooted trees provided the first explicit examples of infinite finitely generated torsion groups, and since then have established themselves as important infinite groups, with numerous applications within group theory and beyond. Among these groups with the most interesting properties are the so-called regular branch groups. In this talk we investigate the normal subgroups in non-torsion regular branch multi-EGS groups, and we show that the congruence completion of these multi-EGS groups have bounded finite central width. In particular, we prove that the profinite completion of a Fabrykowski-Gupta group has width 2. This is joint work with Benjamin Klopsch.

Gareth Tracey (University of Warwick):
How many subgroups are there in a finite group?
Abstract: Counting the number of subgroups in a finite group has numerous applications, ranging from enumerating certain classes of finite graphs (up to isomorphism), to counting how many isomorphism classes of finite groups there are of a given order. In this talk, I will discuss the history behind the question; why it is important; and what we currently know.

The conference website contains abstracts of the talks and further information.

Report by Joshua Maglione and Rachel Quinlan, University of Galway
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4th Irish Linear Algebra and Matrix Theory Meeting
29 April 2025, Mary Immaculate College

The 4$^{\text{th}}$ Irish Linear Algebra and Matrix Theory Meeting was organised by Cian O'Brien, and took place over one day in Mary Immaculate College. This meeting has previously taken place in University College Dublin, Maynooth University, and University of Galway, with the aim of bringing together and stimulating the community of linear algebra and matrix theory researchers in Ireland. More than 20 people registered to attend the meeting, and 18 actively participated on the day.

The conference featured nine invited talks, from colleagues across all career stages, covering a wide range of topics in linear algebra and matrix theory:

Patrick Browne (Technological University of the Shannon: Midwest):
Chord Diagrams and Weight systems
Abstract: In this talk, we explore weight systems in knot theory, i.e. linear functionals on chord diagrams. Chord diagrams, while motivated by singular knots, can be viewed as purely combinatorial objects with rich mathematical structure. The significance of weight systems stems from the fundamental result that every Vassiliev knot invariant

determines and is determined by a weight system. Moreover, Lie algebras provide a powerful framework for constructing these weight systems.

This presentation will introduce the connection between chord diagrams, weight systems, and Lie theory. We'll explore this interplay as preliminary research that may reveal new insights into both knot theory and combinatorial structures. The talk will be accessible to those without specialized background in knot theory or Lie algebras, focusing on the connections between these objects.

Niall Madden (University of Galway):
Enriched FEMs: Stability and Fast Solvers
Abstract: Finite Element Methods produce linear systems of equations which, when solved, yield numerical solutions to differential equations. The stability and efficiency of the FEMs depend on properties of the system matrix.

In this talk I'll outline a strategy for enriching the finite dimensional space on which the FEM is posed in order to improve accuracy, but the main focus will be on the surreptitious impact this has on the system matrix in terms of stability (in an M-matrix type way) and factorizabilty of the system matrix.

Arani Paul (University College Dublin):
Code Equivalence and Conductors
Abstract: Code Equivalence Problems (CEPs) have been discussed and studied for a long time not only for their importance in cryptography and cyber-security, but also because they connect to different areas of mathematics such as the Graph Isomorphism Problems and the Tensor Isomorphism Problems. This talk is particularly focused on the CEP for vector rank metric codes. Although rank metric codes have not been studied as extensively as Hamming metric codes, it has become an important topic of research in recent decades because of its applications in numerous sectors of modern-day digital technology.

Here we give a brief introduction to CEP in vector rank metric codes and discuss a way to classify all the equivalence classes of codes for given parameters. The key concepts are classical objects from linear and abstract algebra, namely conductors and idealisers. We will discuss practical implementations of classification algorithms, and discuss possible future directions.

Rachel Quinlan (University of Galway):
Idempotent Alternating Sign Matrices
Abstract: An alternating sign matrix (ASM) is a square $(0, 1, -1)$-matrix in which the nonzero entries alternate between 1 and -1 and sum to 1, within each row and column. Permutation matrices are examples of ASMs, and ASMs generalize permutation matrices in several apparently natural but unexpected ways. Every multiplicative group of nonsingular ASMs is a group of permutation matrices, but the set of all ASMs of size $n \times n$ also contains multiplicative groups of singular matrices. The identity element E of such a group is an idempotent ASM, it is equal to its own square. In this talk we will discuss some methods for construction of idempotent ASMs, and identify the minimum rank of an idempotent ASM of specified size.

Padraig Ó Catháin (Dublin City University):
Matrices with Specified Automorphisms
Abstract: Combinatorial structures such as strongly regular graphs and projective planes are encoded as incidence matrices, which often have interesting linear algebraic properties. Non-existence results are obtained via algebraic arguments. E.g. the Bruck-Ryser-Chowla theorem gives non-obvious necessary conditions for existence of symmetric designs based on equivalence of quadratic forms, and many non-existence results for difference sets boil down to showing that the eigenvalues of an associated matrix must be norms in a suitable number field.

In this talk, I will discuss (constructive) existence of such matrices, under the assumption of a suitable group of automorphisms. This theory is well known for graphs (i.e. symmetric $\{0, 1\}$-matrices) but rather more subtle when the matrix contains k^{th} roots of unity. In fact, one can construct an explicit basis for the set of all matrices invariant under a given group representation, and construct the eigenvalues of a given invariant matrix in terms of character sums. Time permitting, I will show how to use these methods to build new complex Hadamard matrices.

Helena Šmigoc (University College Dublin):
Arbitrarily Finely Divisible Stochastic Matrices
Abstract: We will consider the class arbitrarily finely divisible stochastic matrices (AFD_+-matrices): stochastic matrices that have a stochastic c^{th} root for infinitely many natural numbers c. This notion generalises the class of embeddable stochastic matrices. In particular, if A is a transition matrix for a Markov process over some time period, then arbitrary finely divisibility of A inside the set of stochastic matrices is the necessary and sufficient condition for the existence of a transition matrix corresponding to this Markov process over infinitesimally short periods.

We will explore the connection between the spectral properties of an AFD_+-matrix A and the spectral properties of a limit point L of its stochastic roots. We will demonstrate a construction of a class of AFD_+-matrices with a given limit point L from embeddable matrices, and examine specific cases, including 2×2 matrices, 3×3 circulant matrices, and offer a complete characterisation of AFD_+-matrices of rank-two.

Badriah Safarji (University of Galway):
Rank Distributions of Matrix Representations of Graphs Over F_2
Abstract: Over a finite field, the number of $n \times n$ matrices of rank r typically increases as r increases, $0 \leq r \leq n$. However, over the field of two elements F, the most frequently occurring rank is not n but $n - 1$. The numbers of symmetric F-matrices of rank n and

$n-1$ coincide if n is odd and differ marginally if n is even. This opens the door to a more thorough investigation of the distribution of the matrix ranks over the field of two elements.

Let Γ be a simple undirected graph. A symmetric matrix M with entries in a field represents Γ if the off-diagonal entries of M correspond to edges of Γ in the sense that the (i, j)-entry of M is non-zero if and only if vertex i and vertex j are adjacent in Γ. The diagonal entries of M are not subject to any constraints, and therefore there are many matrices representing Γ over the field. This project aims to identify and characterize simple connected graphs of order n with more F-matrix representations of rank $n-1$ than rank n, a property rare over other finite fields. We restrict our attention to graphs of order $n \geq 3$ with an induced subgraph isomorphic to the path on $n-1$ vertices. This talk will present results on the rank distributions of matrix representations of such graphs over F.

Bernard Hanzon (University College Cork):
Parametrization of Stable Multivariable Systems: Pivot Structures and Numbered Young Diagrams
Abstract: In this presentation we show how stable linear multivariable systems can be parametrized using orthogonal m-upper $(m+n) \times (m+n)$ Hessenberg matrices, where m stands for the number of inputs and n for the order of the system. To make sure all systems are covered by the parametrization certain column permutations of the Hessenberg matrix will be utilized. The lower part of the permuted (m-upper) Hessenberg matrix will given the pair $[B; A]$; where $(A; B)$ is the controllable pair of the system in state space form. Advantages of this approach to parametrization of stable linear systems will be discussed.

Cian O'Brien (Mary Immaculate College):
The Bruhat Order for Latin Squares
Abstract: Alternating sign matrices arise naturally as a generalisation of permutation matrices in a number of different contexts. For example, they are the unique minimal lattice extension of the permutation matrices under the Bruhat order. In 2018, Brualdi and Dahl defined alternating sign hypermatrices, a 3-dimensional analogue of alternating sign matrices. Latin squares can be thought of as the 3-dimensional analogue of permutation matrices, since the positions of each of the n symbols in an $n \times n$ Latin square correspond to the non-zero entries in some $n \times n$ permutation matrix.

We have extended this idea further, by defining the Bruhat order for Latin squares and studying the resulting poset. This talk presents current work relating to this poset, including 3-dimensional analogues of related combinatorial objects, and a lattice extension of the Latin square poset.

The conference website contains further information.

Report by Cian O'Brien, Mary Immaculate College
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Workshop on Numerical Methods for Problems with Layer Phenomena
24–25 April 2025, University of Galway

The *Workshop on Numerical Methods for Problems with Layer Phenomena* is an annual international workshop focusing on mathematical and numerical aspects of differential equations whose solutions feature boundary or interior layers, such as singularly perturbed problems. Though commonly called 'The Limerick Workshop', recent iterations have been hosted in University of Cyprus (twice), Universidad de Sevilla, University of Limerick (online), Fern Universität in Hagen, and Charles University.

The 21st instance of the workshop was hosted at the University of Galway on 24–25 April 2025. It was organised by Niall Madden, Nanda Poddar, Jekaterina Mosalska, and Seán Tobin. There were twenty in-person participants, with another twenty joining for a special online session. Financial and organisational support was provided by the Irish Mathematical Society, and the School of Mathematical and Statistical Sciences, University of Galway.

The workshop featured fourteen research talks:

- Christos Xenophontos (Cyprus). *On the decomposition of the solution to reaction-diffusion two-point boundary value problems with data of finite regularity.*
- Alex Trenam (Heriot-Watt). *Nodally bound-preserving discontinuous Galerkin methods for charge transport.*
- Seán Kelly (Limerick). *Pointwise-in-time error bounds for a fractional-derivative parabolic problem on quasi-graded meshes.*
- Neofytos Neofytou (Cyprus). *rp-DG FEM for fourth order singularly perturbed problems with two small parameters.*
- Christos Pervolianakis (Jena). *A Stabilized Scheme for an Optimal Control Problem Governed by Convection-Diffusion-Reaction Equation.*
- Nanda Poddar (Galway). *Interplay of Dynamic Boundary Absorption and Layer-like Phenomena in Reactive Solute Transport: A Dual Numerical Approach.*
- Jenny Power (Bath). *Adaptive Regularisation for PDE-Constrained Optimal Control.*
- Niall Madden (Galway). *A tutorial on solving singularly perturbedproblems in Firedrake.*
- Marwa Zainelabdeen (Berlin). *Gradient-robust finite element–finite volume scheme for the compressible Stokes equations.*
- Alan F. Hegarty (Limerick). *Novel meshes for the solution of a problem with interior parabolic layers.*
- Katherine MacKenzie (Strathclyde). *The Bound Preserving Method applied to the 2D Induction Heating Problem.*
- Natalia Kopteva (Limerick). *A posteriori error estimation for convection-diffusion equations.*
- Sebastian Franz (Dresden). *On a posteriori estimation in the energy norm for convection-diffusion problems.*

In addition, Martin Stynes (Beijing) gave a short talk, on Zoom, remembering Lutz Tobiska (Magdeburg), a major figure in the numerical analysis of singularly perturbed problems, who passed away a few weeks earlier.

The conference website contains abstracts of the talks and further information.

Report by Niall Madden, University of Galway
Niall.Madden@UniversityOfGalway.ie

LMS Harmonic Analysis and PDE Network Meeting
1 May 2025, University College Cork

The meeting was organised by Tom Carroll and Spyros Dendrinos (both from University College Cork). The meeting took place fully in person.

The meeting featured a total of four invited talks covering a range of topics in harmonic analysis, complex analysis and differential equations:

Joaquim Ortega-Cerdà (Universitat de Barcelona):
The Hörmander-Bernhardsson extremal function
Abstract: We characterize the function φ of minimal L^1 norm among all functions f of exponential type at most π for which $f(0) = 1$. This function, studied by Hörmander and Bernhardsson in 1993, has only real zeros $\pm\tau_n, n = 1, 2, \ldots$. We identify φ in the following way. We factor $\varphi(z)$ as $\Phi(z)\Phi(-z)$, and show that Φ satisfies a certain second order linear differential equation along with a functional equation, either of which characterizes Φ. Furthermore, we use these facts to establish a series expansion for the zeros and a power series expansion of the Fourier transform of φ, as suggested by the numerical work of Hörmander and Bernhardsson. The dual characterization of Φ arises from a commutation relation that holds more generally for a two-parameter family of differential operators, a fact that is used to perform high precision numerical computations.

This is joint work with Andriy Bondarenko, Danylo Radchenko and Kristian Seip. Link to arXiv

Stefan Buschenhenke (Universität zu Kiel):
Maximal operators for two-dimensional surfaces of finite type and FIO-cone multipliers
Abstract: We report on joint work with Spyros Dendrinos, Isroil Ikromov and Detlef Müller on a new class of 'FIO-cone multipliers'. In previous work, we studied the boundedness range of the maximal average of any smooth compact hypersurface in three-dimensional space, up to a certain "exceptional class", which is linked to the cone multiplier. We encounter a convolution operator, being the composition of two operators: the classic cone multiplier and additionally a certain translation invariant Fourier integral operator (FIO) with non-standard phase functions with singularities near the light cone. We develop a new theory for a class of these "FIO-cone multipliers", that allows phase functions that are in a particular way adapted to the geometry of the cone. Our approach uses the recent breakthrough for the cone multiplier conjecture by Guth, Wang and Zhang.

Myrto Manolaki (University College Dublin):
Boundary behaviour of holomorphic and harmonic functions
Abstract: The study of the boundary behaviour of holomorphic and harmonic functions is of significant importance in many areas in Analysis. In this talk I will present an overview of my research on this topic, focusing on two theorems which complement and strengthen some classical results. The first one concerns Abel's Limit Theorem, which

connects the behaviour of a Taylor series as we approach the boundary from the interior with its behaviour on the boundary itself. The second one strengthens Plessner's and Spencer's theorems about the angular behaviour of holomorphic functions on the unit disc. Moreover, its harmonic analogue in higher dimensions improves classical results of Stein and Carleson. As we will see, these two theorems, which are based on a variety of tools from potential theory, find applications to certain classes of holomorphic functions with wild boundary behaviour. (Based on joint works with Stephen Gardiner, Stéphane Charpentier and Konstantinos Maronikolakis.)

Itamar Oliveira (University of Birmingham):
A phase-space approach to weighted Fourier extension inequalities
Abstract: The goal of the talk is to present a certain ray bundle representation of the Fourier extension operator in terms of the Wigner transform to investigate weighted estimates in restriction theory and their connections to time-frequency analysis and geometric combinatorics.

In joint work with Bennett, Gutierrez and Nakamura, we show how Sobolev estimates for the Wigner transform can be converted into certain tomographic bounds for the Fourier extension operator, which implies a variant of the (recently shown by H. Cairo to be false) Mizohata-Takeuchi conjecture. Together with Bez and the previous three authors, we employed of our phase-space approach in the context of Strichartz inequalities for orthonormal systems in the spirit of the work of Frank and Sabin. If time allows, we will make a further connection between our results and Flandrin's conjecture in signal processing through the study of certain singular integral operators similar to those studied by Lacey, Lie, Muscalu, Tao and Thiele.

The meeting website contains further information. The meeting was also funded by Scheme 3 of the London Mathematical Society and the School of Mathematical Sciences (UCC).

Report by Spyros Dendrinos, University College Cork
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IMS & Irish SIAM Student Chapter Conference 2025
9 May 2025, University of Limerick

On 9th May, the University of Limerick hosted the IMS & Irish SIAM Student Chapter Conference, welcoming over 40 applied mathematicians from industry and academia alike. Originally scheduled to take place in January 2025, the 10th edition of the conference had to be postponed on account of Storm Éowyn. Thankfully, the delay did not dampen the enthusiasm of the participants, who filled the large lecture hall in the Analog Devices Building to listen to two keynote lectures and 11 contributed presentations. A poster session also took place, providing students with an opportunity to present their research to an audience of their peers in a friendly, relaxed atmosphere.

The day opened with remarks from Dr. Doireann O'Kiely, followed by a keynote lecture by Dr. Niall Madden. In his presentation, Dr. Madden described the use of numerical methods in disparate fields such as tidal modelling and preventing aortic aneurysms. The variety of his work drew several thought-provoking questions from a rapt audience. After lunch, the second keynote lecture was delivered by Orla Fitzmaurice, a recent Mathematical Sciences graduate of UL, now working with Analog Devices. Her presentation gave an insight into the various ways large language models were being implemented into Analog Devices' engineering workflows. In addition to outlining the various advantages of large language models, Orla also gave an overview of the challenges in benchmarking the performance of these models.

Following a full conference programme, the organising committee had the difficult decision of choosing the best speaker. After much deliberation, the award was presented to Emmet Lawless for his presentation "A variational approach to portfolio choice", applying methods in control theory to a financial planning problem. Once the conference was brought to a close, a dinner was held in the Pavilion on the UL North Campus. Here, the conference participants had a chance to mix and mingle, forging connections and fostering collaborations among a promising group of early-career researchers.

The conference was a resounding success from start to finish, showcasing the impressive breadth and depth of applied mathematics research taking place at Irish universities. The organising committee are very grateful to SIAM, Analog Devices, the Irish Mathematical Society and also the UL Department of Mathematics and Statistics for their generous financial support. Going forward, the conference will hopefully continue to be a mainstay of the Irish mathematical calendar as an event run by students, for students.

The following speakers contributed talks to the conference programme:

(1) Dr. Anthony Bonfils (University of Limerick)
(2) Lorenzo Pisani (Dublin City University)
(3) Lyudmila Ivanova (Technological University Dublin)
(4) Daire O'Donovan (University College Dublin)
(5) TJ O'Brien (University College Dublin)
(6) Evan Murphy (Dublin City University)
(7) Emmet Lawless (Dublin City University)
(8) Jack Cromwell (Dublin City University)
(9) Ole Cañadas (Dublin City University)
(10) Julius Busse (University College Dublin)
(11) Daniel Devine (Trinity College Dublin)

Titles and abstracts of keynote lectures and contributed talks:

Dr. Niall Madden:
Numerical modelling of solute transport in oscillatory flow, and other things
Abstract: In this talk I'll discuss some recent work on numerical modelling of solute transport in an oscillatory flow. The physical system features particles that are released at a point in a channel bounded above and below by plates. Particles encountering the plates may be either reflected or absorbed. The main focus on the talk will be on how the scenario can be modelled numerically, including

- comparison between numerical models based on classical finite differences and Brownian Dynamic Simulation;
- a discussion on how numerical results might be validated;
- how the numerical modelling process can inform the mathematical model.

Along the way, there will be discussion of how numerical computing (and numerical analysis) can be of value in projects as diverse as modelling Galway Bay, predicting aortic aneurysms, and reducing the reliance on animal testing in testing therapies in ICU settings.

Orla Fitzmaurice:
Generative AI in Practice: Retrieval Augmented Generation and Evaluation Challenges
Abstract: Orla Fitzmaurice graduated from the University of Limerick with a Bachelor of Science in Mathematical Sciences with Statistics in August 2024 and works at Analog Devices in Limerick as a graduate Machine Learning Engineer. Her current work focuses on evaluating application specific Generative AI systems being developed by her team.

Generative AI has revolutionized numerous domains, enabling applications ranging from creative writing to technical problem-solving. Large Language Models (LLMs), a subset of generative AI, are designed to produce natural language output by learning patterns and structures from massive datasets of unlabelled text. Generative Pre-trained Transformers (GPTs) power widely used systems like OpenAI's ChatGPT and Microsoft's Copilot. However, retraining these models to incorporate up-to-date information is time and resource intensive. Retrieval-Augmented Generation (RAG) offers an innovative solution by integrating LLMs with external, domain-specific datasets. RAG can be used as a question-answering based system, it employs prompt engineering and vector search to deliver tailored answers, grounded in truth, without exhaustive retraining.

Despite significant progress in LLM capabilities, the field suffers from notable deficiencies in standardised model evaluation and reporting practices. Ragas is one example of a framework designed to evaluate RAG systems. Ragas employs an evaluation LLM to assess a target model's performance against a dataset of human-verified ground truths. Ragas provides several metrics that allow the user to determine areas where the LLM is not performing as expected. Model evaluation is often conducted without leveraging robust experimental methodologies that have been well established in other scientific disciplines. Current literature frequently reports with a "highest number is best" approach rather than testing for significant results. This presentation will explore the foundational technologies of LLMs and RAG, while examining limitations in LLM evaluation practices.

Dr. Anthony Bonfils:
Finite length and boundary effects in the mode selection of a floating elastic sheet
Abstract: Leave me free, I buckle.
Give me a support, I wrinkle.
Pinned, I am trigonometric.
Clamped, I am eclectic.
My asymptotic analysis will make you tumble.

Lorenzo Pisani:
Quantum effects in black holes
Abstract: The search for a quantum gravity theory is widely regarded as one of the most significant challenges in fundamental physics. While several candidate theories have been proposed, there is no consensus amongst theorists. There are, however, robust and uncontroversial approximations to quantum gravity. One such approximation is semiclassical gravity, which describes the interaction of quantum fields with a classical spacetime metric.

In this framework, the stress-energy tensor, which accounts for the presence of matter in the classical equations, is replaced by its quantum equivalent: a (formally) divergent operator known as the expectation value of the stress-energy tensor of a quantum field in a quantum state. The process of renormalization involves systematically removing the divergence to isolate the physical state-dependent component, which is known as the renormalized stress-energy tensor (RSET). The renormalization procedure is very complicated in black holes spacetimes and practical schemes were only developed in recent years.

We present a mode-sum prescription to directly compute the RSET for scalar fields in the Boulware vacuum state. The method generalizes the recently developed extended coordinate method which was previously only applicable to Hartle-Hawking states and enables the study of semiclassical effects in spacetimes without such states, including extremal black holes and stellar spacetimes.

We demonstrate the accuracy and efficiency of the method by calculating the RSET in both sub-extremal and extremal Reissner-Nordström spacetimes, finding numerical evidence for the regularity of the RSET at the extremal horizon regardless of the field mass or coupling. In addition, we compute the semiclassical backreaction on the background metric by employing the numerical results obtained for the RSET to source the static semiclassical Einstein equations. Our results indicate that extremal horizons are unstable under quantum perturbations: if the RSET is considered as a static perturbation, it will either de-extremalize the black hole or convert it into a horizonless object.

The development of this methodology opens a window to computing semiclassical backreaction in previously unexplored scenarios, particularly in stellar spacetimes.

This talk is based on:
J. Arrechea et al., PhysRevD.111.085009

Lyudmila Ivanova:
Nonlinear Hamiltonian Models for Propagation of Intermediate Internal Ocean Waves in the Presence of Currents
Abstract: A two-dimensional water wave system is examined with two layers separated by a free common interface. The fluids are incompressible and inviscid. The system consists of a lower medium, bounded below by a flatbed, and an upper medium with a free surface, where wind-generated surface waves occur. However, we consider the flat surface approximation, based on the assumption that surface waves have negligible amplitude. In a geophysical context, this represents a model of an internal wave formed within a pycnocline or thermocline in the ocean. In addition, a current profile with depth-dependent currents in each domain is considered. An example of the physical situation described above is clearly illustrated by the equatorial internal waves in the presence of the Equatorial Undercurrent (EUC). We consider wave propagation in the so-called intermediate long wave approximation, where the wavelength is comparable to the depth of the lower layer, which in turn is much shallower than the upper layer. The study is based on the Hamiltonian approach. The equations of motion are formulated as a Hamiltonian system, and the Hamiltonian is determined and expressed in terms of

canonical wave-related variables. A specific scaling is chosen, which leads to the integrable Intermediate Long Wave Equation (ILWE). The limiting behavior is investigated, and connections with other known models are established.

Daire O'Donovan:
Achieving Optimal Locomotion using Self-Generated Waves
Abstract: Horizontal locomotion of a body on the fluid surface can be achieved by interacting with self-generated waves via a vertical bobbing motion. Mathematically, this can be interpreted as a pressure source acting on the surface. To study the conditions for maximal thrust in a chosen direction, an optimal control problem can be posed, where the pressure source is the control and the thrust force is the objective. A bound is then applied to the control to regularise the problem. The work is split into two parts given two different bounds, firstly the norm of the control function, and secondly the power, which is derived from the rate of change of the energy. To begin this work, the problem is reduced to the shallow limit. Given the assumption that the pressure source is periodic, an analytical approach can be taken with variational calculus. Numerical optimisation can be carried out to calculate the optimal pressure given a constraint and can be shown to match the analytical solution, which corresponds with emitting a wave purely in the direction opposite to movement.

TJ O'Brien:
Shell Spacing in GOY-Like Shell Models of Turbulence
Abstract: The Gledzer-Ohkitani-Yamada (GOY) model and its improvement by L'vov et al., the Sabra model, have proved useful in modelling energy cascades in two and three dimensional turbulence. These models are widely used in the literature. However, they are not the only model equations that can be constructed as "improvements" of the GOY model.

First, by revisiting the derivation of the Sabra model, we propose two new model equations based on the GOY and Sabra modelling paradigm. Two heuristic approaches to generate the new equations are presented. These new equations satisfy the same properties of conservation of two quadratic invariants and of phase symmetries, with only slight variations in structure and behaviour. Collectively, these four model equations form what could be considered the class of 'GOY-Like' shell models.

$$\frac{d}{dt}\sum_n |u_n|^2 \propto \begin{cases} \sum_n k_n \Im(\overline{u_{n-1}u_n u_{n-2}}) \\ \sum_n k_n \Im(\overline{u_{n-1}u_n}u_{n-2}) \\ \sum_n k_n \Im(\overline{u_{n-1}}u_n\overline{u_{n-2}}) \\ \sum_n k_n \Im(u_{n-1}\overline{u_n u_{n-2}}) \end{cases}$$

Second, we explore the differences within the class under the assumptions of three dimensional helicity-preserving turbulence. We begin with a review of the phase symmetries and their implication. Next, a general form for the flux formula is analysed. Then, recalling that the shell wavenumber is given in terms of the shell spacing λ by $k_n = k_0\lambda^n$, we compare the models with $\lambda = \mu$ and $\lambda = 1/\mu$, with $\mu > 1$ fixed. This leads to mappings between our GOY-Like models. Certain mappings go from one model to another in the class with reciprocal spacing, indicating that a projection from the typical spacing regime used in literature $\lambda \in (1, \infty)$ to the bounded interval $\lambda \in (0, 1)$ is possible. This amounts to 'flipping' the shell index, where the model then starts with a defined smallest spatial scale at index 0 and grows to large scales as index increases.

Finally, an exploration of the class' behaviour under various shell spacings is conducted. Due to the role of the triangle inequality with respect to a shell model's relation to Navier-Stokes turbulence we focus on the Golden Ratio $\Phi \approx 1.618$ and its reciprocal

$\Phi^{-1} \approx 0.618$. Discussion of other spacings, such as the literature standard ratio $\lambda = 2$, is done in comparison to the former. Additionally, the behaviour of the equations in the limit $\lambda \to 1$ is briefly explored.

Evan Murphy:
A Stochastic Birth-and-Death Approach for Street Furniture Detection in Urban Environments
Abstract: Urban environments are evolving rapidly, and efficient city planning requires accurate and up-to-date information about public spaces. Comprehensive mapping of street assets role in shaping the urban landscape, and enhancing quality-of-life and accessibility in cities. This work aims to contribute to the problem of autonomously maintaining up-to-date data regarding the location, condition, and distribution of these assets.

Existing segmentation modules (such as those discussed in [1]) have proven effective for the extraction of objects from street-view imagery, and providing estimates for camera-to-object distance, bearing, and a measure of confidence. This new work builds upon the model described in [1], wherein pairwise intersections between rays originating from camera positions are considered as favourable candidates for object positions, and form the solution space for a boolean optimisation problem (OP). The solution of the OP is subject to clustering, giving a good prediction for the ground truth object locations.

The alternative strategy considered in this work is to build a configuration of objects by considering a stochastic birth and death (SBD) process [2] led by an energy function that will be constructed from static data terms, and an interaction term. This process models objects as a 2D-coordinate, with a radius r, defining an area of exclusivity. To each pixel in a rendering of the target area, a data energy is assigned, comprising of four terms. The first two terms relate to the detection confidence and depth consistency of nearby pairwise intersections, while the third term penalises clashes with existing infrastructure. The final data energy term is proportional to the radius of the point, penalising "greedy" objects. The data energy of a spawned object is then the sum of all pixels covered by a disc of radius r around the chosen coordinate. An object interaction term is also included, introducing a penalty proportional to the area of overlap between two generated objects. The goal of this approach is to find a configuration of minimal energy, representing the most favourable object locations. Joint work with Vladimir Krylov and Marco Viola.

References

(1) Krylov, V.A., Kenny, E. and Dahyot, R., 2018. Automatic discovery and geotagging of objects from street view imagery. *Remote Sensing, 10*(5), p. 661.
(2) Descombes, X., Minlos, R. and Zhizhina, E., 2009. Object extraction using a stochastic birth-and-death dynamics in continuum. *Journal of Mathematical Imaging and Vision, 33* (3), pp.347-359.

Emmet Lawless:
A variational approach to portfolio choice
Abstract: In this talk we propose a calculus of variations approach to a popular stochastic control problem in finance. Stochastic control can be summarised as a field wherein one tries to solve an optimisation problem that depends on an underlying stochastic process which you can partially control. We focus on the financial planning problem which consists of a single economic agent who has to choose an investment and consumption policy which maximises her expected utility from consumption subject to certain risk preferences. The agent may choose to invest in risky assets (for example the stock market) or a safe asset (for instance a government bond or a bank account) and simultaneously choose a consumption policy which covers required spending. This is the classical

problem faced by pension schemes and financial planners to ensure future financial security for their policy holders.

The main tool used to solve this problem is the so called dynamic programming approach which reduces the infinite dimensional stochastic problem to the study of a (in general) non linear differential equation (DE). This approach although very useful can often times be unsatisfactory as the resulting DE is difficult to analyse. We propose a novel approach to this problem by proving it can be solved by considering a deterministic calculus of variations problem. This circumvents the need to analytically (or numerically) solve the associated DE. This method provides a new set of mathematical tools to analyse such problems and highlights how certain stationary stochastic control problems can be solved via deterministic methods.

This work is part of a joint project with Paolo Guasoni and Ho Man Tai.

Jack Cromwell:
Central Limit Theorem for Random Variables Under Constraints
Abstract: We study sequences of random variables whose joint distribution is supported on constrained surfaces defined by relations $\sum_{j=1}^{n} V_j(X_j) = na$, with general functions V_j. These include hard constraints, such as confining vectors to spheres or ellipsoids. These constraints induce dependency among the random variables and cause them to be non-identically distributed. We develop Central Limit Theorem results on these random variables under Lindeberg conditions tailored to the geometry of the constraint.

Ole Cañadas:
A Class of Mathematical Models for Highly Fluctuating and Random Real World Phenomena
Abstract: Modelling real-world phenomena using mathematical methods and addressing related questions, such as optimal strategies or asymptotic behaviour, has been the focus of mathematicians for decades. Examples include population growth/decline, the optimal route for a salesman, and the efficient distribution of goods in containers. Additionally, the mathematical community has agreed that certain events, like stock prices or weather patterns, should be modelled using methods that incorporate randomness. This is typically done using Itô's classical stochastic calculus, a tool that is nowadays well understood.

However, recent empirical studies of financial and commodity market data suggest that volatility (i.e., price fluctuations) varies more than can be captured by Itô's stochastic calculus. As a result, there has been growing interest in tools that go beyond this classical machinery, such as the theory of rough paths and stochastic Volterra processes. Unfortunately, these tools are thus far not fully understood, leaving many theoretical questions unanswered, such as those regarding monotonicity and long-term behaviour which are essential for developing a robust statistical framework. In this talk, we focus on particular dynamics given in terms of stochastic Volterra processes and present recent results such as comparison principles, limiting distributions, law-of-large numbers, central limit theorem and cut-off phenomena.

This talk is based on joint works in progress with Mohamed Ben Alaya (Rouen University), Luigi Amedeo Bianchi, Stefano Bonaccorsi (both Trento University) and Martin Friesen (DCU).

Julius Busse:
Modelling the spread of particulate pollution in the ocean
Abstract: We explore the influence of the Stokes number, the Stokes drift, and noise (diffusion) on the advection of particles advected by irrotational water waves. We follow a Lagrangian model of particle dynamics, describing the positions of individual particles as governed by a randomly perturbed advection equation.

We ask if adding small-scale turbulence in the finite depth water column increases the transport distance of particles and quantify it. We consider multiple models for turbulence in order to quantify the distance travelled from the initial position where a particle is released to the point where it settles on the seabed.

Simulating the stochastic differential equation (SDE)

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} &= \frac{1}{t_p}(\mathbf{u} - \mathbf{v} - \nu_T \mathbf{e}_z),\\ \frac{\partial \mathbf{x}}{\partial t} &= \mathbf{v} + \sigma W'(t), \text{ where}\\ \mathbf{u} &= \frac{a\omega}{\sinh(kh)} \begin{pmatrix} \cosh\big(k(z+h)\big)\cos(kx - \omega t) \\ \sinh\big(k(z+h)\big)\sin(kx - \omega t)\end{pmatrix},\end{aligned}$$

with $\mathbf{x}(\mathrm{t} = 0) = \mathbf{v}(\mathrm{t} = 0) = \mathbf{0}$ shows that with increased diffusion (realised as an increased noise coefficient a), the mean advection of the particles increases. We seek to quantify this relationship and provide an analytical framework for it by transforming it into a deterministic advection-diffusion like equation using the Fokker–Planck equation. This allows considering diffusion coefficients that depend on the concentration.

Daniel Devine:
Existence and Convergence Results for a System of PDEs
Abstract: In this talk, we will discuss a nonlinear elliptic system of PDEs which has its origins in the study of the dynamics of viscous, heat-conducting fluids. To model viscous heating effects, the system of interest contains source terms with a nonlinear gradient dependence, which presents considerable theoretical challenges. By restricting our attention to solutions which are radially symmetric, the problem becomes far more mathematically tractable. To begin, we will outline some of the progress made since the early 2000s, and then move onto some more recent results. In particular, we will see that all solutions converge monotonically to an explicit solution which we can easily calculate. This talk is based on results jointly obtained with Paschalis Karageorgis, and results jointly obtained with Gurpreet Singh.

The following University of Limerick students presented posters:

(1) Tiernan Brosnan: *Microlocal Analysis of ISAR Imaging*
(2) Jessica Crosse: *Stable local determination of a complex anisotropic conductivity of a medium at the boundary*
(3) Niall Donlon: *Stable Reconstruction of Anisotropic Conductivity in Biological Tissue*
(4) Attiq Iqbal: *Mathematical modelling of drug release from tablets*
(5) Eamonn Organ: *Learning from the weather – A spatial statistics viewpoint*
(6) Mitchell Rae: *Comprehensive Machine Learning Approaches to Modelling State of Charge for LiBs*

The conference website hosts a digital copy of the conference booklet and further information about the event programme and sponsors.

Report by Ben McKeon, University of Limerick
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RIA-ICEDIM Women in the Mathematical Sciences Day
12 May 2025, University of Galway

The 2025 Women in The Mathematical Sciences Day Conference was hosted by the School of Mathematical and Statistical Sciences, University of Galway, on May 12$^{\mathrm{th}}$.

It was organised by the Royal Irish Academy's Physical, Chemical and Mathematical Sciences committee, and the Irish Committee for Equality, Diversity, and Inclusion in Mathematics (ICEDIM). The event included plenary talks, a poster session, and public talks.

Just over 60 participants attended the conference. They were welcomed by Prof. Cathal Seoighe, Head of the School of Mathematical and Statistical Sciences, University of Galway. Opening remarks were provided by Dr Helen Maher (Vice-President for Equality Diversity and Inclusion, University of Galway), and Prof. Louise Allcock (Member of the Royal Irish Academy, and Professor in Zoology).

FIGURE 1. Participants at the 2025 RIA-ICEDIM Women in The Mathematical Sciences Day Conference

The conference speakers were Nicola Fitz-Simon (University of Galway), Róisín Neururer (University College Dublin), Doireann O'Kiely (University of Limerick), Margherita Piccolo (University of Hagen, Germany), and Myrto Manolaki (University College Dublin); abstracts are provided below.

A novel aspect of the 2025 edition of the conference was the inclusion of public talks, which were given by Victoria Sánchez Muñoz (Université Libre de Bruxelles), and attended by over 100 second-level students.

The local organisers were Nina Snigireva and Niall Madden. Financial and organisational support was provided by the Royal Irish Academy, the Irish Mathematical Society, The School of Mathematical and Statistical Sciences of University of Galway, the University of Limerick and, in particular, by Fionnuala Parfrey (RIA) and Romina Gaburro (University of Limierick, Vice-Chair, RIA Physical, Chemical and Mathematical Sciences committee).

Titles and Abstracts

Nicola Fitz-Simon:
Using Statistical Models for Small Area Estimation
Abstract: As an applied statistician, I have worked on many studies that combine data and statistical models to aid decision making, especially in the area of human health. I am currently working on a project for the World Health Organization to estimate risk factors related to diabetes at small area level – for example high blood pressure, smoking, diet, physical activity, overweight and alcohol. The WHO collects data that are designed to provide national estimates of the prevalences of these risk factors. However they also want to use the data to help them see the spatial patterns of risk across the country to

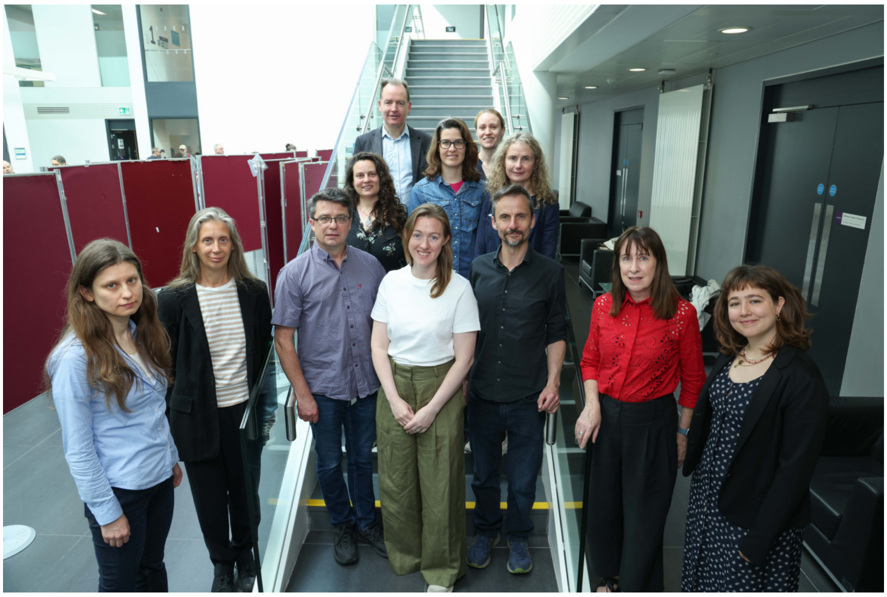

FIGURE 2. Speakers and Organisers. Front Row L-R: Nina Snigireva, Louise Alcock MRIA, Niall Madden, Róisín Neururer, Cathal Seoighe, Helen Maher, Margherita Piccolo Middle Row L-R: Myrto Manolaki, Victoria Sánchez Muñoz, Nicola Fitz-Simon Back Row L-R: Leo Creedon, Doireann O'Kiely Not included: Romina Gaburro.

help them target interventions. The data for each small area on its own are too small to make reliable estimates, but using hierarchical Bayesian statistical models we can borrow statistical strength across areas to make more accurate and precise estimates. A substantial part of this project is on communicating the results to stakeholders, where data visualisation has an important role.

Myrto Manolaki:
The Irish success at the European Girls' Mathematical Olympiad 2025
Abstract: The European Girls' Mathematical Olympiad (EGMO) is the most prestigious mathematical competition for girls, which started in 2012 (Cambridge, UK) and since then is held in April each year. In this talk, after presenting the structure and the historical context of the competition, I will focus on the recent success of the Irish team in Kosovo (one Bronze Medal and three Honourable Mentions, Ireland's second best performance ever).

Róisín Neururer:
The problem of problem-solving in post-primary classrooms: What are the challenges and how might we address them?
Abstract: Successive curriculum reforms in Ireland have led to an increased emphasis on problem-solving within the post-primary mathematics curriculum. However, there is little evidence to suggest classroom practices have significantly changed. In this talk I will share teachers' perspectives on and experiences with these reforms and highlight some of the underlying issues which may be hindering teachers from incorporating problem-solving into their classrooms in a meaningful way. Structured Problem Solving, an approach to teaching mathematics through problem-solving, will be described along with the challenge it poses to teachers. Finally, I will discuss possible supports that might enable teachers to engage more fully with problem-solving in their classrooms.

Doireann O'Kiely:
Three research projects led by University of Limerick women
Abstract: The MACSI research group at the University of Limerick uses mathematical and statistical techniques to solve problems in society, healthcare and industry. In this talk I will outline three very different projects, where mathematics and statistics were used for under-sea imaging, cancer treatment assessment and structural deformation, and profile the University of Limerick women involved in these projects.

Margherita Piccolo:
A Wander into the World of Prime Numbers and Groups
Abstract: Prime numbers have fascinated mathematicians and curious minds for centuries — mysterious, fundamental, and endlessly surprising. In this talk, we'll take a journey through some of the most intriguing ideas in mathematics, starting with prime numbers and the Riemann zeta function - a powerful object that reveals hidden patterns in the distribution of primes. Then, we'll enter the world of groups, the mathematical language of symmetry. Just as integers are built from primes, many groups are built from simple groups — atomic components in the algebraic universe. We'll explore how we study groups through their representations, and how these give rise to zeta functions that help us uncover deep structural patterns. Join me in this exploration of hidden connections, elegant abstractions, and the surprising unity of mathematical ideas.

Victoria Sánchez Muñoz:
Why maths?
Abstract: What's the best route to go home? How to fit best everything inside the backpack? How can we hide messages so that only your friends can read them? Maths has all the answers to these questions! I'll give many (daily life) examples of some of the cool things you can do with maths, and I'll show you that escaping maths is impossible! Because it's everywhere! Even in literature, in videogames, and in art! I will also explain how I used maths to challenge my insurance company, and if the students are interested on knowing what I do now, I'll briefly explain why randomness is super important and my current research on how to generate randomness with quantum stuff. This talk will be non-technical (and hopefully fun), thus suitable for any secondary school student. I'll try to keep it highly interactive, so come along with ideas, with questions, and with some answers to "why maths?".

Further details about the event, including the full programme, list of poster presentations, and photos, can be found at this website.

Report by Nina Snigireva and Niall Madden, University of Galway
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On a certain double integral representation of Catalan's constant and other interesting integration formulae

JAMIL ABREU

Abstract. In this note, we discuss an almost certainly known but unfamiliar double integral representation for Catalan's constant, based on a classical trigonometric integral formula. From this foundation, we also derive some interesting integral identities involving a combination of logarithmic and inverse tangent functions.

1. Catalan's constant

Catalan's constant, often denoted by G, is the alternating sum

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots .$$

It is named after the Belgian mathematician Eugène Catalan (1814-1894), who undertook a comprehensive study of it in 1865. There are many representations of Catalan's constant, both as series and integrals; see Bradley [3]. Many other formulae can be found in classical references such as Gradshteyn and Ryzhik [4] and the three-volume collection by Berndt [2].

The simplest integral representation of G seems to be that coming from the arctangent power series

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

In fact, if we divide by x and integrate from 0 to 1 then we obtain

$$G = \int_0^1 \frac{\arctan x}{x}\, dx. \tag{1}$$

Arguably, the easiest way of justifying interchanging summation and integration above is by writing

$$\frac{\arctan x}{x} = \sum_{n=0}^{N} \frac{(-1)^n x^{2n}}{2n+1} + r_N(x)$$

and noting that, since the series is alternating with terms decreasing in magnitude, we have $|r_N(x)| \leqslant x^{2N+2}/(2N+3)$, so that

$$\lim_{N\to\infty} \int_0^1 r_N(x)\, dx = 0.$$

By substituting $x = \tan\varphi$ into (1) and subsequently setting $\theta = 2\varphi$, we obtain

$$G = \int_0^{\pi/4} \frac{\varphi}{\sin\varphi\cos\varphi}\, d\varphi = \frac{1}{2}\int_0^{\pi/2} \frac{\theta}{\sin\theta}\, d\theta. \tag{2}$$

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Moreover, by noticing that

$$\frac{1}{\sin\varphi\cos\varphi} = \frac{\sec^2\varphi}{\tan\varphi} = \frac{d}{d\varphi}\ln(\tan\varphi),$$

integration by parts in the middle expression in (2) yields

$$G = -\int_0^{\pi/4} \ln(\tan\varphi)\, d\varphi. \tag{3}$$

More generally, the following holds.

Lemma 1.1. *For all* $p = 0, 1, 2, \dots,$

$$\int_0^{\pi/2} \frac{\theta^{p+1}}{\sin\theta}\, d\theta = -2^{p+1}(p+1)\int_0^{\pi/4} \varphi^p \ln(\tan\varphi)\, d\varphi.$$

Proof. By starting with the integral on the right, perform integration by parts (with $u = \varphi^p \ln(\tan\varphi)$ and $dv = d\varphi$), using the derivative

$$\frac{d}{d\varphi}[\varphi^p \ln(\tan\varphi)] = p\,\varphi^{p-1}\ln(\tan\varphi) + \frac{\varphi^p}{\sin\varphi\cos\varphi}.$$

To conclude, make the change of variables $\theta = 2\varphi$. □

For $p = 0$, Lemma 1.1 is just the equality between the right-hand integrals in (2) and (3). For $p = 1, 2$, it well known that

$$\int_0^{\pi/2} \frac{\theta^2}{\sin\theta}\, d\theta = 2\pi G - \frac{7}{2}\zeta(3) \tag{4}$$

and

$$\int_0^{\pi/2} \frac{\theta^3}{\sin\theta}\, d\theta = \frac{3\pi^2}{2}G - 12\beta(4), \tag{5}$$

where $\zeta(3)$ is *Apéry's constant*, namely, the value for $s = 3$ of the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots,$$

and $\beta(4)$ is the value for $s = 4$ of the *Dirichlet beta function*

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots. \tag{6}$$

Note that $\beta(2) = G$. The standard way of deriving formulae (4) and (5) is by using the Fourier series of $\ln(\tan\varphi) = \ln(\sin\theta) - \ln(\cos\varphi)$, see Tolstov [7, Sect. 3.14]. A more general description of the corresponding indefinite integrals in Lemma 1.1 as certain Fourier series can be found in Berndt [2, Part I: p. 261, Entry 14]. By using the Laurent expansion of the co-secant function, one can also express the integrals in Lemma 1.1 as a series involving powers of π and the Bernoulli numbers; see e.g. Sofo and Nimbran [6, Lemma 2.2].

2. An interesting double integral representation of G

There are also some representations of G as double integrals, the most basic being arguably

$$G = \int_0^1\int_0^1 \frac{dx\,dy}{1 + x^2y^2}.$$

This representation can be established directly from (1); see Bradley [3, Formula (40)]. Here, we will prove that

$$G = \int_0^{\pi/2}\int_0^1 \frac{d\theta\,dx}{1+2x\cos\theta+x^2}. \tag{7}$$

The proof of (7) will be based on the following classical formula.

Proposition 2.1. *For all* $0 \leqslant x < 1$,

$$\int_0^{\pi/2} \frac{d\theta}{1+2x\cos\theta+x^2} = \frac{2}{1-x^2}\arctan\frac{1-x}{1+x}. \tag{8}$$

Proof. Using the rational parametrization $\cos\theta = (1-t^2)/(1+t^2)$, the integral on the left in (8) equals

$$\begin{aligned}
\int_0^1 \frac{1}{1+2x\cdot\dfrac{1-t^2}{1+t^2}+x^2}\,\frac{2\,dt}{1+t^2} &= \int_0^1 \frac{2\,dt}{(1+x)^2+(1-x)^2t^2} \\
&= \frac{2}{(1-x)^2}\int_0^1 \frac{dt}{\left(\dfrac{1+x}{1-x}\right)^2+t^2} \\
&= \frac{2}{1-x^2}\arctan\frac{1-x}{1+x},
\end{aligned}$$

where in the last equality we have used

$$\int \frac{dt}{a^2+t^2} = \frac{1}{a}\arctan\frac{t}{a}+C. \qquad \square$$

Now, to prove (7), we integrate (8) over x, from 0 to 1, which yields

$$\begin{aligned}
\int_0^1 \left[\int_0^{\pi/2} \frac{1}{1+2x\cos\theta+x^2}\,d\theta\right] dx &= \int_0^1 \frac{2}{1-x^2}\arctan\frac{1-x}{1+x}\,dx \\
&= \int_0^1 \frac{\arctan y}{y}\,dy \\
&= G,
\end{aligned}$$

where the second identity follows by the change of variables $y = (1-x)/(1+x)$ and the third follows by (1).

We might ask what happens if we interchange the order of integration in the iterated integral above. The conclusion, in brief, is that nothing particularly interesting arises. In fact,

$$\begin{aligned}
\int_0^1 \frac{dx}{1+2x\cos\theta+x^2} &= \int_0^1 \frac{dx}{\sin^2\theta+(x+\cos\theta)^2} \\
&= \frac{1}{\sin\theta}\arctan\frac{x+\cos\theta}{\sin\theta}\Big|_{x=0}^{x=1} \\
&= \frac{1}{\sin\theta}\left[\arctan\frac{1+\cos\theta}{\sin\theta} - \arctan\frac{\cos\theta}{\sin\theta}\right] \\
&= \frac{1}{\sin\theta}\arctan\frac{\sin\theta}{1+\cos\theta} \\
&= \frac{\theta}{2\sin\theta},
\end{aligned}$$

and a further integration over θ, from 0 to $\pi/2$, simply yields (2).

Next, we explore identity (8) in other directions.

3. An elegant integration formula

Consider the following classical formulae, both valid for $0 < \theta < \pi$,

$$\int_0^\infty \frac{\ln x}{1 + 2x\cos\theta + x^2}\,dx = 0, \tag{9}$$

and

$$\int_0^1 \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\,dx = \frac{\theta(\pi^2 - \theta^2)}{6\sin\theta}. \tag{10}$$

Formula (9) appears in Gradshteyn and Ryzhik [4, (4.233-5)] and can be easily verified by changing x to $1/x$, which makes the integral equal to its negative, implying its value is 0. Formula (10) appears in Gradshteyn and Ryzhik [4, (4.261-1)], without proof but with a reference to the 1867 publication *Nouvelles tables d'intégrales définies*, by Bierens de Haan, which in turn refers to an even earlier publication.

We will not try to prove (10) here, but we may notice that changing x to $1/x$ yields

$$\int_0^1 \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\,dx = \int_1^\infty \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\,dx,$$

which implies

$$\int_0^\infty \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\,dx = \frac{\theta(\pi^2 - \theta^2)}{3\sin\theta} \quad (0 < \theta < \pi). \tag{11}$$

If we multiply (8) by $\ln x$, then integrate over x, from 0 to $+\infty$, and interchange the order of integration on the left side, we obtain (using (9)),

$$\int_0^\infty \frac{\ln x}{1 - x^2} \arctan\frac{1-x}{1+x}\,dx = \frac{1}{2}\int_0^{\pi/2} \Big[\int_0^\infty \frac{\ln x}{1 + 2x\cos\theta + x^2}\,dx\Big]\,d\theta = 0.$$

This is also derived by simply changing x to $1/x$ in the integral on the left, with no need of formula (9). The same procedure, this time multiplying (8) by $\ln^2 x$, integrating from 0 to 1, and using (10), yields

$$\begin{aligned}\int_0^1 \frac{\ln^2 x}{1 - x^2} \arctan\frac{1-x}{1+x}\,dx &= \frac{1}{2}\int_0^{\pi/2} \Big[\int_0^1 \frac{\ln^2 x}{1 + 2x\cos\theta + x^2}\,dx\Big]\,d\theta \\ &= \frac{1}{12}\Big[\pi^2 \int_0^{\pi/2} \frac{\theta}{\sin\theta}\,d\theta - \int_0^{\pi/2} \frac{\theta^3}{\sin\theta}\,d\theta\Big].\end{aligned}$$

On the right, the first integral inside brackets equals $2G$, by (2). Combining this with (4) we obtain the interesting formula

$$\int_0^1 \frac{\ln^2 x}{1 - x^2} \arctan\frac{1-x}{1+x}\,dx = \frac{\pi^2 G}{24} + \beta(4). \tag{12}$$

Note that, in light of (11), the corresponding integral from 0 to $+\infty$ is twice that in (12). Moreover, changing $(1-x)/(1+x)$ to x yields the equally interesting

$$\int_0^1 \ln^2\Big(\frac{1-x}{1+x}\Big)\frac{\arctan x}{x}\,dx = \frac{\pi^2 G}{12} + 2\beta(4). \tag{13}$$

4. Final thoughts: the Basel problem

It is in all likelihood an overstatement to assert that the identity (7) is new and has never been highlighted before. It must be observed, however, that it does not appear, for instance, in Bradley's comprehensive list [3], and despite our best efforts, we were unable to find any record of it in the literature. On the other hand, the computation following the proof of (8), which shows that (7) is essentially (2), renders this double integral representation of G quite natural.

The same goes with formulae (12) and (13). There are some close relatives, for instance, in Vălean's books [8, 1.20, 1.21, 1.24, 1.26] and [9, 1.36, 1.37, 1.38, 1.57, 1.58]. By 'close relative' we mean any integral formula involving logarithms multiplied by inverse tangents divided by polynomials. In the event that those identities are already known, we believe and hope that, at least, the evaluations presented here may be a novel and interesting contribution.

The integral in (8) is more often considered over the intervals $[0, \pi]$ or $[0, 2\pi]$. There are many such formulas in various sections of Gradshteyn and Ryzhik [4]. In particular, Gradshteyn and Ryzhik [4, (3.792-1)] is essentially

$$\int_0^\pi \frac{d\theta}{1 + 2x\cos\theta + x^2} = \frac{\pi}{1 - x^2}, \tag{14}$$

valid for $-1 < x < 1$. As with (8), this is easily obtained using rational parametrization. Now, if we multiply (14) by $\ln x$, then integrate over x, from 0 to 1, and interchange the order of integration on the left side, we obtain

$$\int_0^\pi \Big[\int_0^1 \frac{\ln x}{1 + 2x\cos\theta + x^2}\, dx\Big] d\theta = \pi \int_0^1 \frac{\ln x}{1 - x^2}\, dx. \tag{15}$$

As it is well-known, the integral on the right-hand side is related to the so called *Basel problem*, namely, the problem of numerically evaluating the series

$$\zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots .$$

This was first solved by Euler in 1734, who showed $\zeta(2) = \pi^2/6$. The connection between (15) and the Basel problem is

$$\int_0^1 \frac{\ln x}{x^2 - 1}\, dx = \sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \frac{3}{4} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{3}{4}\zeta(2). \tag{16}$$

In fact, the first identity above can be established by expanding $(1 - x^2)^{-1}$ in geometric series and using the formula (obtained with integration by parts)

$$\int_0^1 x^{2n} \ln x\, dx = -\frac{1}{(2n+1)^2},$$

after interchanging integration with summation; the second identity comes from splitting $\sum n^{-2}$ into odd and even indices. The leftmost integral in (16) is known to be $\pi^2/8$, a result obtainable independently of the Basel problem. For more details, see e.g. Abreu [1].

Unfortunately, despite (9), the innermost integral on the left side in (15) is not known as a function of θ in terms of elementary functions; in fact,

$$\int_0^1 \frac{\ln x}{1 + 2x\cos\theta + x^2}\, dx = -\frac{\mathrm{Cl}_2(\pi - \theta)}{\sin\theta} \quad (0 < \theta < \pi),$$

where Cl_2 denotes the *Clausen function of order two*, see Moll and Posey [5]. Thus, the Basel problem is equivalent to

$$\int_0^\pi \frac{\mathrm{Cl}_2(\theta)}{\sin\theta}\, d\theta = \frac{\pi^3}{8}.$$

This connection, despite being possibly familiar to the experts in the field (Clausen functions, polylogarithms, etc.), does not seem to be widely known. For those not familiar with these special functions, or not wishing to delve deeper into these matters, it suffices to say that any elementary evaluation of the double integral in (15), yielding the value $-\pi^3/8$, would constitute a genuinely new solution to the Basel problem.

Finally, by an analogous reasoning, we have

$$\int_0^1 \frac{\ln^2 x}{x^2-1}\,dx = -2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = -\frac{7}{4}\zeta(3), \tag{17}$$

this time using the formula

$$\int_0^1 x^{2n}\ln^2 x\,dx = \frac{2}{(2n+1)^3}.$$

Then, using the elementary identity $\arctan(1/u) = \pi/2 - \arctan(u)$ (valid for $u > 0$) in (12), combined with (17), yields the integration formula

$$\int_0^1 \frac{\ln^2 x}{1-x^2}\arctan\frac{1+x}{1-x}\,dx = \frac{7\pi}{8}\zeta(3) - \frac{\pi^2 G}{24} - \beta(4).$$

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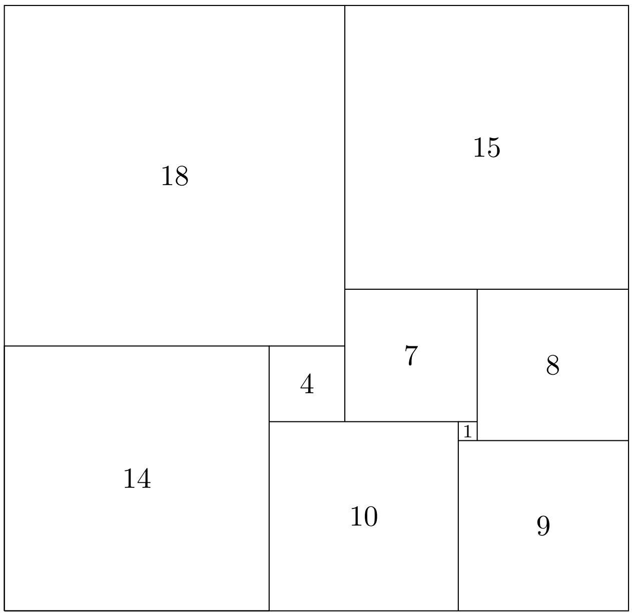
Dissecting Rectangles into Squares

JOE KINGSTON AND DES MACHALE

ABSTRACT. Let n be a positive integer less than 100 which can be expressed as the sum of two or more distinct squares of integers. We ask when a rectangle of area n with sides of integer length can be dissected into different squares with just one of the squares cut, and produce several examples. We also present some rectangular dissections where the cut square satisfies the further constraint that the two pieces are rectangular.

1. Introduction

A classical problem in combinatorial geometry asks if it is possible to dissect a non-square rectangle into a finite number of integer-sided squares, no two of which have the same size. This problem was solved by the Polish mathematician Zbigniew Moron [1] in 1925, who gave an example of a $32 \times 33 = 1056$ rectangle which can be dissected into nine squares of sides $\{18, 15, 14, 10, 9, 8, 7, 4, 1\}$ like so:



He also showed that this is the smallest integer example and that, at least, nine squares are necessary.

For smaller integer-sided rectangles and $n < 9$ squares, we ask when a rectangle can be dissected into squares if we allow some of the squares to be cut. Clearly if we can achieve our objective with just one square cut, then this is the best possible result.

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In this note we produce some examples of this situation, e.g. $30 = 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 5$ is the sum of four distinct squares and we achieve a $5 = 4 + 1$ piece dissection of a 6×5 rectangle so that the pieces can be reassembled to form four distinct squares. In some cases the cut square consists of two rectangular pieces – this situation we refer to as a rectangular dissection (R). It involves an extra constraint which is rarely satisfied.

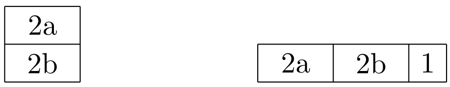
Of course, there are some cases where our objectives cannot be realised. For example, $17 = 4^2 + 1^2$, but a 17×1 rectangle needs at least a five piece dissection to form a 4-square and a 1-square. Also, some integers, for example, 15, are not the sum of distinct squares.

The situation we are looking at for small non-square rectangles appears to differ from that of small squares. See [2]. For example, a dissection of a 5-square to form a 4-square and a 3-square appears to need $4 = 2 + 2$ pieces, based on $5^2 = 4^2 + 3^2$. Intuitively, the unequal length and breadth of a non-square rectangle give more room for manoeuvre. In addition, at least 21 squares are needed to dissect a square into unequal squares.

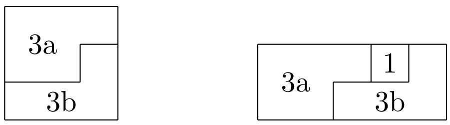
We include the integer equations for which we have failed to find one-cut dissections and where it is not obvious, to us, that no such dissections exist. We would like to hear from readers who have succeeded with some of these. We observe that outside of the one-cut situation, proofs can be extremely difficult and tricky. In this note, we confine ourselves to integer sided rectangles of area less than 100.

2. The Examples

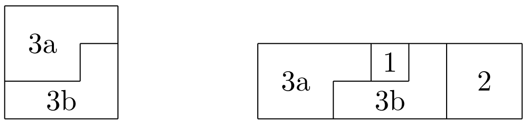
$5 = 2^2 + 1^2 = 5 \times 1$
(R)



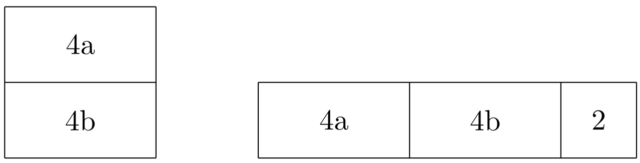
$10 = 3^2 + 1^2 = 5 \times 2$



$14 = 3^2 + 2^2 + 1^2 = 7 \times 2$

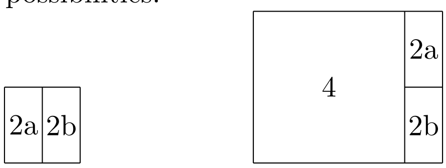


$20 = 4^2 + 2^2 = 10 \times 2$
(R)

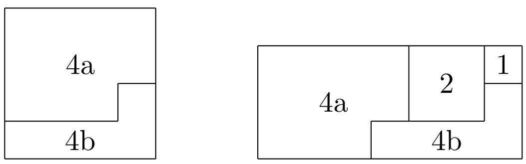


It may be objected that this is merely a 'blow-up' of the 5×1 case, but sometimes increasing the scale leads to new possibilities.

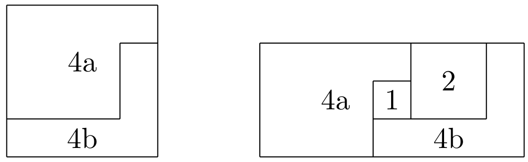
$20 = 4^2 + 2^2 = 5 \times 4$
(R)



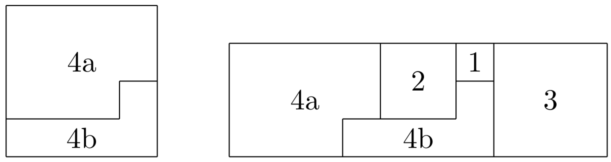
$21 = 4^2 + 2^2 + 1^2 = 7 \times 3$



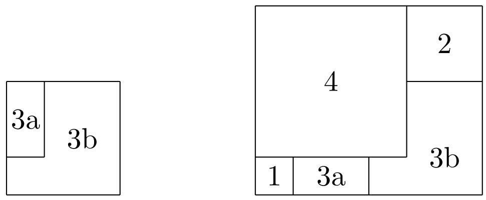
$21 = 4^2 + 2^2 + 1^2 = 7 \times 3$



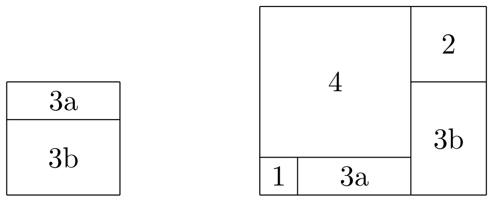
$30 = 4^2 + 3^2 + 2^2 + 1^2$
$= 10 \times 3$ (R)



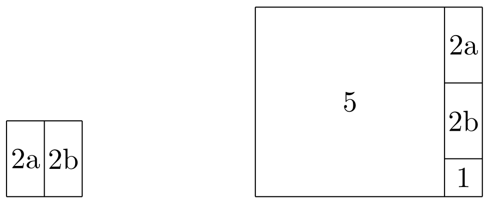
$30 = 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 5$



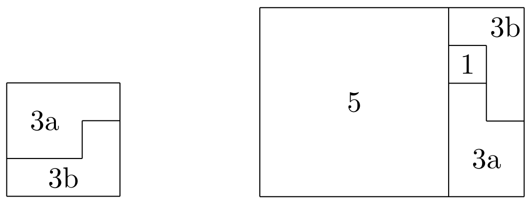
$30 = 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 5$
(R)



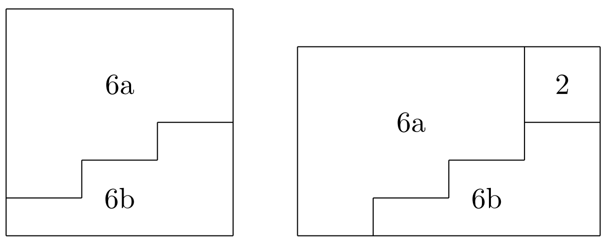
$30 = 5^2 + 2^2 + 1^2 = 6 \times 5$ i
(R)



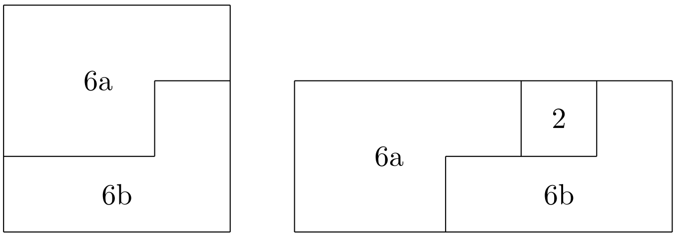
$35 = 5^2 + 3^2 + 1^2 = 7 \times 5$



$40 = 6^2 + 2^2 = 5 \times 8$

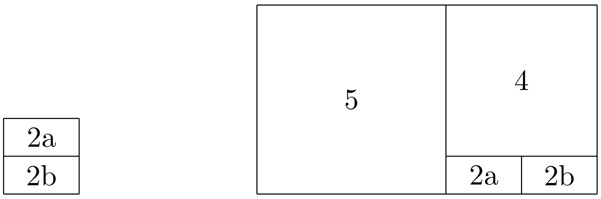


$40 = 6^2 + 2^2 = 10 \times 4$

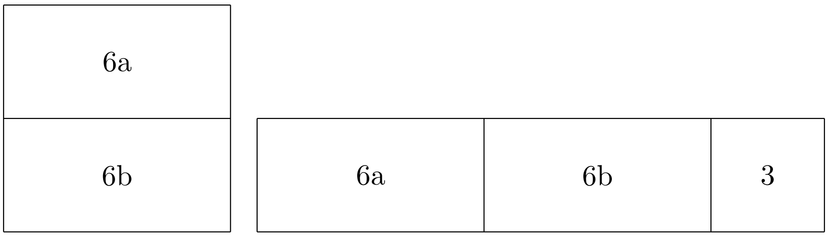


$42 = 5^2 + 4^2 + 1^2 = 6 \times 7$ **Not found**

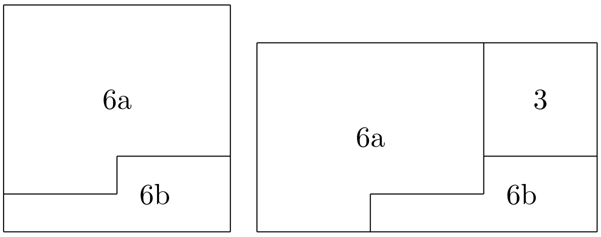
$45 = 5^2 + 4^2 + 2^2 = 5 \times 9$
(R)



$45 = 6^2 + 3^2 = 3 \times 15$
(R)



$45 = 6^2 + 3^2 = 3 \times 15$

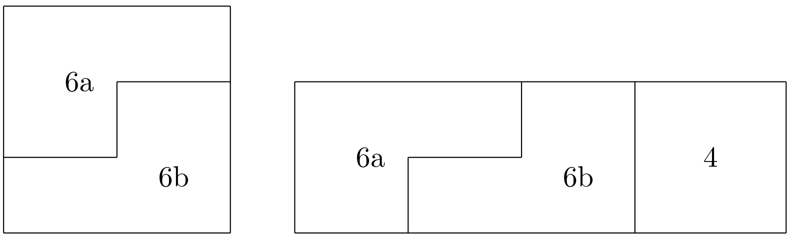


$50 = 7^2 + 1^2 = 5 \times 10$ **Not found**

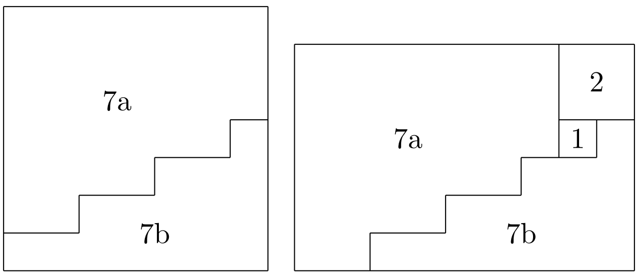
$50 = 6^2 + 3^2 + 2^2 + 1^2 = 5 \times 10$ **Not found**

$50 = 5^2 + 4^2 + 3^2 = 5 \times 10$ **Not found**

$52 = 6^2 + 4^2 = 4 \times 13$

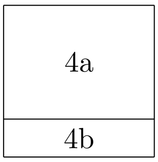


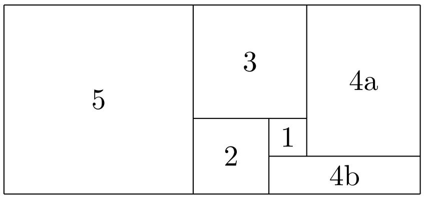
$54 = 7^2 + 2^2 + 1^2 = 6 \times 9$



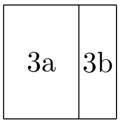
$54 = 5^2 + 4^2 + 3^2 + 2^2 = 6 \times 9$ Not found

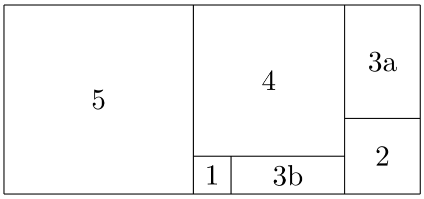
$55 = 5^2+4^2+3^2+2^2+1^2$
$= 5 \times 11$ (R)



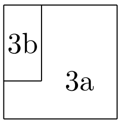


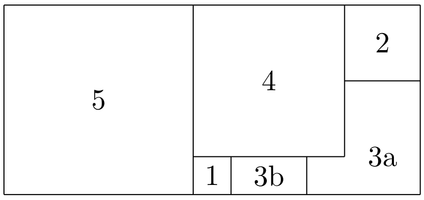
$55 = 5^2+4^2+3^2+2^2+1^2$
$= 5 \times 11$ (R)



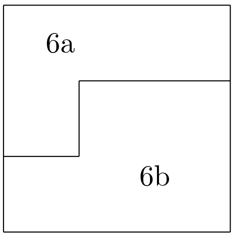


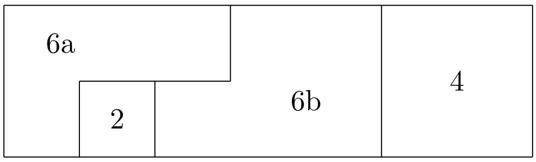
$55 = 5^2+4^2+3^2+2^2+1^2$
$= 5 \times 11$ (R)



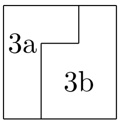


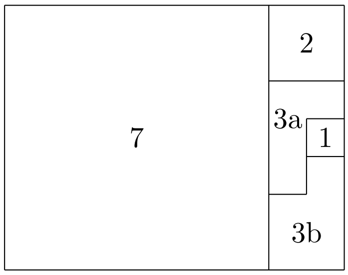
$56 = 6^2 + 4^2 + 2^2$
$= 4 \times 14$ (R)



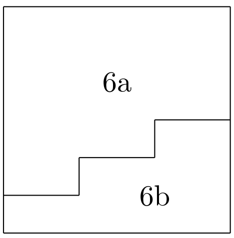


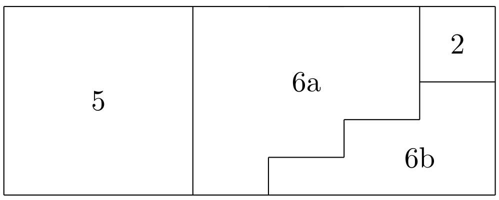
$63 = 7^2 + 3^2 + 2^2 + 1^2 = 7 \times 9$



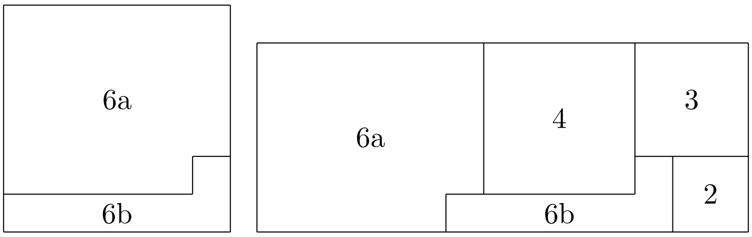


$65 = 6^2 + 5^2 + 2^2$
$= 5 \times 13$ (R)





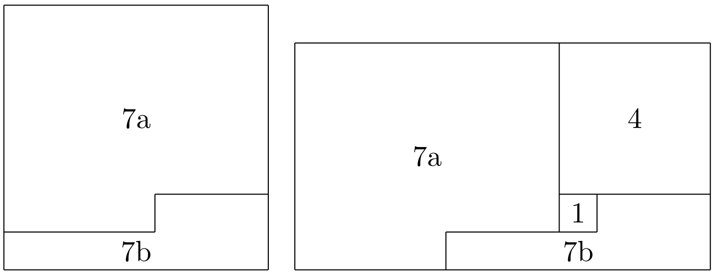
$65 = 6^2 + 4^2 + 3^2 + 2^2$
$= 5 \times 13$ (R)



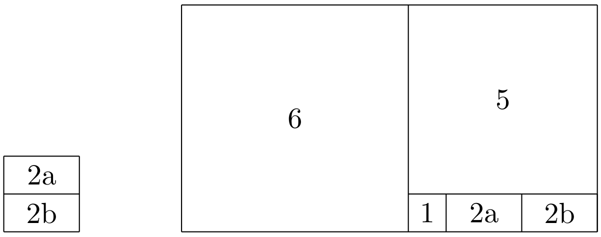
$\mathbf{65 = 8^2 + 1^2 = 5 \times 13}$ **Not found**

$\mathbf{65 = 7^2 + 4^2 = 5 \times 13}$ **Not found**

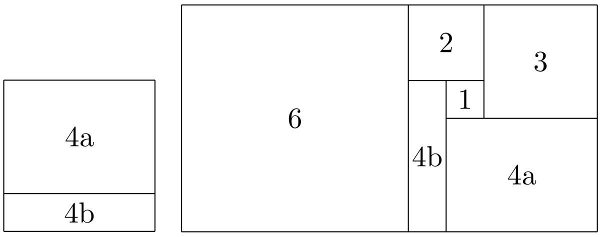
$66 = 7^2 + 4^2 + 1^2 = 6 \times 11$



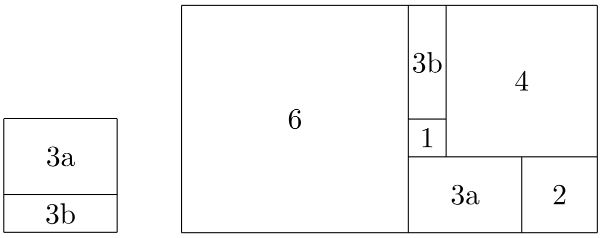
$66 = 6^2 + 5^2 + 2^2 + 1^2 = 6 \times 11$
(R)



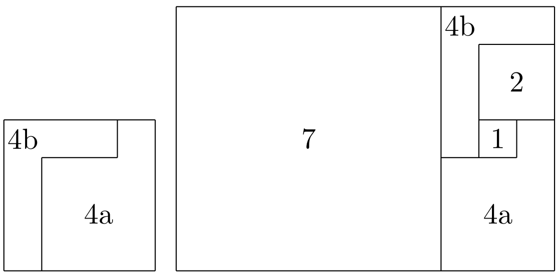
$66 = 6^2 + 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 11$
(R)



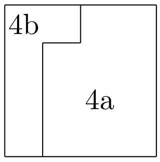
$66 = 6^2 + 4^2 + 3^2 + 2^2 + 1^2 = 6 \times 11$
(R)

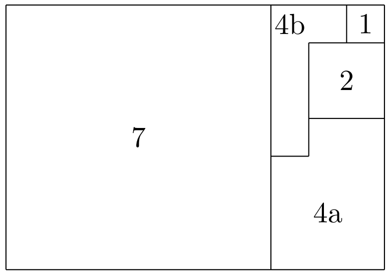


$70 = 7^2 + 4^2 + 2^2 + 1^2 = 7 \times 10$

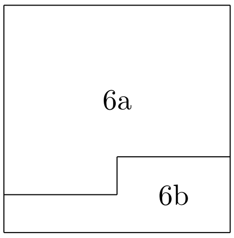


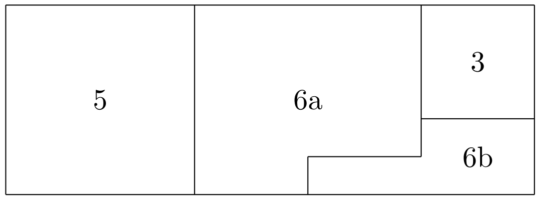
$70 = 7^2 + 4^2 + 2^2 + 1^2$
$= 7 \times 10$ (R)





$70 = 6^2 + 5^2 + 3^2$
$= 5 \times 14$ (R)





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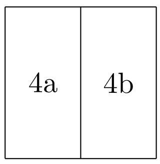
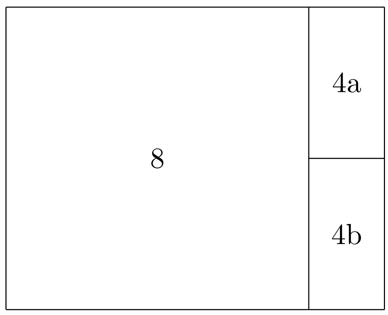
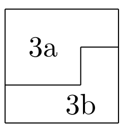
$\mathbf{75 = 6^2 + 5^2 + 3^2 + 2^2 + 1^2 = 5 \times 15}$	**Not found**
$\mathbf{77 = 8^2 + 3^2 + 2^2 = 7 \times 11}$	**Not found**
$\mathbf{77 = 6^2 + 5^2 + 4^2 = 7 \times 11}$	**Not found**
$\mathbf{78 = 8^2 + 3^2 + 2^2 + 1^2 = 6 \times 13}$	**Not found**
$\mathbf{78 = 7^2 + 5^2 + 2^2 = 6 \times 13}$	**Not found**
$\mathbf{78 = 7^2 + 4^2 + 3^2 + 2^2 = 6 \times 13}$	**Not found**
$\mathbf{78 = 6^2 + 5^2 + 4^2 + 1^2 = 6 \times 13}$	**Not found**

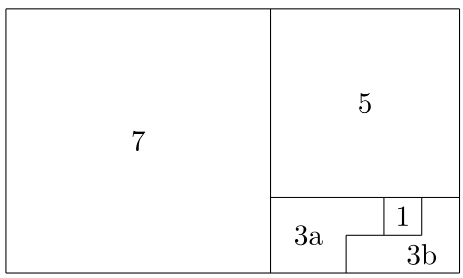
$80 = 8^2 + 4^2 = 8 \times 10$
(R)



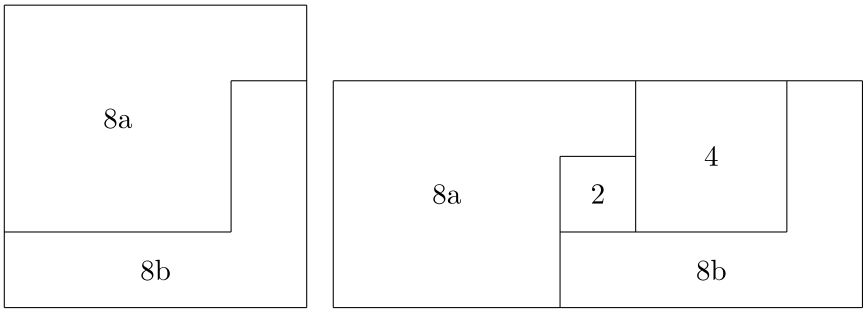


$84 = 7^2 + 5^2 + 3^2 + 1^2$
$= 7 \times 12$ (R)





$84 = 8^2+4^2+2^2$
$= 6 \times 14$ (R)

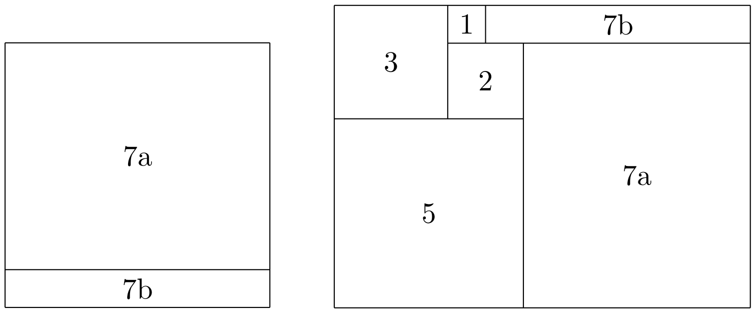


$\mathbf{84 = 7^2 + 5^2 + 3^2 + 1^2 = 6 \times 14}$ **Not found**

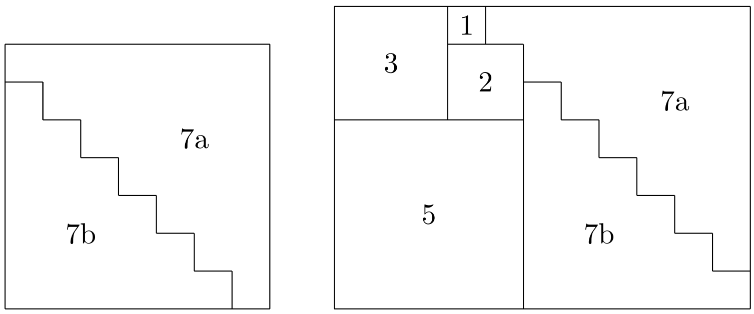
$\mathbf{85 = 9^2 + 2^2 = 5 \times 17}$ **Not found**

$\mathbf{85 = 8^2 + 4^2 +^2 2 + 1^2 = 5 \times 17}$ **Not found**

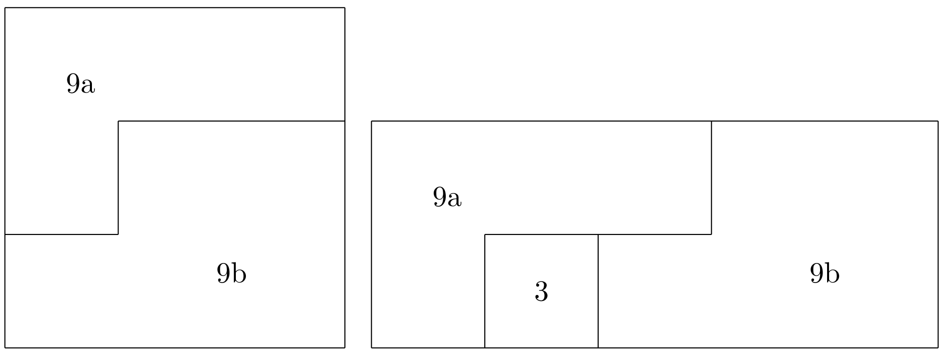
$88 = 7^2+5^2+3^2+2^2+1^2$
$= 8 \times 11$ (R)



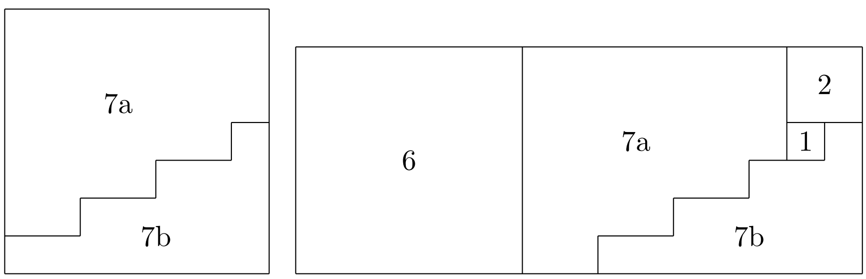
$88 = 7^2+5^2+3^2+2^2+1^2$
$= 8 \times 11$ (R)



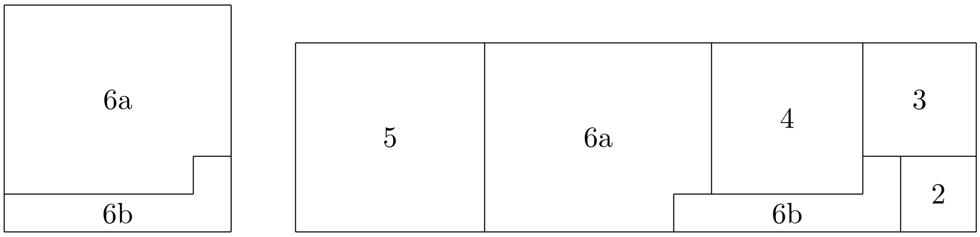
$90 = 9^2 + 3^2$
$= 6\times15$ (R)



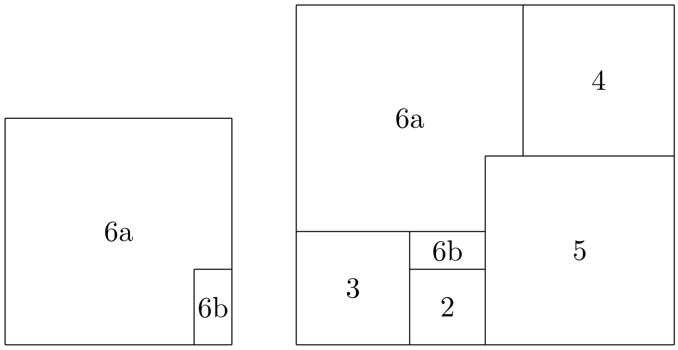
$90 = 7^2+6^2+2^2+1^2$
$= 6 \times 15$ (R)



$90 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 = 5 \times 18$

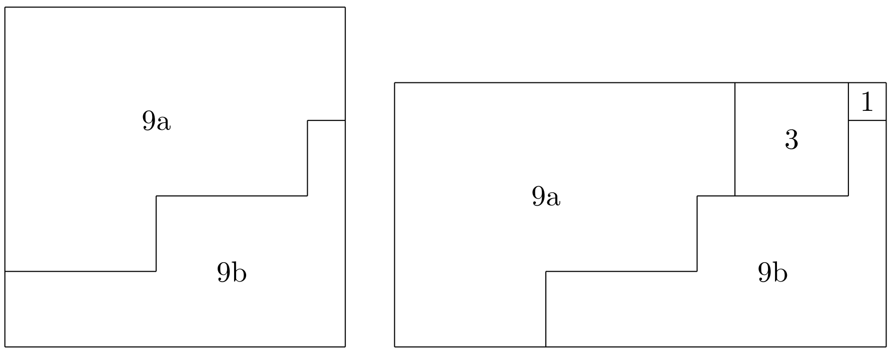


$90 = 6^2{+}5^2{+}4^2{+}3^2{+}2^2$
$= 9 \times 10$ (R)



$\mathbf{90 = 8^2 + 5^2 + 1^2 = 5 \times 18}$ **Not found**
$\mathbf{90 = 8^2 + 4^2 + 3^2 + 1^2 = 5 \times 18}$ **Not found**
$\mathbf{90 = 7^2 + 5^2 + 4^2 = 5 \times 18}$ **Not found**
$\mathbf{90 = 8^2 + 5^2 + 1^2 = 6 \times 15}$ **Not found**
$\mathbf{90 = 8^2 + 4^2 + 3^2 + 1^2 = 6 \times 15}$ **Not found**
$\mathbf{90 = 7^2 + 5^2 + 4^2 = 6 \times 15}$ **Not found**
$\mathbf{90 = 8^2 + 5^2 + 1^2 = 9 \times 10}$ **Not found**
$\mathbf{90 = 8^2 + 4^2 + 3^2 + 1^2 = 9 \times 10}$ **Not found**
$\mathbf{90 = 7^2 + 5^2 + 4^2 = 9 \times 10}$ **Not found**
$\mathbf{90 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 = 6 \times 15}$ **Not found**

$91 = 9^2 + 3^2 + 1^2$
$= 7 \times 13$ (R)



$\mathbf{91 = 7^2 + 5^2 + 4^2 + 1^2 = 7 \times 13}$ **Not found**
$\mathbf{91 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2 = 7 \times 13}$ **Not found**
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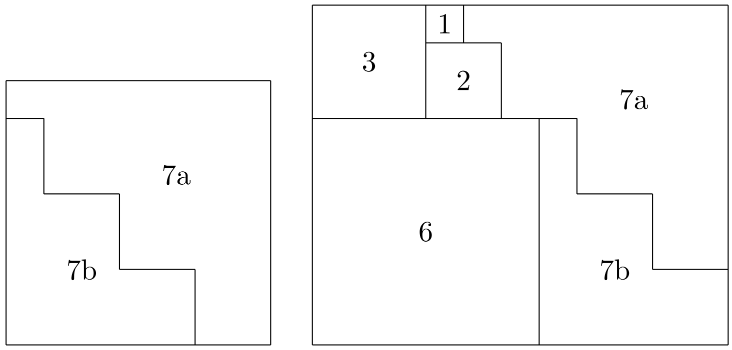
$\mathbf{95 = 7^2 + 5^2 + 4^2 + 2^2 + 1^2 = 5 \times 19}$ **Not found**
$\mathbf{98 = 9^2 + 4^2 + 1^2 = 7 \times 14}$ **Not found**
$\mathbf{98 = 8^2 + 5^2 + 3^2 = 7 \times 14}$ **Not found**
$\mathbf{98 = 7^2 + 6^2 + 3^2 + 2^2 = 7 \times 14}$ **Not found**

$99 = 7^2+6^2+3^2+2^2+1^2$
$= 9 \times 11$ (R)



$\mathbf{99 = 8^2 + 5^2 + 3^2 + 1^2 = 9 \times 11}$ **Not found**

$\mathbf{99 = 7^2 + 5^2 + 4^2 + 3^2 = 9 \times 11}$ **Not found**

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[1] Z. Moron: *On the dissection of rectangles into squares*, Wiadem Mat. (1) 1 (1925), pp. 75—79.

[2] J. Kingston and D. MacHale: *Dissecting Squares*, Mathematical Gazette, Vol 85, Issue 504, November 2001, pp. 403—430.

Joe Kingston taught Mathematics in Hamilton High School, Bandon, for thirty nine years. In the mid-nineties, while studying for a Master's degree at University College, Cork, Des MacHale introduced him to dissections. The introduction should have come with a health warning. Recovery is slow. Other interests include bridge, films, music, puzzles and reading. **Des MacHale** is Emeritus Professor of Mathematics at University College Cork where he taught for forty years. His mathematical interests are in abstract algebra, especially groups and rings, but he has also worked in number theory, Euclidean geometry, combinatorics and the history of mathematics. His other interests include humour, puzzles, words and geology.

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Elementary Proofs of Ring Commutativity Theorems

MICHAEL KINYON AND DES MACHALE

Abstract. Jacobson's commutativity theorem says that a ring is commutative if, for each x, $x^n = x$ for some $n > 1$. Herstein's generalization says that the condition can be weakened to $x^n - x$ being central. In both theorems, n may depend on x. In this paper, in certain cases where n is a fixed constant, we find equational proofs of each theorem. For the odd exponent cases $n = 2k + 1$ of Jacobson's theorem, our main tool is a lemma stating that for each x, x^k is central. For Herstein's theorem, we consider the cases $n = 4$ and $n = 8$, obtaining proofs with the assistance of the automated theorem prover Prover9.

1. Introduction

N. Jacobson's celebrated commutativity theorem for rings [14] and its generalization by I. N. Herstein [12] state:

Jacobson's Theorem. *Let R be a ring such that, for each $x \in R$, there exists an integer $n = n(x) > 1$ such that $x^n = x$. Then R is commutative.*

Herstein's Theorem. *Let R be a ring such that, for each $x \in R$, there exists an integer $n = n(x) > 1$ such that $[x^n - x, y] = 0$ for all $y \in R$. Then R is commutative.*

Jacobson's Theorem is a generalization of Wedderburn's "Little" Theorem that every finite division ring is commutative. Rings satisfying the hypothesis of Jacobson's Theorem have been given various names, such as *potent* rings [2, 23], J-rings [13], and probably others of which we are unaware. Choosing the first one, Jacobson's Theorem can be stated succinctly as *potent rings are commutative*. Rings satisfying the hypothesis of Herstein's Theorem seem to have never been given a separate name as far as we know, probably because the hypothesis is both necessary and sufficient for a ring to be commutative.

As noted in the statements, both theorems allow the exponent n of the power x^n to depend on x. For Jacobson's, there are various proofs in the literature; perhaps the nicest was given independently by J. W. Wamsley [25] and T. Nagahara and H. Tominaga [22].

In this paper we are interested in both theorems in the case where n is a fixed integer not depending on x, that is, rings R for which there exists an integer $n \geq 2$ such that $x^n = x$ for all $x \in R$. There have been various names suggested for such rings: J-rings [17, 18] (also used, as noted, for potent rings [13]), Jacobson rings [10], n-rings [6], n-potent rings [1], and, again, probably others of which we are unaware. Since elements x satisfying $x^n = x$ for some n are often called n-potent elements, we choose the name *n-potent* rings. We will primarily use the name in the interest of simplifying theorem statements.

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The well known class of *Boolean* rings coincides with what we are calling 2-potent rings. Jacobson's Theorem for such rings is a standard exercise with an easy equational proof (see the first alternative proof to Theorem 2.6 below). Indeed, it is a consequence of Birkhoff's Completeness Theorem for Equational Logic [5] that, for each fixed n, an equational proof of Jacobson's Theorem exists.

Among the papers devoted to proofs of Jacobson's Theorem for particular fixed n [8, 11, 26, 27], we would particularly like to single out the *tour de force* of Y. Morita [21], who gave human equational proofs for most even numbers ≤ 50 and all odd numbers ≤ 25.

There is certainly similar interest in equational proofs of Herstein's Theorem for fixed n, although the literature is not as extensive; see, e.g., [7, 19].

It is natural to try to use computer assistance to generate equational proofs of either theorem in the case of fixed n [24, 26, 27]. For Jacobson's Theorem, M. Brandenburg [6] has recently done some very exciting work along these lines.

This paper is in two parts. In §2, we discuss Jacobson's Theorem for certain cases of small n. For n odd, our main tool is a useful result we think is new (Lemma 2.4): *if R is a $(2k+1)$-potent ring then, for all $x \in R$, x^k is central.* This result was first found by Stephen Buckley (unpublished); our proof differs from his. We illustrate how helpful the lemma is for $n = 3, 5$, and 7. We also give some (we believe) new proofs for various even n. All but one of the proofs in §2 were originally human generated; the proof of Lemma 2.2 was first found by an automated theorem prover and subsequently humanized.

In §3, we turn to Herstein's Theorem for the specific cases of $n = 2$, 4, and 8. The proofs were found with the assistance of Prover9, the automated theorem prover developed by W. McCune [20]. In fact, the beginning of the authors' collaboration was an email suggestion by the second author to the first that it would be interesting to find automated proofs of Herstein's Theorem for $n = 4$ and 8, and then to see if such proofs could be suitably humanized.

We would judge the humanization effort to be quite successful for $n = 4$ (Theorem 3.7) and somewhat successful for $n = 8$ (Theorem 3.8). For the latter proof, although it is certainly possible for a patient human to follow the individual steps, the overall pattern is difficult to see. We have no idea how, or even if, a general idea can be extracted from the proof which could be applied to higher powers of 2.

We conclude this introduction by discussing conventions. Rings are assumed to be associative but not assumed to have a unity; that is they are what some, following a suggestion of Jacobson ([15], pp. 155–156), would call a "rng".

The *centre* of a ring R is the subring $Z(R) = \{a \in R : ar = ra, \forall r \in R\}$. Elements of $Z(R)$ are said to be *central*.

In our proofs, especially in §3, we will make heavy use of the commutator $[x, y] = xy - yx$. This is an interesting binary operation in its own right, but here we mainly use it as a computational tool. It turns out that introducing commutators into Prover9 input files and feeding the program basic facts about commutators helps quite a bit in finding proofs. We will discuss the commutator identities we need at the beginning of §3.

2. Jacobson's Theorem for n-potent rings

Our main interest in this section is giving equational proofs that n-potent rings are commutative for various values of n. However, for some preliminary results, there is essentially no extra work involved in giving proofs for reduced rings. A ring R is said to be *reduced* if it has no nilpotent elements. This can be equivalently described by the condition

$$x^2 = 0 \implies x = 0\,,\ \text{for all } x \in R. \tag{1}$$

Every potent ring R is reduced: if $x \in R$ satisfies $x^2 = 0$, let $n = n(x) > 1$ be such that $x^n = x$. Then $x = x^n = x^2x^{n-2} = 0$. Every reduced ring is a subdirect product of domains [3, 16], but it would breach the spirit of this paper to use this rather deep structural result.

Every reduced ring R satisfies the condition

$$xy = 0 \implies yx = 0\,, \tag{2}$$

for all $x \in R$. Indeed, if $xy = 0$, then $(yx)^2 = y(xy)x = 0$ and so $yx = 0$ by (1). Rings satisfying (2) are said to be *reversible* [9].

Reduced rings are neither defined nor characterized by identities. Thus even elementary proofs in reduced rings unavoidably use (1) and hence are not, strictly speaking, equational. However, it is straightforward to convert such proofs to equational ones in n-potent rings. For example, to prove directly that an n-potent ring ($n > 1$) is reversible, note that if $xy = 0$, then $yx = (yx)^n = y(xy)^{n-1}x = 0$.

An idempotent e (that is, an element satisfying $e^2 = e$) of a reduced ring R is central. This is well known and has a short, standard proof: check that $(ex - exe)^2 = 0$ and $(xe - exe)^2 = 0$, so R being reduced implies $ex - exe = 0$ and $xe - exe = 0$, hence $ex = exe = xe$. In fact, the same expressions occur in a mild generalization.

Lemma 2.1. *In reversible rings, idempotents are central.*

Proof. If e is an idempotent in a ring R, then for all $x \in R$, $e(x - ex) = 0$ and $(x - xe)e = 0$. If R is reversible, then (2) implies $(x - ex)e = xe - exe = 0$ and $e(x - xe) = ex - exe = 0$. Thus $xe = exe = ex$, that is, e is central. □

In the reduced case, the proof of the following useful generalization is only a bit more involved than the classic proof and is based on the same idea.

Lemma 2.2. *Let R be a reduced ring. If $c \in R$ satisfies $c^2 = tc$ for some integer t, then c is central.*

Proof. Firstly, for all $x \in R$, $c^2xc = tc \cdot xc = cx \cdot tc = cxc^2$. Thus $c[c, x]c = c(cx - xc)c = 0$. In particular, $(c[c, x])^2 = 0$ and $([c, x]c)^2 = 0$. Since R is reduced, (1) yields both $c[c, x] = 0$ and $[c, x]c = 0$. These imply, respectively,

$$xc[c, x] = 0 \qquad \text{and} \tag{3}$$

$$[c, x]cx = 0\,. \tag{4}$$

Applying (2) to (3), we get

$$[c, x]xc = 0\,. \tag{5}$$

Subtracting (5) from (4), we obtain $[c, x]^2 = 0$. Since R is reduced, $[c, x] = 0$ for all $x \in R$, that is, $c \in Z(R)$. □

Lemma 2.3. *Let R be a reduced ring and let $n > 1$ be an integer. If $c \in R$ satisfies $c^n = c$, then c^{n-1} is a central idempotent.*

Proof. If $n = 2$, then $c^{n-1} = c$, while if $n > 2$, then $c^{n-1}c^{n-1} = c^nc^{n-2} = cc^{n-2} = c^{n-1}$. In either case, c^{n-1} is an idempotent and so Lemma 2.2 applies. □

We now specialize from reduced rings to n-potent rings, our real interest. The following result will turn out to be crucial, and can be viewed as improving Lemma 2.3 in the case of odd n.

Lemma 2.4. *Let $k \geq 1$ be an integer and let R be a $(2k + 1)$-potent ring. Then $x^k \in Z(R)$ for all $x \in R$.*

Proof. Firstly, x^{2k} is a central idempotent by Lemma 2.3. Next we show $x^{3k} = x^k$. The claim is clear if $k = 1$, while if $k > 1$, then $x^{3k} = x^{2k+1}x^{k-1} = xx^{k-1} = x^k$. Finally, $(x^{2k} + x^k)^2 = (x^{2k})^2 + 2x^{3k} + x^{2k} = 2(x^{2k} + x^k)$. By Lemma 2.2 (with $c = x^{2k} + x^k$, $t = 2$), $x^{2k} + x^k \in Z(R)$ and so $x^k = (x^{2k} + x^k) - x^{2k} \in Z(R)$ for all $x \in R$, as claimed. □

For n even, we have the following.

Lemma 2.5. *Let $n > 1$ be an even integer and let R be an n-potent ring. Then $2x = 0$ for all $x \in R$.*

Proof. For all $x \in R$, $-x = (-x)^n = x^n = x$. □

For the remainder of this section, we establish commutativity theorems for n-potent rings for various n. We start with a classic, recalling that 2-potent rings are the same as Boolean rings.

Theorem 2.6. *Every 2-potent ring is commutative.*

Proof. This is the case $n = 2$ of Lemma 2.3. □

As an alternative, here is the classic proof.

Alternative proof 1. Let R be 2-potent. For all $x, y \in R$, $x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$. Cancelling, we have $xy + yx = 0$, and so $xy = -yx = yx$ by Lemma 2.5. □

We also present here the stunning proof of Brandenburg [6].

Alternative proof 2. For all x, y in a ring R,

$$xy - yx = [(x + y)^2 - (x + y)] - (x^2 - x) - (y^2 - y) + [(yx)^2 - yx] - [(-yx)^2 - (-yx)] .$$

If R is 2-potent, then the right hand side equals 0. □

For a plethora of proofs and variations of our next result, see [8].

Theorem 2.7. *Every 3-potent ring is commutative.*

Proof. This is the case $k = 1$ of Lemma 2.4. □

Alternative proof. Let R be 3-potent. By Lemma 2.3, x^2 is central for all $x \in R$, so $2x = 2x^3 = (x^2 + x)^2 - x^4 - x^2$ is central. Next,

$$x^2 + x = (x^2 + x)^3 = x^6 + 3x^5 + 3x^4 + x^3 = x^2 + 3x + 3x^2 + x = 2 \cdot 2(x^2 + x) \in Z(R) .$$

Finally, $x = (x^2 + x) - x^2 \in Z(R)$ for all $x \in R$, that is, R is commutative. □

Lemma 2.8. *Let k be an integer and let R be a ring in which $k(xy + yx) \in Z(R)$ for all $x, y \in R$. Then $kx^2 \in Z(R)$ for all $x \in R$.*

Proof. We have $kx^2 \cdot y + kxyx = x \cdot k(xy + yx) = k(xy + yx) \cdot x = kxyx + y \cdot kx^2$ and so the desired result follows from canceling $kxyx$. □

Lemma 2.9. *If R is a ring in which $x^2+x \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. For all $x, y \in R$, we have $(x+y)^2+x+y \in Z(R)$, that is, $(x^2+x)+(y^2+y)+xy+yx \in Z(R)$. It follows that $xy + yx \in Z(R)$, and thus $x^2 \in Z(R)$ by Lemma 2.8 (with $k = 1$). Therefore $x = (x^2 + x) - x^2 \in Z(R)$ for all $x \in R$, i.e., R is commutative. □

Lemma 2.10. *If n is a power of 2, then each binomial coefficient $\binom{n}{k}$ is even except for $k = 0$ and $k = n$.*

Proof. This is easily established by induction. □

Theorem 2.11. *Every* 4*-potent ring is commutative.*

Proof. Let R be 4-potent. By Lemma 2.5, $2x = 0$ for all $x \in R$. Thus $(x^2+x)^2 = x^4 + x^2 = x^2 + x$ using Lemma 2.10. By Lemma 2.2, $x^2 + x$ is central for all $x \in R$, and so R is commutative by Lemma 2.9. □

Theorem 2.12. *Every* 5*-potent ring is commutative.*

Proof. Let R be 5-potent. By Lemma 2.4, $x^2 \in Z(R)$ for all $x \in R$. Thus $(x^2+x)^2 - x^4 - x^2 = 2x^3$ is central for all $x \in R$. Next $2x = 2x^5 = 2x^3 \cdot x^2 \in Z(R)$ for all x. Finally,

$$\begin{aligned} x^2 + x &= (x^2+x)^5 \\ &= x^{10} + 5x^9 + 10x^8 + 10x^7 + 5x^6 + x^5 \\ &= x^2 + 5x + 10x^4 + 10x^3 + 5x^2 + x \\ &= 2(5x^4 + 5x^3 + 3x^2 + 3x) \in Z(R)\,. \end{aligned}$$

By Lemma 2.9, R is commutative. □

We will postpone the case $n = 6$ for now.

Theorem 2.13. *Every* 7*-potent ring is commutative.*

Proof. Let R be 7-potent. By Lemma 2.4, $x^3 \in Z(R)$ for all $x \in R$. Thus $(x^2+x)^3 - x^6 - x^3 = 3(x^5 + x^4) \in Z(R)$. Multiplying by x^3, we get $3(x^2 + x) \in Z(R)$. It follows that

$$3((x+y)^2 + x + y) - 3(x^2+x) - 3(y^2+y) = 3(xy+yx) \in Z(R)\,.$$

By Lemma 2.8 (with $k = 3$), $3x^2 \in Z(R)$. Thus $3x = 3(x^2+x) - 3x^2 \in Z(R)$ for all $x \in R$.

Next, we have

$$\begin{aligned} x^2 + x &= (x^2+x)^7 \\ &= x^{14} + 7x^{13} + 21x^{12} + 35x^{11} + 35x^{10} + 21x^9 + 7x^8 + x^7 \\ &= x^2 + 7x + 21x^6 + 35x^5 + 35x^4 + 21x^3 + 7x^2 + x\,. \end{aligned}$$

Cancelling $x^2 + x$ from both sides and separating terms, we get

$$0 = 3(2x + 7x^6 + 11x^5 + 11x^4 + 7x^3 + 2x^2) + x + 2x^5 + 2x^4 + x^2\,.$$

Thus $2x^5 + 2x^4 + x^2 + x \in Z(R)$. Replacing x in this last expression with $-x$ and adding gives $4x^4 + 2x^2 \in Z(R)$. Thus $x^4 - x^2 = 4x^4 + 2x^2 - 3(x^4 + x^2) \in Z(R)$

Since both x^3 and $x^4 - x^2$ are central, so is $(x^4 - x^2)^2 + 2(x^3)^2 = x^8 + x^4 = x^2 + x^4$. Therefore $2x^2 = (x^2 + x^4) - (x^4 - x^2) \in Z(R)$, and so $x^2 = 3x^2 - 2x^2 \in Z(R)$.

Finally, $x = x^7 = x^2 \cdot x^2 \cdot x^3 \in Z(R)$ for all $x \in R$, and we are finished. □

Many other cases of $x^n = x$ for small n can be handled by these methods, the more difficult cases being when n is odd (although Lemma 2.4 certainly helps) or a power of 2. We believe our proofs are close to being best possible (in some sense) and would be pleased to hear from any reader who can shorten or improve an any of them.

We conclude this section with one more example, covering the cases $n = 6$, 10, 18, 34, … in a single theorem.

Theorem 2.14. *Let* $j \geq 1$ *be an integer and assume that every* 2^{j+1}*-potent ring is commutative. Let* $i > j$ *be an integer and let* R *be a* $2^i + 2^j$*-potent ring. Then* R *is commutative.*

Proof. We have $2x = 0$ for all $x \in R$ by Lemma 2.5. Now

$$\begin{aligned} x^2 + x = (x^2+x)^{2^i+2^j} \quad &= (x^2+x)^{2^i}(x^2+x)^{2^j} \\ = (x^{2^{i+1}} + x^{2^i})(x^{2^{j+1}} + x^{2^j}) \quad &= x^{2(2^i+2^j)} + x^{2^i+2^{j+1}} + x^{2^{i+1}+2^j} + x^{2^i+2^j} \\ = x^2 + x^{2^i+2^j+2^j} + x^{2^i+2^i+2^j} + x &= x^2 + x^{2^j+1} + x^{2^i+1} + x\,, \end{aligned}$$

using Lemma 2.10 in the third equality. Cancelling, we have $x^{2^j+1} + x^{2^i+1} = 0$, that is, $x^{2^i+1} = x^{2^j+1}$. Finally, we have

$$x = x^{2^i+2^j} = x^{2^i+1}x^{2^j-1} = x^{2^j+1}x^{2^j-1} = x^{2^{j+1}}$$

for all $x \in R$. By assumption, R is commutative. □

Corollary 2.15. *Let R be an (2^i+2)-potent ring where $i > 1$. Then R is commutative.*

Proof. Take $j = 1$ in Theorem 2.14 and apply Theorem 2.11. □

3. Herstein's Theorem for $n = 2, 4, 8$

In a ring R, we have already used ring commutators $[x, y] = xy - yx$ and will do so quite heavily in this section. Among the properties satisfied by commutators, we will need the following:

$$[x,x] = 0\,, \qquad [x,y] = -[y,x]\,, \qquad [x, y+z] = [x,y] + [x,z]$$

for all $x, y, z \in R$. To improve readability, it is useful to introduce, for each $x \in R$, the mapping $\mathrm{ad}(x) : R \to R$ defined by $\mathrm{ad}(x)(y) = [x, y]$ for all $y \in R$. Properties satisfied by this mapping include:

$$\begin{aligned} \mathrm{ad}(x)(y+z) &= \mathrm{ad}(x)(y) + \mathrm{ad}(x)(z)\,, \\ \mathrm{ad}(x+y) &= \mathrm{ad}(x) + \mathrm{ad}(y)\,, \\ \mathrm{ad}([x,y]) &= \mathrm{ad}(x)\mathrm{ad}(y) - \mathrm{ad}(y)\mathrm{ad}(x)\,. \end{aligned}$$

for all $x, y, z \in R$. The first identity says that $\mathrm{ad}(x)$ is an endomorphism of the underlying abelian group $(R, +)$. The second says that $\mathrm{ad}\colon (R, +) \to \mathrm{End}(R, +)$ is a homomorphism of abelian groups. The third is where the symbol "ad" comes from; the identity says that $\mathrm{ad}\colon (R, [\cdot,\cdot]) \to \mathrm{End}(R, +)$ is the *adjoint* representation of the Lie ring associated to R. However, for us, ad is just a notational shorthand; nothing about a ring's Lie ring plays a role outside of just using the identities above in calculations.

As discussed in §1, we are interested in rings R satisfying $[x^n - x, y] = 0$ for some integer $n > 1$ and all $x, y \in R$. We are specifically interested in $n = 2, 4$ or 8, so we will start with the broader assumption that

$$[p(x) - x, y] = 0 \quad \text{for all } x, y \in R \tag{E}$$

where $p(t)$ is a polynomial with integer coefficients such that $p(t) = q(t^2)t^2$, where $q(t) \in \mathbb{Z}[t]$.

Lemma 3.1. *Let R be a ring satisfying (E). Then $2\,\mathrm{ad}(x) = 0$ for all $x \in R$.*

Proof. For all $x \in R$, $-\mathrm{ad}(x) = \mathrm{ad}(-x) = \mathrm{ad}(p(-x)) = \mathrm{ad}(p(x)) = \mathrm{ad}(x)$. □

Roughly speaking, the conclusion of Lemma 3.1 is, in the present setting, what replaces R having characteristic 2 in the n-potent setting for even n. The conclusion can be stated in various equivalent ways, such as $[x, y] = [y, x]$ for all $x, y \in R$.

Lemma 3.2. *Let R be a ring such that $2\,\mathrm{ad}(x) = 0$ for all $x \in R$. Then for each nonnegative integer n,*

$$\mathrm{ad}(x)^{2^n} = \mathrm{ad}(x^{2^n})\,.$$

Proof. For $n = 0$, there is nothing to prove. Assume the goal holds for some $n \geq 0$. For $x, y \in R$, set $u = x^{2^n}$, so that $\mathrm{ad}(u) = \mathrm{ad}(x)^{2^n}$. Then

$$\begin{aligned}\mathrm{ad}(u^2)(y) &= [u^2, y] = u^2 y - y u^2 \\ &= u^2 y - uyu + uyu - yu^2 \\ &= [u, uy] + [u, yu] \\ &= \mathrm{ad}(u)(uy) + \mathrm{ad}(u)(yu) \\ &= \mathrm{ad}(u)(uy) - \mathrm{ad}(u)(yu) \\ &= \mathrm{ad}(u)([u, y]) \\ &= \mathrm{ad}(u)^2(y)\,.\end{aligned}$$

Thus

$$\mathrm{ad}(x^{2^{n+1}})(y) = \mathrm{ad}(u^2) = \mathrm{ad}(u)^2 = (\mathrm{ad}(x)^{2^n})^2(y) = \mathrm{ad}(x)^{2^{n+1}}(y)\,.$$

By induction, we have the desired result. □

Lemma 3.3. *Let R be a ring satisfying (E). Then for all $x, y \in R$,*

$$[x, [x, y]] = 0 \implies [x, y] = 0\,,$$

or equivalently,

$$\mathrm{ad}(x)^2(y) = 0 \implies \mathrm{ad}(x)(y) = 0\,.$$

Proof. Assume $[x, [x, y]] = \mathrm{ad}(x)^2(y) = 0$ for some $x, y \in R$. By Lemmas 3.1 and 3.2, $\mathrm{ad}(x) = \mathrm{ad}(p(x)) = p(\mathrm{ad}(x))$. Since $p(t) = q(t^2)t^2$ for some $q(t) \in \mathbb{Z}[t]$, it follows that $\mathrm{ad}(x)(y) = q(\mathrm{ad}(x)^2)\mathrm{ad}(x)^2(y) = 0$. □

Lemma 3.4. *Let R be a ring satisfying (E). Then for all $x, y \in R$,*

$$[x, [x, y]] = [x, y] \implies [x, y] = 0\,,$$

or equivalently,

$$\mathrm{ad}(x)^2(y) = \mathrm{ad}(x)(y) \implies \mathrm{ad}(x)(y) = 0\,.$$

Proof. Let $x, y \in R$ satisfy $[x, [x, y]] = [x, y]$. Then

$$[[x, y], [[x, y], x]] = -[[x, y], [x, [x, y]]] = -[[x, y], [x, y]] = 0\,.$$

By Lemma 3.3, $[[x, y], x] = 0$. Thus $[x, y] = [x, [x, y]] = 0$. □

Theorem 3.5. *Let R be a ring satisfying $[x^2 - x, y] = 0$ for all $x, y \in R$. Then R is commutative.*

Proof. Since (E) holds, Lemmas 3.1 and 3.2 give $\mathrm{ad}(x)(y) = \mathrm{ad}(x^2)(y) = \mathrm{ad}(x)^2(y)$ for all $x, y \in R$. By Lemma 3.4, $\mathrm{ad}(x)(y) = 0$ for all $x, y \in R$, that is, R is commutative. (See also Lemma 2.9.) □

Lemma 3.6. *Let R be a ring, let n be a positive integer and assume $[x^{2^n} - x, y] = 0$ for all $x, y \in R$. Then $[x^{2^{n-1}} + \cdots + x^2 + x, y] = 0$ for all $x, y \in R$.*

Proof. For all $x \in R$, $\mathrm{ad}(x)^{2^n} = \mathrm{ad}(x^{2^n}) = \mathrm{ad}(x)$, using Lemmas 3.1 and 3.2. Now for all $x \in R$ (and using Lemma 3.1),

$$\begin{aligned}\mathrm{ad}(x^{2^{n-1}} + \cdots + x^2 + x)^2 &= (\mathrm{ad}(x^{2^{n-1}}) + \cdots + \mathrm{ad}(x^2) + \mathrm{ad}(x))^2 \\ &= (\mathrm{ad}(x)^{2^{n-1}} + \cdots + \mathrm{ad}(x)^2 + \mathrm{ad}(x))^2 \\ &= \mathrm{ad}(x)^{2^n} + \mathrm{ad}(x)^{2^{n-1}} + \cdots + \mathrm{ad}(x)^4 + \mathrm{ad}(x)^2 \\ &= \mathrm{ad}(x) + \mathrm{ad}(x)^{2^{n-1}} + \cdots + \mathrm{ad}(x)^4 + \mathrm{ad}(x)^2 \\ &= \mathrm{ad}(x^{2^{n-1}}) + \cdots + \mathrm{ad}(x^2) + \mathrm{ad}(x) \\ &= \mathrm{ad}(x^{2^{n-1}} + \cdots + x^2 + x)\end{aligned}$$

By Lemma 3.4, $\mathrm{ad}(x^{2^{n-1}} + \cdots + x^2 + x) = 0$. This proves the desired result. □

Theorem 3.7. *Let R be a ring satisfying $[x^4 - x, y] = 0$ for all $x, y \in R$. Then R is commutative.*

Proof. This follows from taking $n = 2$ in Lemma 3.6 and using Theorem 3.5. □

We conclude with the result that began this whole endeavor. We are well aware that of all the proofs in this paper, this is the one that most seems like it was generated by an automated deduction tool. We have simplified the proof to the point where it is possible to follow each individual step, but we would certainly agree that it is difficult to see how a human would have found the proof.

Theorem 3.8. *Let R be a ring satisfying $[x^8 - x, y] = 0$ for all $x, y \in R$. Then R is commutative.*

Proof. We will freely use Lemma 3.1 without explicit reference. By Lemma 3.6, $[x^4 + x^2 + x, y] = 0$ for all $x, y \in R$, that is, $\mathrm{ad}(x)^4 + \mathrm{ad}(x)^2 + \mathrm{ad}(x) = 0$.

First, for all $u, v \in R$,

$$\begin{aligned}(\mathrm{ad}(u)^2 + \mathrm{ad}(v)^2)([u,v]) &= (\mathrm{ad}(u)^2 + \mathrm{ad}([u,v]) + \mathrm{ad}(v)^2)([u,v]) \\ &= (\mathrm{ad}(u)^2 + \mathrm{ad}(u)\mathrm{ad}(v) + \mathrm{ad}(v)\mathrm{ad}(u) + \mathrm{ad}(v)^2)([u,v]) \\ &= (\mathrm{ad}(u) + \mathrm{ad}(v))^2([u,v]) \\ &= \mathrm{ad}(u+v)^2([u,v]) \\ &= \mathrm{ad}(u+v)^2([u+v,v]) \\ &= \mathrm{ad}(u+v)^3(v)\,.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{ad}(u+v)(\mathrm{ad}(u)^2 + \mathrm{ad}(v)^2)([u,v]) &= \mathrm{ad}(u+v)^4(v) \\ &= \mathrm{ad}(u+v)^2(v) + \mathrm{ad}(u+v)(v) \\ &= \mathrm{ad}(u+v)([u+v,v] + v) \\ &= \mathrm{ad}(u+v)(v + [u,v])\,.\end{aligned}$$

Rearranging this, we have $\mathrm{ad}(u+v)\mathrm{ad}(v)^2([u,v]) = \mathrm{ad}(u+v)(v + [u,v] + \mathrm{ad}(u)^2([u,v]))$, that is,

$$\mathrm{ad}(u+v)\mathrm{ad}(v)^3(u) = \mathrm{ad}(u+v)(v + \mathrm{ad}(u)(v) + \mathrm{ad}(u)^3(v))\,.$$

In this last equation, replace v with $[u,v]$ to get

$$[u + [u,v], \mathrm{ad}([u,v])^3(u)] = [u + [u,v], \mathrm{ad}(u)(v) + \mathrm{ad}(u)^2(v) + \mathrm{ad}(u)^4 v] = 0\,. \tag{6}$$

Now let $u = [x^2, y^2 + y]$ and $v = y^2 + y$. Then

$$\begin{aligned}[u, v] &= \mathrm{ad}(y^2 + y)^2(x^2) \\ &= (\mathrm{ad}(y)^2 + \mathrm{ad}(y))^2(x^2) \\ &= (\mathrm{ad}(y)^4 + \mathrm{ad}(y)^2)(x^2) \\ &= \mathrm{ad}(y)(x^2) \\ &= [x^2, y]\,,\end{aligned}$$

and $u + [u, v] = [x^2, y^2]$. Plugging all this into (6), we have

$$[[x^2, y^2], \mathrm{ad}([x^2, y])^3([x^2, y^2 + y])] = 0\,. \tag{7}$$

Now

$$\begin{aligned}\mathrm{ad}([x^2, y])^3([x^2, y^2 + y]) &= \mathrm{ad}([x^2, y])^2([[x^2, y], [x^2, y^2] + [x^2, y]]) \\ &= \mathrm{ad}([x^2, y])^3([x^2, y^2]) \\ &= \mathrm{ad}([x^2, y])^3\mathrm{ad}(y)^2(x^2) \\ &= \mathrm{ad}([x^2, y])^3\mathrm{ad}(y)([x^2, y]) \\ &= \mathrm{ad}([x^2, y])^3[[x^2, y], y]) \\ &= \mathrm{ad}([x^2, y])^4(y) \\ &= \mathrm{ad}([x^2, y])^2(y) + \mathrm{ad}([x^2, y])(y) \\ &= \mathrm{ad}([x^2, y])\mathrm{ad}(y)^2(x^2) + \mathrm{ad}(y)^2(x^2) \\ &= [[x^2, y], [x^2, y^2]] + [x^2, y^2]\,.\end{aligned}$$

Thus the left side of (7) simplifies to

$$[[x^2, y^2], [[x^2, y], [x^2, y^2]] + [x^2, y^2]] = [[x^2, y^2], [[x^2, y], [x^2, y^2]] = \mathrm{ad}([x^2, y^2])^2([x^2, y])\,.$$

Therefore (7) reduces to

$$\mathrm{ad}([x^2, y^2])^2([x^2, y]) = 0$$

for all $x, y \in R$. By Lemma 3.3,

$$0 = \mathrm{ad}([x^2, y^2])([x^2, y]) = \mathrm{ad}([x^2, y])\mathrm{ad}(y)^2(x^2) = \mathrm{ad}([x^2, y])^2(y)\,.$$

By Lemma 3.3 again, $\mathrm{ad}([x^2, y])(y) = 0$, that is, $\mathrm{ad}(y)^2(x^2) = 0$. By Lemma 3.3 again, $\mathrm{ad}(y)(x^2) = [x^2, y] = \mathrm{ad}(x)^2(y) = 0$. Applying Lemma 3.3 one last time, $[x, y] = 0$. Therefore R is commutative. □

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Finding Small Solutions of Bivariate Linear Congruences

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Abstract. In this note, we propose an algorithm for computing all solutions of small size of a bivariate linear congruence.

1. Introduction

Let $a_1, \dots, a_k, b, n \in \mathbb{Z}$ with $n \geq 1$. A *linear congruence* in the unknowns $x_1, \dots, x_k$ is an expression of the form

$$a_1x_1 + \cdots + a_kx_k \equiv b \pmod{n}.$$

An ordered k-tuple of integers $(x_1, \dots, x_k)$ that satisfies this congruence is called a *solution*. These solutions are often considered under additional constraints, such as $\gcd(x_i, n) = t_i$ for $1 \leq i \leq k$, where $t_1, \dots, t_k$ are given positive divisors of n. The number of solutions subject to such conditions has been studied by several authors (see [1]). Moreover, small solutions of linear homogeneous congruences and systems have been analysed, with many results extended to number fields (see [2]).

In this note, we focus on solutions of small size to non-homogeneous bivariate linear congruences and describe an algorithm for their computation. Notably, the private key and ephemeral key in several digital signature schemes correspond to solutions of such congruences (see [3, Section 11.5]). We prove the following result:

Theorem 1. *Let q be an odd prime number, and let $A, B \in \{2, \dots, (q-1)/2\}$. Let μ and ν be positive integers such that $\mu \leq A/2$ and $\nu < q/(2A)$. Consider the bivariate linear congruence*

$$y + Ax + B \equiv 0 \pmod{q}. \tag{1}$$

Then, the number of solutions (x, y) satisfying the bounds

$$|x| \leq \mu \left\lfloor \frac{q}{A} \right\rfloor \quad \text{and} \quad |y| \leq \nu A$$

is at most $(2\mu+1)(2\mu+2\nu+1)$. Moreover, all such solutions can be computed in time $O\big(\mu(\mu+\nu)(\log q)^2\big)$ bit operations.

The idea of the proof is to find a "small" list of pairs and select those that satisfy the given bounds. Note that the smaller the quantities μ and ν are, the more efficiently the solutions of the linear congruence that satisfy the given constraints can be calculated.

Let a, $n \in \mathbb{Z}$ and $n > 1$. We denote the remainder when a is divided by n by '$a \bmod n$'.

The paper is organized as follows. Section 2 presents the proof of Theorem 1. In Section 3, we describe an algorithm, based on Theorem 1, that computes efficiently all "small-size" solutions of the congruence (1). Finally, Section 4 provides two examples illustrating the application of this algorithm.

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2. Proof of Theorem 1

Let x_0, $y_0 \in \mathbb{Z}$ satisfy the bounds

$$|x_0| \leq \mu \left\lfloor \frac{q}{A} \right\rfloor \quad \text{and} \quad |y_0| \leq \nu A,$$

and suppose they satisfy the congruence $y_0 + Ax_0 + B \equiv 0 \pmod{q}$. Then, we have the following bound on the absolute value:

$$|y_0 + Ax_0 + B| \leq |y_0| + A|x_0| + B < \frac{q}{2} + \mu q + \frac{q}{2} = (\mu + 1)q.$$

Since q divides $y_0 + Ax_0 + B$, it follows that

$$y_0 + Ax_0 + B = c_1 q, \tag{2}$$

where $c_1 \in \{0, \pm 1, \pm 2, \ldots, \pm \mu\}$.

Next, by Euclidean division, we have $q = Au + v$, where

$$u = \left\lfloor \frac{q}{A} \right\rfloor \quad \text{and} \quad 0 < v < A.$$

This implies the congruence $-Au \equiv v \pmod{q}$.

Multiplying Equation (2) by $-u$ yields

$$-uy_0 + vx_0 + C \equiv 0 \pmod{q},$$

where

$$-uB = -Kq + C \text{ and } 0 \leq C < q. \tag{3}$$

Furthermore, we can bound the absolute value:

$$|-uy_0 + vx_0 + C| \leq u|y_0| + v|x_0| + C < uA\nu + \mu q + q \leq (\nu + \mu + 1)q.$$

Since q divides $-uy_0 + vx_0 + C$, we deduce that

$$-uy_0 + vx_0 + C = c_2 q, \tag{4}$$

with $c_2 \in \{0, \pm 1, \pm 2, \ldots, \pm(\nu + \mu)\}$.

The Equations (2) and (4) constitute a linear system in the unknowns x_0 and y_0. Solving this system, we obtain

$$x_0 = c_1 u + c_2 - \frac{uB + C}{q} \quad \text{and} \quad y_0 = c_1 v - c_2 A + \frac{CA - vB}{q}.$$

Since, by (3), $-uB = -Kq + C$, we have

$$K = \frac{uB + C}{q} = -\left\lfloor \frac{-uB}{q} \right\rfloor.$$

Using this fact, we rewrite the second fraction as

$$\frac{CA - vB}{q} = AK - B.$$

Hence, the solutions can be expressed in the simpler form

$$x_0 = c_1 u + c_2 - K, \quad y_0 = c_1 v - c_2 A + AK - B, \tag{5}$$

where $c_1 \in \{0, \pm 1, \pm 2, \ldots, \pm \mu\}$, $c_2 \in \{0, \pm 1, \pm 2, \ldots, \pm(\nu + \mu)\}$, u and v are the quotient and the remainder of the division of q by A, and $K = -\lfloor (-uB)/q \rfloor$.

Conversely, one can verify that any pair (x_0, y_0) of this form satisfies the original congruence (1). Since

$$c_1 \in \{-\mu, \ldots, 0, \ldots, \mu\} \quad \text{and} \quad c_2 \in \{-(\nu + \mu), \ldots, 0, \ldots, \nu + \mu\},$$

there are at most $(2\mu + 1)(2\mu + 2\nu + 1)$ such solutions satisfying the prescribed bounds on $|x_0|$ and $|y_0|$.

Finally, by [4, Section 3.3], the computation of u, v, K, and hence of the solutions x_0, y_0, can be performed in $O(\mu(\mu+\nu)(\log q)^2)$ bit operations.

3. The Algorithm

The proof of Theorem 1 leads to the following algorithm for computing solutions to the congruence (1) that satisfy the given bounds.

Algorithm: SOLVE-CONGRUENCE
Input: An odd prime q, A, $B \in \{2, \ldots, (q-1)/2\}$, and positive integers μ, ν with $\mu \le A/2$ and $\nu < q/(2A)$.
Output: The solutions (x, y) of Congruence (1) with $|x| \le \mu\lfloor q/A \rfloor$ and $|y| < \nu A$.

(1) Compute integers u and v satisfying $q = Au + v$ and $0 \le v < A$.
(2) Compute positive integers K and C such that $-uB = -Kq + C$ and $0 < C < q$.
(3) For each $i \in \{0, \pm 1, \ldots, \pm\mu\}$, determine all $j \in \{0, \pm 1, \ldots, \pm(\mu+\nu)\}$ such that the quantities

$$x_{i,j} = iu + j - K \quad \text{and} \quad y_{i,j} = iv - jA + AK - B$$

satisfy the inequalities

$$|x_{i,j}| \le \mu \left\lfloor \frac{q}{A} \right\rfloor \quad \text{and} \quad |y_{i,j}| < \nu A.$$

(4) Output all pairs $(x_{i,j}, y_{i,j})$ that satisfy the inequalities specified in the previous step.

Remark 1. If the integers μ and ν are sufficiently small – if, for instance, μ, ν are both less than $(\log q)^2$ – then the above algorithm runs in polynomial time and is therefore practical for computation.

4. Examples

In this section, we work through two examples illustrating the use of the algorithm SOLVE-CONGRUENCE. We remark that, in these examples, the number of solutions satisfying the given bounds is significantly smaller than the upper bound mentioned in Theorem 1.

Example 1. Consider the prime $q = 1073741827$. We shall compute the solutions of the congruence

$$y + 131073x + 25277021 \equiv 0 \pmod{q} \tag{6}$$

with

$$|x| \le 8100 \quad \text{and} \quad |y| \le 12000.$$

We have $A = 131073$, $B = 25277021$, and $\lfloor q/A \rfloor = 8191$. We choose parameters $\mu = \nu = 1$. Thus, we find integers $u = 8191$ and $v = 122884$ such that $q = Au + v$ and $0 < v < A$. Next, we compute $K = 193$ and $C = 188093600$ such that $-uB = -Kq + C$. Finally, we compute $AK - B = 20068$.

We now consider solutions to the Congruence (6) of the form $(x_{i,j}, y_{i,j})$ ($i = 0, \pm 1$, $j = 0, \pm 1, \pm 2$), where

$$x_{i,j} = i8191 + j - 193, \quad \text{and} \quad y_{i,j} = i122884 - j131073 + 20068.$$

We check which of these pairs satisfy the required bounds. For $i = -1, 0$, the values $y_{i,j}$ ($j = -2, -1, 0, 1, 2$) do not meet the given bound. For $i = 1$, only the pair $(x_1, y_1) = (7999, 11879)$ satisfies the bounds. Therefore, the only solution to the congruence (6) within the specified bounds is $(7999, 11879)$.

Example 2. Consider the linear bivariate congruence

$$y + 149x + 475 \equiv 0 \pmod{1013}. \tag{7}$$

We shall compute the solutions of the above congruences $(x, y) \in \mathbb{Z}^2$ with $|x| \leq 90$ and $|y| \leq 149$.

The integer $q = 1013$ is a prime number. We are given $A = 149$, $B = 475$, and observe that $\lfloor q/A \rfloor = 6$. We choose parameters $\mu = 15$ and $\nu = 1$. According to the algorithm, we first determine integers $u = 6$ and $v = 119$ such that $q = Au + v$, with $0 \leq v < A$. Next, we compute integers $K = 3$ and $C = 189$ satisfying $-uB = -Kq + C$, with $0 \leq C < q$. Then, we obtain the solutions

$$(x_{i,j}, y_{i,j}) \quad (i = 0, \pm 1, \ldots, \pm 15,\ j = 0, \pm 1, \ldots, \pm 16)$$

of Congruence (7), where

$$x_{i,j} = i6 + j - 3 \quad \text{and} \quad y_{i,j} = i119 - j149 - 28.$$

For $i = 0$, we find that only $j = 0$ and $j = -1$ yield values of $|y_{0,j}| \leq 149$. Specifically, the corresponding solutions are:

$$(x_{0,0}, y_{0,0}) = (-3, -28), \quad (x_{0,-1}, y_{0,-1}) = (-4, 121),$$

both of which satisfy the imposed upper bounds.

For $j = 0$, we find that only $i = 1$ and $i = -1$, other than $i = 0$, yield values of $|y_{i,0}| \leq 149$. Specifically, the corresponding solutions are:

$$(x_{1,0}, y_{1,0}) = (3, 91), \quad (x_{-1,0}, y_{-1,0}) = (-9, -147).$$

If $i > 0$ and $j < 0$, then

$$y_{i,j} = 119i - 149j - 28 > 149,$$

violating the bound on $y_{i,j}$. Similarly, if $i < 0$ and $j > 0$, then

$$y_{i,j} = 119i - 149j - 28 < -149,$$

which also violates the bound. Therefore, for any $i \neq 0$ and $j \neq 0$, i and j must be of the same sign. Accordingly, for each $i = \pm 1, \ldots, \pm 15$, we examine values of $j = \pm 1, \ldots, \pm 16$ with the same sign to determine whether the corresponding pairs $(x_{i,j}, y_{i,j})$ satisfy the given bounds and solve Congruence (7). We have 59 such solutions that are listed in the table overleaf. Note that this number is considerably smaller than the bound 1023 that is provided by Theorem 1.

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i	j	$(x_{i,j}, y_{i,j})$		i	j	$(x_{i,j}, y_{i,j})$
-15	-12	$(-105, -25)$		0	-1	$(-4, 121)$
-15	-13	$(-106, 124)$		0	0	$(-3, -28)$
-14	-11	$(-98, -55)$		1	0	$(3, 91)$
-14	-12	$(-99, 94)$		1	1	$(4, -58)$
-13	-10	$(-91, -85)$		2	1	$(10, 61)$
-13	-11	$(-92, 64)$		2	2	$(11, -88)$
-12	-9	$(-84, -115)$		3	2	$(17, 31)$
-12	-10	$(-85, 34)$		3	3	$(18, -118)$
-11	-8	$(-77, -145)$		4	3	$(24, 1)$
-11	-9	$(-78, 4)$		4	4	$(25, -148)$
-10	-8	$(-71, -26)$		5	3	$(30, 120)$
-10	-9	$(-72, 123)$		5	4	$(31, -29)$
-9	-7	$(-64, 56)$		6	4	$(37, 90)$
-9	-8	$(-65, 93)$		6	5	$(38, -59)$
-8	-6	$(-57, -86)$		7	5	$(44, 60)$
-8	-7	$(-58, 63)$		7	6	$(45, -89)$
-7	-5	$(-50, -116)$		8	6	$(51, 30)$
-7	-6	$(-51, 33)$		8	7	$(52, -119)$
-6	-4	$(-43, -146)$		9	6	$(57, 149)$
-6	-5	$(-44, 3)$		9	7	$(58, 0)$
-5	-4	$(-37, -27)$		9	8	$(59, -149)$
-5	-5	$(-38, 122)$		10	7	$(64, 119)$
-4	-3	$(-30, 57)$		10	8	$(65, -30)$
-4	-4	$(-31, 92)$		11	8	$(71, 89)$
-3	-2	$(-23, -87)$		11	9	$(72, -60)$
-3	-3	$(-24, 62)$		12	9	$(78, 59)$
-2	-1	$(-16, -117)$		12	10	$(79, -90)$
-2	-2	$(-17, 32)$		13	10	$(85, 29)$
-1	0	$(-9, -147)$		13	11	$(86, -120)$
-1	-1	$(-10, 2)$				

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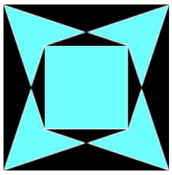

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The Trace and its Extensions in Operator Algebras

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Abstract. We discuss how mathematicians generalize the usual trace on matrices to various finite and infinite-dimensional algebras. We also examine the existence or lack of (faithful) tracial states in the framework of operator algebras.

1. Introduction and Preliminaries

A natural invariant associated to each linear operator T acting on an n-dimensional vector space V is its characteristic polynomial $p_T(\lambda) = \det(T - \lambda I)$, where I is the identity operator on V and λ is a scalar (for simplicity and because of where we are going, we will assume that the field of scalars is $\mathbb{C}$). This polynomial encodes essential information about T; namely its eigenvalues, which are the roots of $p_T(\lambda)$.

In turn, this gives importance to its coefficients, as invariants of the operator. The most well-known of these coefficients is the constant term, that is the determinant $\det T = p_T(0)$. This is equal to the product of the eigenvalues of T, counting multiplicities. Among the other coefficients of $p_T(\lambda)$, the best known is the coefficient of λ^{n-1}. This coefficient is equal to the sum of the eigenvalues of T, counting multiplicities, and it is usually called the *trace* of T, and denoted by $\operatorname{tr}(T)$. Eigenvalues are crucial in understanding the behavior of linear operators, so the trace and the determinant give quick ways to relate a matrix to its eigenvalues without having to compute them.

Via the Jordan form J_T of T, the number $\operatorname{tr}(T)$ can be seen as the sum of the diagonal entries of J_T. A straightforward computation shows that $\operatorname{tr}(ST) = \operatorname{tr}(TS)$ for any two linear operators S and T acting on V, and hence $\operatorname{tr}(STS^{-1}) = \operatorname{tr}(T)$ for all invertible S and all T. From this one can deduce that $\operatorname{tr}(T) = \sum_{i=1}^n a_{ii}$ for any presentation of T as a matrix $A = [a_{ij}]$ with respect to some basis of V. It is not hard to show that tr is the only linear functional on V with the *tracial property*:

$$\operatorname{tr}(ST) = \operatorname{tr}(TS) \qquad \text{for all } S \text{ and } T, \tag{1}$$

up to normalization by a scalar (see Subsection 2.1.1 for a proof of uniqueness).

The trace is particularly meaningful in the case where our finite-dimensional vector space is a Hilbert space $\mathscr{H}$, but its straightforward extension to the infinite-dimensional case cannot work for all bounded operators on $\mathscr{H}$. For example, for the diagonal operator $\operatorname{diag}(1, 1/2, 1/3, \ldots)$ acting on the Hilbert space ℓ^2 of square summable sequences, the sum of its diagonal entries is not finite. Extensions exist, though, and they appear in many flavours. Discussing those extensions is the main goal of this article.

To fix notation, we let $\mathscr{H}$ denote a Hilbert space over the field $\mathbb{C}$ with inner product $\langle \cdot , \cdot \rangle$. We write $\mathbb{B}(\mathscr{H})$ for the $*$-algebra of all linear bounded operators on $\mathscr{H}$; we denote by I the identity operator on $\mathscr{H}$. The space of all compact operators acting on

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$\mathscr{H}$ is denoted by $\mathbb{K}(\mathscr{H})$, which is a closed two-sided ideal of $\mathbb{B}(\mathscr{H})$. In the case where $\dim \mathscr{H} = n$, we can identify $\mathbb{B}(\mathscr{H}) = \mathbb{K}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all complex $n \times n$ matrices. In this latter case the trace takes the form

$$\operatorname{tr}(T) = \sum_{k=1}^{n} \langle Te_k, e_k \rangle,$$

where $\{e_k\}_{k=1}^n$ is any orthonormal basis of $\mathscr{H}$. We denote the normalized trace $\frac{1}{n}\operatorname{tr}$ by $\widehat{\operatorname{tr}}$.

As noted by Albrecht Pietsch [26], the definition of the trace for a square matrix mentioned above has been in use since the 18th century. The term "Spur" for this notion was first introduced by Dedekind [11] within the context of algebraic number theory. In his work on the development of mathematical foundations for quantum mechanics ([32, 33, 34], compiled in [35]), von Neumann defined the trace of a positive operator acting on a Hilbert space and considered the ideal of trace-class operators. Incidentally, von Neumann also defined for the first time the idea of an abstract Hilbert space.

As soon as one tries to extend the notion of trace to the infinite-dimensional setting, issues arise: the only linear functional $\varphi : \mathbb{B}(\mathscr{H}) \to \mathbb{C}$ satisfying the tracial property is the zero functional. This is what led von Neumann to consider the trace-class operators, which form in a sense the largest ideal $\mathbb{T}(\mathscr{H})$ where (1) holds. In fact, $\mathbb{T}(\mathscr{H})$ is the set of all operators $T \in \mathbb{B}(\mathscr{H})$ such that $\|T\|_1 := \sum_{e \in \mathbb{E}} \langle |T|e, e \rangle < \infty$, where $\mathbb{E}$ is any orthonormal basis for $\mathscr{H}$. In addition, we can define the trace of $T \in \mathbb{T}(\mathscr{H})$ as $\operatorname{tr}(T) := \sum_{e \in \mathbb{E}} \langle Te, e \rangle$, and this definition is independent of the choice of basis. The trace in this context appears to be intrinsic, as $\mathbb{T}(\mathscr{H})$ can be seen as the predual of $\mathbb{B}(\mathscr{H})$, in the sense that we have isometric isomorphisms

$$\mathbb{K}(\mathscr{H})^* = \mathbb{T}(\mathscr{H}), \qquad \mathbb{T}(\mathscr{H})^* = \mathbb{B}(\mathscr{H}),$$

where the isomorphisms in both cases are given by the trace; that is, a trace-class operator T is seen as a bounded linear functional on $\mathbb{K}(\mathscr{H})$ via $S \longmapsto \operatorname{tr}(ST)$, and $T \in \mathbb{B}(\mathscr{H})$ is seen as a bounded linear functional on $\mathbb{T}(\mathscr{H})$ via the same duality pairing.

A positive linear functional φ on a C^*-algebra $\mathscr{A}$ is called *tracial* if it satisfies (1) for all $S, T \in \mathscr{A}$. As in [16, Proposition 8.1.1], one can observe that (2) and (3), the latter when $\mathscr{A}$ is unital, are each equivalent to (1):

$$\varphi(X^*X) = \varphi(XX^*), \qquad X \in \mathscr{A}\,. \tag{2}$$

$$\varphi(UXU^*) = \varphi(X), \qquad X \in \mathscr{A} \quad \text{and } U \in \mathscr{A} \text{ a unitary.} \tag{3}$$

There are several papers exploring the characterizations of the tracial functionals on matrices and operator algebras; we mention [3] for further reference.

The study of tracial states, which are tracial positive linear functionals of norm one, is an active area in the theory of operator algebras, particularly in Elliott's Classification Program (see [36] as an initial source of a very large number of references). It is a natural question whether certain classes of C^*-algebras or von Neumann algebras admit a tracial state or not.

Besides the intrinsic interest for operator algebras, such studies have applications in other disciplines. From classifying linear operators to enabling quantum computations and optimizing machine learning models, the trace features both in abstract theory and in real-world applications.

In quantum mechanics, the trace is used in defining the notion of density matrix; such a matrix ρ is a positive semidefinite matrix of trace one. The entropy of a quantum system with density matrix ρ is given by $S = -\operatorname{tr}(\rho \ln \rho)$; see [10, 24]. Moreover, the concept of partial trace in quantum information theory is used to describe subsystems.

The partial trace $\operatorname{tr}_1 : \mathbb{M}_n \otimes \mathbb{M}_m \to \mathbb{M}_m$ is the linear map induced by $\operatorname{tr}_1(A \otimes B) = (\operatorname{tr} A)B$ and the partial trace $\operatorname{tr}_2 : \mathbb{M}_n \otimes \mathbb{M}_m \to \mathbb{M}_n$ is induced by $\operatorname{tr}_2(A \otimes B) = (\operatorname{tr} B)A$. In another setting, the trace is also used in defining the Frobenius inner product on $\mathbb{M}_n$ via $\langle A, B\rangle = \operatorname{tr}(B^*A)$. This inner product is useful in optimization problems over matrices, for example in machine learning where one may minimize some loss function that is expressed using the trace. Another application occurs in random matrix theory, where the trace of random matrices is studied, and results such as the law of large numbers for traces of powers of matrices relate to eigenvalue distributions.

For the readers' convenience we have included a brief summary of the basic theory of C^*-algebras and von Neumann algebras in Appendix A. For any undefined notations or terminologies, readers are referred to [2] for matrix theory and to [16, 22, 31] for the theory of operator algebras.

The main objective of this expository article is to discuss various extensions of the usual trace $\operatorname{tr}\colon \mathbb{M}_n \to \mathbb{C}$ to more general settings in operator algebras. Although the literature contains many interesting and deep results on this topic (see, e.g., [20]), we focus on presenting fundamental facts and some new proofs, and illustrative examples for readers familiar with basic operator algebra theory.

2. Extensions of the usual trace, Uniqueness, and Examples

We aim to explore how to extend the usual trace $\operatorname{tr} : \mathbb{M}_n \to \mathbb{C}$ to positive linear maps satisfying the tracial property by considering changes in the domain $\mathbb{M}_n$, codomain $\mathbb{C}$, or both, to some operator algebras. We also examine the existence or lack of tracial states in the framework of operator algebras.

2.1. **Changing domain.**

2.1.1. *Replacing $\mathbb{M}_n$ with a finite-dimensional C^*-algebra.* As mentioned in the introduction, $\widehat{\operatorname{tr}}$ is the only tracial state on $\mathbb{M}_n$.

Indeed, consider the canonical matrix unit system $\{E_{ij}\} \subseteq \mathbb{M}_n$, which satisfies $E_{ij}E_{kl} = \delta_{jk}E_{il}$. For a tracial state φ, if $i \neq j$ then

$$\varphi(E_{ij}) = \varphi(E_{ij}E_{jj}) = \varphi(E_{jj}E_{ij}) = \varphi(0) = 0;$$

and for any i, j

$$\varphi(E_{ii}) = \varphi(E_{ij}E_{ji}) = \varphi(E_{ji}E_{ij}) = \varphi(E_{jj}).$$

Thus, for $A = [a_{ij}] = \sum_{i,j=1}^n a_{ij}E_{ij} \in \mathbb{M}_n$, we have

$$\begin{aligned}\varphi(A) &= \sum_{i,j=1}^n a_{ij}\,\varphi(E_{ij}) = \sum_{i=1}^n a_{ii}\,\varphi(E_{ii}) = \varphi(E_{11})\sum_{i=1}^n a_{ii} = \varphi(E_{11})\operatorname{tr}(A)\\ &= \frac{1}{n}\operatorname{tr}(A) = \widehat{\operatorname{tr}}(A).\end{aligned}$$

If we replace $\mathbb{M}_n$ with a finite-dimensional C^*-algebra $\mathscr{A} = \bigoplus_{k=1}^m M_{k(m)}$, then there are uncountably many tracial states $\varphi : \mathscr{A} \to \mathbb{C}$, of the form

$$\varphi\left(\bigoplus_{k=1}^m X_k\right) = \sum_{k=1}^m t_k\,\widehat{\operatorname{tr}}(X_k),$$

where $t_k \geq 0$ for all k and $\sum_{k=1}^m t_k = 1$.

2.1.2. *Substituting $\mathbb{C}$ in $\mathbb{M}_n(\mathbb{C})$ with an arbitrary C^*-algebra $\mathscr{A}$ having a tracial state.* Given a tracial state φ on $\mathscr{A}$, we can define a tracial state on $\mathbb{M}_n(\mathscr{A})$ by

$$\varphi_n([a_{ij}]) := \frac{1}{n}\sum_{i=1}^{n}\varphi(a_{ii}). \tag{4}$$

And this is the only way to construct tracial states on $\mathbb{M}_n(\mathscr{A})$: if $\gamma : \mathbb{M}_n(\mathscr{A}) \to \mathbb{C}$ is a tracial state, then there exists a unique tracial state φ on A, defined by $\varphi(a) = \gamma(a \otimes E_{11}))$, such that $\gamma = \varphi_n$. So there is a natural bijective correspondence between tracial states on $\mathscr{A}$ and tracial states on $\mathbb{M}_n(\mathscr{A})$.

2.1.3. *Replacing $\mathbb{M}_n(\mathbb{C})$ with a commutative C^*-algebra.* One may replace $\mathbb{M}_n$ with a commutative C^*-algebra $\mathscr{A}$. In this case, every state is tracial. It is known that $\mathscr{A}$ is isometrically $*$-isomorphic to $C_0(\Omega)$ for some locally compact Hausdorff space Ω. Therefore, any (tracial) positive linear functional on $\mathscr{A}$ can be represented as $\varphi(f) = \int_\Omega f d\mu$ for a unique positive Borel measure μ on Ω such that $\mu(\Omega) = \|\varphi\|$, where $\|\varphi\|$ denotes the operator norm of φ. Hence there exist uncountably many tracial states on a commutative C^*-algebra, as long as it is not one-dimensional. If $\mathscr{A}$ is finite-dimensional, then Ω must be a finite set with, say, n elements; see [22, p. 57]. In such case, the state space is parametrized by the simplex $\{(t_1,\dots,t_n) \in \mathbb{R}^n : \ t_j \geq 0 \ \text{ for all } j, \ \sum_j t_j = 1\}$.

2.1.4. *Finite factors have unique tracial states.* It is a seminal result of Murray and von Neumann [23] that a finite factor has a unique tracial state (the original Murray–von Neumann ideas are developed with detail in [30, Section 1.3]). The unique tracial state is always faithful and normal. A finite-dimensional example of a finite factor is $\mathbb{M}_n$, $n \geq 1$ with the usual tracial state $\widehat{\operatorname{tr}}$. An infinite-dimensional example of a finite factor is the *hyperfinite II_1-factor*, which can be seen as the double commutant (that is, the sot-completion) of $\bigcup_{n\in\mathbb{N}} \mathbb{M}_{2^n}$ (with the embeddings $A \longmapsto A \oplus A$) via the GNS representation of the tracial state $\varphi((A_n))$ extending the natural normalized trace on each subalgebra. The hyperfinite II_1-factor also appears as the sot-closure of the image of the group algebra, via the left-regular representation, of any amenable countable discrete group G with infinite conjugacy classes.

2.1.5. *C^*-algebras without any tracial states.* There exist C^*-algebras $\mathscr{A}$ without any tracial states. A separable example is the simple C^*-algebra $\mathbb{K}(\mathscr{H})$ for any infinite-dimensional separable Hilbert space $\mathscr{H}$. Given a fixed orthonormal basis $(e_i)_{i=1}^\infty$ for $\mathscr{H}$, the corresponding *matrix units* are the operators $\{E_{ij}\}$, where E_{ij} is the rank-one operator that sends e_j to e_i. As in the matrix case, they satisfy the relations $E_{rs}E_{ij} = \delta_{si}\, E_{rj}$. In particular $\{E_{ii}\}$ are pairwise orthogonal rank-one projections. As φ is tracial, $\varphi(E_{ij}) = 0$ for any $i \neq j$, and

$$\varphi(E_{jj}) = \varphi(E_{ji}E_{ij}) = \varphi(E_{ij}E_{ji}) = \varphi(E_{ii}).$$

Then, with $P_n = \sum_{i=1}^n E_{ii}$,

$$n\,\varphi(E_{11}) = \varphi(P_n) \leq \|\varphi\|\,\|P_n\| = 1.$$

As n is arbitrary, this implies that $\varphi = 0$, which is not a state since its operator norm is not equal to one.

The argument above also demonstrates that $\mathbb{B}(\mathscr{H})$ is a nonsimple C^*-algebra without any tracial state. It is known that a C^*-algebra $\mathscr{A}$ has no tracial states if and only if its universal enveloping von Neumann algebra $\pi(\mathscr{A})''$ is properly infinite.

Haagerup proved that if $\mathscr{A}$ is a unital C^*-algebra, then $\mathscr{A}$ has no tracial state if and only if there exist $n \geq 2$ and a finite set $\{A_1,\dots,A_n\} \subseteq \mathscr{A}$ such that $\sum_{i=1}^n A_i^*A_i = 1$ and $\|\sum_{i=1}^n A_iA_i^*\| < 1$ [12, Lemma 2.1]. Pop [27] showed that a C^*-algebra $\mathscr{A}$ has no tracial state if and only if there exists $n \geq 2$ such that any element of $\mathscr{A}$ can be expressed as a sum of n commutators $[A_i, B_i] = A_iB_i - B_iA_i,\ 1 \leq i \leq n$. An interesting question posed by Pop [27] is that if $\mathscr{A}$ has no tracial state, what is the smallest n such that each element of $\mathscr{A}$ can be expressed as a sum of n commutators?

2.1.6. *Existence of a unique tracial state on a nonnuclear C^*-subalgebra of a separable simple C^*-algebra possessing no tracial state.* The Choi algebra is the C^*-algebra generated by two unitary operators U and V acting on an infinite-dimensional Hilbert space $\mathscr{H}$ such that $U^2 = V^3 = 1$. For a construction of U and V, Choi [7] used suitable decompositions $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ and $\mathscr{H}_1 = \mathscr{H}_\alpha \oplus \mathscr{H}_\beta$ subject to the conditions $\dim \mathscr{H}_0 = \dim \mathscr{H}_1 = \dim \mathscr{H}_\alpha = \dim \mathscr{H}_\beta$. He then defined U and V by block operator matrices

$$\begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}_0 \oplus \mathscr{H}_1) \quad \text{and} \quad \begin{bmatrix} 0 & 0 & V_1 \\ V_2 & 0 & 0 \\ 0 & V_3 & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}_0 \oplus \mathscr{H}_\alpha \oplus \mathscr{H}_\beta),$$

where $U_1 : \mathscr{H}_1 \to \mathscr{H}_0, U_2 : \mathscr{H}_0 \to \mathscr{H}_1, V_1 : \mathscr{H}_\beta \to \mathscr{H}_0, V_2 : \mathscr{H}_0 \to \mathscr{H}_\alpha$, and $V_3 : \mathscr{H}_\alpha \to \mathscr{H}_\beta$ are unitaries between corresponding Hilbert subspaces of the same dimensions. This C^*-algebra has a unique tracial state, even though it is a C^*-subalgebra of the Cuntz C^*-algebra $\mathcal{O}_2$, which has no tracial state. To prove the latter fact, recall that $\mathcal{O}_2$ is generated by two isometries S_1 and S_2 such that $S_1S_1^* + S_2S_2^* = I$. If φ is a tracial state on $\mathcal{O}_2$, then $1 = \varphi(I) = \varphi\left(S_1S_1^* + S_2S_2^*\right) = \varphi(S_1S_1^*) + \varphi(S_2S_2^*) = \varphi(I) + \varphi(I) = 2$, a contradiction. A nonunital simple separable C^*-algebra with a unique tracial state is the so-called Jacelon–Razak C^*-algebra; see [14]. An example of a unital separable, nuclear projectionless infinite-dimensional C^*-algebra with a unique tracial state is the Jiang–Su algebra [15].

2.1.7. *Kaplansky's problem.* What happens if one assumes that the tracial property $\varphi(AB) = \varphi(BA)$ holds for specific classes of elements $A, B \in \mathscr{A}$ but not necessarily all elements of $\mathscr{A}$? For instance one could require that $\varphi(A^*A) = \varphi(AA^*)$ for all $A \in \mathscr{A}$. The linearity of φ then implies (1). But what if φ is not required to be linear? A function $\varphi : \mathscr{A} \to \mathbb{C}$ is called a *quasitrace* if it satisfies $\varphi(A^*A) = \varphi(AA^*)$ for all $A \in \mathscr{A}$, it satisfies $\varphi(A + \mathrm{i}B) = \varphi(A) + \mathrm{i}\,\varphi(B)$ for all A, B selfadjoint, and it is linear on each abelian subalgebra of $\mathscr{A}$. Kaplansky [17] asked whether every II_1 AW*-factor is a von Neumann algebra. This would be true if one can prove that every quasitrace is a trace. While still an open problem, Haagerup [12] was able to prove in 1991 that each quasitrace on a unital exact C^*-algebra is a trace. This result has had significant applications to the theory of C^*-algebras.

2.1.8. *Approximately tracial state.* In perturbation theory, one considers situations where (1) we have an object that approximately fulfills a property, and we try to prove that it is close to an object that exactly satisfies that property; (2) there exists a problem for which we do not know the exact solution, but we can find an approximate solution for it; (3) there are objects with an approximate property and we seek an object that exactly meets the property. Here we deal with the third situation.

We may consider $(\mathcal{F}, \varepsilon)$-almost traces for any given finite subset $\mathcal{F}$ of the closed unit ball of $\mathscr{A}$ and any $\varepsilon > 0$. This means that there is a state $\varphi_{\mathcal{F},\varepsilon}$ on $\mathscr{A}$ such that $|\,\varphi_{\mathcal{F},\varepsilon}(A^*A - AA^*)| < \varepsilon$ for all $A \in \mathcal{F}$. It is shown in [19, Lemma 5.4] that a C^*-algebra $\mathscr{A}$ has a tracial state φ if and only if it has $(\mathcal{F}, \varepsilon)$-almost traces for all $\mathcal{F}$ and ε. Indeed, φ can be taken to be an accumulation point of the net $(\varphi_{\mathcal{F},\varepsilon})$ in the weak*-compact unit ball of the dual of $\mathscr{A}$.

2.2. **Changing codomain.**

2.2.1. *An extension of the trace with values in a C^*-algebra.* In seeking an extension of the trace, one may substitute the C^*-algebra $\mathbb{C}$ with an arbitrary C^*-algebra $\mathscr{A}$. If $\varphi : \mathbb{M}_n \to \mathscr{A}$ is a tracial positive linear map, then by repeating the argument in subsection 2.1.1 we get

$$\begin{aligned}\varphi(A) &= \sum_{i,j=1}^{n} a_{ij}\,\varphi(E_{ij}) = \sum_{i=1}^{n} a_{ii}\,\varphi(E_{ii}) = \varphi(E_{11}) \sum_{i=1}^{n} a_{ii} = \varphi(E_{11})\,\mathrm{tr}(A)\\ &= (n\,\varphi(E_{11}))\frac{1}{n}\,\mathrm{tr}(A) = \varphi(I)\widehat{\mathrm{tr}}(A),\end{aligned}$$

where now $\varphi(I)$ is an element of $\mathscr{A}$.

2.3. **Changing both domain and codomain.**

2.3.1. *A generalization of the trace that implies the commutativity of the underlying C^*-algebra.* One can think of replacing $\mathbb{M}_n$ and $\mathbb{C}$ with $\mathbb{M}_n(\mathscr{A})$ and $\mathscr{A}$, respectively, for some unital C^*-algebra $\mathscr{A}$, and then define $\varphi : \mathbb{M}_n(\mathscr{A}) \to \mathscr{A}$ by $\varphi([A_{ij}]) = \sum_{i=1}^{n} A_{ii}$. Then, if I_n denotes the identity element of $\mathbb{M}_n(\mathscr{A})$ and φ satisfies the tracial property (1), we have

$$AB = \frac{1}{n}\,\varphi(ABI_n) = \frac{1}{n}\,\varphi(AI_nBI_n) = \frac{1}{n}\,\varphi(BI_nAI_n) = \frac{1}{n}\,\varphi(BAI_n) = BA.$$

Therefore, $\mathscr{A}$ has to be commutative, every state is tracial, and $\mathscr{A}$ is of the form $C(\Omega)$ for some compact Hausdorff space Ω.

2.3.2. *Replacing $\mathbb{M}_n$ and $\mathbb{C}$ with an arbitrary C^*-algebra and $\mathbb{B}(\mathscr{H})$, respectively.* A linear map $\Phi : \mathscr{A} \to \mathscr{B}$ is called *tracial and positive* if it takes positive elements of $\mathscr{A}$ to those of $\mathscr{B}$ and fulfills the condition (1). A result due to Choi and Tsui [8, pp. 59-60] states that if $\Phi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ is a tracial and positive linear map, then there exist a commutative C^*-algebra $C(X)$, where X is a compact Hausdorff space, and tracial and positive linear maps $\phi_1 : A \to C(X)$ and $\phi_2 : C(X) \to \mathbb{B}(\mathscr{H})$ such that $\Phi = \phi_2 \circ \phi_1$. In particular, any tracial and positive linear map is completely positive.

2.3.3. *Substituting $\mathbb{M}_n$ and $\mathbb{C}$ with a properly infinite von Neumann algebra $\mathscr{M}$ and a unital C^*-algebra $\mathscr{B}$, respectively.* If $\Phi : \mathscr{M} \to \mathscr{B}$ is a unital tracial positive linear map, then Φ is identically zero. The reason is that we can "halve" projections. In particular, there exists a projection $P \in \mathscr{M}$ such that $P \sim I \sim I - P$ [31, Proposition V.1.36]. Hence, there are partial isometries $U, V \in \mathscr{M}$ such that $U^*U = V^*V = I$, $VV^* = P$, and $UU^* = I - P$. By the tracial property of Φ, we have $\Phi(I) = \Phi(P) = \Phi(I - P)$. Therefore, $\Phi(I) = \Phi(P) + \Phi(I - P) = 2\Phi(I)$, whence $\Phi(I) = 0$. Now given any positive element $A \in \mathscr{A}$, we have $A \le \|A\|\, I$. Therefore, $0 \le \Phi(A) \le \|A\|\, \Phi(I) = 0$, and hence, $\Phi(A) = 0$. As any element in $\mathscr{A}$ is a linear combination of four positive elements, it follows that $\Phi = 0$.

2.4. **C^*-algebras and faithful tracial states.**

2.4.1. *A unital C^*-algebra with a faithful tracial state is finite.* Let φ be a faithful tracial state on a unital C^*-algebra $\mathscr{A}$. We show that if $I \sim P$, then $P = I$. To see this, suppose $U^*U = I$. Then, $\varphi(I - UU^*) = \varphi(U^*U - UU^*) = 0$, and because φ is faithful, we infer that $UU^* = I$. This shows that every isometry is a unitary, and in particular the identity I is finite.

2.4.2. *C^*-algebras and von Neumann algebras admitting a faithful tracial state.* As opposed to the case of von Neumann algebras, it is not entirely clear how to characterize a C^*-algebra as *finite*. The naive way is to use the same definition as for von Neumann algebras. This is done for instance on [28], and it is the definition used in 2.4.1 above. The problem with this is that a C^*-algebra may not have enough projections, or it may even fail to have nonzero projections at all; see [18]. This would make C^*-algebras that "feel" infinite be finite, for example $C_0(\mathbb{R}, \mathcal{O}_2)$. A stronger definition is used in [29], where the requirement for finiteness is that all projections are finite, together with the existence of an approximate unit made entirely of projections. With this definition, combining the results from [4] and [12] it is proven that every unital, stably finite, exact C^*-algebra admits a tracial state. Here *stably finite* means that $\mathscr{A} \otimes \mathbb{K}(\mathscr{H})$ contains no infinite projections. Another notion of *finite* was considered by Cuntz and Pedersen in [9]. They consider, instead of equivalence of projections, equivalence of positive elements, where $x \sim y$ in $\mathscr{A}$ if there exists a sequence $\{z_n\} \subset \mathscr{A}$ such that $x = \sum_m z_n^* z_n$ and $y = \sum_n z_n z_n^*$. They say that $\mathscr{A}$ is finite if $0 \leq y \leq x$ and $y \sim x$ implies $x = y$. With this definition of finite, they prove that a separable C^*-algebra $\mathscr{A}$ is finite if and only if it admits a faithful tracial state.

For von Neumann algebras, the situation is simpler. If $\mathscr{M}$ is a finite von Neumann algebra with separable predual, then it has a faithful tracial state. The von Neumann algebra $\mathscr{M}$ is finite precisely when in the central decomposition of $\mathscr{M}$ there exist only types I_n with $n < \infty$ and II_1.

There is a general form for tracial states on finite von Neumann algebras: if $\mathscr{M}$ is a finite von Neumann algebra equipped with a center-valued tracial map $\mathrm{tr}_c : \mathscr{M} \to \mathcal{Z}(\mathscr{M})$, then each tracial state φ on $\mathscr{M}$ is of the form $\varphi = \rho \circ \mathrm{tr}_c$, where ρ is a state on $\mathcal{Z}(\mathscr{M})$. The tracial state φ is normal on $\mathscr{M}$ if and only if the state ρ is normal on $\mathcal{Z}(\mathscr{M})$, as shown in [16, Theorems 8.2.8 and 8.3.6]; see also [5, Theorem 4.1].

2.4.3. *A normal state on a von Neumann algebra gives a faithful normal tracial state on a reduced von Neumann algebra.* Let's consider a similar construction. Let $\mathscr{M}$ be a von Neumann algebra and let φ be a nonzero normal state on $\mathscr{M}$ with support P. Then $P\mathscr{M}P$ is a von Neumann algebra with a faithful state φ. If τ denotes the unique center-valued trace on $\mathscr{M}$, then $\psi = \varphi \circ \tau$ is a faithful normal tracial state on $P\mathscr{M}P$; see [16, Chapter 8] for more details.

2.4.4. *Invertibility in the presence of a faithful tracial state.* If a C^*-algebra $\mathscr{A}$ has a faithful tracial state φ, then the one-sided invertibility of $A \in \mathscr{A}$ implies the two-sided invertibility of A. Indeed, if $BA = I$, then

$$I = (BA)^* BA = A^* B^* BA \leq \|B\|^2 A^* A.$$

This implies that $A^*A \geq \|B\|^{-2} I$, so A^*A is invertible. Let $V = A(A^*A)^{-1/2}$. Then

$$V^*V = (A^*A)^{-1/2} A^*A (A^*A)^{-1/2} = I.$$

We obtain that $\|VV^*\| = \|V\|^2 = \|V^*V\| = 1$; thus $0 \leq VV^* \leq I$. In addition,

$$0 \leq \varphi(I - VV^*) = 1 - \varphi(VV^*) = 1 - \varphi(V^*V) = 1 - 1 = 0.$$

As φ is faithful, $VV^* = I$, so V is unitary (in particular, it is invertible). Thus, $A = V(A^*A)^{1/2}$ is invertible. An analog computation can be made when A is right-invertible.

2.4.5. *Factors with a faithful tracial state.* It is notable that a faithful tracial state φ on a factor $\mathscr{M}$ has the property

$$P \sim Q \Longleftrightarrow \varphi(P) = \varphi(Q)$$

for all projections $P, Q \in \mathscr{M}$. Indeed, if two projections $P, Q \in \mathscr{M}$ are not equivalent, then by the Comparison Theorem in factors [16, Theorem 6.2.7], we may assume $P \prec Q$ (otherwise, we obtain $Q \prec P$ and we can reason the same). That is, $P \sim Q_1 \leq Q$ for some projection Q_1. Therefore, $\varphi(P) = \varphi(Q_1) \leq \varphi(Q)$. If $\varphi(P) = \varphi(Q)$, then $\varphi(Q - Q_1) = 0$ and faithfulness implies that $Q_1 = Q$; this means $P \sim Q$, a contradiction. Thus, $\varphi(P) = \varphi(Q_1) < \varphi(Q)$, and so $\varphi(P)$ and $\varphi(Q)$ are distinct. The converse is clear by the tracial property of φ.

2.4.6. *Examples of nonfaithful tracial states.* Given a unital C^*-algebra $\mathscr{A}$ with a faithful tracial state φ, the extension $\psi : \mathscr{A} \oplus \mathscr{A} \to \mathbb{C}$ defined by $\psi(A, B) = \varphi(A)$ is a nonfaithful tracial state. Furthermore, the restriction of a tracial state on a C^*-algebra to a C^*-subalgebra may fail to be a state. For example, let $\mathscr{A}$ be a C^*-algebra and consider the tracial state $\varphi : \mathscr{A} \oplus \mathbb{M}_n \to \mathbb{C}$ defined by $\varphi(A, B) = \widehat{\operatorname{tr}}(B)$. Then, the restriction of φ to $\mathscr{A}$ is identically 0, which is not even a state. Another example is to consider a non-factor $\mathscr{M}$ with a faithful tracial state φ. Given a nontrivial projection in the center of $\mathscr{M}$, we have $\phi(P) > 0$ by the faithfulness of φ. Then $\psi(A) = \varphi(AP)$ provides a nonfaithful tracial state, since $\Psi(I - P) = 0$. In this situation we can get different faithful tracial states by weighting, in the following sense: for each $t \in [0, 1]$,

$$\psi_t(A) = \frac{t}{\varphi(P)}\, \varphi(AP) + \frac{(1-t)}{\varphi(P^{\perp})}\, \varphi(AP^{\perp}),$$

where $P^{\perp} = I - P$, is a faithful tracial state.

2.4.7. *GNS construction for a tracial state.* Let us now describe a situation where one extends a faithful tracial state on a unital C^*-algebra $\mathscr{A}$ to a faithful normal tracial state on a certain von Neumann algebra. We use the notation in the GNS construction (described in Appendix A). As φ is faithful, $N_{\varphi} = 0$, so $\mathscr{H}_{\varphi}$ is the completion of $\mathscr{A}$ with respect to the norm $\|A\|_{2,\varphi} = \varphi(A^*A)^{1/2}$ induced by the inner product $\langle a, b\rangle = \varphi(b^*a)$. For instance, if $\mathscr{A} = L^{\infty}[0, 1]$ and φ is integration with respect to the Lebesgue measure, then $\mathscr{H}_{\varphi} = L^2[0, 1]$.

In addition, the positive linear functional $\tilde{\varphi} : \pi_{\varphi}(\mathscr{A})'' \to \mathbb{C}$ defined by $\tilde{\varphi}(T) := \langle Tx_{\varphi}, x_{\varphi}\rangle$ is a faithful normal tracial state on the von Neumann algebra $\pi_{\varphi}(\mathscr{A})''$ generated by $\pi_{\varphi}(\mathscr{A})$, since $\tilde{\varphi}(\pi_{\varphi}(A)) = \varphi(A)$ for all $A \in \mathscr{A}$ (see (5)) and $\pi_{\varphi}(\mathscr{A})$ is dense in $\pi_{\varphi}(\mathscr{A})''$ in the strong operator topology. Therefore, $\pi_{\varphi}(\mathscr{A})''$ is a finite von Neumann algebra; see [1, Lemma 2.2] for details. Furthermore, if f is a continuous real-valued function on an interval containing the spectrum of $A \in \mathscr{A}$, then $\varphi(f(A)) = \tilde{\varphi}(\pi_{\varphi}(f(A))) = \tilde{\varphi}(f(\pi_{\varphi}(A)))$. This property is employed in [25] to establish that if f is a monotone (convex) function, then so is $A \mapsto \varphi(f(A))$.

Since φ is a faithful tracial state, the representation $\pi_{\varphi} : \mathscr{A} \to \mathbb{B}(\mathscr{H}_{\varphi})$ is one-to-one, for if $\pi_{\varphi}(A) = 0$, then (5) implies that $\varphi(A^*A) = 0$, and so $A = 0$.

2.4.8. *Constructing a von Neumann algebra with a faithful normal tracial state from a family of C^*-algebras admitting tracial states.* Let J be an infinite set equipped with a nontrivial ultrafilter α, meaning that α is free and there exists a sequence (J_n) in α such that $\cap_n J_n = \varnothing$. Suppose that for each $i \in J$ there exists a unital C^*-algebra $\mathscr{A}_i$ with a tracial state φ_i. Then, the tracial ultraproduct $\prod_{i\in J}^{\alpha} (\mathscr{A}_i, \varphi_i)$ is defined to be the C^*-product $\prod_{i\in J} \mathscr{A}_i$ modulo the ideal of all (A_i) in $\prod_{i\in J} \mathscr{A}_i$ such that $\lim_{i\to\alpha} \|A_i\|_{2,\varphi_i}^2 = \lim_{i\to\alpha} \varphi_i(A_i^* A_i) = 0$. It is established in [13, Theorem 4.1] that a tracial ultraproduct $\prod_{i\in J}^{\alpha} (\mathscr{A}_i, \varphi_i)$ of C^*-algebras is a von Neumann algebra with the faithful normal tracial state $\psi_a((A_i)) := \lim_{i\to\alpha} \varphi_i(A_i)$.

Appendix A. Basics of C* and von Neumann Algebras

A *C^*-algebra* is a complex Banach $*$-algebra $\mathscr{A}$ with an involution such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathscr{A}$. Every C^*-algebra can be realized as a C^*-subalgebra of $\mathbb{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ (Gel'fand–Naimark–Segal; see [22, Theorem 3.4.1]). On $\mathbb{B}(\mathscr{H})$ we can consider the *operator norm*, defined as

$$\|T\| = \sup\{\|Tx\| : \ x \in \mathscr{H}, \ \|x\| = 1\}.$$

An element $A \in \mathscr{A}$ is *selfadjoint* if $A^* = A$ and *positive* if $A = B^*B$ for some $B \in \mathscr{A}$ (equivalently, if $A = A^*$ and $\sigma(A) \subset [0, \infty)$, where $\sigma(A)$ denotes the spectrum of A). We denote by $\mathscr{A}^+$ and $\mathscr{A}^{\rm sa}$ the subsets of positive and selfadjoint operators in $\mathscr{A}$, respectively. For two self-adjoint operators (matrices) A and B, we say that $A \leq B$ whenever $B - A$ is positive (positive semidefinite). A rank-one projection is an operator of the form $e \otimes e$ for some unit vector $e \in \mathscr{H}$, where $(e \otimes e)(f) := \langle f, e\rangle e$ for all $f \in \mathscr{H}$.

By the *commutant* of a set $\mathcal{X} \subseteq \mathbb{B}(\mathscr{H})$, we mean the set $\mathcal{X}' = \{Y \in \mathbb{B}(\mathscr{H}) : \ XY = YX, \ X \in \mathcal{X}\}$. A non-degenerate $*$-subalgebra $\mathscr{M}$ of the algebra $\mathbb{B}(\mathscr{H})$ is called a *von Neumann algebra* acting in the Hilbert space $\mathscr{H}$ if $\mathscr{M} = \mathscr{M}''$. Von Neumann's *Double Commutant Theorem* states that for a non-degenerate $*$-algebra $\mathscr{M}$ we always have $\mathscr{M}'' = \overline{\mathscr{M}}^{\rm sot}$, where *sot* ("strong operator topology") denotes pointwise convergence. The commutative von Neumann algebra $\mathcal{Z}(\mathscr{M}) := \mathscr{M} \cap \mathscr{M}'$ is referred to as the center of $\mathscr{M}$, which in turn is always of the form $L^\infty(\Omega, \mu)$ for some measure space (Ω, μ). A *factor* is a von Neumann algebra with trivial center. If $P \in \mathscr{M}$ is a projection (that is, $P^2 = P$ and $P^* = P$), the corresponding *reduced von Neumann algebra* is defined as $\mathscr{M}_P = \{PX|_{P\mathscr{H}} : \ X \in \mathscr{M}\}$.

For projections $P, Q \in \mathscr{M}$, we denote $P \sim Q$ (Murray–von Neumann equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathscr{M}$; intuitively this says that both projections have the same rank, but there is a dependence on the algebra for the existence of the partial isometry U, so the notion of equivalentce is intrinsic to $\mathscr{M}$. A von Neumann algebra $\mathscr{M}$ is said to be *finite* if $P = Q$ for any equivalent projections $P, Q \in \mathscr{M}$ with $P \leq Q$. Abelian von Neumann algebras are trivially finite. A non-finite projection is said to be *infinite*, and *properly infinite* if it is nonzero and infinite, and for every nonzero central projection $Q \in \mathscr{M}$, either $QP = 0$ or QP is infinite. A von Neumann algebra is said to be *finite* or *properly infinite* if its identity has the corresponding property. It is known that there exists a unique projection P_0 in the center $\mathcal{Z}(\mathscr{M})$ of $\mathscr{M}$ such that P_0 is finite and $I - P_0$ is properly infinite. Hence, we have the direct sum

$$\mathscr{M} = \mathscr{M}P_0 \oplus \mathscr{M}(I - P_0),$$

where $\mathscr{M}P_0$ is finite and $\mathscr{M}(I - P_0)$ is properly infinite.

We say that a projection in $\mathscr{M}$ is *abelian* if the algebra $P\mathscr{M}P$ is commutative. A von Neumann algebra $\mathscr{M}$ is said to be of *type I* if every projection in $\mathcal{Z}(\mathscr{M})$ majorizes a nonzero abelian projection in $\mathscr{M}$. If there is no nonzero finite projection in $\mathscr{M}$, then it is said to be of *type III*. If $\mathscr{M}$ has no nonzero abelian projection and if each nonzero projection in $\mathcal{Z}(\mathscr{M})$ majorizes a nonzero finite projection in $\mathscr{M}$, then it is said to be of *type II*. If $\mathscr{M}$ is type II and finite, then it is said to be of *type* II_1. If $\mathscr{M}$ is of type II and properly infinite, then it is said to be of *type* II_∞.

Every von Neumann algebra $\mathscr{M}$ has a unique *central decomposition* into a direct sum of subalgebras of type I, type II_1, type II_∞, and type *III* [31, Chapter V, Theorem 1.19]. Thus, $\mathscr{M} = \mathscr{M}_{P_I} \oplus \mathscr{M}_{P_{II_1}} \oplus \mathscr{M}_{P_{\mathrm{II}_\infty}} \oplus \mathscr{M}_{P_{\mathrm{III}}}$, where projections $P_I, P_{II_1}, P_{\mathrm{II}_\infty}$, and P_{III} in $\mathcal{Z}(\mathscr{M})$ are such that $P_I + P_{\mathrm{II}_1} + P_{\mathrm{II}_\infty} + P_{\mathrm{III}} = I$; it is possible for one or more of these to be zero.

A linear functional φ on $\mathscr{A}$ is said to be *positive* if $\varphi(X) \geq 0$ for all positive elements $X \in \mathscr{A}$. It is referred to as a *state* if it is positive and its operator norm $\|\,\varphi\,\|$ is equal to one. The positivity-preserving property of a linear functional φ is equivalent to $\|\varphi\| = \varphi(I)$; see [22, Corollary 3.3.5]. It is called *faithful* if it is one to one on $\mathscr{A}^+$. A positive linear functional φ on a von Neumann algebra $\mathscr{M}$ is said to be *normal* if $X_j \nearrow X$ (that is, $\langle X_j z, z\rangle \nearrow \langle Xz, z\rangle$ for all $z \in \mathscr{H}$) with $X_j, X \in \mathscr{M}^{\rm sa}$ implies $\varphi(X) = \sup \varphi(X_i)$.

We briefly introduce the GNS construction corresponding to a given state on a unital C^*-algebra $\mathscr{A}$. Suppose that φ is a state and let $N_\varphi = \{A \in \mathscr{A} : \varphi(A^*A) = 0\}$; this is a norm-closed left ideal of $\mathscr{A}$. An inner product on the quotient space $\mathscr{A}/N_\varphi$ can be defined by

$$\langle A + N_\varphi, B + N_\varphi\rangle := \varphi(B^*A)$$

The completion of this inner product space is denoted by $\mathscr{H}_\varphi$. The linear operator $\pi_\varphi : \mathscr{A}/N_\varphi \to \mathscr{A}/N_\varphi$ defined as $\pi_\varphi(A + N_\varphi)(B + N_\varphi) = AB + N_\varphi$ can be extended to a linear operator on $\mathscr{H}_\varphi$ denoted by the same $\pi_\varphi(A)$. Moreover, $\pi_\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H}_\varphi)$ is a $*$-homomorphism between C^*-algebras; that is, a *representation*. In addition, the unit vector $x_\varphi = I + N_\varphi \in \mathscr{H}_\varphi$ is cyclic (meaning that $\pi_\varphi(\mathscr{A})x_\varphi$ is dense in $\mathscr{H}_\varphi$) and

$$\varphi(A) = \langle \pi_\varphi(A)x_\varphi, x_\varphi\rangle. \tag{5}$$

The triple $(\pi_\varphi, \mathscr{H}_\varphi, x_\varphi)$ is called the *GNS representation* (from Gelfand–Naimark–Segal).

The pair $\{\pi, \mathscr{H}\} = \bigoplus_{\varphi\in S(\mathscr{A})} \{\pi_\varphi, \mathscr{H}_\varphi\}$ is known as the *universal representation* of $\mathscr{A}$. Here, $S(\mathscr{A})$ denotes the set of all states on $\mathscr{A}$. The von Neumann algebra $\mathscr{M} = \pi(\mathscr{A})''$ generated by $\pi(\mathscr{A})$ is said to be the *universal enveloping von Neumann algebra* of the C^*-algebra $\mathscr{A}$ [31, Chap. III, Definition 2.3].

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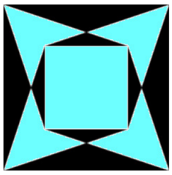

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The Fundamental Theorem of Algebra

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Abstract. This is an expository note about the Fundamental Theorem of Algebra.

1. Introduction

Each nonconstant polynomial with complex coefficients has a complex root. In symbols:

Theorem 1.1. *If $p(z) \in \mathbb{C}[z]$ has positive degree, then there exists $a \in \mathbb{C}$ such that $p(a) = 0$.*

This is one of the foundations on which algebra rests. Burnside and Panton [3][1] state it in article 15, in Chapter II, and use it for most of the rest of Volume I, before giving a proof in article 122. The proof they give is based on the argument principle.

1.1. **Argument principle.** The variation of the argument of the polynomial around a simple closed curve γ on which it does not vanish counts the roots inside:

$$\int_\gamma d\arg(p) = \int_\gamma \frac{p'(z)dz}{p(z)} = 2\pi i n$$

if p has n roots inside γ (where multiple roots are counted a number of times equal to their multiplicity).

Assuming this, Theorem 1.1 follows on applying the principle, taking γ to be a very large circle around 0.

This proof of Theorem 1.1 has the merit of exposing the *real reason* why the theorem is true. The theorem is a consequence of the topological action of polynomials on the plane. More precisely, there are two ingredients: (1) the completeness of the complex plane $\mathbb{C}$, as a metric space; (2) the fact that a polynomial with $b = p(a)$ induces a *positive* map of homotopy groups

$$\pi_1(\{z \in \mathbb{C} : 0 < |z-a| < r\}) \to \pi_1(\mathbb{C} \setminus \{b\})$$

for all sufficiently small positive r. The latter comes down to the fact that the map $z \mapsto z^m$ induces multiplication by m on $\pi_1(\mathbb{C}^\times)$, combined with the remainder theorem.

Usually, people derive the theorem from the argument principle for holomorphic functions, and note that polynomials are entire functions, so that the argument principle applies to them. The argument principle for holomorphic functions depends on Cauchy's Theorem, and hence on the Stokes-Green formula.

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[1]Classic text by two TCD academics, published by Hodges and Figgis, booksellers of happy memory.

1.2. **Issues.** The remainder theorem is elementary algebra, but plane algebraic topology is not. So it is reasonable to ask for proofs of Theorem 1.1 that avoid analysis as much as possible.

It seems obvious to me that you can't avoid analysis altogether, since the completeness of $\mathbb{C}$ is an essential ingredient.

1.3. **From Cauchy-Stokes.** The following proof uses a minimum of complex analysis:

Suppose $p(z) \in \mathbb{C}[z]$ has degree $m > 0$ and has no roots. Assume, as we may, that $p(z)$ is monic. Then $f(z) := z^{m-1}/p(z)$ has $f_{\bar{z}} = 0$ on $\mathbb{C}$. If $D = \mathbb{U}(0, R)$ is the disk of radius R about 0, then by Stokes' Theorem

$$\int_{\partial D} f(z)dz = \int_D df \wedge dz = \int_D (f_z dz + f_{\bar{z}} d\bar{z}) \wedge dz = 0.$$

But, parametrising ∂D by $z = Re^{i\theta}$, we have

$$\begin{aligned}\int_{\partial D} f(z)dz &= \int_{\partial D} \frac{dz}{z(1 + \mathrm{O}(1/R))} = i \int_0^{2\pi} (1 + \mathrm{O}(1/R))d\theta \\ &= 2\pi i(1 + \mathrm{O}(1/R) \to 2\pi i\end{aligned}$$

as $R \uparrow \infty$. This is impossible.

1.4. **Maximum principle.** The maximum principle for polynomials is elementary:

Theorem 1.2. *Let $p(z) \in \mathbb{C}[z]$. Suppose $|p(z)|$ has a local maximum at some point $a \in \mathbb{C}$. Then $p(z) = p(a)$, constant.*

Proof. Suppose p is nonconstant. Composing with translations, we may assume $a = 0$. Applying the remainder theorem, we can factor

$$p(z) - p(0) = z^n g(z),$$

where $n \geq 1$ and $g(z) \in \mathbb{C}[z]$ has $g(0) \neq 0$. Then for small positive r and any $\theta \in \mathbb{R}$, we have

$$p(re^{i\theta}) = p(0) + r^n e^{in\theta} g(0) \left(1 + \mathrm{o}(1)\right).$$

Writing $p(0) = \alpha e^{i\beta}$ and $g(0) = \rho e^{i\phi}$ with $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\rho > 0$ and $\phi \in \mathbb{R}$, this gives

$$p(re^{i\theta}) = \alpha e^{i\beta} + r^n e^{i(n\theta+\phi)} \rho \left(1 + \mathrm{o}(1)\right). \tag{1}$$

So for $\theta = (\beta - \phi)/n$ and all small positive r we have

$$p(re^{i\theta}) = \left(\alpha + r^n \rho\right) e^{i\beta} + \mathrm{o}(r^n),$$

so for arbitrarily small positive r

$$|p(re^{i\theta})| \geq \alpha + r^n\rho - \frac{r^n\rho}{2} > \alpha = |p(0)|,$$

contradicting the assumption that 0 is a local maximum. □

A small twist on the same argument gives the *minimum priciple* away from roots:

Theorem 1.3. *Let $p(z) \in \mathbb{C}[z]$. Suppose $|p(z)|$ has a local minimum at some point $a \in \mathbb{C}$. Then $p(a) = 0$ or $p(z) = p(a)$, constant.*

Proof. Assuming p nonconstant and $p(a) \neq 0$, and proceeding as before, we have Equation (1), where now α is strictly positive. So for $\theta = (\beta - \phi + \pi)/n$ and all small positive r we have

$$p(re^{i\theta}) = (\alpha - r^n\rho)e^{i\beta} + \mathrm{o}(r^n),$$

so for arbitrarily small positive r

$$|p(re^{i\theta})| \leq \alpha - r^n\rho + \frac{r^n\rho}{2} < \alpha = |p(0)|,$$

contradicting the assumption that 0 is a local minimum. □

1.5. **Bolzano-Weierstrass.** The Bolzano-Weierstrass Theorem says that each bounded sequence of real numbers has a convergent subsequence. (See, for instance, [16, Theorem 8.17].) It follows that each bounded sequence of complex numbers has a convergent subsequnce: just apply it to the real parts and then apply it to the imaginary parts of the resulting subsequence. This is enough analysis to give Theorem 1.1.

1.6. **Proof of Theorem 1.1 without winding numbers.** Suppose $p(z) \in \mathbb{C}[z]$ is nonconstant and has no root.

Since $|p(z)| \to +\infty$ as $|z| \to +\infty$, we may choose $R > 0$ such that $|p(z)| \geq 2|p(0)|$ whenever $|z| \geq R$. Let $B := \mathbb{B}(0, R)$.

Let $m := \inf\{|p(z| : |z| \leq R\}$. Then $0 \leq m \leq |p(0)|$.

Suppose $m = 0$. Then we could choose a sequence $(z_n) \subset B$ such that $p(z_n) \to 0$. Passing to a subsequence, we may assume (z_n) converges to some $a \in B$. By continuity of p, $p(a) = 0$, which is impossible. Thus $m > 0$.

Choose a sequence $(z_n) \subset B$ such that $|p(z_n)| \to m$. Passing to a subsequence, we may assume $z_n \to \xi$ for some $\xi \in B$. Then $|p(\xi)| = m$. We cannot have $|\xi| = R$, since otherwise

$$m = |p(\xi)| \geq 2|p(0)| \geq 2m > m.$$

Thus p has a local minimum at ξ, which contradicts Theorem 1.3. □

1.7. **Variation.** A variation on the foregoing proof goes as follows:

Suppose $p(z) \in \mathbb{C}[z]$ is nonconstant and has no root. Then $f := 1/|p(z)|$ is positive and continuous on $\mathbb{C}$, and tends to zero as $|z| \to +\infty$. Thus we may choose $R > 0$ such that $|f(z)| < \frac{1}{2}|f(0)|$ whenever $|z| > R$. Let $B := \mathbb{B}(0, R)$, $D := \mathbb{U}(0, R)$ and $S := B \setminus D$.

There exists some $a \in B$ such that

$$f(a) = \sup_B |f|.$$

By continuity, $|f| \leq \frac{1}{2}|f(0)| \leq \frac{1}{2}|f(a)|$ on S, so $a \in D$. Thus $|p|$ has a local minimum at a, contradicting Theorem 1.3.

1.8. **Proof of Theorem 1.1 using harmonicity.** Harmonic functions may be defined as the twice-differentiable solutions of Laplace's equation, or, equivalently, as the continuous functions having the mean-value property. See [4].

Harmonic functions have a maximum principle.

Theorem 1.4. *Suppose $\Omega \subset \mathbb{C}$ is a connected open set, and $h : \Omega \to \mathbb{R}$ is harmonic on Ω. Then if there is some point $a \in \Omega$ such that*

$$h(a) = \sup_\Omega h,$$

then h is constant on Ω.

This is most conveniently proved by appealing to the mean-value property, and showing that the existence of a global maximum at a implies that the set $h^{-1}(h(a))$ is open-closed relative to Ω.

Now if we had a nonconstant polynomial $p(z)$ having no root, then $u := 2\log|p|$ would be harmonic on $\mathbb{C}$, because $u = \log(p\bar{p})$ and $p_y = p' z_y = ip_x$, $p_{yy} = -p_{xx}$ so a simple calculation gives

$$u_x = \frac{\bar{p}_x}{\bar{p}} + \frac{p_x}{p},$$

$$u_{xx} = \frac{\bar{p}\bar{p}_{xx} - \bar{p}_x^2}{\bar{p}^2} + \frac{pp_{xx} - p_x^2}{p^2},$$

and similar formulas for the y-derivatives, giving

$$\Delta u = u_{xx} + u_{yy} = \frac{\bar{p}\Delta\bar{p} - \bar{p}_x^2 - \bar{p}_y^2}{\bar{p}^2} + \frac{p\Delta p - p_x^2 - p_y^2}{p^2} = 0.$$

We could then argue much as in Subsection 1.6 that, since it is a continuous real-valued function on $\mathbb{C}$ tending to infinity at infinity, u has a global minimum on $\mathbb{C}$ at some point a, and hence the nonconstant harmonic function $-u$ has a global maximum at a, contradicting from Theorem 1.4.

2. Open Mapping Theorems

The argument of Subsection 1.6 (or Subsection 1.7) can also be used by replacing the minimum principle Theorem 1.3 by the open mapping theorem for polynomials, because an open set that meets a circle must have points inside and outside the circle.

The open mapping theorem is:

Theorem 2.1. *Let $p(z) \in \mathbb{C}[z]$ be nonconstant. Then $p(\Omega)$ is open whenever $\Omega \subset \mathbb{C}$ is open.*

2.1. Holomorphic functions.

The open mapping theorem for holomorphic functions is:

Theorem 2.2. *Let $f : \Omega \to \mathbb{C}$ be holomorphic on the connected open set Ω and nonconstant. Then $p(\Omega)$ is open.*

The usual proof uses Rouchés Theorem. (For an alternative, see Section 4, below.)

Theorem 2.1 is an immediate corollary, and this is the standard way to prove it.

2.2. Smooth functions.

The open mapping theorem for vector-valued differentiable functions is:

Theorem 2.3. *Let $f : \Omega \to \mathbb{R}^d$ be continuously differentiable on the connected open set $\Omega \in \mathbb{R}^d$, with nonsingular Frechet derivative at each point. Then $f(\Omega)$ is open.*

This is a corollary of the inverse function theorem for smooth functions, which can be proved by applying Banach's contraction mapping principle.

Notice that an f satisfying the hypotheses must either preserve or reverse orientation on each connected component of Ω, because the sign of the determinant of its derivative cannot change there.

Theorem 2.3 may be used in a proof of Theorem 2.2, as follows:

Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and nonconstant, and Ω is open. Let $C := \{c \in \Omega : f'(c) = 0\}$ be the set of critical points of f. Then C has no accumulation points in Ω. Theorem 2.3 tells us that $f(\Omega \setminus C)$ is open. So it remains to see that for $c \in C$ and sufficiently-small $r > 0$, the image $f(\mathbb{B}(c, r))$ is a neighbourhood of $a := f(c)$.

The set $P := f^{-1}(f(c))$ of preimages of $f(c)$ has no accumulation points in Ω.

Choose $r > 0$ smaller than half the distance from c to the rest of $C \cup P \cup (\mathbb{C} \setminus \Omega)$. Let B be the closed disc $\mathbb{B}(c, r)$, let U be its interior, and S be its boundary circle. Let A be the annulus $U \setminus \{c\}$.

Let $F := f(B)$. Then F is closed, since f is continuous and B is compact. Let $T := \text{bdy}(F)$, so $T \subset F$. Suppose F is not a neighbourhood of a. Then a must belong to T. Since $f(U \setminus \{a\})$ is open, and is contained in F, it does not meet T. Thus $T \subset \{a\} \cup f(S)$. Since $a \notin f(S)$, this means that T has an isolated point at a.

Now A is dense in B, so $D := f(A)$ is a dense subset of F, thus $T \subset \text{bdy}(D)$. Since D is open, it does not meet $\text{bdy}(D)$, so $\text{bdy}(D) \subset F \setminus D \subset T$. Thus $T = \text{bdy}(D)$, and D is a connected open set having a as an isolated boundary point. This implies that D

contains a deleted neighbourhood of a, and then F contains a full neighbourhood of a, contrary to our assumption. Thus $f(\Omega)$ is open. □

This proof does not simplify materially when f is assumed to be a polynomial, in place of an arbitrary holomorphic function.

3. Roots

The fundamental theorem implies that each nonzero $a \in \mathbb{C}$ has m-th roots of each order $m \in \mathbb{N}$, but this fact is more elementary, and can be proved using De Moivre's formula. Proving De Moivre's formula does require some analysis, of course, since we have to introduce the trigonometric functions first. Look at [16], for instance.

4. Formal power series

4.1. Let $\mathcal{F}$ be the ring of all formal power series over $\mathbb{C}$ in one variable, and $\mathcal{F}^\times$ be the group of invertibles under convolution multiplication. Let $\mathcal{G} \subset \mathcal{F}$ be the group of the series that are invertible under formal composition. Let $\mathfrak{F}$, $\mathfrak{F}^\times$ and $\mathfrak{G}$ be the corresponding subsets of series having positive radius of convergence.

Cartan [5] proves the inverse function theorem for convergent series, using a majorization argument:

Theorem 4.1. *Suppose $f \in \mathcal{G} \cap \mathfrak{F}$. Then the compositional inverse of f belongs to $\mathfrak{G}$.*

This has as a corollary the inverse function theorem for holomorphic functions, already mentioned, and this is an interesting alternative to the use of Rouchés Theorem.

4.2. **Roots.**

Proposition 4.2. *Suppose $f = a_0 + a_1 z + \mathrm{HOT} \in \mathcal{F}$ and $a_0 \neq 0$. Then for each $m \in \mathbb{N}$ there exists $g \in \mathcal{F}$ such that $g(z)^m = f(x)$. Moreover, if $f \in \mathfrak{F}$, then each choice of g also belongs to $\mathfrak{F}$.*

(Here, HOT stands for *higher-order terms.*)

Proof. Since a_0 has m-th roots, it suffices to consider the case $a_0 = 1$. The binomial series for the m-th root:

$$r := (1+x)^{1/m} := \sum_{n=0}^{\infty} \binom{\frac{1}{m}}{n} x^n$$

has radius of convergence $1 > 0$, so the composition $g := r \circ (f-1)$ has positive radius of convergence if f does, and satisfies $g^m = f$. □

This gives us another way to prove the open mapping theorem for holomorphic functions:

Suppose f is holomorphic and nonconstant on a neighbourhood N of a. We want to see that $f(N)$ is a neighbourhood of $b = f(a)$. Translating before and after, we may assume $a = b = 0$. The function f has a convergent power series expansion near 0, so for some $m \in \mathbb{N}$, we have

$$f(z) = z^m(a_0 + a_1 z + \mathrm{HOT}) = z^m h(z),$$

with $a_0 \neq 0$. By Proposition 4.2, there is a convergent series $g = b_0 + b_1 z + \mathrm{HOT})$ such that $g^m = h$. Then $f = (zg(z))^m$ near 0. Now by the inverse function theorem, $zg(z)$ maps N onto a neighbourhood N_1 of 0, and since all complex numbers have mth roots, $z \mapsto z^m$ maps N_1 onto a neighbourhood N_2 of 0, so $f = (z^m) \circ (zg(z))$ maps N onto N_2, and we are done.

5. Connectivity

Recall that a map is *proper* if the preimage of each compact set is compact. Proper maps between metric spaces are continuous. A map $f : \mathbb{C} \to \mathbb{C}$ is proper if it is continuous and $|f(z)| \to +\infty$ as $|z| \to +\infty$.

Theorem 5.1. *Suppose $M \neq \emptyset$ and N are connected manifolds, and $f : M \to N$ is continuous, proper, and open. Then $f(M) = N$.*

Proof. $f(M)$ is nonempty, connected, open and closed in N. Since N is connected, $f(M) = N$. □

This gives Theorem 1.1, once we know that nonconstant polynomials are open. This proof sidesteps the use of maxima and minima.

6. Galois Theory

People who like to use as little analysis as possible are drawn to the following proof of Theorem 1, which uses substantial results from Galois theory and group theory. It is found for instance in van der Waerden [17, Kap 11], or [14]. Lang says it is essentially one of Gauss' proofs, and van der Waerden describes it as the second Gauss proof [17, §81, p.252][2].

The analysis is in the following two lemmas.

Lemma 6.1. *Each odd-degree polynomial over $\mathbb{R}$ has a real root.*

Proof. It suffices to consider monic polynomials. If $p(x) \in \mathbb{R}[x]$ is monic and has odd degree, then for all large enough real $x > 0$, $p(x)$ is positive and $p(-x)$ is negative. By the Axiom of Completeness, there exists a least upper bound λ of the set $\{x \in \mathbb{R} : p(x) < 0\}$. Since $p : \mathbb{R} \to \mathbb{R}$ is a continuous function, it follows readily that $p(\lambda) = 0$. □

Lemma 6.2. *Each positive real number has a positive real square root.*

Proof. If $0 < a \in \mathbb{R}$, then $p(x) := x^2 - a$ is negative at $x = 0$ and positive for all large enough real x, so exactly as in the previous lemma, p has a positive real root. □

Corollary 6.3. *Each nonzero complex number has a complex square root.*

Proof. Indeed, let a, b be arbitrary elements of $\mathbb{R}$. We claim that there are $c, d \in \mathbb{R}$ such that $a + bi = (c + di)^2$. The case $a = 0$ is covered by Lemma 6.2 and the observation that -1 and i have complex square roots. For the case of non-zero a, we may assume that a is positive since -1 has a complex square root, and then by Lemma 6.2 that $a = 1$. We have to solve the system

$$\begin{aligned} 1 &= c^2 - d^2, \\ b &= 2cd. \end{aligned}$$

Squaring the second and multiplying both sides of the first by c^2, we get

$$c^4 - c^2 = \frac{b^2}{4}.$$

Completing the square gives

$$(c^2 - \tfrac{1}{2})^2 = \frac{b^2 + 1}{4}.$$

The right-hand side has a positive real square root, say e. Then $\frac{1}{2} + e$ has a positive real square root, say f. So $c = f$ and $d = b/(2c)$ give us real numbers that solve the system. □

[2] It is worth noting that van der Waerden [17, §80] says that the simplest proof of Theorem 1.1 is one that uses complex analysis: a counterexample $p(z)$ would have $1/p(z)$ nonconstant, entire and bounded, contradicting Liouville's Theorem.

Corollary 6.4. *Each monic quadratic over $\mathbb{C}$ has a complex root.*

Proof. Just use the usual quadratic formula and Corollary 6.3. □

Armed with these, we can prove Theorem 1.1, as follows.

Since -1 is not a square in $\mathbb{R}$, $\mathbb{C} := \mathbb{R}[i]$ is a degree 2 extension of $\mathbb{R}$.

Suppose some monic polynomial $p(z) \in \mathbb{C}[z]$ has no root in $\mathbb{C}$. Then there is a proper finite degree extension K of $\mathbb{C}$, which is Galois over $\mathbb{C}$.

Let S be a Sylow 2-subgroup of $\mathrm{Aut}(K/\mathbb{R})$. The fixed subfield $K^S \subset K$ of S is an odd degree extension of $\mathbb{R}$. Pick $\xi \in K^S$ such that $K^S = \mathbb{R}[\xi]$. Then the minimal polynomial of ξ over $\mathbb{R}$ has odd degree, hence has a root in $\mathbb{R}$, and hence has degree one. Thus $S = \mathrm{Aut}(K/\mathbb{R})$.

Thus $\mathrm{Aut}(K/\mathbb{R})$ is a 2-group, hence so is its subgroup $\mathrm{Aut}(K/\mathbb{C})$.

Every 2-group has a subgroup of index two[3], so choose $H \leq \mathrm{Aut}(K/C)$ of index 2. Then the fixed subfield K^H is a degree 2 extension of $\mathbb{C}$. But we can always solve quadratics over $\mathbb{C}$ in $\mathbb{C}$, so $\mathbb{C}$ does not have a degree 2 extension. This contradiction concludes the proof.

7. Pure Algebra

We can avoid analysis completely by changing the question. As already remarked, $\mathbb{R}$ is a convenient fiction, containing a huge set of 'yellow-pack' numbers which are literally indescribable. One can imagine trying to get along without $\mathbb{R}$. Among reasonable alternatives, three come immediately to mind:

- The field $\mathbb{E}$ of 'Euclidean reals'. These are the numbers corresponding to the points that you can construct on a line using straight-edge and compass and a segment on the line with ends labelled 0 and 1. Algebraically, $\mathbb{E}$ is a quadratic closure of $\mathbb{Q}$, i.e. $\mathbb{E}$ has characteristic zero, each quadratic polynomial over $\mathbb{E}$ has a root in $\mathbb{E}$, and no proper subfield of $\mathbb{E}$ has this property.
- The field $\mathbb{G}$ of 'Gaussian reals', or real algebraic numbers. This can be described without reference to $\mathbb{R}$, as follows. Let $\hat{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, and let $i \in \hat{\mathbb{Q}}$ denote one of the square roots of -1. The field automorphism of $\mathbb{Q}[i]$ that sends $i \mapsto -i$ extends to an involutive field automorphism of $\hat{\mathbb{Q}}$, which we denote by $\tau : z \mapsto \bar{z}$. Then $\mathbb{G} \subset \hat{\mathbb{Q}}$ is the subfield fixed by τ, and one sees that $\hat{\mathbb{Q}} = \mathbb{G}[i]$.

  If you try, for a moment, to put yourself in Gauss' shoes, at the time before he had found his first proof of Theorem 1.1, you see that the great man had to grapple with the possibility that, big though it may be, $\mathbb{C}$ might not be large enough to embrace $\hat{\mathbb{C}}$ or even $\hat{\mathbb{Q}}$, and $\mathbb{R}$ might not contain a copy of $\mathbb{G}$.
- The field $\mathbb{D}$ of real numbers that have a definite description. Without getting into technicalities, $\mathbb{D}$ contains $\mathbb{G}$ and also numbers such as π and Euler's e and γ, real and imaginary parts of the values of all explicit elementary functions at all rationals, of zeros of Bessel functions, of Riemann's $\zeta(s)$, and so on. But since there are only a countable number of definite descriptions, $\mathbb{D}$ is much smaller than $\mathbb{R}$, even though it contains all the real numbers anyone might ever care about.

[3]Each p-group is nilpotent [8, Theorem 3.3(iii)], so each 2-group G has $[G,G] < G$ so the abelian group $G/[G,G]$ has a subgroup of index 2, hence its preimage under the surjection

$$G \to \frac{G}{[G,G]}$$

also has index 2.

If we ask what might replace Theorem1.1 if we replace $\mathbb{R}$ and $\mathbb{C} = \mathbb{R}[i]$ by one of these alternatives, then we get nowhere with the Euclidean numbers, because there are cubics over $\mathbb{Q}$ with no solution in $\mathbb{E}$. For $\mathbb{G}$ and $\mathbb{D}$, one can formulate reasonable questions.

The fields $\mathbb{R}$, $\mathbb{G}$ and $\mathbb{D}$ are examples of *formally-real fields* in the sense of Artin and Schreier: This just means that -1 is not a sum of squares in the field.

In logical terms, the *real-closed formally-real fields* share the same first-order properties as the ordered field $\mathbb{R}$, and are the subject of the Grand Artin-Schreier Theorem [19, 6]:

Theorem 7.1 (Grand Artin-Schreier Theorem). *Let F be a field. Then the following are equivalent:*
(i) F is formally real and admits no proper formally real algebraic extension.
(ii) F is formally real, every odd degree polynomial over F has a root, and for each $x \in F^{\times}$, one of $x, -x$ is a square.
(iii) F is formally real and $F(\sqrt{-1})$ is algebraically closed.
(iv) The absolute Galois group of F is finite and nontrivial.

To use this to prove that $\mathbb{G}[i]$ and $\mathbb{D}[i]$ are algebraically-closed, one needs to verify condition (ii) for $F = \mathbb{G}$ and $F = \mathbb{D}$. I don't see any way to do that without applying the fact that it holds for $F = \mathbb{R}$ and hence that (iii) holds for $F = \mathbb{R}$.

8. Wider context

It has been said that Theorem 1.1 is neither fundamental nor algebra. *Algebra* has changed its meaning in the past two centuries, and no longer just means the theory of equations, so the real question is whether the theorem is really fundamental for algebraic equations. The field $\hat{\mathbb{Q}}$ of algebraic numbers is certainly fundamental. It is necessary to deal with all the algebraic numbers, fictions of our imagination though they be. But the field of complex numbers is much larger, even in cardinality, and most complex numbers are even more fictional. In fact, the typical complex number has only generic properties, i.e. it cannot be characterised by a specific finite list of properties. The field $\mathbb{C}$ is convenient, because Theorem 1.1 implies that it contains an isomorphic copy of $\hat{\mathbb{Q}}$ and because we can use the richness of complex analysis on it. It is an interesting consequence of Theorem 1.1 that the field $\mathbb{R}$ has index two in its algebraic closure. However, $\mathbb{R}$ is large and mysterious, and open to the same criticism as $\mathbb{C}$.

8.1. $\mathbb{C}_p$.

Definition 8.1. $|\cdot|$ is a *field norm* on the field F if it satisfies the conditions:
(1) $|x|$ is a nonnegative real number, whenever $x \in F$, and $|x| = 0$ if and only if $x = 0$.
(2) $|x + y| \le |x| + |y|$, whenever $x, y \in F$.
(3) $|xy| = |x| \cdot |y|$, whenever $x, y \in F$.
The norm is *non-archimedean* if it satisfies the stronger condition:
(2') $|x + y| \le \max(|x|, |y|)$, whenever $x, y \in F$.

We remark that one could consider a more general concept, where the values of the norm lie in some totally-ordered abelian group [11].

Only fields of characteristic zero admit a field norm.

Each field of characteristic zero has a subfield isomorphic to $\mathbb{Q}$. Ostrowski [12] proved that the only field norms on $\mathbb{Q}$ are powers of the usual absolute value and powers of the p-adic norms corresponding to primes p.

From the adelic point of view, there is little to choose between $\mathbb{R}$ and any of the p-adic completions $\mathbb{Q}_p$ of the rationals. It is no longer the case that the algebraic closure $\widehat{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is a finite extension, nor is it complete with respect to the (unique!) extension of the p-adic metric, and we can enlarge it to its metric completion, denoted $\mathbb{C}_p$.

Theorem 8.2. [12, Theorem 13, p72][15, Theorem 4.6] *The field $\mathbb{C}_p$ is algebraically-closed.*

The key step in proving this is Krasner's Lemma:

Lemma 8.3. *Suppose K is a field complete with respect to a non-archimedean field norm $|\cdot|$. Suppose $\alpha, \beta \in \hat{K}$. Let L be the Galois completion of $K(\alpha, \beta)$ and let α_j $(j = 1, \dots, m)$ be the conjugates of α under the group G of automorphisms of L that fix $K(\beta)$. Suppose*

$$|\beta - \alpha| < \min\{|\alpha_i - \alpha_j| : i \neq j\}.$$

Then $\alpha \in K(\beta)$.

Proof. Let $\sigma \in G$. Then, since the norm is invariant under σ, we have

$$|\beta - \sigma(\alpha)| = |\sigma(\beta - \alpha)| = |\beta - \alpha|.$$

Thus

$$\begin{aligned} |\sigma(\alpha) - \alpha| &= |\sigma(\alpha) - \beta + \beta - \alpha| \\ &\le \max(|\sigma(\alpha) - \beta|, |\beta - \alpha|) \\ &= |\beta - \alpha| \\ &< |\alpha_j - \alpha|, \ \forall \alpha_j \neq \alpha. \end{aligned}$$

Thus $\sigma(\alpha) = \alpha$, and so $\alpha \in K(\beta)$ since it is fixed by G. □

8.2. **Proof of Theorem 8.2.**

Proof. Fix $\alpha \in \widehat{\mathbb{C}_p}$, nonzero. Let $f(x) \in \mathbb{C}_p[x]$ be the (monic) minimal polynomial of α, and α_j $(j = 1, \dots, n)$ be its roots. Let

$$M := \max(1, |\alpha|^n), \text{ and } m := \min_{i \neq j} |\alpha_i - \alpha_j|.$$

Choose a monic polynomial $g(x) \in \widehat{\mathbb{Q}_p}$ of degree n with all coefficients within $(m/2)^n/M$ (with respect to the field norm) of the corresponding coefficients of $f(x)$. This ensures that

$$|g(\alpha) - f(\alpha)| < \left(\frac{m}{2}\right)^n.$$

Let $\beta_1, \dots, \beta_n$ be the roots of $g(x)$, so that

$$g(x) = \prod_{j=1}^{n} (x - \beta_j).$$

Then

$$\prod_{j=1}^{n} |\alpha - \beta_j)| = |g(\alpha) - f(\alpha)| < \left(\frac{m}{2}\right)^n.$$

It follows that for some j we have $|\alpha - \beta_j| < m/2 < m$, so by Krasner's Lemma it follows that $\alpha \in K(\beta_j)$. Thus $\alpha \in \mathbb{C}_p$. □

8.3. **Spectra.** It is interesting that this proof of Theorem 8.2 is quite different from those we have seen of Theorem 1.1. It uses the extended norm on $\widehat{\mathbb{Q}_p}$. It raises the question whether Theorem 1.1 could be proved in the same way. In fact, if one could prove without assuming the fundamental theorem of algebra that the usual absolute-value norm extends from $\mathbb{C}$ to $\hat{\mathbb{C}}$, then one could deduce the fundamental theorem in various ways. For example, the norm would extend to the metric completion $\overline{\hat{\mathbb{C}}}$, which would then be a Banach algebra (a complete normed complex algebra) and a field, and the Gelfand-Mazur Theorem[1] then yields $\overline{\hat{\mathbb{C}}} = \mathbb{C}$.

Unfortunately, the usual proof of Gelfand-Mazur uses Liouville's Theorem, which may be applied more directly to prove the fundamental theorem. The key ingredient in the proof of the Gelfand-Mazur Theorem is that spectra are always nonempty, for elements of a Banach algebra with unit. The *spectrum* of an element f of a Banach algebra A is defined to be

$$\mathrm{spec}(f) := \{\lambda \in \mathbb{C} : f - \lambda 1 \text{ is noninvertible in } A\}.$$

Theorem 8.4. *Let A be a complete normed algebra with unit over $\mathbb{C}$. If $f \in A$, then $\mathrm{spec}(f) \neq \emptyset$.*

This theorem may be regarded as a *generalisation* of the fundamental theorem of algebra, because each monic complex polynomial of degree n is the characteristic polynomial of a companion $n\times n$ matrix, the set of all $n\times n$ complex matrices forms a Banach algebra, and the spectrum of a matrix in that algebra is the set of its eigenvalues.

Having written the account above, I had a look at Wikipedia [18], and found considerable overlap, along with a good deal of historical information. In particular, it seems that the first correct proof of the full theorem was given by Argand, in 1806, and not by Gauss, as folklore said.

9. Finding the roots

It is one thing to know the number of complex roots of a polynomial, counting multiplicities, but that doesn't butter any parsnips unless you can calculate them all to any desired accuracy. Ever since Galois, we know that this involves more than just the computation of k-th roots. Newton's method, the iteration of

$$q(z) := z - \frac{p(z)}{p'(z)},$$

will approximate any given root of $p(z)$, once you get close enough. For simple roots, it is phenomenally efficient, eventually doubling the number of significant figures at each step. But how can you ensure that you get close enough to each and every root? This problem was solved quite recently by applying twentieth-century advances in the theories of complex dynamical systems, topology, and conformal invariants. The initial breakthrough was made in the doctoral thesis of Scott Sutherland, and a refined and polished account is available in the paper of Hubbard, Schleicher and Sutherland [10]. This is a beautiful exposition of a stunning piece of work. It involves the classical Gauss-Lucas Theorem, a result of F. Riesz about radial limits, the Ahlfors conformal modulus and extremal length. The basic idea is that each root of $p(z)$ lies in a basin of attraction for $q(z)$ that contains a tentacle heading out to infinity, and each sufficiently-large circle meets each of these basins in a set that contains an interval that is not too small. So evenly-spaced points on the circle, provided the spacing is not too large, will do as a set of starting points for iterations of $q(z)$ that will converge to all the roots. In fact, the *same* evenly-spaced points will deliver all the roots of *all possible* polynomials $p(z)$ of a given degree d that have all their roots in the unit disc. By using some fast footwork, the number of starting points can be further reduced by placing them strategically on several circles instead of one. They give an explicit construction of approximately $0.2663 \log d$ circles, each containing $4.1627 d \log d$ points at equal distances.

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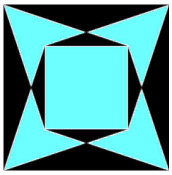

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Richard Susskind: How To Think About AI, Oxford University Press, 2025.

ISBN: 978-019-8941-92-7, GBP 9.75, 123+xiv pp.

REVIEWED BY ZHENWEI LYU

In a time when artificial intelligence is reshaping industries, education, and everyday life, this book stands out as a clear and thoughtful guide to what AI truly is, what it might become, and how we should respond. It is not just about how AI works today, but about how it could evolve over the coming years, and what that means for all of us – individuals, professionals, governments, and society as a whole. Drawing from every part of the book's argument, especially as seen in the conclusion, the author urges us to stop focusing only on tools like ChatGPT or image generators and instead think about where these technologies are leading us. This book is not meant just for AI experts. It is written in a way that anyone interested in the future – teachers, students, business leaders, or policymakers – can understand and learn from. It offers a roadmap for navigating the coming changes with clarity, responsibility, and deep reflection.

One of the book's most useful ideas is the distinction between "process-thinking" and "outcome-thinking". These two ways of thinking help make sense of many of the disagreements and debates people have about AI. Process-thinkers are those who focus on how AI systems work internally – data, algorithms, training models – while outcome-thinkers care more about what AI systems actually do in the real world – the results, the impact, the usefulness. The author shows how many arguments between experts happen because they are thinking in these different ways without realising it. For example, some thinkers worry that AI isn't "really" intelligent because it doesn't think like a human. Others respond that it doesn't matter how it works, as long as it gets the job done. The book doesn't say one view is right and the other is wrong. Instead, it helps us see that both are useful in different ways and that combining them gives us a clearer picture. This framework helps explain why big-name thinkers such as Noam Chomsky and Henry Kissinger often seem to disagree – they are coming from different "AI cultures".

In the middle chapters, especially Chapters 3 through 5, the author clears up many of the most common misunderstandings about AI. One of the strongest points made is about the "AI Fallacy" – the wrong belief that, for AI to be good at something, it must work like a human brain. This mistake leads people to underestimate machines that perform well but don't "think" like we do. The author argues that we should stop comparing AI to the human mind and instead judge it based on performance and outcomes. If a machine can solve complex problems, write useful reports, or help diagnose diseases – even if it does so in a completely different way from us – then it is still powerful and important. Another key idea is what the author calls "not-us thinking". This is when professionals assume that AI might replace or disrupt other people's jobs, but not their own. For example, some doctors and lawyers believe their work is too human, too personal, or too complex to be touched by AI. But the book explains that most clients don't care about the process – they care about the result.

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If an AI system can deliver the same or better results, people will use it, regardless of whether it works like a human or not.

In Chapter 5, the author looks at some of the abilities we often think only humans can have, such as judgement, empathy, and creativity. Many believe that machines will never be able to do these things. But instead of simply agreeing or disagreeing, the author introduces new terms: quasi-judgment, quasi-empathy, and quasi-creativity. These help us understand that AI doesn't need to feel emotions or have imagination in the human sense to perform similar roles. For example, a chatbot can give comforting replies to someone in distress. It may not "feel" the emotion, but the effect on the person can still be real. Similarly, AI can generate music, art, or stories that are creative in output, even if the process is different from how a human creates. The book argues that we need new words and new ways of thinking to describe what AI does. Just like the Industrial Revolution brought new vocabulary – such as "factories" and "middle class" – the AI era needs its own language. Without it, we are stuck using misleading terms such as "hallucination" to describe machine errors, which only adds confusion.

Chapters 6 and 7 shift the focus to how AI will change work. The author identifies three major effects: automation (doing current tasks faster or cheaper), innovation (creating new ways to do things), and elimination (removing certain tasks entirely). These changes will affect not just blue-collar jobs but also many white-collar professions. A powerful example in the book is about neurosurgeons. The author explains that patients don't really want a neurosurgeon – they want to be healthy. If AI can help deliver health more effectively, people will choose that, even if it means human surgeons become less necessary. This is a key point: people care about outcomes, not job titles. This part of the book is especially important for professionals who assume their work is safe from automation. It challenges all of us to rethink our value not in terms of what we do, but in terms of the results we deliver.

One of the most important topics in the book is the possibility of AGI—Artificial General Intelligence. This means AI that can do any intellectual task a human can do. The author doesn't say AGI is definitely coming soon, but suggests it could arrive between 2030 and 2035. This is not a prediction but a warning: we should prepare for this possibility, just in case. The author introduces a helpful way to think about this called "What-if-AGI?" thinking. This is a thought experiment where we imagine what the world would look like if AGI existed. What rules would we need? How would we share the wealth it creates? What jobs would be left for humans? How would we protect people's rights and dignity? Even if AGI never fully arrives, thinking this way helps us prepare for many changes that are already starting. It encourages long-term thinking in a world that usually only looks at the next big app or gadget.

Chapter 8 is one of the most practical and useful parts of the book. It lays out seven types of risks that AI brings – some short-term, some long-term, and all serious. These include political risks (such as misinformation), economic risks (such as job losses), psychological risks (such as identity confusion), and even existential risks (if AI becomes uncontrollable). The author doesn't try to scare the reader but helps us think clearly. This chapter gives readers a framework for understanding risk that is easy to apply and very needed in today's noisy debates. Chapter 9, titled "Harnessing AI," is a strong message that we are late to this conversation. Many of the issues we're facing now were already visible decades ago, but we failed to act. Now, AI systems are becoming more powerful, and we need smart rules, ethical planning, and collaboration across all parts of society.

Later chapters explore some of the biggest and hardest questions: Can machines ever be conscious? What does AI mean for the future of life itself? The author doesn't pretend to have all the answers, but offers deep, balanced thinking. He quotes philosophers including A.C. Grayling and Bryan Magee to help us understand that even human consciousness is still a mystery. So it's okay to be unsure about machine consciousness. One especially powerful idea is in Chapter 12, which discusses the "Great Schism" – a possible future where biological and artificial intelligences split into separate paths. This is not just science fiction. The book treats it as a real possibility and asks: What if our role as humans is to begin the next era of intelligence, even if it goes beyond us?

The conclusion of the book is one of its strongest parts. The author tells us not to get stuck thinking only about today's tools, such as GPT-4. We must use our imagination to think about GPT-7 and beyond. The systems of the future will likely be far more powerful than we expect. If we don't prepare now, we will be caught off guard. The book calls for a new kind of thinking – one that is brave, ethical, and forward-looking. AI is not just another invention. It is a change in how we think, work, live, and understand ourselves. We must be ready.

This book is clear, original, and deeply important. It avoids hype but doesn't downplay the risks. It is not too technical, but it is serious and smart. Most of all, it gives us the tools we need to think clearly about the future of AI. Whether you are a policymaker, teacher, doctor, engineer, student, or simply curious, this book will help you understand where we are headed and how to prepare. It offers a new way of thinking that we all need right now.

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Thomas Waters: The Four Corners of Mathematics, CRC Press, 2025. ISBN: 978-1-032-59498-9, GBP 19.99, 277 pp.

REVIEWED BY PETER LYNCH

This book presents a panoramic view of mathematics – pure and applied – spanning three millennia. It is clearly written with ample illustrations and diagrams and there is an excellent selection of topics, many of which may be new to readers. The book is in four Parts covering the fields of Geometry, Algebra, Calculus and Topology. Each Part comprises four chapters and each is about sixty pages in length. The high point of the book is the final Part, and I will focus most attention on that.

Part I deals with Geometry. The axiomatic approach, which originated in Ancient Greece, was used to great effect in Euclid's *Elements*. Some examples clarify the step-by-step procedure of this approach. The work of Apollonius on conics is reviewed. The development of geometry through the work of Islamic and Indian scholars is traced. The parallel postulate remained a contentious issue until finally it was shown to be inessential. Non-Euclidean geometry sprang independently and simultaneously from three sources, Lobachevsky, Gauss, and Bolyai, greatly broadening our understanding of the subject. Through his study of general curves and surfaces, Gauss initiated differential geometry. Chapter 3 gives an excellent presentation of how the line element of Gauss encapsulated the essence of geometry and led on to the n-dimensional manifolds of Riemann, so crucial for Einstein's later work. Fractals are treated in Chapter 4 in enough detail to enable readers to generate images never seen before.

The second Part is on Algebra. Various primitive number systems are described, ultimately displaced by the Hindu-Arabic numerals. Diophantus used symbols for numbers and introduced some key ideas that are of interest today, but the first systematic account of algebraic methods was that of al-Khwarizmi. This work eventually reached Europe, triggering a flurry of mathematical activity in Italy, where solutions of cubics and quartics were found. The intrigue and skulduggery accompanying these mathematical advances is recounted in the book. From this work there emerged complex numbers and, eventually, the Fundamental Theorem of Algebra. The relationship between the winding number and the roots of a polynomial is well described in the text.

Next comes an account of the difficulties in solving quintic equations, the findings of Abel and the tragic story of Galois, whose work gave rise to modern group theory. There follows an account of the fundamental group of a manifold, and the foundational work of Emmy Noether on abstract algebra.

Chapter 8 opens with a bold claim: "While the 19^{th} century was the century of Calculus and the 20^{th} the century of Topology, the 21^{st} century will be the century of Linear Algebra". The rationale for this is the rapidly increasing importance of Artificial Intelligence and the need to handle ever-larger data sets. Quaternions and the many developments following from Hamilton's discovery, are discussed. Several interesting applications illustrate the importance of the resulting mathematical advances.

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Part III is on Calculus, described in the opening sentence as "the greatest idea Mathematics ever had". Archimedes made some vital early contributions, building on ideas of Eudoxus. Two great problems passed down from Greece were to find areas bounded by curves and to determine tangent lines to curves. Descartes and Fermat made substantial inroads but it was the work of Leibniz and Newton that established the fundamental basis of the subject. The lack of rigour in handling infinitesimals, highlighted by Bishop Berkeley, was ultimately resolved in the 19$^{\text{th}}$ century. Two little letters, ϵ and δ, have struck terror in maths students' hearts ever since.

Chapter 10, on the Solar System, is an interesting whistle-stop tour from the Greeks, via Ptolemy, Copernicus and Kepler to Newton's *Principia*. The n-body problem is introduced, and the discoveries of Poincaré and his homoclinic tangle lead us into chaos. The chapter ends with a brief look at General Relativity.

Maxima and minima are considered in Chapter 11. Here we must consider functions of several variables, and partial derivatives are introduced. The problem of Johann Bernoulli, to find the curve on which a particle will slide to the lowest point in minimum time, was solved by Newton but, more importantly, it later inspired Euler to develop what he called the Calculus of Variations. Both Euler and Lagrange found the equations for a general solution of such problems – the Euler-Lagrange Equations – which are central in analytical dynamics. Poincaré's analysis of the 3-body problem is reviewed, as is the remarkable and delightful theorem of Emmy Noether that links mechanical invariants and symmetries. The chapter ends with a discussion of geodesics on a triaxial ellipsoid, a problem that was first solved by Jacobi.

Partial Differential Equations, which form the subject matter of Chapter 12, are "at the very heart of modern mathematics". The origins of the three classical PDEs – the wave equation, Laplace's equation and the heat equation – are treated. The solution of these stimulated profound mathematical developments. A fourth order PDE featured in the research of Monsieur Antoine Le Blanc, aka Sophie Germain, whose tale is told. The chapter ends with a return to General Relativity, the wave equation emerging from Einstein's Field Equations and the detection in 2015 of gravitational waves, which were triggered by an orbiting pair of black holes more than a billion years ago.

Part IV, on Topology, opens with the usual example of the Bridges of Königsberg, but moves quickly to an excellent discussion of the Gauss-Bonnet Theorem, which provides a strong connection between geometry and topology. A clear sketch gives a good idea of the proof of this beautiful result. Then the key topological equivalence relations, homotopy and homeomorphism, are introduced.

In Chapter 14, entitled "Degree", curves in the plane and in higher-dimensional spaces are discussed. The winding number and the rotation number are defined and used to classify plane curves. Then the Poincaré-Hopf Index Theorem is described and some surprising and delightful connections are made linking Euler's characteristic ($V - E + F$), the Hairy Ball Theorem and the Gauss-Bonnet Theorem. This sounds formidable, but the treatment in Waters' book is a model of lucid exposition. The chapter ends with the statement that "A hairy 18-sphere would have to have a 'tuft' somewhere, but we could comb a hairy 19-sphere flat". You cannot tell when you might need to know that!

Chapter 15 is on Homology or "using algebra to count holes". The going gets tougher but the author manages well to strike a good balance between clarity and rigour. The ideas here are not often found in a "popular maths" book and are all the more welcome for that. Betti numbers are defined and evaluated for the sphere and torus. The Euler-Poincaré characteristic is introduced and shown to be equal to the alternating sum of Betti numbers. This explains why the closing statement of the previous chapter must hold! Finally, simplicial complexes are introduced. Waters recalls a suggestion of Emmy Noether: to define the quotient group of the group of cycles by the group of cycles homologous to zero as the Betti Group. This led to the general formulation of homology groups and the emergence of Algebraic Topology as a major branch of maths.

Having warmly praised this worthy book, I have three minor grumbles. The title is poorly chosen; the branches of mathematics in the four parts are not *corners*, but *pillars* upon which rest many other results. The references are excellent but, with 137 entries, an additional short list of "highly-recommended" sources would be very helpful. The index is comprehensive, but many entries point to in excess of a dozen pages, plunging the reader into multi-dimensional space; these topics need to be sub-divided.

In summary, I can heartily recommend this well-researched and well-written book as a valuable and accessible introduction to some of the principal branches of mathematics. In my youth I read every maths book in Dun Laoghaire Public Library, and many more from elsewhere, but none compared, in quality or scope, to the book under review. I believe that it could be recommended to any young student hoping to embark on a mathematical career. He or she is sure to learn some new and delightful mathematical truths, and should thoroughly enjoy the process of discovery.

Peter Lynch is emeritus professor at UCD. He is interested in all areas of mathematics and its history. He writes a regular mathematics column in *The Irish Times* and has published three books in the *That's Maths* series. His new book, *AweSums: the Majesty and Magnificence of Mathematics*, is to be published by World Scientific in 2026. Peter's website is `https://maths.ucd.ie/~plynch` and his mathematical blog is at `http://thatsmaths.com`.

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PROBLEM PAGE

J.P. MCCARTHY

Problems

Thanks to all those who responded to Problem Page 95's call for more problems. However, *no* solutions to Problem Page 94 have been received: we will keep those problems open until Summer 2026.

The first of this edition's problems comes courtesy of Des MacHale of University College Cork.

Problem 96.1. Prove, using group theory, the following results in number theory:

(1) If m and n are natural numbers, then $m!n!$ divides $(m+n)!$.
(2) If p is prime and n a natural number, then $n!$ divides

$$(p^n - 1)(p^n - p)\cdots(p^n - p^{n-1}).$$

The second problem was sent in by Yagub N. Aliyev, of ADA University, Baku, Azerbaijan.

Problem 96.2. Let $a > 0$. Suppose that two distinct normals to the parabola $2y = ax^2$ intersect the parabola again at A. Prove that the y-coordinate of A is strictly greater than $4/a$.

Finally a problem from Finbarr Holland of University College Cork.

Problem 96.3. Where Γ is the gamma function, determine the limit

$$\lim_{p\to 0^+} \frac{1}{p^2}\left(1 - \frac{p\,\Gamma^2(p)}{2\,\Gamma(2p)}\right).$$

We invite readers to submit problems and solutions. Please email submissions to imsproblems@gmail.com in any format (preferably LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. If possible, please include solutions with your submissions.

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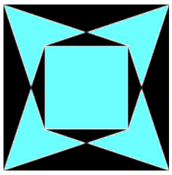

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