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Pierre Py: Lectures on Kähler Groups, Princeton University Press, 2025. ISBN:978-0-691-24715-1, USD 65.00, 382+xiii pp.

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Py's book is the first survey in almost thirty years on this exciting area of research at the intersection of algebraic geometry, Hodge theory, topology, analysis of harmonic maps, complex analysis, and geometric group theory. The heart of the book is a focus on relating morphisms from Kähler groups to holomorphic factorizations of maps of Kähler manifolds, often after covering, so perhaps maps of noncompact Kähler manifolds. This book would already be indispensable for its many appendices, which provide exceptionally clear introductions to the many strands of geometry that the book winds together, including ends of spaces, ends of groups, groups acting on trees, and harmonic maps. The reader is assumed fluent in differential, Riemannian and complex geometry, Hodge theory and Kähler geometry. This book would be difficult for a doctoral student, but is accessible to researchers in Kähler geometry.

Recall that a *Kähler manifold* is a complex manifold with a Riemannian metric which is compatible in the sense that (a) the metric is Hermitian for the complex structure, and (b) the parallel transport of the metric is complex linear. A compact complex manifold is a *smooth projective variety* if it admits a holomorphic embedding into a complex projective space. Kähler manifolds first arose in algebraic geometry: every smooth projective variety is a compact Kähler manifold. Compact Kähler manifolds satisfy various well known relations on their cohomology groups which are not satisfied by some compact complex manifolds. The original motivation to introduce Kähler manifolds was to clarify these relations, for application to smooth projective varieties. However in classical algebraic geometry, compact Kähler manifolds naturally arise which are not smooth projective varieties. Moreover, there are compact Kähler manifolds which cannot even be deformed to smooth projective varieties.

A group is finitely presented precisely when it is the fundamental group of a compact complex manifold [4], [6]. A *Kähler group* is a group arising as the fundamental group of a compact Kähler manifold. Serre proved that every finite group is a Kähler group, and asked for the classification of Kähler groups. The known constructions of Kähler groups are quite diverse, as the book explains.

A *projective group* is a group arising as the fundamental group of a smooth projective variety. Clearly every projective group is Kähler, but it is not known if there are Kähler groups which are not projective, perhaps the most important open question about Kähler groups. Surprisingly, this book does not survey the results on this question. As examples:

(1) Compact oriented surfaces arise as Riemann surfaces, so surface groups are Kähler.
(2) Finite groups are Kähler, after Serre.
(3) Products of Kähler groups are Kähler.

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(4) Any torus of even dimension admits the structure of an abelian variety, so free abelian groups of even rank are Kähler.
(5) Take any bounded symmetric domain in complex Euclidean space, for example the unit ball, and quotient by any torsion-free discrete cocompact group π of biholomorphisms. The quotient is a smooth projective variety with fundamental group π.

An obstruction: if π is the fundamental group of a compact Kähler manifold M then in group cohomology $H^1(\pi, \mathbb{R}) = H^1(M, \mathbb{R})$. This vector space has even dimension. To see this: each element of $H^1(M, \mathbb{R})$ is a harmonic 1-form. Hodge theory shows that each harmonic 1-form on a Kähler manifold is the real part of a holomorphic 1-form. Holomorphic 1-forms constitute a complex vector space, so $H^1(M, \mathbb{R})$ has even dimension. On the other hand, the Hopf surface $M = (\mathbb{C}^2 - \{0\})/(z \sim 2z)$ is a compact complex manifold with $\pi_1(M) = \mathbb{Z}$, so $H^1(M, \mathbb{R}) = \mathbb{R}$, so M is not Kähler.

The hard Lefschetz theorem yields more stringent restrictions on Kähler groups. The rational homotopy type of a compact Kähler manifold is a formal consequence of its cohomology ring, which restricts Massey products on the group cohomology of Kähler groups. Harmonic map theory restricts the actions of Kähler groups on symmetric spaces and trees [3]. One of the most important steps in the book: Py simplifies [3] and generalizes the result, simplifying the theory of linear fractional representations of Kähler groups. Every morphism from a Kähler group to a surface group arises from a holomorphic map of the Kähler manifold to a Riemann surface. A Kähler group acting on a tree factors through a holomorphic map to a Riemann surface, so from a surface group acting on the tree.

The book proves these results, with complete proofs, making use of extensive appendices on essential related material. Prior to this important and comprehensive book, the study of Kähler groups had been without a guide book for nearly 30 years [1], while crucial steps were made. For example, Delzant's 2010 work on BNS invariants for Kähler groups is explained clearly here.

In addition, the author proves a number of new results. For example, the Castelnuovo–de Franchis theorem which says that, on any compact Kähler manifold, any two linearly independent holomorphic 1-forms which wedge to zero are both pulled back by the same holomorphic map to a compact Riemann surface. The author proves a powerful L^2 generalization on noncompact Kähler manifolds, which plays a central role in the subsequent theory.

The author gives a clear introduction to the theory of ends of covering spaces of compact Kähler manifolds. He outlines nonabelian Hodge theory, and explains how Corlette and Simpson apply it to study linear fractional representations of Kähler groups, generalizing their approach to apply to compact Kähler manifolds.

Unfortunately, the author does not discuss the Shafarevich conjecture (that the universal covering space of a smooth projective variety is holomorphically convex). Nonetheless, the methods that are explained here are closely related to the progress that has been made on that conjecture.

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