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PROBLEM PAGE

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Problems

The first of this issue's problems comes courtesy of Des MacHale of University College Cork:

Problem 97.1. A prime number p is called *isolated* if $p-2$ and $p+2$ are both composite. For example, 37 is an isolated prime.

Show that there are infinitely many isolated primes.

The second problem was sent in by Seán M. Stewart of the King Abdullah University of Science and Technology, Saudi Arabia:

Problem 97.2. Evaluate

$$\int_0^1 \frac{x\,\tan^{-1}(x)}{\sqrt{1-x^2}\,(1+\sqrt{1-x^2})}\,dx.$$

Finally a problem from Finbarr Holland of University College Cork.

Problem 97.3. Let n be a positive integer, and denote by $H_n = [h_{ij}]$ the square matrix of order $n \times n$ where

$$h_{ij} = \frac{1}{\max(i,j)}, \qquad (i,j = 1,2\ldots,n).$$

Prove that

$$\det H_n = (n!)^{-2},$$

and determine H_n^{-1}.

I would like to thank Anthony G. O'Farrell for pointing out that Problem 71.3 has never been solved in these pages. Problems 80.2 and 94.3 share a similar status; however, unlike Problem 71.3, solutions of these were provided by the proposers. We invite readers to attempt all these problems: any solution to 71.3 will be published in Issue 99, while solutions of 80.2 and 94.3 will be published (one way or another) in Issue 98. These older problems may be accessed via `https://www.irishmathsoc.org/bulletin/index.php?file=byvol`.

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Solutions

Here are solutions to two problems from *Bulletin* Issue 94, all of which were solved by the North Kildare Problem Club (and the proposers). We provide the solutions of the NKPC. Problem 94.3 remains at large.

Problem 94.1. Let X, Y be independent and identically distributed random variables with values in a finite group G. Let $H < G$ be a subgroup such that $\mathbb{P}[X \in H] \in (1/2, 1)$. Prove that

$$\mathbb{P}[XY \in H] < \mathbb{P}[X \in H].$$

Solution 94.1. If we fix $x \in G$, then the map $y \mapsto xy$ is bijective. Because H is a subgroup, if x happens to be in H, then the image of H under this map is H, and because H is a subgroup we see that $xy \in H$ iff $y \in H$. Likewise for multiplication on the other side. So, we end up with a table:

	$x \in H$	$x \in G \setminus H$
$y \in H$	$xy \in H$	$xy \in G \setminus H$
$y \in G \setminus H$	$xy \in G \setminus H$	undetermined

If we let $p = \mathbb{P}[X \in H]$, then we see that the maximum probability for $\mathbb{P}[XY \in H]$ is $p^2 + (1-p)^2$, but $p > p^2 + (1-p)^2$ when $p \in (1/2, 1)$. □

Problem 94.2 was solved by the proposer Des MacHale, and North Kildare Problem Club. Again, we provide the solution of the NKPC:

Problem 94.2. If G is a group with centre Z and $|G/Z| = n!$, for some integer $n > 1$, show that G/Z is non-abelian.

Solution 94.2. Let's say that p is a *solo prime factor* of an integer $n \in \mathbb{N}$ if p is prime, p divides n, but p^2 does not divide n. Note that 2! has a solo prime factor $p = 2$. If $n > 2$ is an integer, then by Bertrand's postulate there is a prime p such that $n/2 < p < n$. Hence p divides $n!$, but no multiple $2p, 3p, \ldots$ occurs among $1, 2, \ldots, n$. Thus $p^2 \nmid n!$. Therefore $n!$ has a solo prime factor when $n > 1$.

Suppose G is a group with centre Z, and the order $|G/Z|$ is finite, and has a solo prime factor. We claim that G/Z is non-abelian.

Suppose, on the contrary, that G/Z is abelian. Then the commutator $[g, h] = g^{-1}h^{-1}gh$ belongs to Z whenever $g, h \in G$. Thus, for any $g_1, g_2, h \in G$, we have

$$\begin{aligned}
[g_1, h][g_2, h] &= g_1^{-1}h^{-1}g_1h[g_2, h] \\
&= g_1^{-1}[g_2, h]h^{-1}g_1h \\
&= g_1^{-1}g_2^{-1}h^{-1}g_2hh^{-1}g_1h \\
&= (g_2g_1)^{-1}h^{-1}(g_2g_1)h \\
&= [g_2g_1, h].
\end{aligned}$$

Hence, by induction,

$$[g, h]^m = [g^m, h],$$

whenever $g, h \in G$ and $m \in \mathbb{N}$. It follows that if $g^m \in Z$, then $[g, h]^m = 1$. Interchanging the rôles of g and h, we conclude that if $g^m \in Z$ and $h^r \in Z$, then

$$[g, h]^m = [g, h]^r = 1.$$

This tells us that if the order of gZ in G/Z is relatively prime to the order of hZ, then g commutes with h.

Choose a solo prime factor p of $|G/Z|$. Since G/Z is abelian, it is a product of cyclic groups whose orders have product $|G/Z|$, so it contains a cyclic group of order p. Choose $g \in G$ such that gZ has order p, i.e. $g \notin Z$ and $g^p \in Z$. Since G/Z is finite abelian and p occurs only once in $|G/Z|$, we may write

$$G/Z = \langle gZ \rangle \times K,$$

where gZ has order p and $|K| = |G/Z|/p$. Hence every $hZ \in G/Z$ has the form

$$hZ = (gZ)^s(kZ)$$

for some integer s and some $kZ \in K$. In particular, the order of kZ is relatively prime to p. Thus $h = g^s kz$ for some $z \in Z$, and it follows that h commutes with g. But this means that $g \in Z$, a contradiction. The claim is proven.

If we assume that G is finite, then the claim becomes an almost obvious consequence of some general facts: The assumption that G/Z is abelian implies that G is nilpotent, and each nilpotent finite group is the direct product of its Sylow subgroups. Then the centre Z of G is the internal direct product of the centres of the Sylow subgroups of G, and the quotient G/Z is the product of the corresponding quotients. But a p-group cannot have a centre of index p, so the order of G/Z cannot have a solo prime factor.

Example: let D be a dihedral group of order eight and let E be the group of 3×3 upper-triangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathrm{GF}(3)$, the field with three elements. Then with $G = D \times E$, we get $|Z| = 6$ and $|G/Z| = 2^2 \cdot 3^2$ and G/Z is abelian. This shows that the condition that $|G/Z|$ have no solo prime factor is essential. $\square$

Here are solutions to the problems from *Bulletin* Issue 95. Again, all solved by the North Kildare Problem Club (and the proposers).

Problem 95.1 was also solved by Yagub N. Aliyev, ADA University, Baku, Azerbaijan; Riccardo Della Martera; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; and Kee-Wai Lau, Hong Kong, China. We present the solution submitted by Kee-Wai Lau:

Problem 95.1. Suppose $p \geq 2$ is an integer, and

$$u_n = \frac{p^{pn}\,(n!)^p}{(pn)!n^{(p-1)/2}}, \qquad (n = 1, 2, \ldots).$$

Prove that $(u_n)_{n\geq 1}$ is a strictly decreasing sequence, and determine its limit.

Solution 95.1. For any real $x > 0$, let

$$f(x) = \sum_{k=1}^{p-1} \ln(px+k) - \frac{(p-1)(\ln(x(x+1)) + 2\ln p)}{2}.$$

Then

$$f'(x) = p\sum_{k=1}^{p-1} \frac{1}{px+k} - \frac{p-1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right), \text{ and}$$

$$f''(x) = \frac{p-1}{2}\left(\frac{1}{x^2} + \frac{1}{(x+1)^2}\right) - p^2\sum_{k=1}^{p-1} \frac{1}{(px+k)^2}.$$

Since

$$\frac{1}{(px+k)^2} < \frac{1}{(px+k)^2 - \frac{1}{4}} = \frac{1}{px+k-\frac{1}{2}} - \frac{1}{px+k+\frac{1}{2}},$$

so, telescoping:

$$\sum_{k=1}^{p-1} \frac{1}{(px+k)^2} < \frac{1}{px+\frac{1}{2}} - \frac{1}{px+p-\frac{1}{2}} = \frac{4(p-1)}{(2px+1)(2px+2p-1)}.$$

Therefore

$$\begin{aligned} f''(x) &> \frac{p-1}{2}\left(\frac{1}{x^2} + \frac{1}{(x+1)^2} - \frac{8p^2}{(2px+1)(2px+2p-1)}\right) \\ &= \frac{(p-1)\left[(4p^2+4p-2)x^2 + (4p^2+4p-2)x + 2p - 1\right]}{2x^2(x+1)^2(2px+1)(2px+2p-1)} > 0. \end{aligned}$$

Since $\lim_{x\to\infty} f'(x) = 0$, we have $f'(x) < 0$. Note that $e^{f(n)} = \frac{u_n}{u_{n+1}}$ and

$$\lim_{x\to\infty} e^{f(x)} = \lim_{x\to\infty} \frac{\prod_{k=1}^{p-1}(px+k)}{(x(x+1))^{(p-1)/2}p^{p-1}} = 1,$$

so $\lim_{x\to\infty} f(x) = 0$, and hence $f(x) > 0$.

It follows that $\frac{u_n}{u_{n+1}} = e^{f(n)}$ is greater than one, so that $(u_n)_{n\geq 1}$ is a strictly decreasing sequence. By using Stirling's approximation that $N! \sim \sqrt{2\pi}\, N^{N+1/2}e^{-N}$ as N tends to infinity, we readily obtain that

$$\lim_{n\to\infty} u_n = \frac{(2\pi)^{(p-1)/2}}{\sqrt{p}}. \qquad \square$$

Here is a solution of the NKPC:

Problem 95.2. Let $\mathcal{A}$ be a regular polytope in n-dimensional Euclidean space $\mathbb{E}^n$ with $n \geq 2$. Let O be the centroid of $\mathcal{A}$, and $\mathcal{S}$ a hypersphere in $\mathbb{E}^n$ with O in its interior. Let $\{A_i\}_{i\in I}$ be the set of vertices of $\mathcal{A}$. For all $i \in I$, say ray OA_i meets $\mathcal{S}$ at B_i, and the opposite ray of OA_i meets $\mathcal{S}$ at C_i.

Prove that

$$\sum_{i\in I} |OB_i| = \sum_{i\in I} |OC_i|.$$

Solution 95.2. Let $\mathcal{A}$ be any regular polytope in $\mathbb{E}^n$, having vertex set $\{A_i\}_{i\in I}$, and let O be the centroid of its vertices. Let $\mathcal{S}$ be the hypersphere with centre P and radius r and suppose O lies inside this sphere. For all $i \in I$, say ray OA_i meets $\mathcal{S}$ at B_i, and the opposite ray of OA_i meets $\mathcal{S}$ at C_i. Without loss of generality, we assume $|OA_i| = 1$ for each $i \in I$.

Consider the triangle PB_iC_i and its plane. The point O lies on the line $a_i := B_iC_i$. Let M_i be the midpoint of a_i.

Since B_iC_i is a chord of the hypersphere $\mathcal{S}$, the line PM_i is perpendicular to a_i. Hence OM_i is the projection of OP onto the line a_i. Taking directed lengths along a_i in the direction $\overrightarrow{OA_i}$, we therefore have

$$|OB_i| - |OC_i| = 2\,\overrightarrow{OP} \cdot \overrightarrow{OA_i}.$$

Equivalently,

$$|OC_i| - |OB_i| = 2\,\overrightarrow{PO} \cdot \overrightarrow{OA_i}.$$

Adding, we get

$$\sum_{i\in I}|OC_i| - \sum_{i\in I}|OB_i| = 2\,\overrightarrow{PO}\cdot\sum_{i\in I}\overrightarrow{OA_i}.$$

However this equals zero because O is the centroid of the vertices, so

$$\sum_{i\in I}\overrightarrow{OA_i} = \overrightarrow{0}.$$

Therefore

$$\sum_{i\in I}|OB_i| = \sum_{i\in I}|OC_i|. \qquad \square$$

Problem 95.3 was also solved by Kee-Wai Lau, Hong Kong, China. We give Kee-Wai Lau's solution:

*Problem 95.3***.** In ΔABC, show that:

$$\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}+\left(\frac{a}{c}+\frac{c}{a}\right)\cos^2\frac{B}{2}+\left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2}\le\frac{3}{2}\left(\frac{R}{r}+1\right)$$

Solution 95.3. Since

$$\begin{aligned}\cos^2\frac{A}{2} &= \frac{1+\cos A}{2} = \frac{(a+b+c)(b+c-a)}{4bc},\\ \cos^2\frac{B}{2} &= \frac{(a+b+c)(c+a-b)}{4ca},\\ \cos^2\frac{C}{2} &= \frac{(a+b+c)(a+b-c)}{4ab},\end{aligned}$$

we obtain, after simplification,

$$\begin{aligned}&\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}+\left(\frac{a}{c}+\frac{c}{a}\right)\cos^2\frac{B}{2}+\left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2}\\ &=\frac{a+b+c}{4}\left[\frac{(b^2+c^2)(b+c-a)}{b^2c^2}+\frac{(c^2+a^2)(c+a-b)}{c^2a^2}+\frac{(a^2+b^2)(a+b-c)}{a^2b^2}\right]\\ &=\frac{(a+b+c)(ab+bc+ca)}{2abc}. \qquad (1)\end{aligned}$$

Now let

$$s=\frac{a+b+c}{2}.$$

We need the following known results:

$$ab+bc+ca = s^2+4rR+r^2, \qquad (2)$$
$$abc = 4srR, \qquad (3)$$
$$s^2 \le 4R^2+4rR+3r^2, \qquad (4)$$
$$R \ge 2r. \qquad (5)$$

Identities (2) and (3) are well known. Inequality (4) appears in item 5.8 on p. 50 of the book by O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969. Inequality (5) is Euler's inequality.

By (1), (2), and (3), the left-hand side equals

$$\begin{aligned}\frac{(a+b+c)(ab+bc+ca)}{2abc} &= \frac{2s(ab+bc+ca)}{2abc}\\ &= \frac{s(ab+bc+ca)}{abc}\\ &= \frac{ab+bc+ca}{4rR}\\ &= \frac{s^2+4rR+r^2}{4rR}.\end{aligned}$$

Hence the required inequality is equivalent to

$$\frac{s^2+4rR+r^2}{4rR} \le \frac{3}{2}\left(\frac{R}{r}+1\right),$$

which is equivalent to

$$s^2 \le 6R^2 + 2rR - r^2. \tag{6}$$

Now the right-hand side of (6) can be written as

$$6R^2 + 2rR - r^2 = 4R^2 + 4rR + 3r^2 + 2(R-2r)(R+r).$$

By Euler's inequality (5), we have $R - 2r \ge 0$, and hence

$$2(R-2r)(R+r) \ge 0.$$

Therefore, by (4),

$$\begin{aligned}s^2 &\le 4R^2 + 4rR + 3r^2\\ &\le 4R^2 + 4rR + 3r^2 + 2(R-2r)(R+r)\\ &= 6R^2 + 2rR - r^2.\end{aligned}$$

Thus (6) holds, and consequently

$$\begin{aligned}&\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2} + \left(\frac{a}{c}+\frac{c}{a}\right)\cos^2\frac{B}{2} + \left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2}\\ &\quad \le \frac{3}{2}\left(\frac{R}{r}+1\right).\end{aligned}$$

This completes the proof. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (preferably raw LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by the end of October. The solution to a problem is published two issues after the issue in which the problem first appeared. If possible, please include solutions with your submissions.

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