Irish Math. Soc. Bulletin
Number 97, Summer 2026, 5–18
ISSN 0791-5578

From the Basel Problem to the Sandham–Au-Yeung Quadratic Series: An Arcsine Pathway

JAMIL ABREU AND ANA CLAUDIA PEREIRA

Abstract. We relate the Basel problem to two classical nested sums, known as Euler sums, through expansions of powers of the arcsine function. Using only elementary methods, we then show how one of these Euler sums leads to the evaluation of the celebrated Sandham–Au-Yeung quadratic series. Finally, we present an elementary approach to Stewart's three-term relation, namely, an identity relating the Sandham–Au-Yeung quadratic series with two linear sums of weight four.

1. Introduction

The Basel problem, first posed by the Italian mathematician Pietro Mengoli (1626–1686) in 1650, later highlighted by Jakob Bernoulli (1654–1705) in Basel in 1689, and finally solved by Leonhard Euler (1707–1783) in 1734, asks for the evaluation of the infinite sum

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}. \tag{1}$$

For a gentle presentation of Euler's original work, see Dunham [11, Ch. 3]. For a more serious treatment, see the chapter entitled "Euler's Solution of the Basel Problem — The Longer Story" in Bradley et al. [6], as well as Sandifer's analysis of Euler's original paper in [26, Ch. 21].

Euler's first argument, not entirely rigorous, was based on comparing two expressions for $(\sin x)/x$, one from the Maclaurin series of $\sin x$, the other from his now famous *infinite product formula for the sine function*, namely,

$$\frac{\sin x}{x}=\Big(1-\frac{x^2}{\pi^2}\Big)\Big(1-\frac{x^2}{4\pi^2}\Big)\Big(1-\frac{x^2}{9\pi^2}\Big)\Big(1-\frac{x^2}{16\pi^2}\Big)\cdots.$$

Later, Euler gave several other proofs using various methods. The quest for new solutions to the Basel problem continues to this day, with new proofs of (1) still appearing from time to time; see e.g. Campbell and Levrie [8].

In more recent years, another sum has attracted considerable attention, so much so that it is on its way to acquiring a status similar to that of the Basel problem in the mathematical subconscious. Discovered and rediscovered several times throughout the second half of the twentieth century, it appears to have been first proposed in 1948 by the Irish mathematician Henry F. Sandham (1917–1963) as a problem in *The American Mathematical Monthly* (see [25]), in the form

$$1+\Big(\frac{1+\frac{1}{2}}{2}\Big)^2+\Big(\frac{1+\frac{1}{2}+\frac{1}{3}}{3}\Big)^2+\Big(\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{4}\Big)^2+\cdots=\frac{17\pi^4}{360}. \tag{2}$$

A response came one and a half years later, with the German mathematician Martin Kneser (1928–2004) providing a solution based on a triple summation (see [19]).

2020 *Mathematics Subject Classification.* 11M06, 33B10, 40A25, 40C10.
Key words and phrases. Basel problem, arcsine power series, Euler sums, Sandham–Au-Yeung series.
Received on 31-03-2026, revised on 19-05-2026.
DOI: 10.33232/BIMS.0097.5.18.

Some decades later, it was numerically rediscovered in 1993 by Enrico Au-Yeung (then an undergraduate student in the University of Waterloo), based on computations of 500,000 terms (to five-digit accuracy), and was subsequently proved by Borwein and Borwein [3] using Fourier series. Between these two events, the sum appeared on other occasions, largely unnoticed. We refer the interested reader to Borwein and Borwein's article just cited, and especially Stewart [28] for further historical background.

There is a visible similarity between (1) and (2): both involve summing the inverses of squares of integers, the difference being that in the latter each term is weighted by the square of a *harmonic number*. It turns out that both series are connected in a more profound and mathematically structured way. This article is about this connection.

2. Revisiting some classical results

Let us begin by recalling the classical expansion

$$\arcsin x = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{x^{2n+1}}{2n+1}, \tag{3}$$

valid for $-1 \leqslant x \leqslant 1$. In fact, convergence for $-1 < x < 1$ is inherited from the binomial series, wherefrom (3) can be obtained by integration, and convergence at ± 1 can be inferred from Abel's limit theorem, cf. Knopp [20, p. 177]. The binomial coefficient appearing in (3) is the so-called *central binomial coefficient*:

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = 4^n \cdot \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}.$$

Thus,

$$\frac{\binom{2n}{n}}{4^n} = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \quad \text{and} \quad \frac{4^n}{\binom{2n}{n}(2n+1)} = \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)}.$$

Next in the line is

$$\frac{(\arcsin x)^2}{2!} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{x^{2n}}{(2n)^2} \quad \left[= \sum_{n=0}^{\infty} \frac{4^n}{\binom{2n}{n}(2n+1)} \frac{x^{2n+2}}{2n+2} \right]. \tag{4}$$

Of course, (4) can be obtained by writing

$$\frac{(\arcsin x)^2}{2!} = \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}}\, dt$$

and using the classical expansion

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{x^{2n+1}}{2n+1}. \tag{5}$$

This, in turn, can be derived in many ways; see, for example, Spiegel [27] and Lehmer [21]. Lehmer's article, in particular, presents many applications of (5) that still resonate today.

One natural question is what happens if we make the change of variables $x = \sin\theta$ and then integrate these identities over the interval $[0, \pi/2]$. To pursue this approach, we will need the Wallis integrals

$$\int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta = \int_0^{\pi/2} \sin^{2n+1}\theta \, d\theta = \frac{4^n}{\binom{2n}{n}(2n+1)} \tag{6}$$

and

$$\int_0^{\pi/2} \cos^{2n}\theta \, d\theta = \int_0^{\pi/2} \sin^{2n}\theta \, d\theta = \frac{\binom{2n}{n}}{4^n} \frac{\pi}{2}. \tag{7}$$

Example 2.1. The substitution $x = \sin\theta$ into (3) yields

$$\theta = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{\sin^{2n+1}\theta}{2n+1}.$$

By integrating over θ from 0 to $\pi/2$ and using (6), we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

This is the solution of Choe [9], and Kimble [18], to the Basel problem. △

Example 2.2. The substitution $x = \sin\theta$ into (4) yields

$$\frac{\theta^2}{2} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{\sin^{2n}\theta}{(2n)^2}.$$

By integrating over θ from 0 to $\pi/2$ and using (7), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is Spiegel's solution [27] to the Basel problem. △

Going back to the expansions of the arcsine function, less known than both (3) and (4) are

$$\frac{(\arcsin x)^3}{3!} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big] \frac{x^{2n+1}}{2n+1}, \tag{8}$$

(the sum inside brackets is often denoted by $h_n^{(2)}$ or $O_n^{(2)}$, and called the n-th odd generalized harmonic number of order 2), as well as

$$\frac{(\arcsin x)^4}{4!} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}(2n+1)} \Big[\sum_{k=1}^{n} \frac{1}{(2k)^2}\Big] \frac{x^{2n+2}}{2n+2}. \tag{9}$$

The sum inside brackets equals $\frac{1}{4}H_n^{(2)}$, where $H_n^{(2)} = \sum_{k=1}^{n} 1/k^2$ is the n-th generalized harmonic number of order 2.

These expansions of the powers of $\arcsin x$ have an interesting history of their own. Euler was likely aware of them since the expansion (5) appears explicitly in his 1755 *Institutiones Calculi Differentialis* and, later, in his 1768 *Institutiones Calculi Integralis* he develops the expansion of $y = e^{a \arcsin x}$, which naturally leads to the arcsine expansions. This method found a place in the classic 1886 differential calculus text by Joseph Edwards, where it appears as a problem with a clearly delineated solution in the cases of $(\arcsin x)^{2,3}$. Srinivasa Ramanujan (1887–1920) seems to have been interested in these matters, as evidenced by Berndt [2, Ch. 9, Prop. 15], where the case of $(\arcsin x)^4$ is treated in detail. More recently, Muzaffar [23], Borwein and Chamberland [5], and Nimbran et al. [24], among others, have revisited these expansions and made extensive use of them in the investigation of various *Euler sums*; see the next section for an explanation of this notion.

Here is how Euler's method works in a nutshell: by writing

$$y = e^{a \arcsin x} = \sum_{k=0}^{\infty} a_k x^k, \tag{10}$$

one can first show that

$$(1-x^2)y'' - xy' - a^2 y = 0. \tag{11}$$

Inserting the powers series into the differential equation, it is then a routine matter to find a recursive relation between a_{k+2} and a_k; clearly, $a_0 = 1$ and $a_1 = a$ (see the Appendix below). Then, by expanding $e^{a \arcsin x}$ in powers of a and equating coefficients of a^N on both sides, one deduces the expansion of $(\arcsin x)^N$ for each $N \geqslant 1$. We provide more details in the Appendix.

We may now ask the same question as before: what happens if we make the change of variables $x = \sin\theta$ and then integrate (8) and (9) over the interval $[0, \pi/2]$? This question is addressed in the next section.

3. Beyond Basel: the arcsine gateway to an Euler sum

Let us introduce some notation. The *Riemann zeta function* is defined for $p > 1$ by

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Thus, Euler's identity (1) is simply $\zeta(2) = \pi^2/6$. Clearly, $\zeta(1)$ is the harmonic series, which is well known to diverge. Its partial sums, which appear in the numerators of (2), are denoted by

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

H_n is called the nth harmonic number; it is often convenient to put $H_0 = 0$.

It turns out that, for the expansions (8) and (9), the same procedure as employed in Examples 2.1 and 2.2 leads to more interesting results (in the sense of being less familiar to a broader audience). We therefore state these results as propositions rather than examples.

We will need the following classical identity

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^p} = \Big(1 - \frac{1}{2^p}\Big)\zeta(p), \tag{12}$$

which can be obtained by splitting $\sum_{n=1}^{\infty} n^{-p}$ into even and odd indices.

Proposition 3.1.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\Big[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2}\Big] = \frac{5\pi^4}{384}.$$

Proof. The substitution $x = \sin\theta$ into (8) yields

$$\frac{\theta^3}{3!} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n}\Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big]\frac{\sin^{2n+1}\theta}{2n+1}.$$

By integrating over θ from 0 to $\pi/2$ and using (6), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}\Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big] = \frac{\pi^4}{4!2^4}. \tag{13}$$

The sum on the left equals

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}\Big[\sum_{k=1}^{n+1} \frac{1}{(2k-1)^2}\Big] - \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4}$$
$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big] - \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.$$

By (12), and using $\zeta(4) = \pi^4/90$, the rightmost sum equals $\pi^4/96$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big] = \frac{5\pi^4}{384}. \qquad \square$$

We believe that Proposition 3.1 is known, although we were unable to determine where it may have first appeared. Identity (13) was observed in Nimbran et al. [24] (using essentially the same argument), but those authors did not carry the evaluation through to the form stated here. Li and Chu [22] investigated numerous sums via integration; in particular they evaluated many sums involving odd harmonic numbers, including the following alternating sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\Big[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2}\Big] = \frac{7}{4}\zeta(3) - \frac{\pi G}{2},$$

where G is Catalan's constant. Leaving aside the question of originality, our interest here lies instead in what emerges from the expansion of $(\arcsin x)^4$.

Proposition 3.2.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\Big(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\Big) = \frac{7\pi^4}{360}.$$

Proof. The substitution $x = \sin\theta$ in (9) gives

$$\frac{\theta^4}{4!} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}(2n+1)}\Big[\sum_{k=1}^{n} \frac{1}{(2k)^2}\Big]\frac{\sin^{2n+2}\theta}{2n+2}. \tag{14}$$

By integrating both sides over θ from 0 to $\pi/2$ and using the Wallis integral (see (7))

$$\int_0^{\pi/2} \sin^{2n+2}\theta \, d\theta = \frac{(2n+2)!}{4^{n+1}[(n+1)!]^2}\frac{\pi}{2} = \frac{(2n+2)(2n+1)}{(2n+2)^2}\frac{\binom{2n}{n}}{4^n}\frac{\pi}{2},$$

we obtain

$$\sum_{n=1}^{\infty} \frac{1}{(2n+2)^2}\Big[\sum_{k=1}^{n} \frac{1}{(2k)^2}\Big] = \frac{\pi^4}{5!2^4},$$

or, after multiplying both sides by 2^4,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}\Big[\sum_{k=1}^{n} \frac{1}{k^2}\Big] = \frac{\pi^4}{120}.$$

As in the end of the proof of Proposition 3.1, the left-hand side equals

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}\Big[\sum_{k=1}^{n+1} \frac{1}{k^2}\Big] - \sum_{n=1}^{\infty} \frac{1}{(n+1)^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}\Big[\sum_{k=1}^{n} \frac{1}{k^2}\Big] - \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

The right most sum is $\zeta(4) = \pi^4/90$, so that the above combined with the previous identity yields

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\Big[\sum_{k=1}^{n} \frac{1}{k^2}\Big] = \frac{7\pi^4}{360}. \qquad \square$$

The sum in Proposition 3.2 is classical. It belongs to a family of nested sums studied by Euler in the 1770's, nowadays called *Euler sums.* Its "proof from the book" seems to consist simply of substituting $p = 2$ into the folklore identity (attributed to Euler)

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^p} = \frac{1}{2}(\zeta(p)^2 + \zeta(2p)),$$

valid for all $p > 1$, and then using $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$; see Vălean [30, pp. 384–385] for a related, more general result. Above,

$$H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p}$$

is called the *n-th generalized harmonic number of order p*. Actually, Euler investigated general sums of the form

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} \quad \Big[= \sum_{n=1}^{\infty} \frac{1}{n^q} \sum_{k=1}^{n} \frac{1}{k^p} \Big],$$

a problem already posed by Christian Goldbach (1690–1764) in his correspondence with Euler dating back to 1742. Part of this story is recounted in Borwein and Bradley [4], who also collected thirty-two proofs of the simplest Euler sum, namely,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).$$

A more recent proof of Proposition 3.2 has been given by Vălean [29], using Abel's summation formula. In his article, Vălean's purpose was actually to provide a new derivation of the Sandham–Au-Yeung quadratic series

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n} \right)^2 = \frac{17}{4} \zeta(4).$$

In the remainder of this article, drawing on earlier contributions in the literature, we show how Proposition 3.2 relates to the above as well as to another classical Euler sum.

4. Intermezzo: mathematical tools

Later in this article, we will need the integration by parts formula

$$\int_0^1 x^{n-1} \ln^k x \, dx = \frac{(-1)^k k!}{n^{k+1}} \quad (n = 1, 2, 3 \ldots; k = 0, 1, 2, \ldots). \tag{15}$$

This is a classical and well-known formula, though the reader may still find an interesting treatment of it in Vălean [30, pp. 57–58]. By rearranging and summing over k, then interchanging summation with integration, and summing a geometric series, it follows that

$$\zeta(k+1) = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} = \frac{(-1)^k}{k!} \sum_{n=1}^{\infty} \int_0^1 x^{n-1} \ln^k x \, dx = \frac{(-1)^k}{k!} \int_0^1 \frac{\ln^k x}{1-x} \, dx.$$

Moreover, replacing x by $1 - x$ yields

$$\zeta(k+1) = \frac{(-1)^k}{k!} \int_0^1 \frac{\ln^k (1-x)}{x} \, dx. \tag{16}$$

We will also need two important tools, which we state in the next two lemmas.

Lemma 4.1. *For all integers $n \geqslant 1$,*

$$\int_0^1 x^{n-1} \ln(1-x) \, dx = -\frac{H_n}{n} \tag{17}$$

and

$$\int_0^1 x^{n-1} \ln^2(1-x) \, dx = \frac{2}{n} \sum_{k=1}^{n} \frac{H_k}{k}. \tag{18}$$

Proof. By noting that $nx^{n-1} = (d/dx)(x^n - 1)$, integration by parts yields

$$\begin{aligned}\int_0^1 x^{n-1}\ln(1-x)\,dx &= \frac{x^n-1}{n}\ln(1-x)\Big|_0^1 - \frac{1}{n}\int_0^1 \frac{1-x^n}{1-x}\,dx \\ &= -\frac{1}{n}\int_0^1 (1+x+x^2+\cdots+x^{n-1})\,dx \\ &= -\frac{H_n}{n}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^1 x^{n-1}\ln^2(1-x)\,dx &= \frac{x^n-1}{n}\ln^2(1-x)\Big|_0^1 - \frac{2}{n}\int_0^1 \frac{1-x^n}{1-x}\ln(1-x)\,dx \\ &= -\frac{2}{n}\sum_{k=1}^n \int_0^1 x^{k-1}\ln(1-x)\,dx \\ &= \frac{2}{n}\sum_{k=1}^n \frac{H_k}{k},\end{aligned}$$

where the last identity follows from the first part. □

Both identities in Lemma 4.1 are often used in working with Euler sums. They are often proved via double integration; see, for instance, Vălean [29] or Stewart [28]. As for the integration by parts argument used above, we initially thought it was new, but later found that it already appears in Vălean [30, pp. 59–60]. The identity (17) is old, going back at least to 1867, appearing for instance as Problem 1031 on Page 214 of Wolstenholme's problem book [32]; in fact, it is presented there in the form

$$\int_0^1 (1-x)^{n-1}\log\left(\frac{1}{x}\right)dx = \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

As for identity (18), we did not find it in Wolstenholme's book after a brief search, and we do not know when it first appeared in the literature.

Next, we present an auxiliary result of independent interest. It is well known to experts and admits many different proofs. For instance, Vălean [29] derives it using Abel's summation-by-parts formula. We include a simple proof based on induction.

Lemma 4.2. *For all* $n = 1, 2, \ldots,$

$$H_n^2 + H_n^{(2)} = 2\sum_{k=1}^n \frac{H_k}{k}.$$

Proof. The identity is clear for $n = 1$, both sides being equal to 2. Assuming it to hold for $n-1$, we have

$$\begin{aligned}H_n^2 + H_n^{(2)} &= \left(H_{n-1} + \frac{1}{n}\right)^2 + H_{n-1}^{(2)} + \frac{1}{n^2} \\ &= H_{n-1}^2 + H_{n-1}^{(2)} + 2\frac{H_{n-1}+1/n}{n} \\ &= 2\sum_{k=1}^{n-1}\frac{H_k}{k} + \frac{2H_n}{n} \\ &= 2\sum_{k=1}^{n}\frac{H_k}{k},\end{aligned}$$

which concludes the induction step. □

Finally, we assume the reader to be familiar with the Cauchy product of power series

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)x^n.$$

For instance, the Cauchy product $(\sum_{n=1}^{\infty} x^n/n)(\sum_{n=0}^{\infty} x^n)$ of the logarithmic series for $-\ln(1-x)$ with the geometric series yields the classical expansion

$$-\frac{\ln(1-x)}{1-x} = \sum_{n=1}^{\infty} H_n x^n. \tag{19}$$

From this, and writing

$$\ln^2(1-x) = -2\int_0^x \frac{\ln(1-t)}{1-t}\,dt,$$

we deduce the logarithmic series

$$\ln^2(1-x) = 2\sum_{n=1}^{\infty} \frac{H_n}{n+1} x^{n+1}, \tag{20}$$

valid for $-1 \leqslant x < 1$; see Bromwich [7, §62] for an alternative argument.

5. The Sandham–Au-Yeung quadratic series and a close cousin

Using (16) with $k = 3$, and then the logarithmic series, yields

$$\begin{aligned} 6\zeta(4) &= -\int_0^1 \frac{\ln^3(1-x)}{x}\,dx \\ &= \sum_{n=1}^{\infty}\int_0^1 \frac{x^{n-1}}{n}\ln^2(1-x)\,dx \\ &= \sum_{n=1}^{\infty}\frac{2}{n^2}\sum_{k=1}^{n}\frac{H_k}{k}, && \text{(by (18))} \end{aligned}$$

or, by Lemma 4.2,

$$6\zeta(4) = \sum_{n=1}^{\infty}\frac{H_n^2}{n^2} + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}. \tag{21}$$

The above argument captures the essential idea in Vălean [29], although it does not appear there in exactly this form. Having previously derived the second term (Proposition 3.2 above) using Abel's summation formula, he obtained the following result, which was the main goal of his article.

Proposition 5.1 (Sandham–Au-Yeung quadratic series)**.**

$$\sum_{n=1}^{\infty}\Big(\frac{H_n}{n}\Big)^2 = \frac{17}{4}\zeta(4) \quad \Big[= \frac{17\pi^4}{360}\Big].$$

Using (20), Stewart [28] argued that

$$\begin{aligned} 6\zeta(4) &= -\int_0^1 \frac{\ln^3(1-x)}{x}\,dx \\ &= -2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx \\ &= 2\sum_{n=1}^{\infty}\frac{H_{n-1}H_n}{n^2}, && \text{(by (17))} \end{aligned}$$

so that, using $H_{n-1} = H_n - 1/n$,

$$3\zeta(4) = \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} - \sum_{n=1}^{\infty} \frac{H_n}{n^3}. \tag{22}$$

This, combined with Proposition 5.1, yields:

Proposition 5.2.

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{\pi^4}{72}.$$

The sum in Proposition 5.2 is classical, its "proof from the book" consisting of putting $q = 3$ in the identity

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q} = \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{n=1}^{q-2} \zeta(n+1)\zeta(q-n). \tag{23}$$

In fact, putting $q = 3$ in the above yields

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{2}\zeta(4) - \frac{1}{2}\zeta(2)^2 = \frac{\pi^4}{72},$$

where we again used the well-known values $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. Identity (23) was given by Euler (without proof) in 1776; for a comprehensive discussion of Euler's original work, see Harada [17]. Basu and Apostol [1] derived this identity by a careful analysis of some general telescoping properties of infinite series, and Furdui and Sîntămărian [15] found a calculus-based proof of it. Among the most classical, well-known references to both results (Propositions 5.1 and 5.2) is Borwein and Borwein [3], who derived them from the integral formula (obtained via contour integration)

$$\frac{1}{\pi}\int_0^{\pi} \theta^2 \ln^2\left(2\cos\frac{\theta}{2}\right) d\theta = \frac{11\pi^4}{180} \quad \left[= \frac{11}{2}\zeta(4)\right].$$

As with Vălean's identity (21), Stewart originally conceived (22) as a means of deriving the Sandham–Au-Yeung series, taking Proposition 5.2 as given via its "proof from the book". In fact, from a mathematical research point of view, linear Euler sums — such as those on the left-hand side of (23) — are well understood in view of Euler's identity, whereas nonlinear Euler sums, such as the Sandham–Au-Yeung series, are the true "targets" we seek to evaluate. Stewart even ventured to say that (22) is "perhaps the simplest approach" to the Sandham–Au-Yeung series. Since there are many proofs of Proposition 5.2 — possibly simpler than the one presented here using (23) — such as the one given in Furdui and Sîntămărian [14], we believe Stewart has a point.

Later in his article, and using dilogarithms, Stewart derived the three terms relation

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = 2\sum_{n=1}^{\infty} \frac{H_n}{n^3} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2}. \tag{24}$$

This is a curious identity indeed ("intriguing", as Stewart wrote), but it is admittedly weaker than (21) and (22) taken together, since it requires the evaluation of two series to yield a third. In particular, the results in Propositions 3.2 and 5.2, taken together, yield the Sandham–Au-Yeung series. Nevertheless, we believe there is room for an alternative, more elementary approach to this relation.

6. An elementary approach to Stewart's identity

We can derive Stewart's identity (24) in an elementary way (meaning that no dilogarithms are involved) by computing the integral

$$\int_0^1 \frac{\ln^2(1-x)\ln x}{(1-x)x}\,dx \tag{25}$$

in two different ways. For this, we will also need the expansion

$$\frac{\ln^2(1-x)}{1-x} = \sum_{n=1}^{\infty}[H_n^2 - H_n^{(2)}]x^n. \tag{26}$$

This can be obtained as a Cauchy product. In fact, using (20) and the geometric series,

$$\frac{\ln^2(1-x)}{1-x} = \Big(2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}x^n\Big)\Big(\sum_{n=0}^{\infty}x^n\Big) = 2\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\frac{H_{k-1}}{k}\Big)x^n.$$

Then, (26) follows from the identity

$$H_n^2 - H_n^{(2)} = 2\sum_{k=1}^{n}\frac{H_{k-1}}{k},$$

which can be proved by induction or obtained directly from Lemma 4.2.

Now, on the one hand, by using (26) and (15) (with $k=1$), the integral (25) equals (after interchanging summation with integration)

$$\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})\int_0^1 x^{n-1}\ln x\,dx = -\sum_{n=1}^{\infty}\frac{H_n^2 - H_n^{(2)}}{n^2}.$$

On the other hand, changing x to $1-x$ and then using (19) and (15) (with $k=2$), the integral (25) also equals

$$\int_0^1 \frac{\ln^2 x\ln(1-x)}{x(1-x)}\,dx = -\sum_{n=1}^{\infty}H_n\int_0^1 x^{n-1}\ln^2 x\,dx = -2\sum_{n=1}^{\infty}\frac{H_n}{n^3}.$$

Stewart's identity (22) follows by equating the rightmost sums of the two previous identities.

We arrived at the above derivation some days before discovering Stewart's article. At the time, we were more engaged with the paper of Li and Chu [22], as well as with the pioneering work of P. J. De Doelder [10], where similar logarithmic integrals appear. In retrospect, this may have been fortunate, for we can only wonder whether we would have pursued the matter so persistently had we encountered Stewart's paper earlier.

7. On De Doelder's article [10]

Many results on Euler sums were anticipated by De Doelder [10], who expressed his findings in terms of the ψ function (the logarithmic derivative of the Gamma function, usually referred to as the *digamma function*). De Doelder begins his paper with

> In a protracted correspondence with J. Gilles (Belgium) on [3][1] we discussed a lot of integrals related with dilogarithms and polylogarithms. In this correspondence also series containing $\psi(k)-\psi(1)$ appeared, where $\psi(x) = d/dx\log\Gamma(x)$ and $k\in\mathbb{N}$.

[1]This refers to a bibliographical entry in [10], namely, Lewin's book *Polylogarithms and Associated Functions* (North-Holland, 1981).

The relation between ψ and the harmonic numbers is $\psi(n) = H_{n-1} - \gamma$, where γ denotes the Euler-Mascheroni constant, hence $\psi(n) - \psi(1) = H_{n-1}$; see e.g. Whittaker and Watson [31, p. 254, Ex. 16]. Note then that (17) can be written as

$$\int_0^1 x^{n-1} \ln(1-x)\, dx = -\frac{\psi(n+1) - \psi(1)}{n} = -\frac{\psi(n+1) + \gamma}{n},$$

which is how it appears, for instance, in Gradshteyn and Ryzhik [16, Entry (4.293-8)].

We find, for instance, that Proposition 5.2 above is essentially De Doelder's identity (7), namely,

$$\sum_{k=1}^{\infty} \frac{\psi(k) - \psi(1)}{k^3} = \frac{\pi^4}{360}.$$

This is used two pages later to derive the Sandham–Au-Yeung series — see De Doelder's identity (9) — in the form

$$\sum_{k=1}^{\infty} \frac{(\psi(k) - \psi(1))^2}{k^2} = \frac{11\pi^4}{360}.$$

Both results, and others, are obtained via generating functions of various sequences involving $\psi(k) - \psi(1)$, expressed in terms of certain logarithmic integrals.

It is now widely acknowledged that De Doelder was aware of neither the previous evaluations of the Sandham–Au-Yeung series nor Euler's identity (23). Despite the fact that the term "Euler sum" (having been coined some three years later) does not appear in De Doelder's article, we venture to say that it is among the best starting points for learning about the subject, particularly as far as generating functions and integration methods are concerned.

Appendix: finding $(\arcsin x)^N$ with Euler's method

The power series method to obtain the expansions of $(\arcsin x)^N$ was originally outlined by Euler [13, Ch. III, § 186], and later reformulated by Edwards [12, pp. 87–90], with a brief appearance in Bromwich [7, p. 188, Ex. 1]. In this appendix, we provide a brief description of the cases $N = 1, 2, 3, 4$ based on Berndt's exposition in [2, p. 263]; in this respect, an alternative title to the present appendix might be "A friendly presentation of the Ramanujan-Berndt approach to the expansion of $(\arcsin x)^{1,2,3,4}$".

Firstly, if $y = e^{a \arcsin x}$ then

$$y' = \frac{ay}{\sqrt{1-x^2}}$$

and

$$y'' = \frac{ay}{\sqrt{1-x^2}} + \frac{axy}{(1-x^2)\sqrt{1-x^2}}.$$

Thus,

$$\begin{aligned}(1-x^2)y'' &= a\sqrt{1-x^2}\, y' + \frac{axy}{\sqrt{1-x^2}} \\ &= a^2 y + xy',\end{aligned}$$

which is the differential equation (11). Now, by inserting the power series (10) into the differential equation (11), we obtain the recursive relation

$$a_{k+2} = \frac{k^2 + a^2}{(k+2)(k+1)} a_k \quad (k = 0, 1, 2, \dots).$$

See Berndt [2, p. 263] for some additional details on this point. By induction, it is not difficult to see that

$$a_{2n+1} = \frac{a(a^2+1)(a^2+3^2)\cdots(a^2+(2n-1)^2)}{(2n+1)!}, \quad n = 1, 2, 3, \ldots$$
$$a_{2n} = \frac{a^2(a^2+2^2)(a^2+4^2)\cdots(a^2+(2n-2)^2)}{(2n)!}, \quad n = 1, 2, 3, \ldots.$$

Since $y = y(x) = e^{a \arcsin x}$ clearly satisfies $y(0) = 1$ and $y'(0) = a$, we see that $a_0 = 1$ and $a_1 = a$. Thus, on the one hand,

$$\begin{aligned} y &= e^{a \arcsin x} \\ &= 1 + ax + \frac{a^2}{2!}x^2 + \frac{a(a^2+1^2)}{3!}x^3 + \frac{a^2(a^2+2^2)}{4!}x^4 \\ &\quad + \frac{a(a^2+1^2)(a^2+3^2)}{5!}x^5 + \frac{a^2(a^2+2^2)(a^2+4^2)}{6!}x^6 \\ &\qquad + \frac{a(a^2+1^2)(a^2+3^2)(a^2+5^2)}{7!}x^7 + \frac{a^2(a^2+2^2)(a^2+4^2)(a^2+6^2)}{8!}x^8 + \cdots \end{aligned}$$

On the other hand, expanding the exponential $y = e^{a \arcsin x}$ as a power series in a yields

$$y = 1 + (\arcsin x)a + \frac{(\arcsin x)^2}{2!}a^2 + \frac{(\arcsin x)^3}{3!}a^3 + \frac{(\arcsin x)^4}{4!}a^4 + \cdots.$$

Now, we compare the coefficients in the powers of a in both sides. Clearly, the coefficient of a is

$$\begin{aligned} \arcsin x &= x + \frac{1}{3!}x^3 + \frac{1^2 \cdot 3^2}{5!}x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + \cdots \\ &= \frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots. \end{aligned}$$

Moreover, the coefficient in a^2 is

$$\begin{aligned} \frac{(\arcsin x)^2}{2!} &= \frac{x^2}{2!} + \frac{2^2}{4!}x^4 + \frac{2^2 \cdot 4^2}{6!}x^6 + \frac{2^2 \cdot 4^2 \cdot 6^2}{8!}x^8 + \cdots \\ &= \frac{x^2}{2} + \frac{2}{3}\frac{x^4}{4} + \frac{2 \cdot 4}{3 \cdot 5}\frac{x^6}{6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\frac{x^8}{8} + \cdots, \end{aligned}$$

whilst the one in a^3 is

$$\begin{aligned} \frac{(\arcsin x)^3}{3!} &= \frac{x^3}{3!} + \frac{1^2+3^2}{5!}x^5 + \frac{3^2 \cdot 5^2 + 1^2 \cdot 5^2 + 1^2 \cdot 3^2}{7!}x^7 + \cdots \\ &= \frac{1}{2}\left[\frac{1}{1^2}\right]\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\left[\frac{1}{1^2} + \frac{1}{3^2}\right]\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right]\frac{x^7}{7} + \cdots. \end{aligned}$$

Finally, we can also see that the coefficient of a^4 is

$$\begin{aligned} \frac{(\arcsin x)^4}{4!} &= \frac{x^4}{4!} + \frac{2^2+4^2}{6!}x^6 + \frac{4^2 \cdot 6^2 + 2^2 \cdot 6^2 + 2^2 \cdot 4^2}{8!}x^8 + \cdots \\ &= \frac{2}{3}\left[\frac{1}{2^2}\right]\frac{x^4}{4} + \frac{2 \cdot 4}{3 \cdot 5}\left[\frac{1}{2^2} + \frac{1}{4^2}\right]\frac{x^6}{6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2}\right]\frac{x^8}{8} + \cdots. \end{aligned}$$

We leave it to the reader to verify that these expansions agree with the expressions given earlier. The reader may notice the curious fact that, with regard to the second expression in each of the above expansions, as we pass from one odd power to the next (even one), all the numbers (except the 1's in the numerators within brackets) increase by one. This pattern persists for all higher powers.

Acknowledgments. The authors wish to thank the anonymous referee for many suggestions which improved the overall presentation of this article.

References

[1] A. Basu and T. M. Apostol: *A New Method for Investigating Euler Sums*, Ramanujan J. (4) (2000), 397–419.

[2] B. C. Berndt: *Ramanujan's Notebooks: Part I*, Springer-Verlag, New York (NY), 1985.

[3] D. Borwein and J. M. Borwein: *On An Intriguing Integral and Some Series Related to* $\zeta(4)$, Proc. Amer. Math. Soc. (123) 4 (1995), 1191–1198.

[4] J. M. Borwein and D. M. Bradley: *Thirty-Two Goldbach Variations*, Int. J. Number Theory (2) 1 (2006), 65–103.

[5] J. M. Borwein and M. Chamberland: *Integer Powers of Arcsin*, Int. J. Math. Math. Sci. (2007) Article ID 19381, 10 pp.

[6] R. E. Bradley, L. A. D'Antonio and C. E. Sandifer (eds.): *Euler at 300: An Appreciation*, Spectrum (MAA Tercentenary Euler Celebration), vol. 5, Mathematical Association of America (MAA Press), Washington, DC, 2007.

[7] T. J. I'A. Bromwich: *An Introduction to the Theory of Infinite Series*, 2nd edition, Macmillan, 1926.

[8] J. Campbell and P. Levrie: *The Modern History of the Basel Problem*, Euleriana (6) 1 (2026), 70–96.

[9] B. R. Choe: *An Elementary Proof of* $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, Amer. Math. Monthly (94) 7 (1987), 662–663.

[10] P. J. De Doelder: *On some series containing* $\psi(x) - \psi(y)$ *and* $(\psi(x) - \psi(y))^2$ *for certain values of* x *and* y, J. Comput. Appl. Math. (37) (1991), 125–141.

[11] W. Dunham: *Euler — The Master of Us All*, The Dolciani Mathematical Expositions, vol. 22, The Mathematical Association of America, Washington, DC, 1999.

[12] J. Edwards: *An Elementary Treatise on the Differential and Integral Calculus with Aplications and Numerous Examples*, 2nd edition, Macmillan, London and New York, 1892.

[13] L. Euler: *Institutionum Calculi Integralis: Volumen Primum*, Academiae Imperialis Scientiarum Petropolitanae, St. Petersburg, 1768.

[14] O. Furdui and A. Sîntămărian: *A new proof of the quadratic series of Au-Yeung*, Gaz. Mat. Ser. A (37 (116)) 1–2 (2019), 1–6.

[15] O. Furdui and A. Sîntămărian: *A famous formula of Euler revisited*, Gaz. Mat. Ser. A (43 (122)) 3–4 (2025), 1–9.

[16] I. S. Gradshteyn and I. M Ryzhik: *Table of Integrals, Series, and Products*, 8th edition, Academic Press, San Diego, CA, 2015.

[17] R. Harada: *On Euler's Formula for Double Zeta Values*, Kyushu J. Math. (72) (2018), 15–24.

[18] G. Kimble: *Euler's Other Proof*, Math. Mag. (60) 5 (1987), 282.

[19] M. Kneser: *A summation problem. Solution to Problem 4305*, Amer. Math. Monthly (57) 4 (1950), 267–268.

[20] K. Knopp: *Theory and Application of Infinite Series*, Dover Publications, Inc, New York (NY), 1990. (Originally published by Blackie & Son, Ltd (1951), in accordance with the Fourth German edition (1947).)

[21] D. H. Lehmer: *Interesting Series Involving the Central Binomial Coefficient*, Amer. Math. Monthly (92) 7 (1985), 449–457.

[22] C. Li and W. Chu: *Evaluating Infinite Series Involving Harmonic Numbers by Integration*, Mathematics (12) 4 (2024) Article No. 589, 21 pp.

[23] H. Muzaffar: *Some Interesting Series Arising from the Power Series Expansion of* $(\sin^{-1} x)^q$, Int. J. Math. Math. Sci. (14) (2005), 2329–2336.

[24] A. S. Nimbran, P. Levrie and A. Sofo: *Harmonic-binomial Euler-like sums via expansions of* $(\arcsin x)^p$, RACSAM (116) (2022), 1–23.

[25] H. F. Sandham: *Advanced problem 4305*, Amer. Math. Monthly (55) 7 (1948), 431.

[26] C. E. Sandifer: *The Early Mathematics of Leonhard Euler*, Spectrum (MAA Tercentenary Euler Celebration), vol. 1, Mathematical Association of America (MAA Press), Washington, DC, 2007.

[27] M. R. Spiegel: *Some Interesting Series Resulting from a Certain Maclaurin Expansion*, Amer. Math. Monthly (60) 4 (1953), 243–247.

[28] S. Stewart: *In the Shadow of Euler's Greatness: Adventures in the Rediscovery of an Intriguing Sum*, Math. Intelligencer (43) 3 (2021), 82–91.

[29] C. I. Vălean: *Reviving the quadratic series of Au-Yeung*, J. Class. Anal. (6) 2 (2015), 113–118.

[30] C. I. Vălean: *(Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, Springer, Cham (Switzerland), 2019.
[31] E. T. Whittaker and G. N. Watson: *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions*, 2nd edition, Cambridge University Press, Cambridge (UK), 1915.
[32] J. Wolstenholme: *A Book of Mathematical Problems on Subjects Included in the Cambridge Course*, Macmillan, 1867.

Jamil Abreu received his Ph.D. in Mathematics from the State University of Campinas, Brazil, in 2013. He is currently a Professor of Mathematics at the Federal University of Lavras in Brazil. His research interests include functional analysis, stochastic analysis, and partial differential equations. He further engages with classical mathematics and, to a lesser extent, the history and philosophy of mathematics.

Ana Claudia Pereira received her Ph.D. in Mathematics from the Federal University of São Carlos, Brazil, in 2009. She is currently a Professor of Mathematics at the Federal University of Lavras, Brazil. Her research interests include partial differential equations and nonlinear dynamical systems.

(Both authors) Federal University of Lavras, Department of Mathematics and Applied Mathematics, Avenida Central UFLA, S/n, Lavras MG, Brazil
E-mail address, J. Abreu: jamil.abreu@ufla.br
E-mail address, A.C. Pereira: anaclaudia@ufla.br