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The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

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Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

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EDITORIAL

There has been a wonderfully gratifying response to the email (21$^{\text{st}}$ August 2025) from Rachel Quinlan, President of the IMS, inviting the submission of papers for this anniversary issue of the Bulletin. The articles, book reviews, and problems herein are a testament to the vibrancy and wide range of mathematical activity on this island and among the wider IMS community. If you had in mind an article of relevance to the 50$^{\text{th}}$ anniversary of the IMS that, for whatever reason, didn't quite make the submission deadline for this issue, please do consider Winter 2026; after all, this anniversary does last a full calendar year!

Of particular note in this issue, since both authors are young mathematicians straddling the second-level to beginning university transition, is an article by Owen Barron and Odhran Keenan on designing challenging Olympiad-style geometry problems. Their beautifully written article is a credit to them. This editor would be delighted to receive further suitable articles written by students. And so, if you are a lecturer supervising a final year project or working in any context with students, do please encourage them to write an account of their work for the Bulletin. Remember that the Bulletin publishes articles 'written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community'.

This issue also includes articles on the history of mathematics: van Weerden on Hamilton, MacHale on Boole, and Curran's history of the mathematics learning centre at the University of Limerick; mathematics education research is represented by the work of Tyrrell-Nic Dhonncha on gestures in the classroom and that of Guerin and Walsh on the impact of a professional training programme for postgraduate tutors; Hegarty, Hill, Madden and Phelan present a timely snapshot of the mathematics staff gender balance and Athena Swan; mathematical research is represented by the work of Abreu and Pereira on the summation of various series, of Kelly and Tang on the numerical solution of stochastic delay differential equations, and of Monaco and Perrotta on a model of wealth distribution in an economy.

Rachel Quinlan, President of the IMS, in her letter in these pages, highlights the origins of the IMS and the many changes in the mathematical ecosystem that have taken place over the past 50 years. This editor echos the sentiments Rachel expresses therein and, in particular, how *community* must feature centrally in the Society – in this we have, perhaps, one advantage over the larger societies. The IMS is a community and the 'September Meeting' (its moniker becoming an ever more endearing misnomer) is the embodiment of that community. I, for one, will be at Trinity College Dublin for the 50th birthday celebrations and hope to see as many members there, as can make it, as an expression of that community.

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E-mail address: `ims.bulletin@gmail.com`

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The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor.

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Letters to the Editor

Caoga Bliain ag Fás: Cumann Matamaitice na hÉireann

From Rachel Quinlan

Dear Editor,

The 50th anniversary in 2026 of the foundation of the Irish Mathematical Society provides excellent grounds for celebration. The history of the IMS reflects a period of rapid expansion in mathematical research and education on the island of Ireland. Looking back at records of the earliest meetings, archived at `irishmathsoc.org`, we see that the vision and purpose of the founding members remains entirely current and as relevant as ever. The constitution asserts that "the Society is incorporated for the purpose of promoting and extending the knowledge of mathematics and its applications". The 1978 newsletter reports progress towards this goal, which included the first conference of the IMS in May 1978, visits by distinguished international speakers and initial steps towards the establishment of the Irish Mathematical Olympiad. The same report highlights another issue that has not changed since 1976, that the IMS depends on membership fees and voluntary involvement to carry out its activities. The annual fee has admittedly changed from the original £2 for regular members and 50p for students, but the generosity, commitment and engagement of members continues to sustain the IMS.

The evolution of the Society since those early days can be seen as a resounding validation of the insights and aspirations that guided its first steps. The series of early meetings became the annual meeting, running in its current format since 1988, and hosted so far by sixteen different institutions. The Irish Mathematical Olympiad was inaugurated in 1988 and has taken place every year since, with 14,000 students taking part in the first round in 2025. Since 2005, the IMS has awarded the Fergus Gaines Cup to the most successful participant in this contest. The newsletter evolved into the Bulletin of the IMS, which was established in 1986 and will publish Issue 100 in 2027 (another numerical milestone).

The mathematical landscape in Ireland has of course changed dramatically since 1976. PhD education in mathematics in Ireland was a rarity until at least the 1990s; now our country is an attractive destination for aspiring research mathematicians from all over the world. The growth of our community has been accompanied by a diversification of activity and of people, which is reflected in the programmes of the annual conferences. The commitment of our community to sound practice in education, and the emergence of mathematics education (especially at university level) as a research area within the mathematical sciences, has led to many informative and valuable talks and discussions at IMS conferences. The establishment of ICEDIM (the Irish Committee for Equality, Diversity and Inclusion in Mathematics) aims to ensure that all voices are heard in our organisation, and to counter the negative effects of patterns of exclusion and underrepresentation that persist in higher education and in mathematics. Since 2024, ICEDIM has overseen the organisation of the annual Women in Mathematics Day Ireland Conference, which typically takes place on or near May 12 as part of the global celebration of International Women in Mathematics Day.

The Irish Committee for Mathematics Education was established in 2021, to advise the IMS Committee on matters concerning mathematics education. Amongst the intiatives of this committee are a series of online seminars on research developments of practical relevance, and a review of the quality of mathematics textbooks in use in post-primary schools.

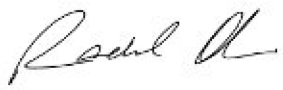
As we look back at the last 50 years and ahead to the next 50, it is appropriate to recognize all the people whose efforts have brought our Society to this moment. They include the founding officers Finbarr Holland (President), Michael Hayes (Vice-President), Ted Hurley (Secretary) and Trevor West (Treasurer) and all the founding members. They established the newsletter, bulletin and conference series and generated the momentum that ensured the persistence and robustness of the IMS. We thank all the colleagues who organised all the annual meetings over the years, and all the speakers, poster presenters and audience participants. We thank all contributors to the Bulletin, including the editors and editorial board members, and all authors and referees, as well as colleagues who arranged printing and distribution on paper and online. Thanks to everyone who maintained the IMS online resources and communications and everyone who served on the committee and subcommittees and as local representatives. Every member who has participated in the activities of the IMS in any way has made a valuable contribution to the collective achievement that we celebrate this year.

Central to the goal of “promoting and extending the knowledge of mathematics and its applications” is the need for community. Mathematical knowledge cannot be promoted or extended in a vacuum, and the drive to share, exchange and co-create knowledge is a human one. Although conferences and collaborations surely enhance our productivity and efficiency, they are motivated more by the human experience of doing mathematics, of searching for insights, of fitting together the parts of our understanding and sharing the pleasure of progress. Whatever the challenges of the decades ahead, that human experience will not go out of fashion and the IMS will continue to flourish.

We hope to see you in Trinity College Dublin for the 50th birthday celebration, and we thank our colleagues there for their outstanding organisational efforts. The last word goes to the 1978 report on the achievements of the first year, a call to action that is as pressing now as then.

> The aims of the Society are to promote the teaching and research of Mathematics, particularly in Ireland. Being a relatively small Society we need the active support of a good percentage of our members, especially in the initial formative stage. So let us hear from <u>you</u>!

Yours sincerely



Rachel Quinlan, President of the IMS
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NOTICES FROM THE SOCIETY

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(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers' Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€250
Ordinary member	€40
Lifetime member	€400
Student member	€20
DMV, IMTA, NZMS, MMS or RSME reciprocity member	€20
AMS reciprocity member	$25
LMS reciprocity member (paying in Euro)	€20
LMS reciprocity member (paying in Sterling)	£20

(3) The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$40.
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The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

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(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate.

(6) Those members who have reached 75 years of age, and who have been members in good financial standing with the Society for the previous 15 years, are entitled upon notification to the Treasurer to have their subscription rate reduced to €0.

(7) Subscriptions normally fall due on 1 February each year.

(8) Cheques should be made payable to the Irish Mathematical Society.

(9) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets three times each year.

(10) Please send the completed application form, available at

`https://www.irishmathsoc.org/business/imsapplicn_2024.pdf`

with one year's subscription, either by post or by email, to:

Dr Dana Mackey
School of Mathematics and Statistics
Technological University of Dublin
Central Quad, Grangegorman
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`subscriptions.ims@gmail.com`

Deceased Members

It is with regret that we report the deaths of members:

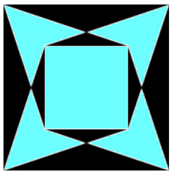
Thomas Ambrose of Kevin Street/DIT, who died on 23 December 2023.

David Judge of UCD who died on 19 September 2024.

Don McQuillan of UCD who died on 4 January 2025.

David Wilkins of Trinity College Dublin, who died on 3 February 2026.

David Redmond of Maynooth University, who died on 21 June 2026.



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From the Basel Problem to the Sandham–Au-Yeung Quadratic Series: An Arcsine Pathway

JAMIL ABREU AND ANA CLAUDIA PEREIRA

Abstract. We relate the Basel problem to two classical nested sums, known as Euler sums, through expansions of powers of the arcsine function. Using only elementary methods, we then show how one of these Euler sums leads to the evaluation of the celebrated Sandham–Au-Yeung quadratic series. Finally, we present an elementary approach to Stewart's three-term relation, namely, an identity relating the Sandham–Au-Yeung quadratic series with two linear sums of weight four.

1. Introduction

The Basel problem, first posed by the Italian mathematician Pietro Mengoli (1626–1686) in 1650, later highlighted by Jakob Bernoulli (1654–1705) in Basel in 1689, and finally solved by Leonhard Euler (1707–1783) in 1734, asks for the evaluation of the infinite sum

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}. \tag{1}$$

For a gentle presentation of Euler's original work, see Dunham [11, Ch. 3]. For a more serious treatment, see the chapter entitled "Euler's Solution of the Basel Problem — The Longer Story" in Bradley et al. [6], as well as Sandifer's analysis of Euler's original paper in [26, Ch. 21].

Euler's first argument, not entirely rigorous, was based on comparing two expressions for $(\sin x)/x$, one from the Maclaurin series of $\sin x$, the other from his now famous *infinite product formula for the sine function*, namely,

$$\frac{\sin x}{x}=\Big(1-\frac{x^2}{\pi^2}\Big)\Big(1-\frac{x^2}{4\pi^2}\Big)\Big(1-\frac{x^2}{9\pi^2}\Big)\Big(1-\frac{x^2}{16\pi^2}\Big)\cdots.$$

Later, Euler gave several other proofs using various methods. The quest for new solutions to the Basel problem continues to this day, with new proofs of (1) still appearing from time to time; see e.g. Campbell and Levrie [8].

In more recent years, another sum has attracted considerable attention, so much so that it is on its way to acquiring a status similar to that of the Basel problem in the mathematical subconscious. Discovered and rediscovered several times throughout the second half of the twentieth century, it appears to have been first proposed in 1948 by the Irish mathematician Henry F. Sandham (1917–1963) as a problem in *The American Mathematical Monthly* (see [25]), in the form

$$1+\Big(\frac{1+\frac12}{2}\Big)^2+\Big(\frac{1+\frac12+\frac13}{3}\Big)^2+\Big(\frac{1+\frac12+\frac13+\frac14}{4}\Big)^2+\cdots=\frac{17\pi^4}{360}. \tag{2}$$

A response came one and a half years later, with the German mathematician Martin Kneser (1928–2004) providing a solution based on a triple summation (see [19]).

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Some decades later, it was numerically rediscovered in 1993 by Enrico Au-Yeung (then an undergraduate student in the University of Waterloo), based on computations of 500,000 terms (to five-digit accuracy), and was subsequently proved by Borwein and Borwein [3] using Fourier series. Between these two events, the sum appeared on other occasions, largely unnoticed. We refer the interested reader to Borwein and Borwein's article just cited, and especially Stewart [28] for further historical background.

There is a visible similarity between (1) and (2): both involve summing the inverses of squares of integers, the difference being that in the latter each term is weighted by the square of a *harmonic number*. It turns out that both series are connected in a more profound and mathematically structured way. This article is about this connection.

2. Revisiting some classical results

Let us begin by recalling the classical expansion

$$\arcsin x = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{x^{2n+1}}{2n+1}, \tag{3}$$

valid for $-1 \leqslant x \leqslant 1$. In fact, convergence for $-1 < x < 1$ is inherited from the binomial series, wherefrom (3) can be obtained by integration, and convergence at ± 1 can be inferred from Abel's limit theorem, cf. Knopp [20, p. 177]. The binomial coefficient appearing in (3) is the so-called *central binomial coefficient*:

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = 4^n \cdot \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}.$$

Thus,

$$\frac{\binom{2n}{n}}{4^n} = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \quad \text{and} \quad \frac{4^n}{\binom{2n}{n}(2n+1)} = \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)}.$$

Next in the line is

$$\frac{(\arcsin x)^2}{2!} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{x^{2n}}{(2n)^2} \quad \Big[= \sum_{n=0}^{\infty} \frac{4^n}{\binom{2n}{n}(2n+1)} \frac{x^{2n+2}}{2n+2} \Big]. \tag{4}$$

Of course, (4) can be obtained by writing

$$\frac{(\arcsin x)^2}{2!} = \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}}\, dt$$

and using the classical expansion

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{x^{2n+1}}{2n+1}. \tag{5}$$

This, in turn, can be derived in many ways; see, for example, Spiegel [27] and Lehmer [21]. Lehmer's article, in particular, presents many applications of (5) that still resonate today.

One natural question is what happens if we make the change of variables $x = \sin\theta$ and then integrate these identities over the interval $[0, \pi/2]$. To pursue this approach, we will need the Wallis integrals

$$\int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta = \int_0^{\pi/2} \sin^{2n+1}\theta\, d\theta = \frac{4^n}{\binom{2n}{n}(2n+1)} \tag{6}$$

and

$$\int_0^{\pi/2} \cos^{2n}\theta\, d\theta = \int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\binom{2n}{n}}{4^n} \frac{\pi}{2}. \tag{7}$$

Example 2.1. The substitution $x = \sin\theta$ into (3) yields

$$\theta = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{\sin^{2n+1}\theta}{2n+1}.$$

By integrating over θ from 0 to $\pi/2$ and using (6), we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

This is the solution of Choe [9], and Kimble [18], to the Basel problem. △

Example 2.2. The substitution $x = \sin\theta$ into (4) yields

$$\frac{\theta^2}{2} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{\sin^{2n}\theta}{(2n)^2}.$$

By integrating over θ from 0 to $\pi/2$ and using (7), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is Spiegel's solution [27] to the Basel problem. △

Going back to the expansions of the arcsine function, less known than both (3) and (4) are

$$\frac{(\arcsin x)^3}{3!} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big] \frac{x^{2n+1}}{2n+1}, \tag{8}$$

(the sum inside brackets is often denoted by $h_n^{(2)}$ or $O_n^{(2)}$, and called the n-th odd generalized harmonic number of order 2), as well as

$$\frac{(\arcsin x)^4}{4!} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}(2n+1)} \Big[\sum_{k=1}^{n} \frac{1}{(2k)^2}\Big] \frac{x^{2n+2}}{2n+2}. \tag{9}$$

The sum inside brackets equals $\frac{1}{4}H_n^{(2)}$, where $H_n^{(2)} = \sum_{k=1}^{n} 1/k^2$ is the n-th generalized harmonic number of order 2.

These expansions of the powers of $\arcsin x$ have an interesting history of their own. Euler was likely aware of them since the expansion (5) appears explicitly in his 1755 *Institutiones Calculi Differentialis* and, later, in his 1768 *Institutiones Calculi Integralis* he develops the expansion of $y = e^{a \arcsin x}$, which naturally leads to the arcsine expansions. This method found a place in the classic 1886 differential calculus text by Joseph Edwards, where it appears as a problem with a clearly delineated solution in the cases of $(\arcsin x)^{2,3}$. Srinivasa Ramanujan (1887–1920) seems to have been interested in these matters, as evidenced by Berndt [2, Ch. 9, Prop. 15], where the case of $(\arcsin x)^4$ is treated in detail. More recently, Muzaffar [23], Borwein and Chamberland [5], and Nimbran et al. [24], among others, have revisited these expansions and made extensive use of them in the investigation of various *Euler sums*; see the next section for an explanation of this notion.

Here is how Euler's method works in a nutshell: by writing

$$y = e^{a \arcsin x} = \sum_{k=0}^{\infty} a_k x^k, \tag{10}$$

one can first show that

$$(1-x^2)y'' - xy' - a^2 y = 0. \tag{11}$$

Inserting the powers series into the differential equation, it is then a routine matter to find a recursive relation between a_{k+2} and a_k; clearly, $a_0 = 1$ and $a_1 = a$ (see the Appendix below). Then, by expanding $e^{a \arcsin x}$ in powers of a and equating coefficients of a^N on both sides, one deduces the expansion of $(\arcsin x)^N$ for each $N \geqslant 1$. We provide more details in the Appendix.

We may now ask the same question as before: what happens if we make the change of variables $x = \sin\theta$ and then integrate (8) and (9) over the interval $[0, \pi/2]$? This question is addressed in the next section.

3. Beyond Basel: the arcsine gateway to an Euler sum

Let us introduce some notation. The *Riemann zeta function* is defined for $p > 1$ by

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Thus, Euler's identity (1) is simply $\zeta(2) = \pi^2/6$. Clearly, $\zeta(1)$ is the harmonic series, which is well known to diverge. Its partial sums, which appear in the numerators of (2), are denoted by

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

H_n is called the nth harmonic number; it is often convenient to put $H_0 = 0$.

It turns out that, for the expansions (8) and (9), the same procedure as employed in Examples 2.1 and 2.2 leads to more interesting results (in the sense of being less familiar to a broader audience). We therefore state these results as propositions rather than examples.

We will need the following classical identity

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^p} = \Big(1 - \frac{1}{2^p}\Big)\zeta(p), \tag{12}$$

which can be obtained by splitting $\sum_{n=1}^{\infty} n^{-p}$ into even and odd indices.

Proposition 3.1.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\Big[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2}\Big] = \frac{5\pi^4}{384}.$$

Proof. The substitution $x = \sin\theta$ into (8) yields

$$\frac{\theta^3}{3!} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n}\Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big] \frac{\sin^{2n+1}\theta}{2n+1}.$$

By integrating over θ from 0 to $\pi/2$ and using (6), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}\Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big] = \frac{\pi^4}{4!2^4}. \tag{13}$$

The sum on the left equals

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}\Big[\sum_{k=1}^{n+1} \frac{1}{(2k-1)^2}\Big] - \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4}$$
$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big] - \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.$$

By (12), and using $\zeta(4) = \pi^4/90$, the rightmost sum equals $\pi^4/96$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\Big[\sum_{k=1}^{n} \frac{1}{(2k-1)^2}\Big] = \frac{5\pi^4}{384}. \qquad \square$$

We believe that Proposition 3.1 is known, although we were unable to determine where it may have first appeared. Identity (13) was observed in Nimbran et al. [24] (using essentially the same argument), but those authors did not carry the evaluation through to the form stated here. Li and Chu [22] investigated numerous sums via integration; in particular they evaluated many sums involving odd harmonic numbers, including the following alternating sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\Big[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2}\Big] = \frac{7}{4}\zeta(3) - \frac{\pi G}{2},$$

where G is Catalan's constant. Leaving aside the question of originality, our interest here lies instead in what emerges from the expansion of $(\arcsin x)^4$.

Proposition 3.2.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\Big(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\Big) = \frac{7\pi^4}{360}.$$

Proof. The substitution $x = \sin\theta$ in (9) gives

$$\frac{\theta^4}{4!} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}(2n+1)}\Big[\sum_{k=1}^{n} \frac{1}{(2k)^2}\Big]\frac{\sin^{2n+2}\theta}{2n+2}. \tag{14}$$

By integrating both sides over θ from 0 to $\pi/2$ and using the Wallis integral (see (7))

$$\int_0^{\pi/2} \sin^{2n+2}\theta\, d\theta = \frac{(2n+2)!}{4^{n+1}[(n+1)!]^2}\frac{\pi}{2} = \frac{(2n+2)(2n+1)}{(2n+2)^2}\frac{\binom{2n}{n}}{4^n}\frac{\pi}{2},$$

we obtain

$$\sum_{n=1}^{\infty} \frac{1}{(2n+2)^2}\Big[\sum_{k=1}^{n} \frac{1}{(2k)^2}\Big] = \frac{\pi^4}{5!2^4},$$

or, after multiplying both sides by 2^4,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}\Big[\sum_{k=1}^{n} \frac{1}{k^2}\Big] = \frac{\pi^4}{120}.$$

As in the end of the proof of Proposition 3.1, the left-hand side equals

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}\Big[\sum_{k=1}^{n+1} \frac{1}{k^2}\Big] - \sum_{n=1}^{\infty} \frac{1}{(n+1)^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}\Big[\sum_{k=1}^{n} \frac{1}{k^2}\Big] - \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

The right most sum is $\zeta(4) = \pi^4/90$, so that the above combined with the previous identity yields

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\Big[\sum_{k=1}^{n} \frac{1}{k^2}\Big] = \frac{7\pi^4}{360}. \qquad \square$$

The sum in Proposition 3.2 is classical. It belongs to a family of nested sums studied by Euler in the 1770's, nowadays called *Euler sums.* Its "proof from the book" seems to consist simply of substituting $p = 2$ into the folklore identity (attributed to Euler)

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^p} = \frac{1}{2}(\zeta(p)^2 + \zeta(2p)),$$

valid for all $p > 1$, and then using $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$; see Vălean [30, pp. 384–385] for a related, more general result. Above,

$$H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p}$$

is called the *n-th generalized harmonic number of order p*. Actually, Euler investigated general sums of the form

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} \quad \Big[= \sum_{n=1}^{\infty} \frac{1}{n^q} \sum_{k=1}^{n} \frac{1}{k^p} \Big],$$

a problem already posed by Christian Goldbach (1690–1764) in his correspondence with Euler dating back to 1742. Part of this story is recounted in Borwein and Bradley [4], who also collected thirty-two proofs of the simplest Euler sum, namely,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).$$

A more recent proof of Proposition 3.2 has been given by Vălean [29], using Abel's summation formula. In his article, Vălean's purpose was actually to provide a new derivation of the Sandham–Au-Yeung quadratic series

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4).$$

In the remainder of this article, drawing on earlier contributions in the literature, we show how Proposition 3.2 relates to the above as well as to another classical Euler sum.

4. Intermezzo: mathematical tools

Later in this article, we will need the integration by parts formula

$$\int_0^1 x^{n-1} \ln^k x \, dx = \frac{(-1)^k k!}{n^{k+1}} \quad (n = 1, 2, 3 \dots; k = 0, 1, 2, \dots). \tag{15}$$

This is a classical and well-known formula, though the reader may still find an interesting treatment of it in Vălean [30, pp. 57–58]. By rearranging and summing over k, then interchanging summation with integration, and summing a geometric series, it follows that

$$\zeta(k+1) = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} = \frac{(-1)^k}{k!} \sum_{n=1}^{\infty} \int_0^1 x^{n-1} \ln^k x \, dx = \frac{(-1)^k}{k!} \int_0^1 \frac{\ln^k x}{1-x} \, dx.$$

Moreover, replacing x by $1 - x$ yields

$$\zeta(k+1) = \frac{(-1)^k}{k!} \int_0^1 \frac{\ln^k(1-x)}{x} \, dx. \tag{16}$$

We will also need two important tools, which we state in the next two lemmas.

Lemma 4.1. *For all integers $n \geqslant 1$,*

$$\int_0^1 x^{n-1} \ln(1-x) \, dx = -\frac{H_n}{n} \tag{17}$$

and

$$\int_0^1 x^{n-1} \ln^2(1-x) \, dx = \frac{2}{n} \sum_{k=1}^{n} \frac{H_k}{k}. \tag{18}$$

Proof. By noting that $nx^{n-1} = (d/dx)(x^n - 1)$, integration by parts yields

$$\begin{aligned}\int_0^1 x^{n-1}\ln(1-x)\,dx &= \frac{x^n-1}{n}\ln(1-x)\Big|_0^1 - \frac{1}{n}\int_0^1 \frac{1-x^n}{1-x}\,dx \\ &= -\frac{1}{n}\int_0^1 (1+x+x^2+\cdots+x^{n-1})\,dx \\ &= -\frac{H_n}{n}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^1 x^{n-1}\ln^2(1-x)\,dx &= \frac{x^n-1}{n}\ln^2(1-x)\Big|_0^1 - \frac{2}{n}\int_0^1 \frac{1-x^n}{1-x}\ln(1-x)\,dx \\ &= -\frac{2}{n}\sum_{k=1}^n \int_0^1 x^{k-1}\ln(1-x)\,dx \\ &= \frac{2}{n}\sum_{k=1}^n \frac{H_k}{k},\end{aligned}$$

where the last identity follows from the first part. □

Both identities in Lemma 4.1 are often used in working with Euler sums. They are often proved via double integration; see, for instance, Vălean [29] or Stewart [28]. As for the integration by parts argument used above, we initially thought it was new, but later found that it already appears in Vălean [30, pp. 59–60]. The identity (17) is old, going back at least to 1867, appearing for instance as Problem 1031 on Page 214 of Wolstenholme's problem book [32]; in fact, it is presented there in the form

$$\int_0^1 (1-x)^{n-1}\log\Big(\frac{1}{x}\Big)\,dx = \frac{1}{n}\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\Big).$$

As for identity (18), we did not find it in Wolstenholme's book after a brief search, and we do not know when it first appeared in the literature.

Next, we present an auxiliary result of independent interest. It is well known to experts and admits many different proofs. For instance, Vălean [29] derives it using Abel's summation-by-parts formula. We include a simple proof based on induction.

Lemma 4.2. *For all $n = 1, 2, \ldots,$*

$$H_n^2 + H_n^{(2)} = 2\sum_{k=1}^n \frac{H_k}{k}.$$

Proof. The identity is clear for $n = 1$, both sides being equal to 2. Assuming it to hold for $n-1$, we have

$$\begin{aligned}H_n^2 + H_n^{(2)} &= \Big(H_{n-1} + \frac{1}{n}\Big)^2 + H_{n-1}^{(2)} + \frac{1}{n^2} \\ &= H_{n-1}^2 + H_{n-1}^{(2)} + 2\frac{H_{n-1}+1/n}{n} \\ &= 2\sum_{k=1}^{n-1}\frac{H_k}{k} + \frac{2H_n}{n} \\ &= 2\sum_{k=1}^n \frac{H_k}{k},\end{aligned}$$

which concludes the induction step. □

Finally, we assume the reader to be familiar with the Cauchy product of power series

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)x^n.$$

For instance, the Cauchy product $(\sum_{n=1}^{\infty} x^n/n)(\sum_{n=0}^{\infty} x^n)$ of the logarithmic series for $-\ln(1-x)$ with the geometric series yields the classical expansion

$$-\frac{\ln(1-x)}{1-x} = \sum_{n=1}^{\infty} H_n x^n. \tag{19}$$

From this, and writing

$$\ln^2(1-x) = -2\int_0^x \frac{\ln(1-t)}{1-t}\,dt,$$

we deduce the logarithmic series

$$\ln^2(1-x) = 2\sum_{n=1}^{\infty} \frac{H_n}{n+1}x^{n+1}, \tag{20}$$

valid for $-1 \leqslant x < 1$; see Bromwich [7, §62] for an alternative argument.

5. The Sandham–Au-Yeung quadratic series and a close cousin

Using (16) with $k = 3$, and then the logarithmic series, yields

$$\begin{aligned} 6\zeta(4) &= -\int_0^1 \frac{\ln^3(1-x)}{x}\,dx \\ &= \sum_{n=1}^{\infty}\int_0^1 \frac{x^{n-1}}{n}\ln^2(1-x)\,dx \\ &= \sum_{n=1}^{\infty}\frac{2}{n^2}\sum_{k=1}^{n}\frac{H_k}{k}, && \text{(by (18))} \end{aligned}$$

or, by Lemma 4.2,

$$6\zeta(4) = \sum_{n=1}^{\infty}\frac{H_n^2}{n^2} + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}. \tag{21}$$

The above argument captures the essential idea in Vălean [29], although it does not appear there in exactly this form. Having previously derived the second term (Proposition 3.2 above) using Abel's summation formula, he obtained the following result, which was the main goal of his article.

Proposition 5.1 (Sandham–Au-Yeung quadratic series)**.**

$$\sum_{n=1}^{\infty}\Big(\frac{H_n}{n}\Big)^2 = \frac{17}{4}\zeta(4) \quad \Big[= \frac{17\pi^4}{360}\Big].$$

Using (20), Stewart [28] argued that

$$\begin{aligned} 6\zeta(4) &= -\int_0^1 \frac{\ln^3(1-x)}{x}\,dx \\ &= -2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx \\ &= 2\sum_{n=1}^{\infty}\frac{H_{n-1}H_n}{n^2}, && \text{(by (17))} \end{aligned}$$

so that, using $H_{n-1} = H_n - 1/n$,

$$3\zeta(4) = \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} - \sum_{n=1}^{\infty} \frac{H_n}{n^3}. \tag{22}$$

This, combined with Proposition 5.1, yields:

Proposition 5.2.

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{\pi^4}{72}.$$

The sum in Proposition 5.2 is classical, its "proof from the book" consisting of putting $q = 3$ in the identity

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q} = \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{n=1}^{q-2} \zeta(n+1)\zeta(q-n). \tag{23}$$

In fact, putting $q = 3$ in the above yields

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{2}\zeta(4) - \frac{1}{2}\zeta(2)^2 = \frac{\pi^4}{72},$$

where we again used the well-known values $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. Identity (23) was given by Euler (without proof) in 1776; for a comprehensive discussion of Euler's original work, see Harada [17]. Basu and Apostol [1] derived this identity by a careful analysis of some general telescoping properties of infinite series, and Furdui and Sîntămărian [15] found a calculus-based proof of it. Among the most classical, well-known references to both results (Propositions 5.1 and 5.2) is Borwein and Borwein [3], who derived them from the integral formula (obtained via contour integration)

$$\frac{1}{\pi}\int_0^{\pi} \theta^2 \ln^2\left(2\cos\frac{\theta}{2}\right) d\theta = \frac{11\pi^4}{180} \quad \left[= \frac{11}{2}\zeta(4)\right].$$

As with Vălean's identity (21), Stewart originally conceived (22) as a means of deriving the Sandham–Au-Yeung series, taking Proposition 5.2 as given via its "proof from the book". In fact, from a mathematical research point of view, linear Euler sums — such as those on the left-hand side of (23) — are well understood in view of Euler's identity, whereas nonlinear Euler sums, such as the Sandham–Au-Yeung series, are the true "targets" we seek to evaluate. Stewart even ventured to say that (22) is "perhaps the simplest approach" to the Sandham–Au-Yeung series. Since there are many proofs of Proposition 5.2 — possibly simpler than the one presented here using (23) — such as the one given in Furdui and Sîntămărian [14], we believe Stewart has a point.

Later in his article, and using dilogarithms, Stewart derived the three terms relation

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = 2\sum_{n=1}^{\infty} \frac{H_n}{n^3} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2}. \tag{24}$$

This is a curious identity indeed ("intriguing", as Stewart wrote), but it is admittedly weaker than (21) and (22) taken together, since it requires the evaluation of two series to yield a third. In particular, the results in Propositions 3.2 and 5.2, taken together, yield the Sandham–Au-Yeung series. Nevertheless, we believe there is room for an alternative, more elementary approach to this relation.

6. An elementary approach to Stewart's identity

We can derive Stewart's identity (24) in an elementary way (meaning that no dilogarithms are involved) by computing the integral

$$\int_0^1 \frac{\ln^2(1-x)\ln x}{(1-x)x}\,dx \tag{25}$$

in two different ways. For this, we will also need the expansion

$$\frac{\ln^2(1-x)}{1-x} = \sum_{n=1}^{\infty}[H_n^2 - H_n^{(2)}]x^n. \tag{26}$$

This can be obtained as a Cauchy product. In fact, using (20) and the geometric series,

$$\frac{\ln^2(1-x)}{1-x} = \Big(2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}x^n\Big)\Big(\sum_{n=0}^{\infty}x^n\Big) = 2\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\frac{H_{k-1}}{k}\Big)x^n.$$

Then, (26) follows from the identity

$$H_n^2 - H_n^{(2)} = 2\sum_{k=1}^{n}\frac{H_{k-1}}{k},$$

which can be proved by induction or obtained directly from Lemma 4.2.

Now, on the one hand, by using (26) and (15) (with $k=1$), the integral (25) equals (after interchanging summation with integration)

$$\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})\int_0^1 x^{n-1}\ln x\,dx = -\sum_{n=1}^{\infty}\frac{H_n^2 - H_n^{(2)}}{n^2}.$$

On the other hand, changing x to $1-x$ and then using (19) and (15) (with $k=2$), the integral (25) also equals

$$\int_0^1 \frac{\ln^2 x\ln(1-x)}{x(1-x)}\,dx = -\sum_{n=1}^{\infty}H_n\int_0^1 x^{n-1}\ln^2 x\,dx = -2\sum_{n=1}^{\infty}\frac{H_n}{n^3}.$$

Stewart's identity (22) follows by equating the rightmost sums of the two previous identities.

We arrived at the above derivation some days before discovering Stewart's article. At the time, we were more engaged with the paper of Li and Chu [22], as well as with the pioneering work of P. J. De Doelder [10], where similar logarithmic integrals appear. In retrospect, this may have been fortunate, for we can only wonder whether we would have pursued the matter so persistently had we encountered Stewart's paper earlier.

7. On De Doelder's article [10]

Many results on Euler sums were anticipated by De Doelder [10], who expressed his findings in terms of the ψ function (the logarithmic derivative of the Gamma function, usually referred to as the *digamma function*). De Doelder begins his paper with

> In a protracted correspondence with J. Gilles (Belgium) on [3][1] we discussed a lot of integrals related with dilogarithms and polylogarithms. In this correspondence also series containing $\psi(k)-\psi(1)$ appeared, where $\psi(x) = d/dx \log\Gamma(x)$ and $k\in\mathbb{N}$.

[1]This refers to a bibliographical entry in [10], namely, Lewin's book *Polylogarithms and Associated Functions* (North-Holland, 1981).

The relation between ψ and the harmonic numbers is $\psi(n) = H_{n-1} - \gamma$, where γ denotes the Euler-Mascheroni constant, hence $\psi(n) - \psi(1) = H_{n-1}$; see e.g. Whittaker and Watson [31, p. 254, Ex. 16]. Note then that (17) can be written as

$$\int_0^1 x^{n-1} \ln(1-x)\, dx = -\frac{\psi(n+1) - \psi(1)}{n} = -\frac{\psi(n+1) + \gamma}{n},$$

which is how it appears, for instance, in Gradshteyn and Ryzhik [16, Entry (4.293-8)].

We find, for instance, that Proposition 5.2 above is essentially De Doelder's identity (7), namely,

$$\sum_{k=1}^{\infty} \frac{\psi(k) - \psi(1)}{k^3} = \frac{\pi^4}{360}.$$

This is used two pages later to derive the Sandham–Au-Yeung series — see De Doelder's identity (9) — in the form

$$\sum_{k=1}^{\infty} \frac{(\psi(k) - \psi(1))^2}{k^2} = \frac{11\pi^4}{360}.$$

Both results, and others, are obtained via generating functions of various sequences involving $\psi(k) - \psi(1)$, expressed in terms of certain logarithmic integrals.

It is now widely acknowledged that De Doelder was aware of neither the previous evaluations of the Sandham–Au-Yeung series nor Euler's identity (23). Despite the fact that the term "Euler sum" (having been coined some three years later) does not appear in De Doelder's article, we venture to say that it is among the best starting points for learning about the subject, particularly as far as generating functions and integration methods are concerned.

Appendix: finding $(\arcsin x)^N$ with Euler's method

The power series method to obtain the expansions of $(\arcsin x)^N$ was originally outlined by Euler [13, Ch. III, § 186], and later reformulated by Edwards [12, pp. 87–90], with a brief appearance in Bromwich [7, p. 188, Ex. 1]. In this appendix, we provide a brief description of the cases $N = 1, 2, 3, 4$ based on Berndt's exposition in [2, p. 263]; in this respect, an alternative title to the present appendix might be "A friendly presentation of the Ramanujan-Berndt approach to the expansion of $(\arcsin x)^{1,2,3,4}$".

Firstly, if $y = e^{a \arcsin x}$ then

$$y' = \frac{ay}{\sqrt{1-x^2}}$$

and

$$y'' = \frac{ay}{\sqrt{1-x^2}} + \frac{axy}{(1-x^2)\sqrt{1-x^2}}.$$

Thus,

$$\begin{aligned}(1-x^2)y'' &= a\sqrt{1-x^2}\, y' + \frac{axy}{\sqrt{1-x^2}} \\ &= a^2 y + xy',\end{aligned}$$

which is the differential equation (11). Now, by inserting the power series (10) into the differential equation (11), we obtain the recursive relation

$$a_{k+2} = \frac{k^2 + a^2}{(k+2)(k+1)} a_k \quad (k = 0, 1, 2, \dots).$$

See Berndt [2, p. 263] for some additional details on this point. By induction, it is not difficult to see that

$$a_{2n+1} = \frac{a(a^2+1)(a^2+3^2)\cdots(a^2+(2n-1)^2)}{(2n+1)!}, \quad n = 1, 2, 3, \ldots$$
$$a_{2n} = \frac{a^2(a^2+2^2)(a^2+4^2)\cdots(a^2+(2n-2)^2)}{(2n)!}, \quad n = 1, 2, 3, \ldots.$$

Since $y = y(x) = e^{a \arcsin x}$ clearly satisfies $y(0) = 1$ and $y'(0) = a$, we see that $a_0 = 1$ and $a_1 = a$. Thus, on the one hand,

$$\begin{aligned} y &= e^{a \arcsin x} \\ &= 1 + ax + \frac{a^2}{2!}x^2 + \frac{a(a^2+1^2)}{3!}x^3 + \frac{a^2(a^2+2^2)}{4!}x^4 \\ &\quad + \frac{a(a^2+1^2)(a^2+3^2)}{5!}x^5 + \frac{a^2(a^2+2^2)(a^2+4^2)}{6!}x^6 \\ &\qquad + \frac{a(a^2+1^2)(a^2+3^2)(a^2+5^2)}{7!}x^7 + \frac{a^2(a^2+2^2)(a^2+4^2)(a^2+6^2)}{8!}x^8 + \cdots \end{aligned}$$

On the other hand, expanding the exponential $y = e^{a \arcsin x}$ as a power series in a yields

$$y = 1 + (\arcsin x)a + \frac{(\arcsin x)^2}{2!}a^2 + \frac{(\arcsin x)^3}{3!}a^3 + \frac{(\arcsin x)^4}{4!}a^4 + \cdots.$$

Now, we compare the coefficients in the powers of a in both sides. Clearly, the coefficient of a is

$$\begin{aligned} \arcsin x &= x + \frac{1}{3!}x^3 + \frac{1^2 \cdot 3^2}{5!}x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + \cdots \\ &= \frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots. \end{aligned}$$

Moreover, the coefficient in a^2 is

$$\begin{aligned} \frac{(\arcsin x)^2}{2!} &= \frac{x^2}{2!} + \frac{2^2}{4!}x^4 + \frac{2^2 \cdot 4^2}{6!}x^6 + \frac{2^2 \cdot 4^2 \cdot 6^2}{8!}x^8 + \cdots \\ &= \frac{x^2}{2} + \frac{2}{3}\frac{x^4}{4} + \frac{2 \cdot 4}{3 \cdot 5}\frac{x^6}{6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\frac{x^8}{8} + \cdots, \end{aligned}$$

whilst the one in a^3 is

$$\begin{aligned} \frac{(\arcsin x)^3}{3!} &= \frac{x^3}{3!} + \frac{1^2+3^2}{5!}x^5 + \frac{3^2 \cdot 5^2 + 1^2 \cdot 5^2 + 1^2 \cdot 3^2}{7!}x^7 + \cdots \\ &= \frac{1}{2}\Big[\frac{1}{1^2}\Big]\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\Big[\frac{1}{1^2} + \frac{1}{3^2}\Big]\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\Big[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\Big]\frac{x^7}{7} + \cdots. \end{aligned}$$

Finally, we can also see that the coefficient of a^4 is

$$\begin{aligned} \frac{(\arcsin x)^4}{4!} &= \frac{x^4}{4!} + \frac{2^2+4^2}{6!}x^6 + \frac{4^2 \cdot 6^2 + 2^2 \cdot 6^2 + 2^2 \cdot 4^2}{8!}x^8 + \cdots \\ &= \frac{2}{3}\Big[\frac{1}{2^2}\Big]\frac{x^4}{4} + \frac{2 \cdot 4}{3 \cdot 5}\Big[\frac{1}{2^2} + \frac{1}{4^2}\Big]\frac{x^6}{6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\Big[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2}\Big]\frac{x^8}{8} + \cdots. \end{aligned}$$

We leave it to the reader to verify that these expansions agree with the expressions given earlier. The reader may notice the curious fact that, with regard to the second expression in each of the above expansions, as we pass from one odd power to the next (even one), all the numbers (except the 1's in the numerators within brackets) increase by one. This pattern persists for all higher powers.

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George Boole and Geometry

DES MACHALE

George Boole FRS (1815–1864) was the first professor of mathematics at Queen's College Cork, now University College Cork, from 1849 to 1864. Though a seemingly quiet, gentle and peace-loving man, he was involved in several controversies and disputes during that period. His problem, if problem it was, was that he could not stand idly by if he saw what he felt were abuses in the university system without speaking out against them in public. For many years he was at loggerheads with the then president of QCC, Sir Robert Kane FRS, which culminated in a Royal Commission being set up to investigate the state of the Irish university system ([5, pp. 171-174, 201-203] details Boole's whistleblowing, Kane's Behaviour, and the admonition of Boole; the author's biography [2] may be used as a general reference for Boole-related assertions and facts mentioned in this article).

Kane was an eminent scientist, chemist and environmentalist who had written several groundbreaking works including a monumental *Elements of Chemistry* and *The Industrial Resources of Ireland*, a book way before its time and still relevant today. Kane had also worked on the Bordeaux Mixture, a chemical fungicide which could have averted the calamitous Irish potato famine, but sadly not in time.

In Boole's opinion, and many others agreed with him, Kane was not content to be just a university president and administrator but interfered too much in the internal affairs of academic departments even down to the content of courses - he felt he knew better than the professors who had been appointed. Kane's own appointment as president was probably due to the fact that he was a Catholic, one of two senior staff chosen to counteract the nineteen other Protestant professorial appointments. These choices caused great controversy locally as QCC was intended largely for the higher-level education of the Catholic middle classes (all male of course). At the Synod of Thurles in 1851, QCC was condemned as a 'Godless College' by the Irish bishops, led by Archbishop John MacHale of Tuam. Incidentally, Kane also refused to live in Cork, because it is said that Lady Kane much preferred the social life and other delights of Dublin.

The report of the resultant Royal Commission of 1857 largely vindicated Boole's whistle-blowing activities but he was reprimanded for sending anonymous letters to the newspapers exposing the problems of university life in Cork. But there was a distinct tension between the two men for the rest of their lives. To add to the complication, Boole was allied to the Vice-President Professor John Ryall, the uncle of his future wife, Mary Everest, in a large anti-Kane faction at QCC. However, conflict between Boole and Kane had started much earlier. Kane had received from Colonel Portlock, Superintendent at Woolwich in England, a letter claiming that students who went to Woolwich from Queen's College Cork were not properly instructed in the elements of geometry.[1] Kane placed the letter in Boole's hands but took no further action. The

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[1]This letter seems no longer to exist, nor has the author ever seen it.

message however was clear - Kane, the practical scientist, considered the applications of elementary geometry to be of much more importance than the niceties of algebra for which Boole was now widely famous.

Boole was rightly furious and countered that he had never neglected to impart instruction in elementary and more advanced geometry. However, he stated that he was of the opinion that geometry received too much attention in university courses. Boole also added that he could recall only one student of his who had gone from Cork to Woolwich and that student had received the lowest possible mark at the sessional examinations.

The purpose of this article is to back Boole's assertion that geometry was not neglected at Queen's College Cork, and in fact formed a major part of the curriculum there. The mathematical courses of the 1850's in the QCC calendar clearly mention 'The Elements of Euclid with deductions from the propositions'. We cannot of course tell what was taught in class by Boole, and he was the only teacher in the mathematics department, but surely examination papers set by Boole during this time must reflect the content of these courses at all levels. Luckily, these examination papers still survive in the archives of the Boole Library at University College Cork [1], and we now present a selection of those questions on geometry, which may be of independent interest.

(1) Explain the nature and object of the science of geometry and define the following terms viz. Axiom, Postulate, Problem, Theorem; also the terms Line, Angle, and Surface.
(2) Define a rhombus. Show that the diagonals of a rhombus bisect each other at right angles.
(3) Find the locus of the centre of a circle which passes through a given point and touches a given straight line.
(4) Give Euclid's definition of proportion and show that it virtually involves the arithmetical definition.
(5) Show how to describe (construct) an equilateral triangle equal in area to a given triangle.
(6) Deduce expressions for the radii of the escribed circles of a triangle whose sides are a, b and c.

There was a strong emphasis on trigonometry too, especially with applications to surveying for agricultural students:

(7) Show how to find the distance between two inaccessible objects on a horizontal plane.
(8) State the chief difference in principle and performance between the sextant and the theodolite.

Boole seems to have had a particular fondness for conic sections, and there were whole examination papers on this topic, presumably following such a course.

(9) Show how to draw a tangent at a given point on a parabola.
(10) Find the equation of a tangent to an ellipse and show that it makes equal angles with the focal distances.

Boole seems to have included a course on spherical trigonometry, not much in vogue nowadays. Typical examination questions were:

(11) Prove that the sum of the sides [sic] of a spherical triangle is less than four right angles.
(12) Show how to find the distance between two points on the Earth's surface, given their latitude and longitude.

Other topics included a three-dimensional helix, alternative axioms to Euclid, coordinate plane geometry, and complete quadrilaterals. It would be difficult to imagine a student without a thorough knowledge of both basic and advanced geometry answering any of these questions. So, we find George Boole not guilty of the charge of having neglected the teaching of geometry at Queen's College Cork.

Other course and examination topics included calculus in abundance, differential equations, algebraic equations, commercial and mathematical arithmetic, and a strong emphasis on axioms and definitions in algebra. However, it is perhaps of more significance to list topics and subjects which do not seem to have been included in Boole's courses at Queen's College Cork.

(1) There is no mention of Matrix Theory, initiated by Boole's friend Arthur Cayley around this time. Not even determinants appear anywhere.
(2) Group Theory, also formalised by Cayley in 1853, is conspicuous by its absence.
(3) Quaternions, discovered by Hamilton in 1843, and on which Boole had written a paper in 1848, do not feature anywhere.
(4) Number Theory, particularly the results of Fermat, Lagrange and Gauss do not appear anywhere.
(5) Perhaps most inexplicably, Boole's own creations, including invariant theory, operator methods for solving both difference equations and differential equations and particularly his algebra of logic of 1847, seem not to feature at all in his courses.

Admittedly, none of these topics had yet reached mainstream mathematics, but one wonders if he discussed them with his more able students. Or were they considered too difficult or too radical for undergraduates in the 1850's? Maybe he was scared of what President Kane might say, if he found out that Boole was filling students' heads with silly algebraic notions. But then if Kane had had his way, we might never have had the digital revolution!

One wonders what George Boole would have made of the current Department of Mathematics at University College Cork where he laboured singlehandedly for fifteen years and to which he probably sacrificed his life by walking from Ballintemple in a blinding November rainstorm and lecturing all day in soaking clothes. Doubtless he would have been proud of his many successors and mathematical graduates who have contributed so much to the subject worldwide. He would have been delighted too that his old chair is now termed the George Boole Chair of Mathematics.

And we have not forgotten him either - there is the magnificent Boole Library at UCC, the Boole Window in the Aula Maxima at UCC, the Boole window in Lincoln Cathedral, Boole statues in both Lincoln and Cork, plaques on his many houses and the Boole Crater on the moon. Finally, there is Boolean Algebra, known to every mathematician on the planet, and the basis of the most extraordinary inventions of mankind - the electronic computer and digital technology.

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Gender Balance among Staff in Irish Mathematical Sciences Departments, 2020–2025

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Abstract. This article summarises the gender balance in staff cohorts in the mathematical sciences departments of several Irish higher education institutions, based on data collected as part of a project funded by the HEA's Gender Equality Enhancement Fund. This can be used to provide sectoral averages to which departments can compare their own data; and so aid them in preparing applications for Athena Swan Awards. In addition, data are presented for research, professional and academic staff, with comparisons between 2020 and 2025. We observe that, in the five-year period considered, there has been a significant increase in the proportion of women holding research and senior academic posts. However, we also observe that the proportion of academic posts held by women has decreased.

1. Introduction

This article presents aggregated data for the number and proportion of women and men employed in mathematical sciences departments in a selection of Irish higher education institutions (HEIs). Such data is useful for a variety of purposes, but especially for benchmarking in Athena Swan Award applications.

1.1. **Athena Swan.** The Athena Swan programme was launched in the UK in 2005 as an accreditation framework for institutions to assess their own progress and plans towards supporting gender equality in higher education. It was originally intended to advance the careers of academic women in science, technology, engineering, mathematics, and medicine (STEMM): initially, "*SWAN*" abbreviated "Scientific Women's Academic Network". Although the brand remains, the programme now has a much broader remit. Since 2015, it includes non-STEMM departments and supports all categories of staff and students.

In 2015, the programme was also expanded to include Ireland, accrediting awards for institutions and departments[1], and managed by Athena Swan Ireland.

Athena Swan is primarily associated with its awards. An award is made to an institution or department that makes an application that, in the assessment of a panel, meets certain criteria. To achieve a Bronze Award, the fundamental criteria are as follows:

(a) The application provides accurate and detailed quantitative and qualitative data on the unit's organisation, culture, and gender balance.

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[1]We follow Athena Swan terminology, and refer to such units as "departments" even if they are officially titled "Schools".

(b) The application demonstrates that the unit has collected and analysed that data, and reflected on the results. The process specifically requires that data be compared with relevant external benchmarking data.
(c) The report contains an Action Plan that is deemed to be sufficiently robust so that progress can be made on any issues raised.

Silver Awards recognise that the unit in question has provided evidence that its Action Plan has delivered significant results. A Gold Award is given to those institutions and departments that demonstrate a record of achievement in gender equality over a long period, and support others in achieving their goals.

Since applications are required to compare their data with the norms for that sector, this article provides such data for mathematical sciences departments in Irish HEIs. To date there has been no reliable set of data for the mathematical sciences sector in Ireland. However, data are available that provide an insight into the gender balance in Higher Education in Ireland: a 2022 report from the Higher Education Authority (HEA) states that, in 2020, 46% of both research and academic posts were held by women [1], and an interactive dashboard from the HEA shows that in 2024 in STEMM departments, 48.9% of research posts and 40.1% of academic posts were held by women [2]. As shown in the remainder of this article, the balance in the mathematical sciences is somewhat different.

1.2. **Institutions that contributed data to this report.** The aggregated data we present here are largely based on data collected for departments' own Athena Swan applications. The following are the departments (and their representatives) that contributed to this work, and provided data from their Athena Swan awards.

1. The School of Mathematical Sciences, Dublin City University (Eabhnat Ní Fhloinn and Ronan Egan). Athena Swan Bronze Award (2022).
2. The Department of Mathematics and Statistics, Maynooth University (Niamh Cahill and David Malone). Athena Swan Bronze Award (2023).
3. The School of Mathematics and Statistics, Technological University Dublin (Dana Mackey). Athena Swan Bronze Award (2022).
4. The School of Mathematical Sciences, University College Cork (Tony Fitzgerald). Athena Swan Bronze Award (2022).
5. The School of Mathematics and Statistics, University College Dublin (Isabella Gollini and Pauline Mellon). Athena Swan Bronze Award (2023).
6. The School of Mathematical and Statistical Sciences, University of Galway (Niall Madden). Athena Swan Bronze Award (2021).
7. The Department of Mathematics and Statistics, University of Limerick (Alan Hegarty). Athena Swan Bronze Award (2015 and 2019) and Silver Award (2023).

2. Methodology

The data we focus on here compares current data from 2025 to data from 2020, which is the last year for which we have a complete set of data from all contributing institutions, and offers an appropriate time interval to show any effects of HEIs' gender equality initiatives.

The 2020 data was collected from each university's Athena Swan Awards applications, with more recent data collected from applications in preparation or from data provided directly by departments. All data is aggregated, so no institution or department is identified.

The data provided for each department included the number of women and men in Academic, Research, and Professional, Managerial and Support Service (PMSS) cohorts; here the PMSS terminology follows that used by Athena Swan, and includes technical staff.

Furthermore, Academic staff numbers were further broken down into the career structure grades *Lecturer*, *Senior Lecturer*, and *Professor*. There is some variation between institutions in how these groups are named. Here we have followed the terminology used in the University of Galway as detailed in, for example, its 2024 Gender Pay Gap report [4]. We report on the *Lecturer Below the Bar*, *Lecturer Above the Bar* and *Lecturer* grades simply as “Lecturer”; the grade titled “Senior Lecturer/Associate Professor” is referred to as “Senior Lecturer”, while the *Professor In*, *Personal Professor*, and *Established Professor* roles are combined into a single “Professor” grade. We map career structures of other institutions onto that of the University of Galway by identifying the greatest overlap in pay scales. Specifically,

- grades reported as Assistant Professor, Lecturer Above the Bar, Lecturer Below the Bar, Teaching Assistant, Teaching Fellow, Non-Senior Lecturer, University Tutor, Hourly Paid Lecturer, and Assistant Lecturer, are all aggregated as *Lecturer*;
- grades reported as Associate Professor, Senior Lecturer I, Senior Lecturer II, and Senior Lecturer III, are combined under the heading *Senior Lecturer*; and
- grades reported as Established Professor, Full Professor and Personal Professor are aggregated as *Professor*.

3. Results

3.1. **All cohorts.** In Table 1, we present the aggregated gender makeup of staff in mathematical sciences departments across our contributing HEIs in 2020 and 2025. The data are also visualised in Figure 1.

	2020		2025	
Category	F	M	F	M
Academic	58 (27%)	155 (73%)	56 (24%)	178 (76%)
PMSS	23 (77%)	7 (23%)	19 (66%)	10 (34%)
Research	6 (17%)	29 (83%)	22 (34%)	42 (66%)

Table 1. Staff cohorts, by gender, in 2020 and 2025

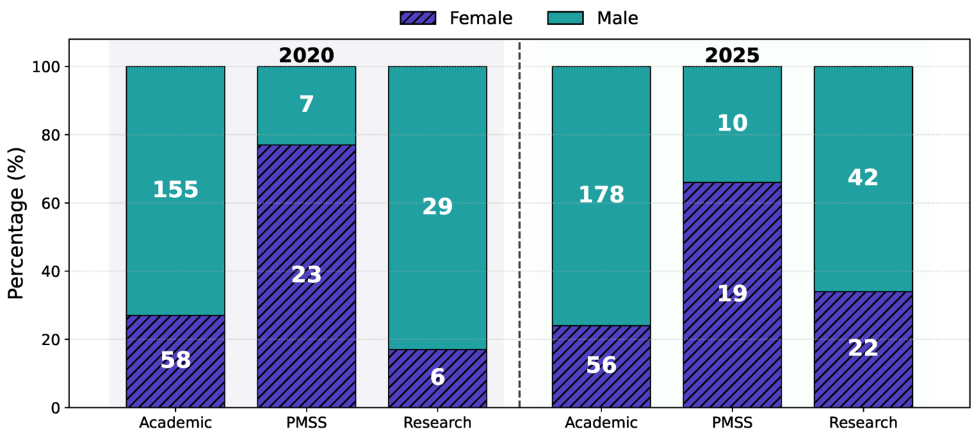

Figure 1. Staff cohorts, by gender, in 2020 and 2025

A number of observations can be made regarding this data. The over-representation of women in the PMSS category reflects broader trends that persist in Irish higher education institutions and disciplines [1]. The data suggest a slight shift towards equality, with the proportion of such posts being held by men increasing from 23% in 2020, to 34% in 2025. However, given the small number of individuals represented, this may not represent a definite trend.

Next we observe that there was a very large increase in the number of research staff employed in these departments: from 35 in 2020 to 64 in 2025, a jump of over 80%. Furthermore, the number of women in such posts grew from 6 (17% of the total) to 22 (34% of the total).

These trends, however, are not reflected among academic staff. Although the total number of people in academic posts grew by 10%, there was a slight reduction in the number of posts held by women. We investigate this more deeply in the next section.

3.2. **Academic grades.** As discussed in Section 2, we break down the Academic cohort into groups listed as Lecturer, Senior Lecturer, and Professor. The results are shown in Table 2, visualised in Figure 2.

	2020		2025	
Category	F	M	F	M
Lecturer	40 (29%)	98 (71%)	33 (24%)	105 (76%)
Senior Lecturer	15 (36%)	27 (64%)	13 (24%)	42 (76%)
Professor	3 (9%)	30 (91%)	10 (24%)	32 (76%)

TABLE 2. Academic staff grades by gender, 2020 and 2025

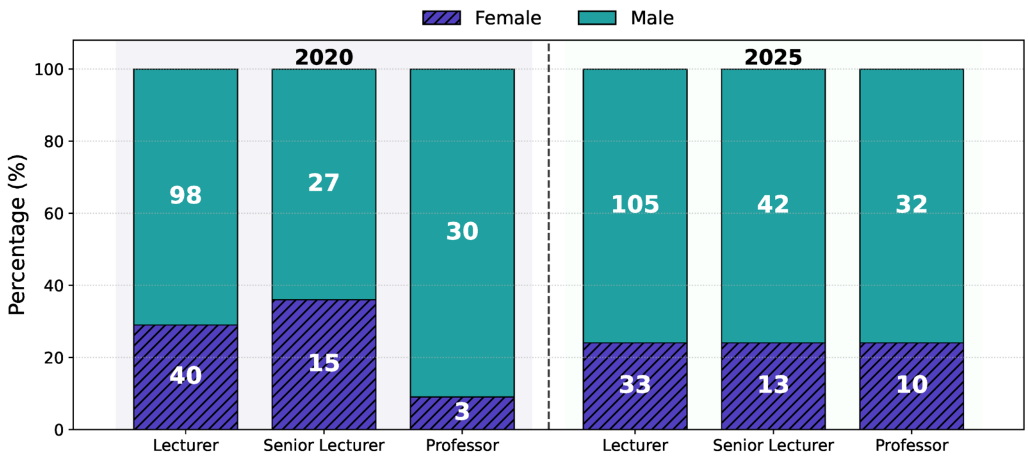

FIGURE 2. Academic staff grades by gender, 2020 and 2025 (by percentage)

Perhaps the most eye-catching statistic in Table 2 is that, in 2025, women hold 24% of posts in each of the reported grades. This masks some variation observed between institutions; however, to retain anonymity we don't present a breakdown by institution. Nonetheless, it supports the assertion that men and women may have equal opportunity in career progression, since the representation of each gender matches that of the overall cohort (as reported in Table 1).

It should be noted, however, that, at 24%, there are far fewer women proportionally in the mathematical sciences compared to the STEMM average (reported as 40.1% [2]).

Next we observe that, from 2020 to 2025, the number of female professors increased from 3 (and 9% of the total), to 10, suggesting an improvement on the "leaky pipeline" (a commonly used metaphor for the progressive loss of women from senior academic roles [3]). That is, more women are successfully progressing through the academic ranks and reaching professorial positions.

In an effort to make this clearer, in Figure 3 we present the same data as in Figure 2, but showing more clearly the absolute numbers in each grade. This makes the "leaky pipeline" of 2020 more obvious: there are far fewer women in Senior Lecturer roles than in Lecturer ones, and fewer again at Professorial level. In fact, in 2020, 69% of all women employed in academic posts were in Lecturer posts, compared to 26% and 5% in Senior Lecturer and Professor, respectively. In 2025, the corresponding percentages were 59%, 23% and 18%.

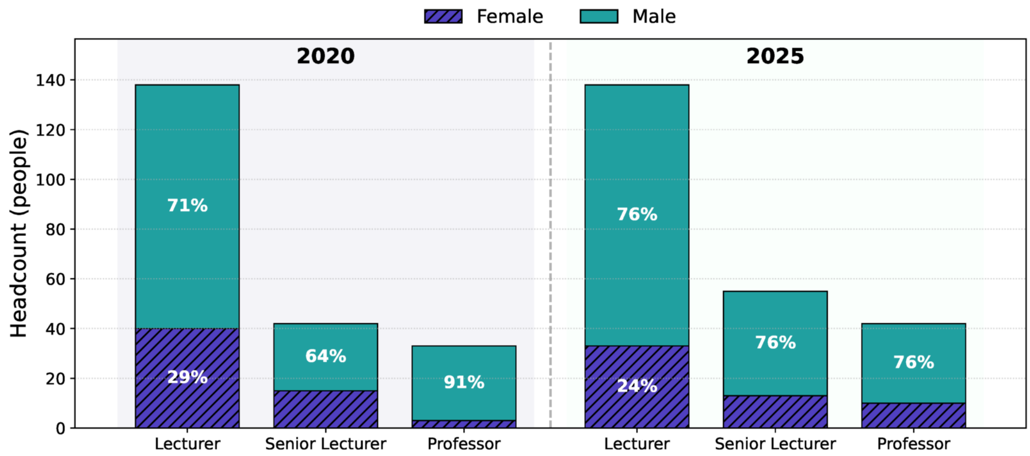

FIGURE 3. Academic staff grades by gender, 2020 and 2025 (by headcount)

Finally, we noted at the end of Section 3.1, that the total number of women in academic posts decreased slightly over the five year period, from 58 to 56. From the data presented in this section, we see that the increase of seven in the number of female Professors is more than offset by the decrease in the number of female Lecturers (also seven) and Senior Lecturers (two).

4. Conclusions

The data presented suggest that, while female academic staff have more equal opportunities for progression in 2025 compared to 2020, there are still issues regarding the recruitment of female academics.

We include here a small selection of action plan items listed across the HEIs' most recently published Athena Swan applications:

- Appoint a female professor.
- Increase the number of female applicants for research and academic positions.
- Conduct analyses of recruitment and promotion trends for academic staff and of recruitment trends for research staff.
- Increase staff awareness of maternity and parental leave supports.
- Ensure students have at least one female lecturer (role model theory).
- Increase the number of PhD students in general, with at least 40% being female.

Since many of these actions are already in train, and may have impacted on the results here, it is possible that they are having a noticeable effect, especially on the number of senior female academics.

However, much remains to be done. It is speculative, but not unreasonable, to suggest that fewer women were recruited to Lecturer posts between 2020 and 2025, compared to the period before 2020. If such a trend exists, it is possible that the growing numbers of women in research roles may contribute to reversing it.

The recent increase in the number of women in research roles should not be taken for granted, however: internationally, there has been a decrease in the percentage of female doctoral students in mathematical fields, (for example, from 32.6% in 2018 to 29.4% in 2021 across OECD countries [5]). The same study claims that the corresponding percentage in Ireland increased from 19.2% to 36% across this time period.

In terms of policy, Ireland appears to be performing comparatively well – by international standards. For example, the Inspire project concluded that Ireland has "widespread uptake of systematic and comprehensive gender equality efforts, with well-developed, solid and comprehensive legal and policy frameworks" [6]; the SheFigures index shows Ireland is one of the top overall performers in the OECD in its progress towards gender equality in Research and Innovation and has the highest proportion of HEIs taking measures towards gender equality [5]; and the Irish Research Council Gender Strategy and Action Plan notes that "women are accessing research funding across all career stages at rates which compare very well, and in many cases exceed the rates for men" [7]. Research funding from the HEA is linked to an institution's ability to reduce gender inequality [8], and the HEA Act 2022 requires each HEI to produce a gender equality statement [9].

4.1. **Further work.** While it seems that there has been some reduction in gender inequality in career progression in the mathematical sciences, the apparent imbalance in recruitment into academic roles needs to be monitored. Ireland's policies, and the recent increase in the number of women in research posts, should lead to more gender balanced recruitment, but it may take some time for this to be realised.

Further, while the number of female academics has remained almost unchanged from 2020 to 2025, this, of course, ignores details around potential imbalances in retirements, resignations, and recruitment. Teasing this out further would be interesting and useful, but would depend on obtaining data related to career progression, recruitment, and applications for posts. Some institutions provide such data in Athena Swan applications, but not enough for a detailed study.

While our data from the seven institutions around the country should be largely representative of the country as a whole, factoring in the data of other institutions, especially the more recently formed technological universities, should give an even more comprehensive understanding of the national landscape.

Furthermore, the authors are keen to devise a framework for analysing participation rates in postgraduate and undergraduate mathematical sciences programmes, and welcomes input from the Irish mathematical community. We believe that such a study would be timely: the Society of Actuaries in Ireland has observed the number of women entering the actuarial profession has declined, proportionally, between 2006 and 2020, from approximately 50% to 30% [10]. The authors of that study link this to the decline in the proportion of female students obtaining the highest attainment grade in Leaving Certificate mathematics examinations, as further examined by the Society in 2025 [11]. It would be valuable to establish if there was a similar decline in the number of female students entering mathematical degree programmes.

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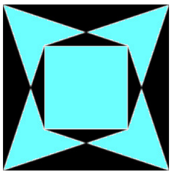

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Creating Geometry Problems

OWEN BARRON AND ODHRAN KEENAN

Abstract. A method of creating Olypmiad-style geometry problems is described, and is illustrated by two example problems.

1. Introduction

Over the past year, we have created geometry problems together while attending the Irish Team Selection Test training camps and other events. We have found that, although it may seem intimidating at first, the process of doing this is very accessible with some knowledge of Euclidean Geometry techniques and tools such as *Geogebra*. In this article, we will show the method by which we devised two of our problems, to 'demystify' the process of creating Olympiad Geometry problems.

To get started, we'll need some tools.

(1) *Geogebra* or another drawing software.
While not strictly necessary, using *Geogebra* can make claims much easier to spot, and much more certain. It's possible to draw new circles and lines instantly, as well as to hide unnecessary ones.
(2) A starting point.
This is a configuration on which you would like to base your problem. It could come from your imagination, another problem, or an interesting theorem.
(3) Pen and Paper.
For sketching and calculating! Used in angle and length calculations, as well as to record properties.

Now that we have our toolbox, we can begin discussing problems.

2. Problem 1

2.1. **Problem Statement.** Let triangle ABC have incentre I, incircle ω and circumcircle Ω. Let E and F be the tangency points of ω on AC and AB respectively. Choose S to be the point on Ω satisfying $\angle ASI = 90°$. Let N be the midpoint of the arc BC containing A, and P the intersection of NI with BC. See Figure 1.

Prove that AP, SI, and EF concur.

2.2. **Motivation.** We would like to note that much of the discussion in this problem and the next hinges on some knowledge of projective geometry. Some information on these techniques can be found in this handout [1]. Importantly, we will denote the cross-ratio of four points A, B, C, and D on a line or a circle, as defined in the above handout, as $(A, B; C, D)$.

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Many thanks to Anca Mustata for input and advice on making the problems, as well as for teaching us geometry over the past few years.

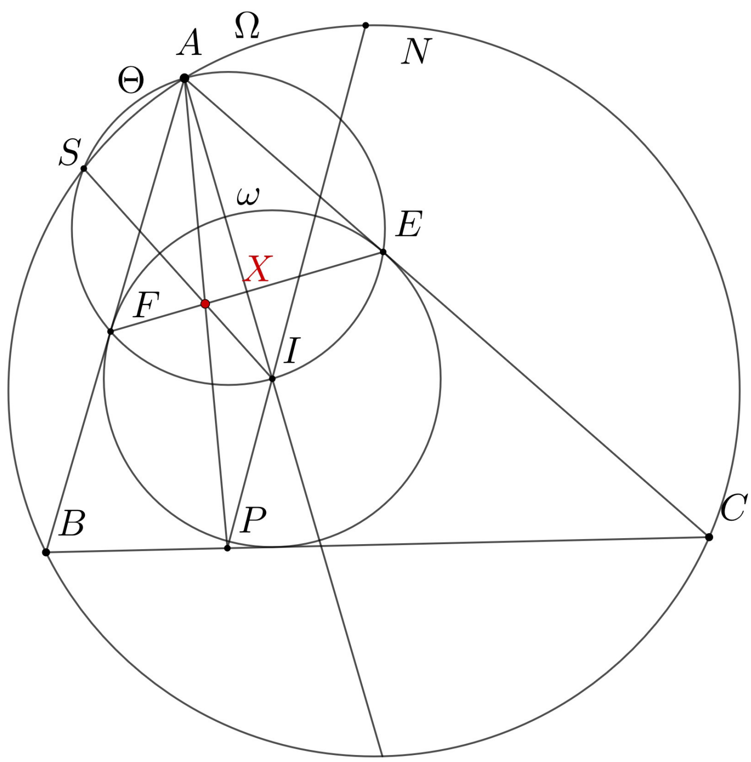

FIGURE 1. The setting of Problem 1

We started this problem with the following observation: when we take the incentre I of $\triangle ABC$, the circumcentre of $\triangle BIC$ lies on the A-angle bisector. The proof of this is left to the reader: try angle chasing. This implies that, when we take the circle of diameter AI, Θ, it will be tangent to the circumcircle of $\triangle BIC$, which we denote by Γ as shown in Figure 2.

Now we include some extra points. When we have a tangency, it seems natural to add in the tangent line. We do this and label its intersection with BC as T. By the so-called 'radical axis theorem', we know that AS, BC, and the common tangent to Θ and Γ concur at T.

We have two more 'set-up' points. We add in Q, the intersection of the other tangent from T to Γ. The reason behind this is symmetry: Γ is an important circle, and only including one tangent does not make full use of it. Our final point is N, as we defined in the initial problem statement. The motivation for adding N is the following fact: it is the intersection of the tangents to Γ at B and C. We leave the proof of this fact again to the reader.

At this point, we are able to notice something interesting in Figure 2: it seems as if N, I, and Q are collinear. Moreover, if we recall that P is the intersection of IQ and BC from the problem statement, it looks as if EF, SI, and AP concur.

We will begin by proving that N, I, and Q are collinear. This is relatively straightforward: we know that $(I, Q; B, C) = -1$, and this implies that N is on IQ as it is the intersection of the tangents to Γ at B and C.

We can also use projective techniques to prove the concurrency. Let Y be the intersection of AT and EF, and X be the intersection of SI and EF. First, $(A, I; E, F) = -1$ as $AEIF$ is a kite. Projecting through S onto EF gives $(X, Y; E, F) = (A, I; E, F) = -1$.

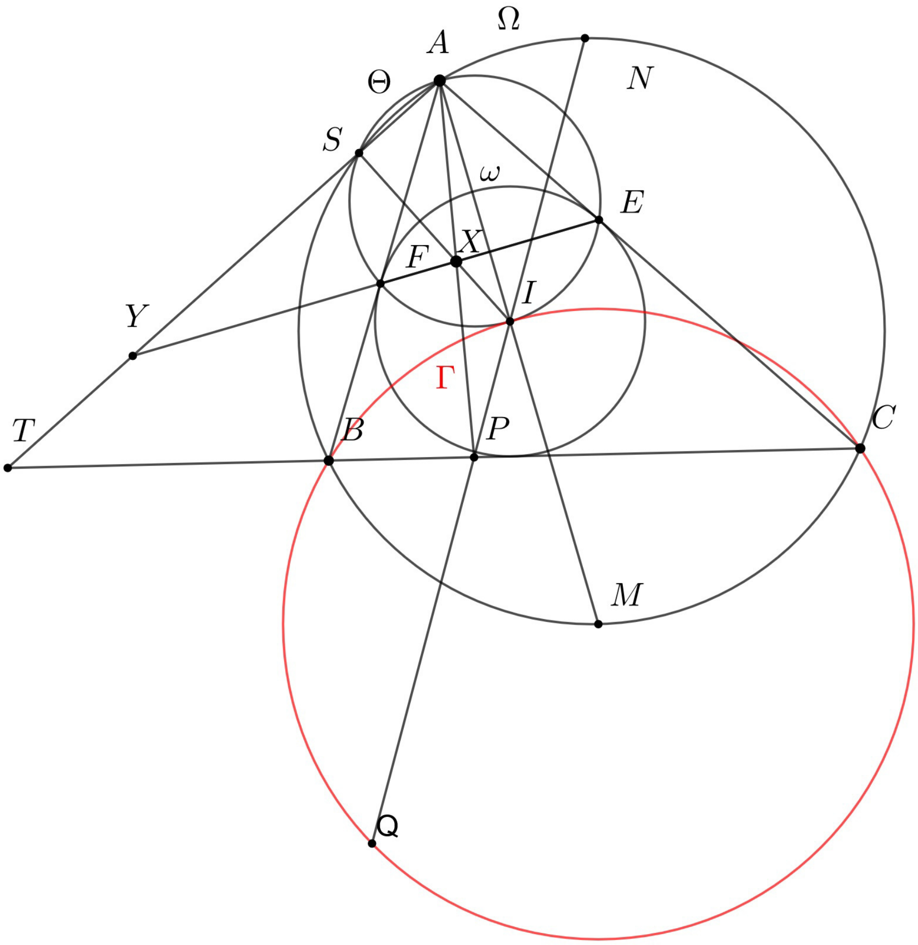

FIGURE 2. Extra circle and points for Problem 1

Now let X' be the intersection of EF and AP. Notice also that $(T, P; B, C) = -1$ and, by projecting through A onto EF, we get that $(Y, X'; E, F) = (T, P; B, C) = -1$. But this implies $X = X'$, and so we have proved the concurrency!

But what makes this problem interesting is how well we can hide this concurrency. Compare the problem statement to our final diagram above: very few of the extra points we have defined are included. This means students are challenged to 'find' the original configuration to solve the problem. The act of hiding intermediate steps can make problems more difficult and help students feel as if they are discovering the problem rather than solving it.

3. PROBLEM 2

3.1. **Problem Statement.** Let ABC be a triangle with incenter I. Define D, E, F to be the intersections of the incircle with sides BC, AC, AB respectively. Let D' be the antipode of D with respect to the incircle. Let the line parallel to BC through D' meet AB,AC at points B', C' respectively. Let the lines BC', and the line through I perpendicular to $B'C$, meet at G. See Figure 4.

Show that $\angle EGF = \frac{1}{2}\angle A$.

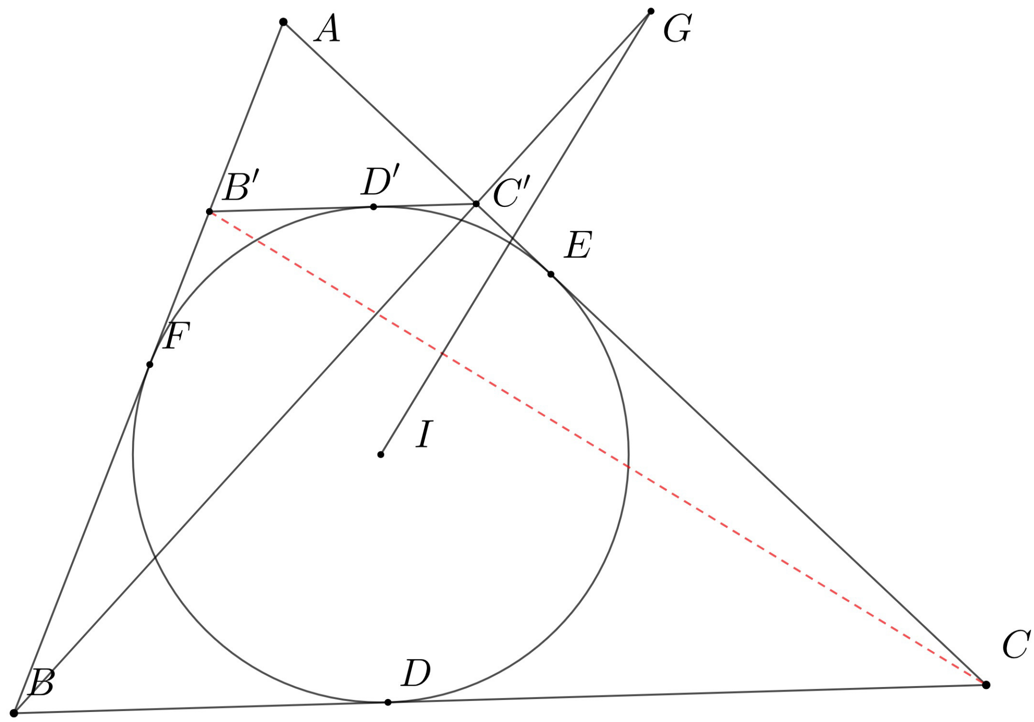

Figure 3. The setting of Problem 2

3.2. **Motivation.** As with most problems, This began with an interesting property noticed in a previous problem. In trying to solve one problem, one often encounters many interesting but typically irrelevant properties that end up unused in a final solution.

The source of the original problem is lost to our memory but, in attempting a solution, we began to notice some strange concurrencies. Originally we saw that lines BC', FD' and DE seemed to concur at point G. This leads to some follow up questions: are there any more lines passing through this point and why do they concur?

Following these questions, we began to play around with more possible lines. We can first note that all of our initial lines are either tangents to the incircle or directly related to tangency points of the incircle. This gives a major hint of the underlying feature at play in this problem (Poles and Polars).

Now, we have twofold motivation to consider the line through I perpendicular to $B'C$. Firstly, with enough experience using Poles and Polars, this configuration can be automatically recognised due to the number of tangencies. Secondly, we have already used line BC' in which both B and C' were intersections of tangents, hence why not consider the line through I (as every point so far can be defined using I and triangle ABC) perpendicular to the opposite of BC', CB'?

If these properties hold true we are starting to have a nice configuration! We will now confirm the concurrency of the four lines. We do this by using point redefinition: Let FD' and DE meet at G'. We will try prove that $G' = G$.

Here is where prior knowledge of Poles and Polars comes into effect. We note that the Polar of B' is FD' due to tangency and the Polar of C is DE again due to the tangency. See Figure 4.

By La Hire's Theorem we know that since G' lies on the polar of B', B' lies on the polar of G'. Applying this to C as well we get that both C and B' lie on the polar of G', and therefore CB' is the polar of G'. Hence as $B'C$ is the polar of G', the line $G'I$ must be perpendicular to $B'C$.

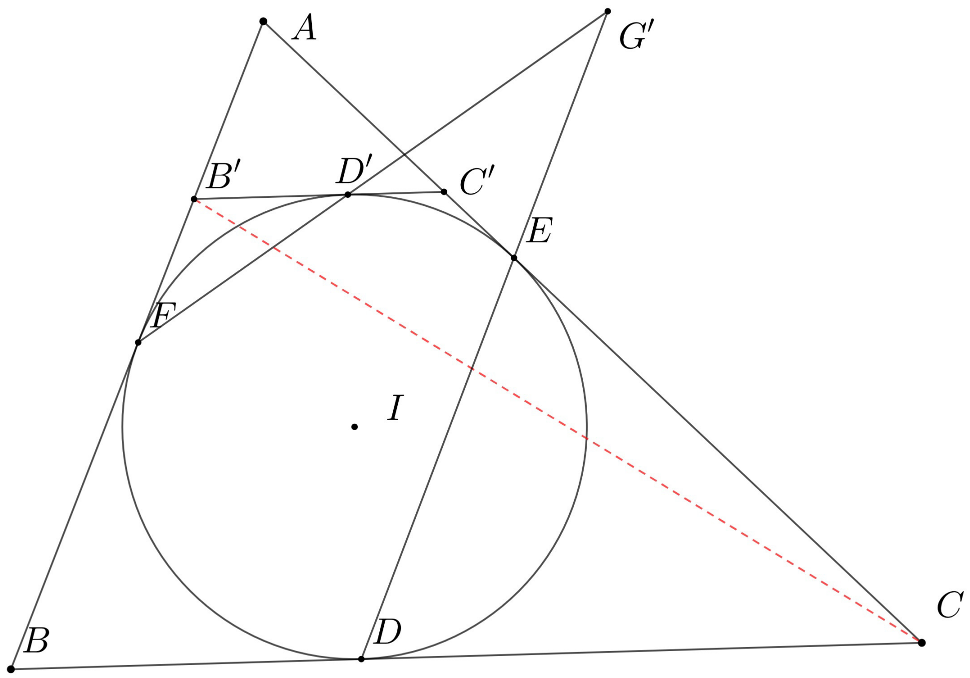

Figure 4. Intermediate diagram for Problem 2

Proving that G', C' and B are collinear turns out to be slightly more tricky. This can be achieved using complex numbers or Menelaus's Theorem, and it shows $G' = G$. The details of this step are left to the reader.

Again, we can note that a lot of geometry is simply seeing the obvious properties. This is a skill in and of itself and can be used in both solving and creating geometry problems. To finish our problem, we simply have

$\angle GFD = \pi/2$ as $\angle D'FD = \pi/2$ hence:

$$\angle EGF = \pi/2 - \angle FDE = \pi/2 - \angle AEF = \pi/2 - (\pi/2 - 1/2\angle A) = 1/2\angle A$$

In writing the problem statement, we try to remove as much information as possible. In this case, our simplification leads to only the intersection of two lines in the problem statement.

Much more can be done with this configuration to create other interesting problems. For example, we have the following properties: If G^* is defined as the symmetric point of G, then

- G^*, A and G are collinear;
- $G^*AG \parallel BC$;
- A is the centre of circle GG^*FE.

4. Conclusion

In making geometry problems a few things are required:

- An interesting property or initial observation;
- An understanding of why the property/properties work;
- A good diagram for proving and expanding upon the problem;
- Ideally, a clever way of concealing your initial construction.

Many observations can simply come from solving other problems in which extraneous properties are found. As was stated earlier, there are many properties even in this

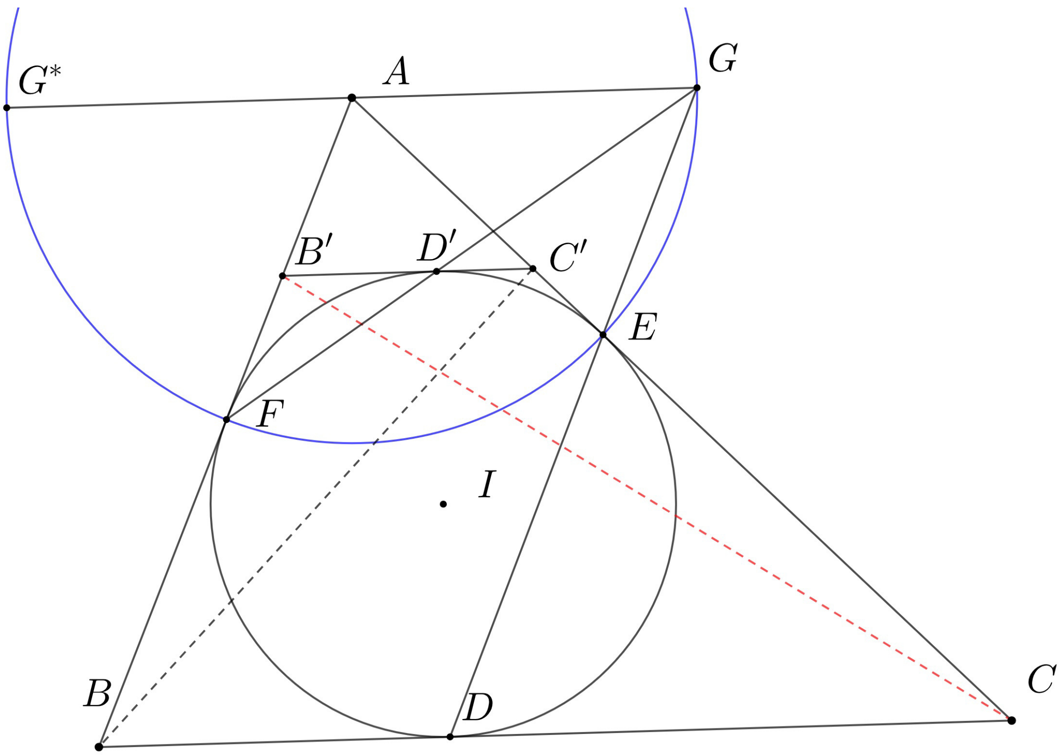

FIGURE 5. Final diagram for Problem 2

problem which were not used and as a result this problem could be expanded upon in the future.

The better you can hide or conceal the initial diagram as seen in Problem 1, the more interesting and rewarding the problem becomes.

We hope that this article has been informative and, above all, enjoyable to read. The creation of geometry problems is an art, but it is not one with an especially high barrier to entry: Hopefully, some readers will be inspired to try it out.

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Gesture in the University Mathematics Classroom

BRIAN TYRRELL-NIC DHONNCHA

Abstract. One of the enduring challenges in undergraduate mathematics education is understanding how students have internalised complex mathematical concepts. Often, however, students do reveal their conceptualisations – though not through speech or writing, but through their bodily actions. In this way, aspects of students' mathematical meaning-making become externally visible in the *gestures* they produce. In this paper, I outline the process of gestural analysis through examples drawn from undergraduate mathematical interactions. These episodes illustrate how attention to embodied cognition can enrich our understanding of mathematical thinking, and offer educators new ways to perceive and support students' conceptual development.

1. Introduction

When we teach mathematics, what is going on in our students' heads? This might be a question that every mathematics educator has wrestled with. Descartes argued the mind and body are dichotomous: distinct, though connected, entities [7]. This would imply that to speak to the mind, we should put aside the body – however recent advances in cognitive science challenge this partitionist view. Neurobiologists have found that even in the absence of external (physical) stimuli, our sensorimotor systems can remain active (e.g. [6]) – suggesting the brain may operate *thought* and *action* through the same neural circuitry. This idea is central to the theory of *embodied cognition*, which proposes that abstract thought (including mathematical reasoning) is grounded in bodily experience. Neurophysiology is one component of embodied cognition – though, admittedly, a hard one to observe in the classroom. In contrast, for the teaching and learning of mathematics, a very accessible manifestation of embodied cognition are the *gestures* of students and teachers.

Gestures are spontaneous, idiosyncratic, symbolic movements of the hands and arms that accompany speech [17], offering a nonverbal perspective into a speaker's mind. Gesture has been widely studied in teaching and learning contexts (particularly in mathematics education for children; see [9] for review) as a tool to aid education. Gestural analysis can be especially valuable in the context of service courses for STEM undergraduates (such as calculus, linear algebra, and differential equations for non-mathematics majors) where large numbers of students encounter complex, layered mathematical ideas for the first time, and must navigate them carefully. These courses offer an opportunity to bring gesture research into practical use: reading students' gestures can help instructors spot conceptual difficulties, and support clearer communication between teacher and learner. Gesturing itself can also help students think through ideas, and furthermore serve as a foundation for abstraction and formalisation in higher mathematics. By reflecting on some pedagogical experiences at a large, public, primarily undergraduate

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institution on the U.S. Western coast, I will expose gestural analysis as an active and engaging (and enjoyable) educational aid for teachers and students alike.

2. Taxonomy

Researchers classify the gestures used in mathematical discourse by a number of different criteria. One could frame McNeill's taxonomy [17] as the 'standard' framework, insofar as it has been examined and expanded on by several researchers in mathematics education (e.g. [8, 19]). In keeping with the holistic perspective of embodied cognition, McNeill proposes dimensions – rather than rigid categories – for labelling gestures: *deictic*, *iconic*, and *metaphoric*. (Edwards [8] later refined this framework for mathematics, proposing a split of the *iconic* dimension into *iconic-physical* and *iconic-symbolic*.)

Deictic or *pointing* gestures indicate objects, locations, and can include abstract referents: for example, after solving a problem, *pointing* upwards while saying "I have it!". (Here the gesture is not aimed at a physical object, but the abstract concept of the problem's solution.) In mathematics, we all use deictic gestures frequently: pointing at a step of a proof or calculation (to link in a verbal explanation), to highlight information in text or on graphs, to focus attention, or to indicate confusion. To add emphasis, a more pronounced gesture may be useful – a repeated tap on the board, a revolution of the index finger, or any simple, rhythmic hand movement synchronised with speech [3]. These "*beat*" gestures are used to communicate mathematical significance, which can be lost if only expressed verbally, and in general might not be recognised by undergraduates [16]. Given the difference in proof comprehension, reading and engagement between experts and novices (e.g. [12]), providing emphasis and direction using gesture might provide what Goldin-Meadow calls a "window to the mind" – but of the *lecturer*, for the student.

3. Iconic-Physical Gestures

In a first course on multivariable calculus, I structured a lesson around the use of *iconic-physical* gestures to probe students' concept images of *surfaces* – an approach I ultimately found more effective than relying solely on the students' mathematical language. Iconic-physical gestures "depict semantic content directly via the shape or motion trajectory of the hand(s)" [3], directly evoking characteristics of "a concrete object or event" [17]. (For example, using one's straightened hand as a knife to 'cut' an imagined circle into portions, when discussing fractions [8].) After a lecture on traces and contour maps, my students were presented with an in-class exercise from *Calculus, Eighth edition*, J. Stewart (2015) replicated in *Figure 1 (left)*: *A contour map of a function is shown below. Use this to make a sketch of the function.* A sketch of the function is included in *Figure 1 (right)*.

Contour maps are a vital tool for representing three-dimensional data in a two-dimensional format, and will be frequently employed by these students in their mathematics, physics, and engineering courses. In presenting them first as a collection of cross sections of a surface in $\mathbb{R}^3$, students can approach them as a physically realisable and gesturable object; even if the contour map represents abstract data, the students can benefit from grounding the problem in terms of their bodily experiences [18]. The learning outcome of this section is to reconstruct a surface in three-dimensional space from the data of a two-dimensional contour map: whether students have done so correctly can be determined by the teacher from the *iconic-physical* gestures they produce in support of their description of the surface.

Most commonly, students incorrectly identify this surface as a *cone*, which one can identify from the iconic-physical gesture in *Figure 2 (left)*. Here the student traces the shape of the surface in their imagined gesture space, and the straight trajectory of the

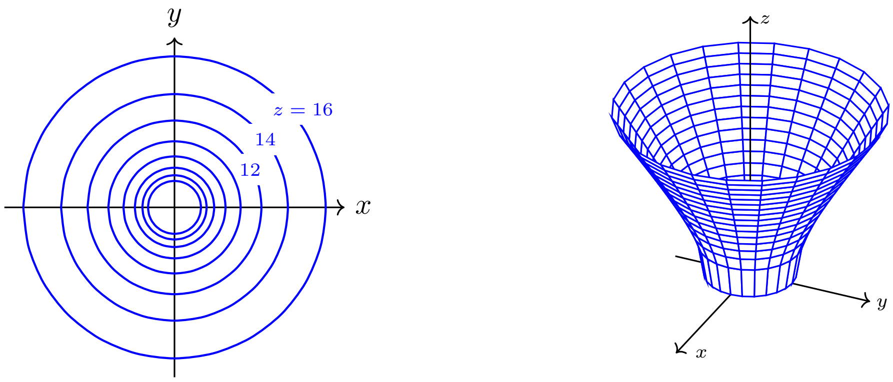

FIGURE 1. Contour map (Left) and graph (Right) of a hyperboloid of one sheet.

hands indicates a cone. This is immediately noticeable as different to the gesture one naturally produces when tracing the shape of a hyperboloid of one sheet, in *Figure 2 (right)*. As suggested by Alibali and colleagues [4], this misconception (or 'trouble spot') can be readily identified by the instructor, and – instead of further verbalisation, *corrected by gesture* – to re-establish shared understanding. This *gestural mimicry* [13] between student and teacher uses a resource accessible to the entire classroom to develop mathematical meaning, at a more intuitive level for the mathematical novice.

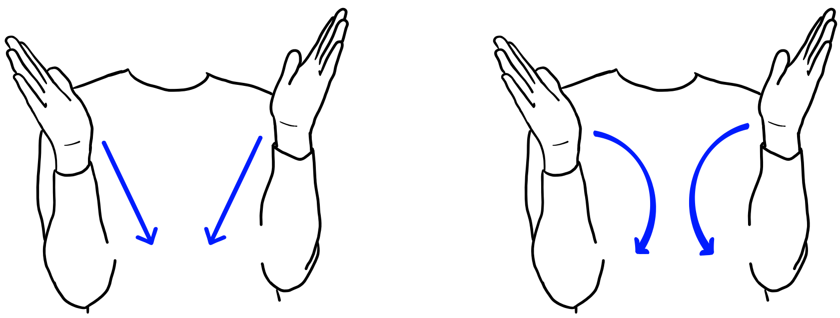

FIGURE 2. Iconic-physical gestures corresponding to the surface in *Figure 1*: a cone (Left) versus a hyperboloid of one sheet (Right).

The student's subsequent – often subconscious – *grounding action* of embodying the mathematical data of *Figure 1* when corrected beneficially influences future attempts at such problems: according to the theory of *action-cognition transduction* [1], thoughts and actions are a two-way street, with one influencing the other. By connecting *Figure 1 (left)* with *Figure 2 (right)*, I teach non-verbally – a practice that, as Hord and colleagues suggest, might be particularly salient for students with learning disabilities, such as dyslexia [11]. Undergraduate students with dyslexia report high-level difficulties (re)organising and (re)structuring information mentally [22]; difficulties that can make processing mathematical information verbally or in text especially demanding. Given a 364% increase over the last 15 years in the number of students with disabilities registered for support services in Irish HEIs (the largest faction of which are students with dyslexia) [2], this is not an abstract issue, but an urgent and growing concern for Irish higher education.

4. Iconic-Symbolic Gestures

When teaching material on matrices, elementary row operations, and row-echelon form, watch for an *iconic-symbolic* gesture used by students when reasoning to each other their row-reduction steps (see *Figure 3*). Iconic-*symbolic* gestures, as opposed to iconic-*physical* gestures, refer to "a symbolic, written inscription, which in turn represents a specific mathematical entity or procedure" [8]. They are a nod by Edwards that, in mathematics, a 'concrete referent' may not be a physical object or event – the 'crossing out' gesture when simplifying fractions in numerator and denominator [8], for example, is iconic-symbolic.

In my linear algebra class, for a displayed matrix (on the board, in their prewritten notes, or in a book — a setting where the students could not or would not physically write their calculations or steps), when nullifying all elements of the matrix below a pivot element, students would move their finger down in a "C" shape while describing the row reduction operation they are performing (see *Figure 3*). This was taken by all parties to mean 'perform (verbally expressed) operation to (verbally expressed) row'.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ -3 & 2 & -1 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 8 & 8 & 12 \end{bmatrix}$$

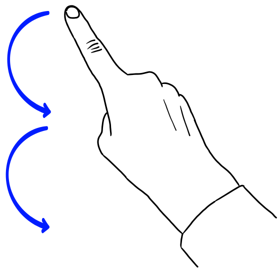

Figure 3. (Left) Steps towards reducing a matrix to row-echelon form. (Right) Accompanying iconic-symbolic gesture.

This is a deictic gesture, but can have another meaning as well – in *Figure 3*, often students would not be pointing towards R_1 (at the top of the "C") nor did the gesture conclude pointing specifically at R_2 or R_3. In addition, the accompanying speech often specified the rows involved more accurately than pointing from a distance. This gesture captures the *action* of performing an elementary row operation – moving from one step of the row reduction algorithm to the next.

It is also noteworthy that this gesture was sometimes performed *without* accompanying speech, when a student was self-explaining. This suggested the gesture has value as an *offloading mechanism* [9]: instead of mentally performing individual, isolated calculations, the student can store somatically the overall goal of "$R_3 \to R_3 + 3R_1$" through pointing at R_1, and with the "C" gesture, incorporating R_3. The intensity of activity in the university classroom – thinking, listening, writing, reasoning – would suggest that any opportunities to 'free up' mental space and reduce the cognitive pressure on students should be taken by the educator!

5. Grounded Blending

After drawing a curve on a blackboard x-y plane, there is a gesture that often appears when discussing the *tangent line* to a point on the curve: a straightened hand, hovering above the drawn curve at a roughly appropriate slope (*Figure 4, right*). Usually I do not need to specify that the hand is a stand-in for the line: there are a sufficient number of common features (shape, gradient, position, direction) between the mathematical concept and the iconic-physical gestural representamen that the association is automatic and natural. The formal mechanism to systematically analyse such internal mental representations is known as *conceptual blending*, though when applied to hand gestures tends to be labelled *grounded blending* [20]. 'Blending' is an internal process where an abstract mental space is combined with (the mental representation of) a real, physical space, along generic features common to both spaces. The visual schema commonly used (e.g. [8, 20]) to represent the conceptual blend of (1) mathematical objects and properties with (2) a mental representation of the hand and its location in space, is shown below (*Figure 4, left*). By decomposing the grounded blend into its constituent mental spaces, I specify a *meaning* for the gesture: the conceptual structure that underlies the physical hand movement.

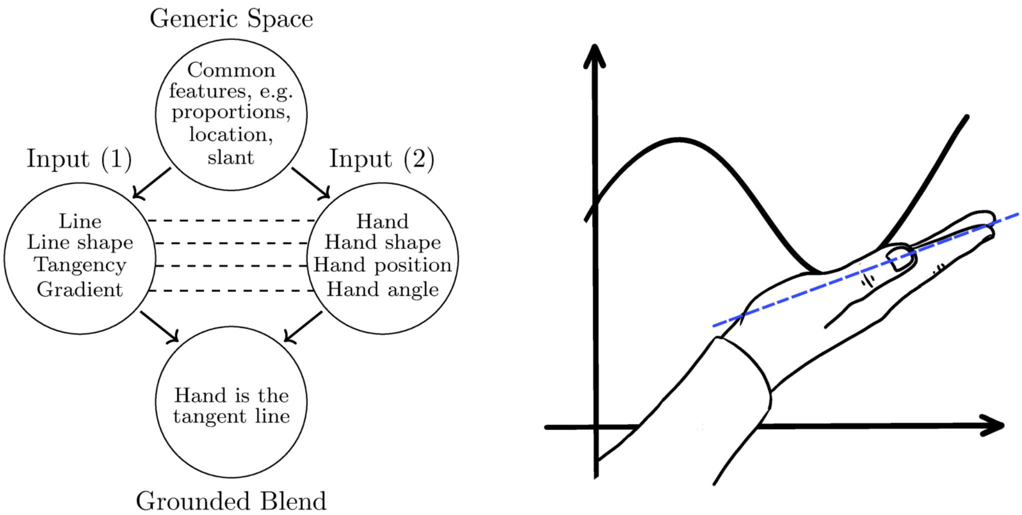

FIGURE 4. (Left) Grounded blend of (Right) the iconic-physical gesture of the hand representing the tangent line to a curve.

In any pedagogical setting – a lecture, tutorial, office hours, etc. – gestures are frequently used as communication tools between the student and the educator. Interpreting the meaning of a gesture is thus crucial to effective teaching and learning. As the final example shows, there are sometimes multiple layers behind a gesture, and teasing out the details becomes a fascinating exercise into students' mathematical conceptions.

6. Metaphoric Gestures

The discussion of *series* (and their convergence or divergence) constitutes a significant portion of the second or third course in single-variable calculus which STEM undergraduate students complete in the U.S. Successful execution of convergence arguments requires the student developing *analytical proof schemes*, of the Harel and Sowder hierarchy [10]: for a given problem, students must first determine an appropriate series convergence test (by "anticipating the outcomes of proposed transformations" [19] – *operational thought*) then perform a chain of *logical inferences* to utilise the test and draw an appropriate conclusion. While a particular example is not wholly *general*, considering the use and execution of a convergence test in abstract does require an awareness that these arguments may be applied across several members of some class of series. Thus, determining whether a student is engaging analytical proof schemes, as opposed to *empirical* or *external conviction* schemes, becomes essential to promoting successful and rigorous engagement with the material.

So, can one 'read' from a student's gestures that they have developed a sophisticated understanding of series? Similar to Nathan and colleagues [19], I argue that certain gestures can indicate when a student is operating analytical proof schemes. A recurrent gesture I noticed when discussing material one-on-one with a student, appearing across several examples, was a sequence of descending horizontal slices (depicted in *Figure 5*). This was typically presented by the student after selecting a series convergence test to employ; the gesture served to substitute (or supplement a summary of) verbal calculations and confirmations necessary to apply the test – and notably *did not* reference prior work or written inscription. This can be categorised as a *metaphoric* gesture – the slices had no physical referent in the students' speech, and no inherent mathematical relevance to series – representing to the student the concept of *proof*, through the metaphor "proving is motion" (cf. [20]).

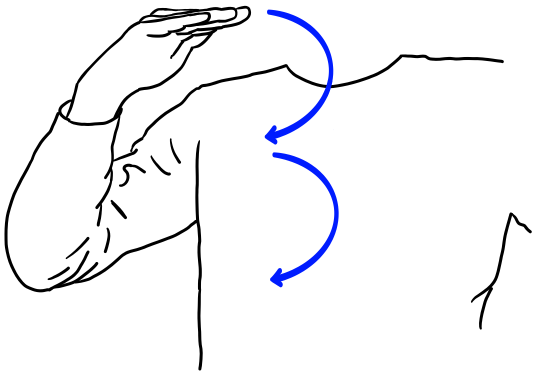

FIGURE 5. Descending horizontal slices made while discussing series convergence.

Metaphoric gestures depict semantic content through metaphor, presenting abstract concepts in a tangible, embodied form [3]. Parrill and Sweetser propose that metaphoric gestures can be modelled as the composition of two conceptual blends: the first linking an iconic gesture with a concrete subject (the *source* of the metaphor), and the second linking this subject with an abstract concept (the *target* of the metaphor) [20]. In this way, a metaphoric gesture is in fact iconic – but the icon the gesture references has no immediate referent correlating to a physical object, inscription, mathematical entity, or procedure, and instead symbolises the tangible component of an abstract relationship which the speaker has internalised through a metaphor. These types of gesture are commonly observed in mathematical discussion [8] as the language used to describe mathematical ideas can be highly metaphoric [15].

By providing a grounded blend decomposition of the *Figure 5* gesture, I argue the gesture provides an *iconic* reference to the process of navigating a mathematical proof – presented in text as a sequence of paragraphs, each outlining and validating a step in a single cohesive argument. *Figure 6, top left* illustrates how the (iconic) grounded blend between this gesture and the textual presentation of the proof may be established. This textual format also serves as a convenient vehicle for grounding the metaphor "proving is motion": the premises, intermediate deductions, and conclusion of the abstract logical argument can be mapped onto physical locations within the text, with premises appearing first, deductions unfolding in subsequent paragraphs, and the conclusion stated at the end. In this way, the student may establish a mental space (*Figure 6, top right*; "Conceptual Blend (Metaphoric)") where the source and target of the metaphor are integrated. The result (*Figure 6, bottom*) is a conceptualisation of the gesture as representing the process of proof.

Using the *Figure 6* blending model to analyse the *Figure 5* gesture subsequently allows us to argue that the student, in utilising this gesture, is employing analytical proof

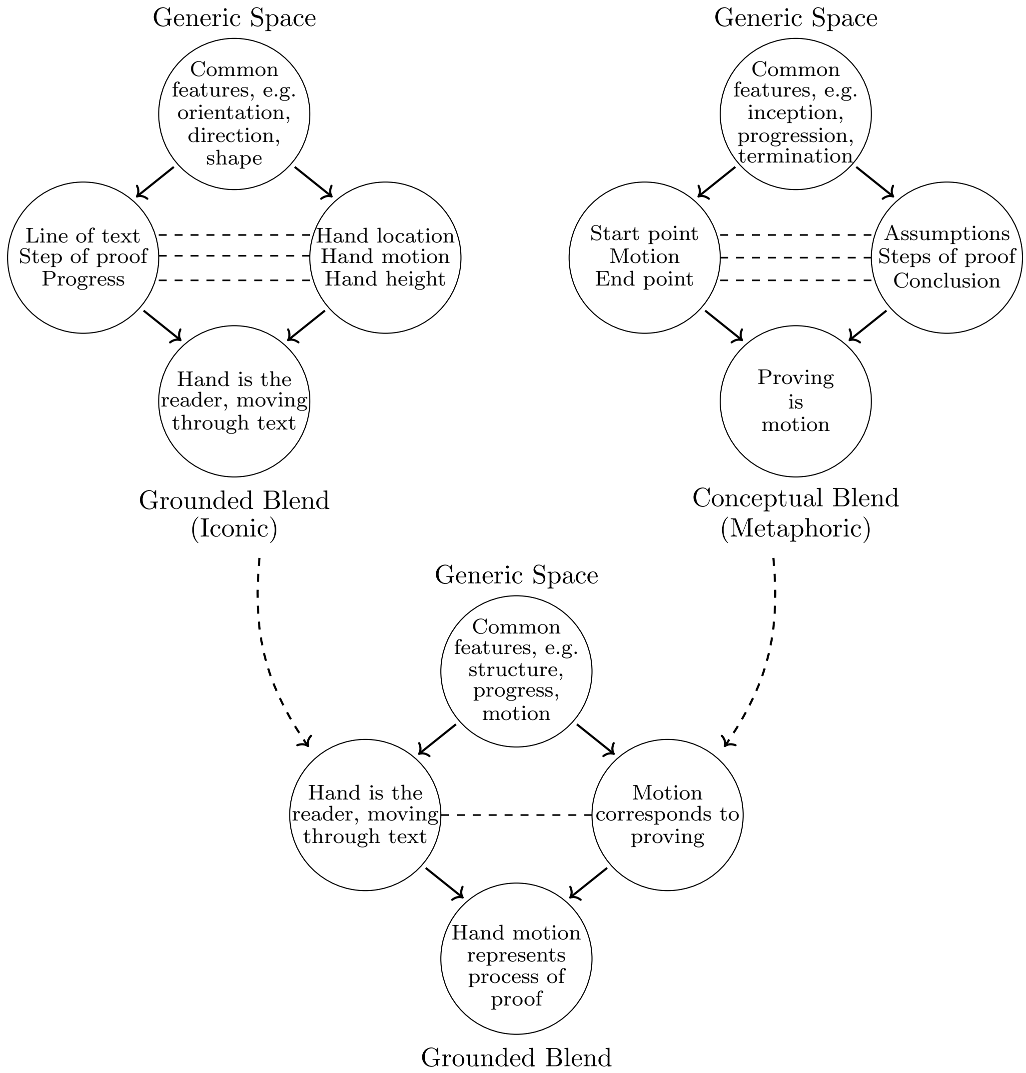

FIGURE 6. Assignment of meaning to the *Figure 5* gesture, using Parrill and Sweetser's blending model [20].

schemes. The *Figure 5* gesture illustrates a *chain of logical inference* by representing individual steps in a proof as consecutive horizontal slices. The gesture supports *operational thinking* (an "application of mental operations that are goal oriented and anticipatory" [10]) though its direction and dynamism: the gesture has a definite termination point (near the abdomen), and embodies expected incremental progress to that point through a sequence of steps. Finally, the prevalence of the *Figure 5* gesture across multiple examples of series and convergence tests indicates that the student considers this process a *general* one.

7. Discussion

The brain is the locus of physical movement, perception, and abstract thought: gestural analysis gives educators a unique perspective on the silent (and possibly subconscious) thinking and reasoning processes of undergraduate students. I have strived to make gestural analysis a standard part of my teaching routine, and will actively encourage my students to make gestures to facilitate their learning: to hold ideas in their hands, to offload cognitive demands, and as a means of confidence building and self-expression. Gesture can be used to bridge a mathematical language barrier: to recognise the idea a student wishes to communicate, when the student's mathematical register is incomplete. It further provides a "window to the mind" [9], offering insight into students' reasoning processes and the ways they construct mathematical knowledge, as understood through the theory of grounded blends. It is a particular joy to tease out the details of why a student may frequently make a certain (metaphorical) gesture: what metaphor does this embody? What understanding of mathematics does this reveal? And how may my own understanding of mathematics change? Even at research-level talks in abstract mathematics, one may find that the speaker's hands can say more than their voice.

Gesture is one element of a complex system where multiple modalities intersect to facilitate interpretation, integration, and action [18]. These examples further the argument that teacher-student communication is dependent on extra-verbal interactions, and mindfulness of such expressions can lead to more effective and fruitful instruction, as reviewed by Roth [21]. Embodiment can be seen as one tool a student and teacher have innate access to, when expressing thoughts to themselves and to others – especially useful if that expression is pre-emptively hindered by communication issues or learning disabilities.

In a related article, Kokushkin and I examine some of the cognitive challenges and navigational strategies that undergraduate students with dyslexia identify, when reflecting on their mathematical experience [22]. From interviews, we found a persistent theme of *gesturing* as a strategy students identify in mathematics to navigate cognitive load, and organise and structure information at a high level. These strategies are not unique to students with learning disabilities (e.g. [14]), hence when we make use of the embodied resources in the classroom's *semiotic bundle* (the resources used to communicate meaning – such as symbols, gestures, phrases, or relationships – in use by the classroom; [5]), we can engage more students in mathematics at a level where reading and writing skills are not the focus, opening up more inclusive ways of learning. I encourage the reader to reflect on their embodied mathematical experiences and incorporate their own insights – alongside the examples presented in this article – into future teaching, thinking, and learning.

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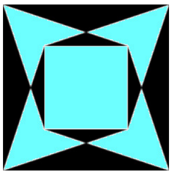

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Marriage and Quaternions : Hamilton's Eureka Moment

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Abstract. This is a discussion of Hamilton's discovery of the quaternions, placed in the context of that joyous day. When during the twentieth century very negative and caricatural views on Hamilton and on his marriage became prevalent, his elated reports about his eureka moment lost their credibility. To show that the quaternions were born into a good and happy marriage, this article will start with a discussion of Hamilton's marriage as found in his two main biographies, published by Robert Graves in the 1880s, and by Thomas Hankins in 1980, from which it appears that Hankins was strongly influenced by the gossip which evolved after Graves' biography. Next, Hamilton's account of the discovery will be given, in which also Lady Hamilton plays her role. It will be followed by a chronology which shows that Hamilton's reports about the discovery are coherent and consistent. Then Hankins' reconstruction of the discovery will be discussed; he doubted some of Hamilton's claims, and in his version, there is no eureka moment. Yet, Hankins' version appeared to be based on an erroneous assumption of chronology, and therefore is less coherent than Hamilton's reports. This article will conclude with a lacuna in both Graves' and Hankins' biographies, namely that, next to his intellect, Hamilton also used his astounding memory for his private life. And with a plea for a new biography of Hamilton.

1. Introduction

It is common knowledge that, when in 1843 Sir William Rowan Hamilton (1805-1865) found the quaternions, Lady Hamilton (1804-1869) was walking with him along the Royal Canal near Dublin. Some authors of biographical sketches of Hamilton ignored this fact, due to the gossip about Hamilton that evolved in the twentieth century [11] perhaps believing the marriage was unhappy, or just unimportant.[1] Others felt sorry for her, having to wait and watch her husband carve some formula on Broome Bridge.[2]

This article will relate that exceptional event at the Royal Canal, and some hitherto little known details will be added. Hamilton's reports about how this discovery came to be appeared to be so consistent that it was possible to make a chronology of that now famous walk, a chronology that permits a determination of the near-exact moment of the discovery in which, in a flash of insight, triplets were replaced by quaternions. It will be asserted that the discovery happened within the context of a happy marriage, and that Lady Hamilton's presence that afternoon was a reason for delight.

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I am grateful to Emma Rothwell and Rebecca Cairns of the Royal Irish Academy Library, for very valuable information about the minutes of the RIA council meetings of 2 and 16 October 1843. And to Estelle Gittins of Trinity College Dublin Library, for the surprising scan of Hamilton's pocket-book.

[1]See for instance J. Kuzmanovich, *William Rowan Hamilton* [users.wfu.edu], where Lady Hamilton is not mentioned. In a recent addition to the gossip, she "could not stand the strain of living with [Hamilton] and returned to live with her parents." J. S. Avery, *Lives in Mathematics*, 2021 [transcend.org].

[2]See for instance J. Baez, blog entry 2010, *Re: State-Observable Duality (Part 1)* [golem.ph.utexas.edu], "I pity his wife for having to stand by and watch him have his self-absorbed 'moment of genius'." Here, perhaps even more curious, it is mentioned that Lady Hamilton "eventually died."

To set such a description in context, it must be preceded by a discussion of Hamilton's two main biographies, the first one written by Robert Perceval Graves and published in the 1880s, the second one by Thomas Leroy Hankins, published in 1980. Graves' problems with the marriage of his adored friend will be discussed, and it will be shown that, strongly influenced by the contemporary gossip, Hankins had a less positive view on Hamilton than Graves had. Hankins openly doubted Hamilton's reports about how he had discovered the quaternions, yet it will be argued that Hankins' reconstruction of the discovery, in which he even changed details, was based on an erroneous interpretation of the chronology of that day. This article will conclude with aspects of Hamilton's ways of thinking that were missed by both biographers, but which throw a very different light on Hamilton and his marriage.

1.1. **The biographies and their consequences.** Graves' enormous biography consists of more than 2000 pages, filled with letters by Hamilton and his correspondents, to which Graves added comments. He mentioned that the Hamiltons had been "attached" spouses, and he called Lady Hamilton a "good woman" [2, 335]. Yet he was very critical of her; he believed that she should have made Hamilton stop drinking alcohol [2, 505], but had been too "weak of body and mind" to do so [10, 218-219, 381]. His criticisms and laments led to a train of gossip, in which during the twentieth century Hamilton became ever more alcoholic and even depressed, and instead of having been too weak to regulate her husband's drinking habits, Lady Hamilton became incapable of handling the entire household.

Hankins' biography is different; consisting of about 500 pages, Hankins gave his views on Hamilton and his work. He barely distinguished between opinions, conclusions and facts and, since in 1980 it was not yet possible to easily check statements online, his views came to be seen as the truth. This biography was especially detrimental for Lady Hamilton; Hankins began his section about her by stating, "Helen Bayly remains a shadowy figure in Hamilton's life in spite of all the references to her in the manuscripts" [9, 114].[3] Following the contemporary gossip [11, 16], he assumed that "when ill, or when Hamilton was away, when the burden of managing the observatory household became too great for her, she often went to stay with her mother at Nenagh or to [her sisters at] Dunsinea or Scripplestown" [9, 116],[4] something Graves never claimed. For three longer periods of time she indeed was away from home, but it had nothing to do with the household, and after 1842 she never again was away for longer than a few days.[5] Hankins closed the section by turning Graves' criticisms into a fact, "If [Hamilton] felt her weakness of health and character as a burden, it was a burden that he willingly accepted" [9, 126].

Thereafter the gossip about Lady Hamilton became extreme; nowadays she is described as a "pathologically shy and timid", "semi-invalid", "mentally limited" "local lass", a "country preacher's daughter" with "neurotic and probably slovenly tendencies" who was "unable to provide an orderly domestic regime," causing her husband to live in "squalor."[6] Such scorn further aggravated the views on Hamilton's private life, and consequently on his reports about the discovery of the quaternions; the fact that his wife had been present lost its lustre. This article aims to show that it was the trust

[3]There is some unavoidable overlap with the 2018 articles *A most gossiped about genius*, [11], and *Helen Bayly and Catherine Disney*, [doi.org/10.33232].

[4]Communal sleeping still was customary then [2, 273], and with only candles as sources of light, it is quite understandable that when Hamilton was away, Lady Hamilton preferred sleeping with her relatives in the neighbouring houses over sleeping alone in a necessarily very dark observatory.

[5]For why Lady Hamilton was away from home three times, and that it had nothing to do with the household, see [10, 178-188, 204-211].

[6]For these quotes, see the blog entry of 25 April 2020 [annevanweerden.nl].

and commitment between husband and wife that allowed for the peaceful atmosphere during their walk along the canal, in which Hamilton had his eureka moment.

1.2. **A psychological discovery.** One of the reasons for the stark difference between the positive view on Hamilton's marriage that emerges from his letters, and the current very negative view on it, is that both Graves and Hankins missed the fact that Hamilton, in 1832, discovered how to cope with feelings of loss [10, 119-120].

In August 1824 Hamilton, having just turned 19, had fallen in love with Catherine Disney and, in February 1825, he wrote a Valentine poem for her [1, 174]. Yet apparently becoming aware of Catherine's feelings for Hamilton, her family decided to advance the presumably already agreed marriage with her brother-in-law, William Barlow [12, 38]. Hamilton did not know that Catherine was coerced into this marriage, and assumed she married gladly; in a 'Farewell' poem, which he did not send to her, he prayed that her lot on earth would be a "younger heaven" [1, 185], as he had also written in the Valentine poem [1, 176]. Hamilton was devastated, and tried for years to bear his melancholy with calm, which at times was very difficult.

In December 1831 Hamilton fell in love with Ellen de Vere. After her indirect but certain rejection, he became even more melancholic than he had been before [1, 515], even to the extent that friends and family became worried [1, 528]. On 15 August 1832 he wrote to his friend Lady Campbell, who knew about Catherine Disney [12, 51],[7] "I had another affliction of the same kind and indeed of the same degree, except that my mind had been a little better disciplined to receive it [...]. I find that I am even now indulging too much my habit of dwelling upon painful recollections, instead of exerting myself to the utmost to control them by other thoughts" [1, 601], [10, 117].

Indeed, shortly before he wrote this letter, Hamilton had started to realise that he was living a "passion-wasted life" [1, 595]. He discovered how to cope with such feelings; that "action" instead of "meditation" was the "divine specific against Despondence" [10, 271],[8] which is what any therapist today would recommend to someone like Hamilton, a self-assured man who had an attached and secure childhood but did not know how to overcome such a loss. But that kind of talk therapy did not exist yet, and it indicates that Hamilton also used his enormous intellect for matters of life outside mathematics. In October 1832, his health having been greatly restored by this "power of hope" [2, 3], he found conical refraction, for which he would be knighted in 1835,[9] and fell in love with Helen Bayly [10, 121-122]. Although thereafter Hamilton could be troubled, worried, distressed, and at times melancholic, he never again drowned in such feelings.

That was shown again in October and November 1853, when Catherine Disney was finally able to tell him that she had wanted to marry him but had been coerced to marry Barlow, and died some weeks later [12, 83]. Hamilton became utterly distressed, now knowing that he had lost her to an unhappy and even forced marriage, yet this time he did not become melancholic; he started to write very many letters, according to Hankins to "unburden himself of grief" [9, 353]. Hankins concluded that it had "salutary effects" [9, 357], and apparently around June 1854 Hamilton was his happy self again [10, 307]; in those socially very strict years not having been able to talk about it, writing all these letters had been his "action instead of meditation" against despondence.

[7] In those times, reputations were easily destroyed. Only some people very close to Hamilton knew her name until in 1885 Graves revealed it to the world [2, 708], [3, 662].

[8] Hamilton wrote this to Catherine Disney in 1848 as a consolation [2, 611]. It did not help; at that time he still did not know that she had been forced to marry, but could not reveal that [12, 69]. In February 1833 Hamilton also gave this advice to his then wife-to-be, "the will has an influence [against gloomy fits]: there is such a thing as indulging in dejection, and on the other hand a persevering resolution against such indulgence effects much in time" [2, 22]. In Helen Bayly's case, this may have been very helpful advice; no biographer doubted the happiness of their early years.

[9] 'Mrs. Hamilton' then became 'Lady Hamilton', as she will be called hereafter.

But his friend Aubrey Thomas de Vere, Ellen de Vere's younger brother whom Hamilton had "won from the wreck" of her rejection [1, 511], as well as Graves and Hankins, missed this life-changing discovery. De Vere philosophised about passion and suffering [1, 615]; he loved the poems following Hamilton's loss of his sister, and about his love for Helen Bayly and the fear she also would reject him. In 1879 he wrote to Graves, "I suppose if [Hamilton] had not married [Helen Bayly in 1833] that remarkable outburst of poetry which characterized the year 1832 would have continued, and left large and important poetical results behind it."[10] That Graves missed it is surprising, because he knew that Hamilton more often had been occupied with "analysing his own character" [2, 247], [10, 189-190]. Yet he sometimes openly questioned Hamilton's views on himself [10, 190-192], and at times even disagreed with the choices Hamilton made for himself [10, 115], though always within the context of his adoration for his friend. That Hankins missed it appears from his conclusion that Hamilton "was saved from despair by always being able to work" [9, 112, 116].

2. The Hamilton marriage

William Rowan Hamilton and Helena Maria Bayly had met in the townland of Scribblestown, about eight kilometres northwest of Dublin city centre, where Hamilton lived at Dunsink Observatory since October 1827. Helen Bayly was from Nenagh, Co. Tipperary, and apparently regularly visited two of her elder sisters, Jane and Penelope Bayly, who also lived at Scribblestown [2, 2]. The sisters had married two brothers, Henry and William Rathborne; Jane and Henry lived at Dunsinea, Penelope and William at Scripplestown House. Both houses were only about a kilometre away from the observatory, and Hamilton loved to walk in their fields [1, 443], [2, 183].[11]

The Bayly family was from Ballinaclough, Debsborough, a few kilometres southeast of Nenagh,[12] and it is said about them that "unlike the national stereotype of the time of landlords being part of an elitist landed gentry, the Bayly's were known in Ballinaclough as good and fair landlords. They were considered to be the poor man's friend."[13] Lady Hamilton's father, Henry O'Neale Bayly, was an alumnus of Trinity College Dublin,[14] and, having become a clergyman, had lived since 1803 with his wife Anne Penelope née Grueber in Nenagh, where he was rector[15] and where Helen Bayly was born in 1804. Apparently around 1820 they built Bayly Farm, close to Debsborough House, where Henry's brother William Bayly lived.

Henry and Anne had twenty-three children, but not all names are known; some of them may have died young.[16] Henry died in 1826 at Scripplestown, the home of his daughter Penelope Rathborne, Anne died in 1837 at Dunsink Observatory, the home of her daughter Helen Hamilton.[17] From Graves' biography it seems that Hamilton had befriended especially William and Penelope Rathborne; although the Hamiltons also

[10]This letter is not online. It is kept at the Hesburgh Libraries of the university of Notre Dame. See the blog entry of 31 March 2018 [annevanweerden.nl].

[11]For a map of Scribblestown, with the observatory and the neighbouring houses, see [10, 158].

[12]For the townland Ballinaclough or Ballynaclogh, see Placenames Database of Ireland [logainm.ie]. For the Bayly Estate, including Bayly Farm and Debsborough, see Landed Estates [landedestates.ie]. For *Lady Helena Maria Hamilton's Bayly descent*, see [annevanweerden.nl].

[13]Silvermines Historical Society [silvermineshistoricalsociety.com].

[14]*Admissions Records* Trinity College Dublin p. 24 [doi.org/10.48495], also *Alumni Dublinenses* p. 51 [doi.org/10.48495].

[15]In the Anglican church, then the Established Church, rectors, vicars and curates were parish priests, with rectors the highest in status. Which priests were ordained depended on the parish.

[16]See William Edwin Hamilton's *Peeps at my Life* [archive.org]. For Helen Bayly's known siblings, see A. Young's 2021 blog entry *N is for Nenagh* [anneyoungau.wordpress.com].

[17]The British Newspaper Archive, 4-7 February 1826 [britishnewspaperarchive.co.uk], and 5-6 October 1837 [britishnewspaperarchive.co.uk].

were in close contact with the Rathbornes of Dunsinea [2, 273], [3, 102], Hamilton called William Rathborne his "private friend" [2, 101]. In 1831, Hamilton sent an invitation to Penelope Rathborne for tea [1, 442], asking her to bring her mother and sister with her; this is the first time Helen Bayly is mentioned in Graves' biography.

2.1. **Requisites.** Starting his section about Helen Bayly [9, 114],[18] Hankins gave parts of letters written by Graves in 1873 to Ellen de Vere, from which it appeared that Graves had come to loath Lady Hamilton; he wrote about her "poverty of mind" [9, 114, 414]. That may have started when, after Hamilton's death, Graves began to read the family papers to prepare the biography; he had been far from negative about the marriage in his 1842 *Portrait* of Hamilton,[19] and even as late as 1858, when he was visiting the observatory, he had "drank the health of Lady Hamilton" [3, 548].[20] Now knowing she had only written a few, rather short letters, the highly literary Graves must have been dismayed. From her letters it was very clear that she did not like writing them, she just wrote them because Hamilton wanted to receive letters from her [9, 119].

For Hamilton that was enough. Literacy was not what he had been looking for in a wife; his highest values were piety and truthfulness, and his requirement was that she was interested in his work [10, 152, 182].[21] He fell in love with Ellen de Vere because of her enthusiasm about astronomy [1, 459, 460], and he had stopped contemplating marriage with Louisa Disney, a younger sister of Catherine, because of her disinterest in science [9, 56]. He must have been well aware that he could be so deep into his "mathematical trances", as he called them when writing to Aubrey de Vere [1, 517], or in the "pains and pleasures of thought-birth," as he wrote in an ante-nuptial letter to Helen Bayly [2, 22], that she would often be unable to reach him even if he was close-by, or would have to go to bed alone; then a quite uncommon expectation of marriage.

Helen Bayly was everything he asked for: pious, truthful, and interested in his work [2, 14, 121], moreover, she had people she trusted in the neighbourhood. Yet having visited her sisters regularly, she must have seen how Hamilton lived his daily life at the observatory. Next to periods in which he was immersed in his mathematics and worked through the night, he was regularly visited by eminent people [1, 573], who usually brought their own staff [10, 175], which made managing that household a difficult task. Hamilton's three sisters Grace, Eliza and Sydney still took care of it [1, 603],[22] but when she would marry him, his sisters would leave the observatory, as was common then, and Helen must have doubted whether she would be able to cope with that.

She could very suddenly fall very ill [10, 197-198], and the medical treatments of those days such as calomel, raising blisters and shower baths[23] hardly alleviated her situation. Because visits often lasted for days, and sometimes even weeks, having to undergo such treatments while having eminent guests, or being a guest at someone else's

[18]The woman in the photo indicated as Helen Bayly [9, 123], was Hamilton's daughter and youngest child Helen Eliza Amelia.

[19]*Our Portrait Gallery – Sir William R. Hamilton*, The Dublin University Magazine, 19 (1842), 94-110 [babel.hathitrust.org].

[20]For placing Hamilton's remark on that page in context, see [10, 475].

[21]That Helen Bayly was truthful can for instance be seen in the ante-nuptial letters, when she refused to become "patient Griselda," which Hamilton called "courageous and candid" [10, 150]. Moreover, in a sonnet written in November 1832, Hamilton called Catherine Disney, Ellen de Vere and Helen Bayly "Sweet Piety, Enthusiasm, Truth," respectively [2, 4].

[22]The fourth and youngest sister was Archianna. Their mother Sarah Hamilton née Hutton died in May 1817 [1, 14], their father Archibald in December 1819. From 1818, Archianna lived with maternal relatives in Ballinderry [1, 65, 190].

[23]Shower baths had been installed at the observatory [9, 415]; bathing was recommended for "preserving health," and for "the cure of diseases." See p. 1211 of *An Encyclopædia of Domestic Economy* [archive.org]. For the shower baths, see p. 1220.

house, may have been a terrifying prospect. She also could not guarantee to uphold a high society Victorian life, full of dinners, parties and balls, and she asked for a "retired life at the Observatory" [10, 141, 387]; when he accepted that, she decided to marry him. She thus gave him a comparatively quiet life at the observatory, and created the circumstances in which he could do his mathematics the way he had to.

2.2. **"Layers of love and Algebra".** William Hamilton and Helen Bayly married on 9 April 1833 in Ballinaclough Church, less than two kilometres south of Bayly Farm.[24] Hamilton was very happy; a week before the marriage he had written from Bayly Farm to his former pupil Lord Adare, "I fear I may have forgotten something amid my alternate layers of love and Algebra, as I have lately been busy in both" [2, 42]. About two weeks after the wedding they went to Dunsink Observatory [2, 43], where the new Mrs. Hamilton had a difficult start. She fell ill in the summer and, not yet knowing she was pregnant since August [10, 170], in September she became "alarmingly ill" [2, 60], while all that time the observatory was being renovated [10, 162].

But eventually she felt at home at Dunsink; in January 1834 uncle James Hamilton wrote from Trim, "Your last letter was very gratifying, excepting the account you give of the health of your amiable wife. ... I know really scarcely anyone who seems more universally to have "won golden opinions" from her extensive circle of acquaintance, as letters from, and casual communication with, almost every friend we have testify. Of her present guests and your aunt she has won hearts – and hearts not easily won. I am preparing to yield mine, or rather have done so" [2, 72].[25]

3. An "obscuration" according to Graves

In the aforementioned 1879 letter to Graves, De Vere had also remarked, referring to a sonnet written by Hamilton on 16 June 1837 [2, 199], that "up to that time, A.D. 1837, Hamilton's affection for his wife had not waned. Indeed I do not know that it ever did,"[26] and that was something Graves could not ignore in his biography. Having introduced her at the beginning of the second volume as having been of "pleasing ladylike appearance" [2, 2], when describing 1840, Graves wrote that "[Hamilton] remained to the end of his life an attached husband, as Lady Hamilton remained an attached wife, as well as a good woman" [2, 335]. Yet he wrapped every positive remark about her in warnings and indications of doom, of which the most strange is his perceived "obscuration" of Hamilton, leading to a twisted and far-fetched lament [2, 334-335], [11, 5].

In November 1839 Lady Hamilton became pregnant again, and around the same time she started to suffer from what was called a "nervous illness".[27] From spring to August 1840 she lived in lodgings in Dublin, where Hamilton frequently visited her [10, 205]. After the birth of their daughter, who was born at home in August, she lived at Scripplestown, presumably to breastfeed her baby, and from February 1841 until January 1842 with a sister in England [2, 333], [10, 205-209]. Graves believed that from 1840, while it "gradually spread over some years, and for considerable time scarcely to be taken note of," Hamilton's personality changed, and that he became a "solitary worker" [2, 335]. Stating that before this "obscuration", in Hamilton's home "the strictest temperance reigned" [2, 505], Graves saw a "relaxation of domestic order" as the cause of the obscuration, the relaxation having been "a want of governance" by Lady

[24] Ballinaclough Church [historicgraves.com].

[25] After the deaths of his parents, "Uncle James" from Trim and his cousin, "Cousin Arthur" Hamilton from Dublin, became father figures for Hamilton.

[26] See the blog entry of 3 May 2020 [annevanweerden.nl].

[27] For Lady Hamilton's so-called "nervous illness," and what Hamilton's ante-nuptial promises about a retired life at the observatory had to do with it, see [10, 384-398].

Hamilton,[28] which led to the "long unfelt and unrecognised danger" of the "insidious habit" of drinking porter, in order to "dissipate chill" when at night "he desisted from the work of observing" [2, 334-335]. Graves had "learned on good authority" that such a habit caused "cravings" [2, 506], and he therefore believed that it led to a "humiliating" event in 1846, where Hamilton seemingly drank too much [2, 335, 507].

Graves' assumption that around 1840 Hamilton still regularly observed at night is strange because, from the first volume of his biography, it is obvious that after his early years at Dunsink Hamilton had to stop observing regularly because it "rather seriously affected his health" [1, 285, 358, 409]. Moreover, he must have known that Hamilton did not drink porter to "dissipate chill" because in 1866, the year after Hamilton's death, his eldest son, William Edwin, sent memoranda to Graves, and from one of them it appeared that, at some unknown time, Hamilton started to "take porter in small sips" when after many hours of concentrated mathematical work he became tired, but wanted to finish "a task in which good progress had been made" [3, 239], [10, 428-429].[29] It had nothing to do with observing in the cold; this was the highly concentrated mathematical work Lady Hamilton knew about before she agreed to marry him. She obviously granted her husband his "fascinating investigations" [1, 364], [2, 480], including his sipping porter. Graves furthermore combined two completely different periods, the first lasting from late 1839 until January 1842, the second from October 1843 until February 1846, into one continued process of "obscuration".

The first period started with Lady Hamilton's illness; Hamilton was very worried, and could hardly work. Moreover, in December 1840 his beloved Cousin Arthur Hamilton died [2, 331]. Early in 1841 Graves, who lived in England, visited Hamilton at the observatory, because he was writing his aforementioned *Portrait* of Hamilton, and he must have seen that Hamilton was not doing well. In May 1841 Hamilton wrote a depressed letter to his former tutor Charles Boyton, in which he mentioned to have been "driven from mathematics" [2, 332-334], and in August 1841, in a letter to Graves, Hamilton mentioned "many anxieties, more trying perhaps than the imaginative afflictions of early life" [2, 346].[30] This letter must have added to Graves' worries about Hamilton, which would explain why he believed that Hamilton's personality was changing [2, 335].

Graves wrote that when, in January 1842, Lady Hamilton had recovered so well that she could come home again [10, 211], Hamilton "immediately, with renewed cheerfulness, resumed his mathematical studies" [2, 361]. It is very likely that they had talked about the causes of her illness [10, 390], and even though every now and then Hamilton mourned the loss of Cousin Arthur, life was good, and the marriage happy again. Hamilton's sister Sydney, who had run the household during Lady Hamilton's absence, stayed until spring 1843 [2, 334, 414], more than a year after Lady Hamilton's return.

The second period started when, in October 1843, Hamilton discovered the quaternions. Combining that time frame with Graves' statements that after 1840 Hamilton became a solitary worker, but that the "obscuration" took some years and was hardly noted at first, it becomes very plausible to suggest that it actually started with the quaternions. After his discovery, the frequency of Hamilton's mathematical trances greatly increased, and he did become a solitary worker because no one could keep up with him. For a time he was nearly unstoppable; in November 1844 even his friend John Graves, a former college classmate and eldest brother of Robert and Charles Graves,[31] became worried that he would become overworked [2, 463], [10, 403].

[28] It must be noted that the "relaxation" was not about managing the household staff, that had been a temporary problem, when Lady Hamilton still was "expected soon to return" [2, 334].

[29] Porter is a dark malt flavoured beer, full of sugars. It is easy to imagine that when he became tired, Hamilton felt better because of the sugars in the porter.

[30] Here Hamilton doubtless referred to having lost Catherine Disney and Ellen de Vere.

[31] John Thomas Graves lived in England, and was a jurist and a mathematician.

Not having been a mathematician, Graves either did not recognise the importance of the quaternions, or he may have thought he needed this lament in order to prepare his readers for the "humiliating" event which happened in February 1846 at a meeting of the Geological Society, in which Hamilton seemed to have drunk too much, and which started the Dublin gossip about him [10, 414-417]. After the event, following advice from Graves' brother Charles, Hamilton abstained from alcohol for two years, but between 1848 and most likely 1851 [10, 424] he again drank in public, when enjoying social gatherings after long abstemious periods of hard work, and therefore perhaps a little less accustomed to the effects of alcohol than he had been before. It led to a further intensification of the Dublin gossip, and another lament by Graves [2, 632].

Around 1851, Charles Graves warned Hamilton about his reputation for a second time, and although this time he did not abstain, from then on Hamilton only drank very moderately [10, 422-428]. Even Graves was not entirely unsatisfied, writing that Charles Graves' "strengthening counsel was not without good effect" [2, 632], yet he again wrapped this positivity in criticisms. Combined with omitting to state explicitly that although Hamilton sometimes drank much[32] he was never drunk [9, 224],[33] he not only destroyed Lady Hamilton's reputation, but with it that of his admired friend.

To put Graves' lamentations into perspective, it must be realised that Graves never saw Hamilton skip meals or work through the night. He hardly knew what Hamilton's daily life looked like; from 1833 to 1864, therefore during nearly the entire duration of Hamilton's marriage, Graves lived in England, and when he did visit the observatory it was as an eminent guest. Moreover, he found himself in a very difficult position, having to publish, in the 1880s, during the times of temperance,[34] a biography about a friend who had loved the "pleasures of the table" [2, 527], [10, 418]. It was not an option to leave the 1846 event out; because of the Dublin gossip everyone knew about it, and he would lose his credibility. But at the same time, as an active clergyman, he may not have seen any possibility of defending his friend while simultaneously supporting his brother Charles Graves in his attempts to diminish the use of alcohol in Ireland.[35]

Graves therefore only once defended Hamilton, when he wrote about the period of Hamilton's resumption of drinking between 1848 and circa 1851, "Suffice it now to say that a most exaggerated notion of his weakness, of the degree to which he yielded to it, and of the number of his lapses, became prevalent" [2, 506]. Yet Graves did something remarkable when he wrote his pages about what had happened at the Geological Society in 1846; he gave a part of the account which Hamilton had sent him, that he had taken "what he was told was only a moderate quantity of wine" [2, 506]. Graves thus allowed his readers to conclude that Hamilton had not been drunk without literally saying so.

On six or seven pages, scattered through the biography, Graves lamented about his friend's alleged "cravings", and how it was all Lady Hamilton's fault. He could not know that readers outside Dublin, and living long after everyone featuring in the biography had died, would only remember his laments, that his other 2000+ pages, about what a wonderful man Hamilton had been, were soon forgotten. That was the unavoidable consequence of Graves' choice to write a biography which was neither about

[32] At the 1842 meeting in Manchester of the British Association, Hamilton met Carl Jacobi [2, 386-387]. They were "left with a deep mutual respect for each other's drinking prowess." See p. 3 of *Sir William Rowan Hamilton (1805-1865)* by J. H. Reid, 1965 [articles.adsabs.harvard.edu]. Very unusually for 1965, Reid called the Hamilton marriage "very happy."

[33] For a discussion about the 1846 event at the Geological Society, and that Hamilton had not been drunk, see *A biographer's opinion as primary source* [youtube.com].

[34] See *Temperance Movement in Ireland* [en.wikipedia.org]. Or, *The influence of the 1880s temperance struggles on Hamilton's posthumous reputation* [youtube.com].

[35] In the 1840s, Charles Graves was a fellow mathematician at Trinity College Dublin; in the 1880s, when the biography was published, he was Bishop of Limerick.

Hamilton's work, nor about his daily life. It was instead one long defence against the Dublin gossip [1, v], in which Graves tried to show how all eminent people he could think of [2, 656-657] admired his morally high-standing friend.

4. Hankins' views on Hamilton's private life

Following the publication of Graves' biography, in an ever worsening train of gossip, Lady Hamilton came to be described as a semi-invalid who could not manage the slovenly servants. It had caused Hamilton to become sad and lonely and, giving in to his cravings, he had started to drink at a dangerous level [11, 14-16]. Yet it still was all Lady Hamilton's fault until, in 1980, Hankins took it to a higher level. Graves had been unhappy with some important choices Hamilton made in his life; he should have married Ellen de Vere [10, 115], should have forswore alcohol, and should not have given himself over to his mathematical trances at the cost of the "healthful order and beauty [of] the course of his daily life" [1, 506]. Yet further aggravating Graves' view on Hamilton, Hankins was certain; Hamilton only loved Catherine Disney [9, 39, 347].

Hankins apparently did not realise that until 1848 Hamilton had not known how extremely unhappy Catherine Disney was [12, 70], that only shortly before her death in 1853 could she tell him that she had wanted to marry him, but that she had been forced by her family to marry Barlow. And not having recognised Hamilton's 1832 psychological discovery, that he had found that "action instead of meditation" was the remedy against despondence, Hankins took Hamilton's understandably very distressed writings after these two shocking revelations for an everlasting truth.

He moreover dismissed what Hamilton himself wrote about his feelings for Ellen de Vere and Helen Bayly. That, as mentioned above, what he felt for Ellen de Vere had been an "affliction of the same kind and indeed of the same degree" as that of Catherine had been. And that his ante-nuptial poems show how much in love he was with Helen Bayly [2, 3-7], how afraid he was that she would not accept him [2, 7-16], and how happy when she did; finally being certain she would marry him he wrote, "how full of silence is deep happiness" [2, 27], [10, 157]. How honest Hamilton was in his poems appears from the fact that, while holding to the conviction that poetry must come from true feelings [10, 80], he sent his ante-nuptial poems to Samuel Coleridge, in his accompanying letter calling them "love poems" [2, 37], [10, 146].

Hankins instead concluded that Hamilton had been a hopeless romantic [9, 347], and in one stroke added a new branch to the gossip; it was concluded from his biography that Hamilton's drinking alcohol had all been about Catherine Disney. Completely passing over Graves' remarks that the "obscuration" had "gradually spread over some years," and had been "long unfelt and unrecognised," in his section 'From Algebraic Couples to Quaternions', Hankins assumed that already in 1840 or 1841, when "seriously depressed" by his wife's absence and the death of Cousin Arthur, Hamilton started to "suffer from his first difficulties with alcoholism" [9, 287].[36]

In his section 'Catherine Disney Barlow', Hankins returned to the subject, now indicating 1844 instead of 1840 or 1841 as the year of the first problems with alcohol;

> [In 1844] Hamilton first began to have trouble with alcohol. The following year, in 1845, his old college friend Thomas Disney paid a visit to the observatory and brought Catherine with him. [...] He left little record of the 1845 visit, but it must have been upsetting in the extreme. In February [1846] his problems with alcohol became public knowledge when he collapsed and became violent at the meeting of the Geological Society [9, 348].

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[36] Hankins did not literally give a year, but thereafter he quoted a sentence from a letter by Hamilton to John Graves, written in October 1840 [2, 328], and Cousin Arthur died in December 1840.

By putting the sentence about 1846 so directly after the one about 1845, Hankins implied a direct connection between Catherine's visit in 1845 and the event in 1846.[37] However, such a continuous path of drinking more and more, lasting from 1844 to 1846, or even from 1840 to 1846, is not likely at all.

From about 1839 [2, 284] until early 1846 [9, 238], Hamilton had been attracted by the "high church" Oxford Movement, which "argued for the reinstatement of some older Christian traditions of faith and their inclusion into Anglican liturgy and theology."[38] Graves had written that, in the summer of 1845, Hamilton had experienced "religious contrition" over having drunk more than he had judged right, even though according to Graves he "was perfectly clear in mind and in possession of all his powers" [2, 491], [10, 409]. Moreover, in his section 'In and Out of the Oxford Movement', Hankins wrote about 1845, referring to Graves' third volume [3, 236],

> This was the year that Hamilton's friends noted his excitability over religious matters. He became much more strict in all religious observances. [...] These were what Hamilton's son William Edwin later called his "high church days," when Hamilton felt strongly about proper religious observance. Even their greyhound "Smoke" was subject to special religious discipline and received a beating for chewing on an old Book of Common Prayer in the library, the implication being that chewing a sacred book deserved a more severe punishment than chewing a nonsacred one [9, 238-239].

The idea that an openly truthful man as Hamilton, adhering so strict to religious observances that he even punished his beloved dog, had been drinking too much from 1840 or 1844 until February 1846, is utterly illogical. Not in the least because during his "high church days" Hamilton "read prayers in the centre of his assembled household" [3, 236], which means that if in that particular period he had been drinking too much, perhaps even seeking to do so secretly, everyone in his household, or living on the Dunsink premises for that matter [2, 311], would have judged him to have been a hypocrite, but no trace of that was found, rather the contrary [10, 49 fn. 64, 502].

When describing the February 1846 event at the Geological Society, Graves had repeated Hamilton's own view on what had happened, allowing his readers to come to their own conclusions. But in Hankins' version, all these nuances were gone. Moreover, making some mistakes led to a far more drunken impression than Graves had given. Hankins assumed that during the dinner, Hamilton had "talked excitedly" about "a plan to use the instruments at the observatory to detect earth movements" [9, 223], but what had happened was that that morning Hamilton had received the confirmation of this "suggestion of a new connexion between astronomy and geology" [2, 506], which explains why he had been so excited. Hankins further wrote that Hamilton had "fainted and tumbled down the stairs" [9, 223], yet that is not in Graves' description [2, 506-507], to which Hankins referred [9, 429]. Nor was Graves "close to the events" as Hankins suggested; he lived in England, and Hamilton had sent him an account of what had happened. Hankins did not repeat the "moderate quantity of wine," instead he suggested that the fact that thereafter Hamilton stopped drinking alcohol "would indicate that alcohol had been the cause of his collapse" [9, 223]. Not even Graves suggested that; shortly after the event Hamilton followed the advice of Charles Graves to stop drinking alcohol, not for his health, but for his reputation [2, 507, 527], [10, 415]. Hamilton then abstained for two years.

[37] That Hankins made this connection may not seem completely obvious here, but he directly made that connection when he described Hamilton's 1848 visit to Parsonstown, see p. 57.

[38] For the Oxford Movement, see Wikipedia [en.wikipedia.org].

In July 1848, Catherine Disney contacted Hamilton. That led to an upsetting correspondence, in which Catherine told Hamilton how terribly unhappy she was [10, 269-275]. After the conclusion of this correspondence, Hamilton visited Parsonstown, where Lord Rosse had built his enormous "Leviathan" telescope. After a few days having calmed down again, Hamilton drank a glass of champagne with Airy, then Astronomer Royal of England [10, 284, 418-420]. Hamilton left Parsonstown on 12 September, and on 5 October 1848 Catherine wrote to Hamilton that she wanted to commit suicide. But in 1853 Hamilton mentioned that he had received her suicide note in Parsonstown, and comparing these dates, Hankins supposed that Catherine wrote her note on 5 September instead of 5 October, that "Hamilton must have confused the two months in his remembrances" [9, 449]. He then concluded that Hamilton's distress about the suicide note had led him to end his abstinence, that "the internal torture of Catherine's suicide attempt must have been the real culprit – if a culprit was needed" [9, 350].

There are serious problems with Hankins' scenario. From letters Hamilton then wrote, it can be inferred that he did not read the note before 11 September, and by then Airy had already left Parsonstown [10, 278-285]. Hamilton left on the 12th, but it is very likely that he later returned to Parsonstown; on 16 September Saturn's moon Hyperion was discovered, and from 21 September to 5 October the circumstances to see it with the "Leviathan" were ideal.[39] It is therefore perfectly possible that Hamilton then and there received Catherine's note, and his drinking champagne thus had nothing to do with distress about it; he most likely had judged he had abstained long enough.

Despite his assertions, Hankins did not believe that Hamilton became an alcoholic,

> Hamilton remained on the edge of alcoholism for the rest of his life, but his condition was never as bad as the Dublin gossips would have it. There were no more incidents like the one at the Geological Society, and there is no indication that he ever lost his capacity for sustained mathematical work. The gigantic volumes of the *Lectures on Quaternions* and the *Elements of Quaternions* could never have been written by a man in an alcoholic stupor [9, 224].

But just as happened to Graves' positivity, which was wrapped in criticisms, Hankins' defences were sandwiched between doubts about Hamilton's private life [9, 224, 378-379]. Hankins' confident sentences about "trouble with alcohol," and having "remained on the edge of alcoholism," were the ones which were best remembered.

4.1. **Views on the marriage.** After the publication of Hankins' biography, the view on Hamilton and his marriage became caricatural. For instance Hankins' suggestion that Hamilton did not speak of love in connection with Helen Bayly when in 1832 he wrote to De Vere about her [9, 113], led to views of Hamilton as a hopeless case, drowning in alcohol because he had not married Catherine Disney. Yet not having spoken of love is only literally true for the first letter to De Vere, written on 7 November 1832 [1, 622], because five days later Hamilton wrote a second one,

> very lately I have begun again to associate Hope with Love, in the case of a person whom I have long known and deeply respected for eminent truth of character. [...] I felt myself induced by our friendship, and bound by our agreement, to apprise you (as I lately did) as soon as I perceived the dawn of a new love. How the seed has grown into a tree, the sonnets that I have just composed will show you [2, 3, 4-6].

In his section about Helen Bayly, having first given Graves' harsh opinion about her, Hankins defended her by commenting that "Graves' judgment was probably not fair" because he "idolized Hamilton, and was all too ready to blame his hero's weaknesses on

[39] See the blog of 22 February 2020 [annevanweerden.nl].

the lack of "controlling influence" from his wife" [9, 114]. But then he went even further than Graves had done in his biography. Graves had written that in 1832 Hamilton's "tenderer and warmer feelings" for Helen Bayly came "at a time when he had felt obliged to suppress his former passion" [2, 2]; Hankins wrote her off as a serious factor in Hamilton's life by stating that she "knew that Hamilton married her "on the rebound," so to speak, from two previous romances," which she could not ever replace [9, 115].

Hankins drew this latter conclusion from what he called "a revealing sonnet" [9, 114-115], but in which, on the contrary, Hamilton showed Helen Bayly why he would not hide his former loves from her, "Praising a new love to dispraise the old, As if I had before but false tales told" [2, 4]; after all, his former "loves" were not relationships which had gone awry, he had loved and lost them. Hamilton was completely open about his former loves, and he doubtless was of the opinion that he had to be; Augustus De Morgan, who corresponded with Hamilton from 1841 until his death in 1865, remarked about Hamilton's extreme honesty, "In the matter of right and wrong, Hamilton was very simple-minded. To say he was truthful would be only a part of the truth; his aptitude to entertain misgivings [...] made him often think it right to express his opinions to avoid the possibility of being misunderstood" [3, 218]. Throughout Graves' biography it is obvious that Hamilton valued truth very highly, and his absolute clarity about loving Helen Bayly as he had loved Catherine Disney and Ellen de Vere was why she was certain of his love for her.

Their mutual trust can be felt in Hamilton's description of the walk along the Royal Canal, that his wife "talked with [him] now and then, yet an under-current of thought was going on in [his] mind" [2, 434]. It is not really known what a eureka moment is or where it comes from, but it seems to emerge more often when someone is relaxed and a little distracted after a period of intense thought as, allegedly, Archimedes was in his bath. In Hamilton's case, it is easy to imagine how, when his wife joined him on his walk, the "under-current" which led to his eureka moment would have been interrupted if this marriage had been as bad as the gossip has it. Yet it was their shared faith and spirituality [10, 197], and high value in truth, that defined this marriage and created the peaceful atmosphere which can be found in the poems Hamilton wrote for and about his "wedded and beloved one", his "gentle" wife [2, 50, 270], and in the report about his discovery of the quaternions as they were walking together along the Canal.

5. The discovery of the quaternions; Hamilton's account

Hamilton left eight known reports about his discovery[40] of the quaternions,[41] of which the shortest one consists of two pages of a pocket-book, Figure 5.3, written on the afternoon of 16 October 1843 [2, 436]. There is an entry in Hamilton's 'Note-book 24·5', or as Graves called it, 'A 1829' [2, 439], dated 16 October 1843, written after Hamilton had returned home, hereafter called *Evening entry* [4, 103-105]; a letter to John Graves written on 17 October 1843, the day after the discovery, hereafter called *Letter to John Graves* [5, 489-495]; a paper in the *Proceedings of the Royal Irish Academy* which was communicated on the 13th of November 1843, hereafter called *Proceedings* [6, 424-434];

[40] In the *Letter to John Graves*, Hamilton wrote, as a footnote to the word "discovered", "The reader is requested to pardon this expression for the reason mentioned in another note" [5, 490]. Such a note was not found, but Hamilton invariably used the words "discovered" and "discovery" for the quaternions.

[41] "Previously, in 1819 or 1820, Gauss had set down the multiplication table for quaternions, but did not publish it; nor did he develop the algebra" [4, xv]. The exact reference given in [4], "Sitz. Göttingen Nachrichten 1898, p. 8" was not found, but on pp. 130-131 of *Mathematische Annalen* 1899, Band 51, Felix Klein wrote, "[Gauss] gibt die expliciten Formeln für die Zusammensetzung zweier Scalen (also die Multiplication zweier Quaternionen), wobei er die symbolische Schreibweise $(abcd)\,.\,(\alpha\beta\gamma\delta) = (ABCD)$ benutzt und ausdrücklich bemerkt, daß es sich dabei um einen nicht commutativen Process handelt!" [gdz.sub.uni-goettingen.de]. See also footnote 47.

an entry in one of Hamilton's manuscript books, written 18 July 1848, hereafter called *Book C., 1848* [2, 437-439]; a letter to Peter Guthrie Tait, written 15 October 1848, hereafter called *Letter to Tait* [2, 435-436]; a part of the preface to Hamilton's *Lectures on Quaternions*, dated June 1853, hereafter called *Preface* [7, (43)-(47)]; and a letter which Hamilton wrote to his son Archibald Henry on 5 August 1865, about a month before his death, hereafter called *Letter to Archibald* [2, 434-435].

Hamilton's version of how he discovered the quaternions will be described within the context of the above discussed positive view on his marriage. It would indeed have been logical for Graves to conclude that the "obscuration" he perceived had begun in 1843, after Hamilton discovered the quaternions and became captivated by them. Yet Graves was not a mathematician, and searching for personal or emotional reasons for the change in Hamilton,[42] he believed, as mentioned before, see p. 52, that it began in 1840, and that it "gradually spread over some years, and [was] for considerable time scarcely to be taken note of." But that means that, regardless whether Graves was right or not, in 1843 it did not yet play a role, or it was, in the worst possible scenario, still "scarcely visible." This justifies placing the discovery in the context of a happy marriage.

5.1. **Reminiscing.** Since 1830 [1, 366, 419], Hamilton had searched, "at intervals" [2, 439], for triplets, an extension of the complex numbers, $a + ib$, by an added second imaginary term, $a + ib + jc$; it was expected that these triplets could be used for geometrical descriptions in three dimensions, similarly to imaginary numbers in two dimensions. Hamilton wanted to "discover the laws of [their] multiplication" [2, 434], and he started his last search in the second half of September 1843. In 1865, a month before his death, Hamilton wrote in the *Letter to Archibald*,

> If I may be allowed to speak of myself in connexion with the subject [of the multiplication of vectors], I might do so in a way which would bring you in, by referring to an ante-quaternionic time, when you were a mere child, but had caught from me the conception of a Vector, as represented by a Triplet. [...] Every morning in the early part of [October], on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, "Well, Papa, can you multiply triplets"? Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only add and subtract them" [2, 434].

That changed when, on Monday 16 October 1843, Hamilton was walking along the Royal Canal on his way to preside over a council meeting of the Royal Irish Academy [2, 434-437], of which he had become president in 1837 [2, 211]. Council meetings were held "on the first and third Mondays of every month, from November to June, inclusive, at half past three, P.M. or when specially summoned; and the Academy at large [meets] on the second Monday of November and December, and on the second and fourth Mondays of January, February, April, May, and June, at Eight, P.M.," as was announced every year in the *Dublin Almanac*.[43]

The council meeting of 16 October 1843, held during the annual adjournment, thus was such a 'summoned meeting'; those meetings were held on the same weekdays and times as the regular council meetings. Having to open the meeting at half past three, see Figure 5.1, Hamilton had left home in the early afternoon. The route he used to take while walking to Dublin is well known;[44] first walking across the sloping fields to

[42]Here Graves slightly differed in opinion with his brother John Graves, who did understand the mathematics, [2, 462-463], [10, 404].

[43]See the *Dublin Almanac* for 1843, p. 167 [babel.hathitrust.org]. Time is in Dublin local time, UTC -25m21s [dublincity.ie].

[44]As the 'Hamilton Walk', it is walked yearly in commemoration of Hamilton's eureka moment [wikipedia.org]. For the route, see the map which was made for it [google.com/maps].

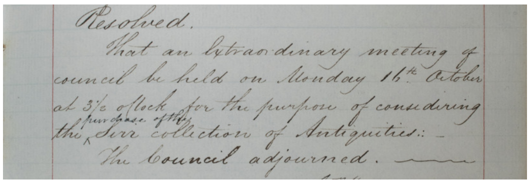
Resolved.
That an extraordinary meeting of
council be held on Monday 16th October
at 3½ o'clock for the purpose of considering
the purchase of the Sirr collection of Antiquities.: –
The Council adjourned.

FIGURE 5.1. Minutes of the RIA council meeting of 2 October 1843. "Resolved. That an extraordinary meeting of council be held on Monday 16th October at 3 1/2 o'clock for the purpose of considering the purchase of the Sirr collection of Antiquities.: – The council adjourned."
By permission of the Royal Irish Academy © RIA

the south, after having passed Dunsinea, where Lady Hamilton's sister Jane Rathborne lived, he turned left and walked along the Royal Canal. Hamilton must have been very deep in thought because he did not hear his wife coming, nor did he know where she came from, as appears from the *Letter to Archibald*, given below. This leads to the likely scenario, that when Hamilton passed by Dunsinea, Lady Hamilton saw him, and because she apparently also wanted to go to Dublin, she decided to catch up with him.

In the *Letter to Archibald*, Hamilton described their walk and his discovery,

> I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse – unphilosophical as it may have been – to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely,
>
> $$i^2 = j^2 = k^2 = ijk = -1,$$
>
> which contains the Solution of the Problem, but of course, as an inscription, has long since mouldered away [2, 434-435].

FIGURE 5.2. Sir William Rowan and Lady Helena Maria Hamilton depicted at Broome Bridge on 16 October 1843. A sand sculpture by Daniel Doyle, 2012, Dublin. The reverse side, in fact the front side, shows Hamilton "trying to understand dimensions" [steemit.com]. *Reproduced with permission.*

In the 1858 *Letter to Tait* Hamilton wrote,

> [The quaternions] started into life, or light, full grown, on the 16th of October, 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge, which my boys have since called the Quaternion Bridge.[45] That is to say, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them ever since. I pulled out on the spot a pocket-book, which still exists, and made an entry, on which, at the very moment, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come. But then it is fair to say that this was because I felt a problem to have been at that moment solved – an intellectual want relieved – which had haunted me for at least fifteen years before [2, 435-436].

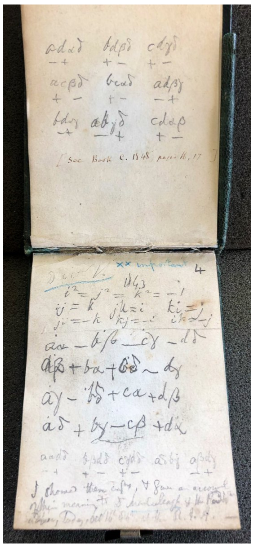

FIGURE 5.3. The pocket-book showing, at the top of the lower page, the "fundamental equations." "xx important" was written by Graves when he was preparing the biography, with his "blue pencil" as Hankins called it [9, xxi]. Beneath the terms on the upper page, Hamilton later inserted a note about his *Book C., 1848* [2, 437].
Courtesy of the Board of Trinity College, the University of Dublin.

And on 18 July 1848 Hamilton wrote in his *Book C., 1848* [2, 437],

> The pocket-book remains, given me by Helen in 1840, in which, while I was walking to town on the day thus referred to, I did actually pencil at the time, and just as I reached the Bridge here mentioned, the notes

$$\begin{gathered} i^2 = j^2 = k^2 = -1 \\ ij = k \qquad jk = i \qquad ki = j \\ ji = -k \qquad kj = -i \qquad ik = -j. \end{gathered} \tag{5.1}$$

5.2. **Quaternions.** In the *Letter to John Graves*, written on 17 October, Hamilton gave his course of thought the day before, and most clearly described his discovery. He had aimed for a system of triplets which would be "analogous" to "Mr. Warren's geometrical

[45]It has been suggested that calling the bridge "Brougham Bridge" was a family in-joke, see p. 618 of Charles Mollan, *It's Part of What We Are* [google.com/books].

representation of imaginary quantities,"[46] essentially the now common representation in a plane. With $i^2 = j^2 = -1$, the multiplication of two triplets, $(a+ib+jc)(x+iy+jz) = ax - by - cz + i(ay + bx) + j(az + cx) + ijbz + jicy$, raised the question what to do with ij. Hamilton considered $ij = 1$, or $ij = -1$, but then his prerequisite, that "the sum of the squares of the coefficients of 1, i, and j in the product [should be equal] to the product of the corresponding sums of squares in the factors" [5, 490], would not hold; his system would not obey this multiplicative property of lengths, which Hamilton called the "law of the moduli," and a geometrical representation as Mr. Warren's would not be possible.

Hamilton then calculated a square as the simplest case of a product, while suppressing both ij and ji, $(a+ib+jc)(a+ib+jc) = a^2 - b^2 - c^2 + 2iab + 2jac$. To obey the law of the moduli, it was necessary that the product of the sums of squares in the factors, here $(a^2 + b^2 + c^2)^2$, was equal to the sum of the squares of the coefficients of 1, i, and j in the product, which indeed is the case, $(a^2 - b^2 - c^2)^2 + (2ab)^2 + (2ac)^2 = (a^2 + b^2 + c^2)^2$. Therefore, in case of a square,

> the condition respecting the moduli is fulfilled, if we suppress the term involving ij altogether. [...] Behold me therefore tempted for a moment to fancy that $ij = 0$. But this seemed odd and uncomfortable, and I perceived that the same suppression of the term which was de trop might be attained by assuming what seemed to me less harsh, namely that $ji = -ij$. I made therefore $ij = k$, $ji = -k$, reserving to myself to inquire whether k was $= 0$ or not. For this purpose I next multiplied $a + ib + jc$, as a multiplier, into $x + ib + jc$ as a multiplicand, keeping still, as you see, the two factor lines in one common plane with the unit line [5, 490-491].

For such triplets, $ij = k$ and $ji = -k$ indeed cancels the k term; $(a+ib+jc)(x+ib+jc) = ax - b^2 - c^2 + i(ab + bc) + j(ac + cx) + k(bc - cb)$.

> [...] Confirmation of $ij = -ji$; but no information yet of the value of k. Try boldly then the general product of two triplets, and seek whether the law of the moduli is satisfied when we suppress the k altogether. Is $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2$? No, the first member exceeds the second by $(bz - cy)^2$. But this is just the square of the coefficient of k, in the development of the product $(a + ib + jc)(x + iy + jz)$, if we grant that $ij = k$, $ji = -k$ [5, 492].

This is where, in the *Letter to John Graves*, Hamilton ended the pre-discovery calculations, yet in the *Evening entry*, which he wrote for himself, there are some extra steps.

Before discussing them, it must be remarked that there is a seemingly confusing part in the *Evening entry* as given in [4], namely, that Hamilton first wrote down that he had conceived "that the product ij might be equated to a new imaginary, say k," and then that he "still [...] thought it possible that ij might be equal to zero" [4, 104]. He then wrote, "I believe that I remember now the order of my thought" and corrected himself, starting anew with the two triplets having their factor lines in one common plane with the unit line; this is written out in the quote on p. 63. The confusion disappears when it is realised that Hamilton was writing this for himself, and therefore had no reason to strike errors through, nor could he tear out pages of his notebooks; he usually also wrote on the reverse side of the pages [9, 399].

[46] John Warren, 1828, *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*, [archive.org]. Hamilton's geometrical contemplations have been omitted here, they can be found in, for instance, [5, 10-13] and [6, 424-434].

In the *Evening entry* already having concluded that $ij = -ji$, Hamilton remarked, "trying to remember this evening my course of thought in the morning," that he had "even thought it likely though odd" that $ij = 0$. In the *Letter to John Graves* Hamilton showed how $ij = 0$ came from squaring a triplet, but in the *Evening entry* he wrote,

> I believe that I remember now the order of my thought. The equation $ij = 0$ was recommended by the circumstance that $(ax - y^2 - z^2)^2 + (a + x)^2(y^2 + z^2) = (a^2 + y^2 + z^2)(x^2 + y^2 + z^2)$ [4, 104].

This indeed follows from $(a+iy+jz)(x+iy+jz)$, as Hamilton also wrote in the *Letter to John Graves*, the only difference is that there he used $(a+ib+jc)(x+ib+jc)$ to show that the k term, or the ij term, "still vanished," and he did not so succinctly order the terms.

Hamilton continued, just as in the *Letter to John Graves* using $ij = k$, $ji = -k$ for two general triplets and supposing the k term is cancelled again,

> I therefore tried whether it might not turn out to be true that
>
> $$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2;$$
>
> but found that this equation required, in order to make it true, the addition of $(bz - cy)^2$ to the second member. This forced on me the non-neglect of ij; and suggested that it might be equal to k, a new imaginary. For a while I thought it likely that k^2 might be equal to 1; but this supposition $k^2 = 1$, combined with $i^2 = j^2 = -1$, would give
>
> $$(a + ib + jc + kd)(\alpha + i\beta + j\gamma + k\delta) = a\alpha - b\beta - c\gamma + d\delta + i(\&c.) + j(\&c.) + k(\&c.); \tag{5.2}$$
>
> in which the part multiplied by k must involve the terms $a\delta + d\alpha$, unless we choose to give up [...] certain very simple and fundamental principles of ordinary algebra [...]; but in the square of the modulus, (that is, in the sum of the 4 squares of the coefficients of 1, i, j, k) [which he later wrote out on the lower page of the pocket-book, Figure 5.3], this part of the coefficient of k would give the term $+2ad\alpha\delta$, while, if $k^2 = 1$, the coefficient of 1 would give the same term $2ad\alpha\delta$ repeated; thus there would be no mutual destruction of these two terms. But if we suppose $k^2 = -1$, the real part (so to speak) is $a\alpha - b\beta - c\gamma - d\delta$; and this gives $-2ad\alpha\delta$, destroying $+2ad\alpha\delta$, which latter arises from $a\delta + d\alpha$. Thus I was led to assume not only $i^2 = -1$ and $j^2 = -1$, but also $k^2 = -1$, and $ij = k$, $ji = -k$ [4, 104].

It thus appears that already having assumed $ij = k$, $ji = -k$, which he had also mentioned in the *Letter to John Graves*, Hamilton now had added $k^2 = -1$; it can therefore be concluded that finding that the square of the new imaginary k should also be equal to -1, led to the flash of insight.

5.3. **The discovery.** Hamilton described the moment of discovery in the *Letter to John Graves*, which he continued writing,

> And here there dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triplets; or transferring the paradox to algebra, must admit a third distinct imaginary symbol k, not to be confounded with either i or j, but equal to the product of the first as multiplier, and the second as multiplicand; and therefore was led to introduce quaternions such as $a + ib + jc + kd$, or (a, b, c, d) [5, 491-492].

Although in the *Letter to John Graves* the moment of discovery is very clear, in the *Evening entry* that is far less so. Yet comparing these two reports, both times the "new imaginary" follows from recognising the extra term $(bz - cy)^2$. It can therefore be suggested that, already feeling certain about $ij = k$, $ji = -k$, while still at home Hamilton had recognised the extra term which, as he wrote in the *Evening entry*, "forced on him the non-neglect of ij; and suggested that it might be equal to k, a new imaginary."

This seems the most likely stage in the calculations at which Hamilton had to lay down his pen and perhaps take the cat from his shoulder [3, 235]. He may have had luncheon, and started walking, meanwhile contemplating that the square of this k might be equal to 1 because $i^2 = j^2 = -1$ and therefore perhaps $ij = 1$ or $ij = -1$, as he wrote in the *Letter to John Graves* [5, 490]; at that time, k was still a product of i and j, and he was still searching for triplets. He therefore started to calculate the still unnamed quadruplet, equation (5.2), thus including the extra term which was connected to k. There was nothing really new in trying a quadruplet because already in 1841, Hamilton had written to De Morgan, "if my view of algebra be just, it must be possible, some way or other, to introduce not only triplets but polyplets" [3, 247].

It can be assumed that these calculations were the "under-current" of thought Hamilton mentioned in the *Letter to Archibald*, quoted on p. 60; after Lady Hamilton had joined him, she talked with him "now and then," leaving enough room for such calculations, to which he was completely used, and still nothing was new. But just before reaching Broome Bridge he realised that the $2ad\alpha\delta$ term would not be cancelled if $k^2 = 1$, but it would if $k^2 = -1$, and then such a quadruplet would obey the law of the moduli; k was a "third distinct" imaginary, on equal footing with i and j, and these quadruplets were what he had been looking for all that time.

Now having assumed "not only $i^2 = -1$ and $j^2 = -1$, but also $k^2 = -1$, and $ij = k$, $ji = -k$," as he mentioned in the last line of the quote on p. 63, continuing the *Evening entry* Hamilton described the contemplations following the eureka moment, or the "sparks" as he later would call them in the *Letter to Tait*,

> I then thought it likely that it might be proper to assume $ik = -j$, because $ik = iij$ and $i^2 = -1$. If so, since $ji = -ij$, it seemed likely that $ki = -ik = j$ which might also follow from $k = -ji$. In like manner, it appeared possible (or at least natural to be assumed) that $kj = ijj = -i$, $jk = -jji = i$. The multiplication-assumptions, or definitions, were therefore collected to be $i^2 = j^2 = k^2 = -1$; $ij = k$, $jk = i$, $ki = j$; $ji = -k$, $kj = -i$, $ik = -j$ [4, 104].[47]

He wrote these equations in his pocket-book, and very unphilosophically, cut the fundamental formula, $i^2 = j^2 = k^2 = ijk = -1$, on a stone of the bridge.

5.4. **The second part of the walk.** After Hamilton had cut the formula on Broome Bridge, they walked to the cab stand in the "neighbourhood of the turnpike," as Hamilton mentioned in *Book C., 1848* [2, 437].[48] The turnpike was near the Cross Guns

[47]From *Carl Friedrich Gauss* Werke, Band VIII, 1900, *Mutationen des Raumes* pp. 358, 359, it is obvious that Gauss found for a multiplication of his Scalen the same 16 terms as the ones Hamilton wrote on the lower pocket-book page [gallica.bnf.fr]. Gauss' Scalen are antisymmetric to the quaternions; they would obey $i^2 = j^2 = k^2 = kji = -1$, therefore, the only difference is that the Scalen would yield a left-handed coordinate system instead of a right-handed one. But Gauss did not publish it, and indeed did not develop it into an algebra, see footnote 41.

[48]After the Hamilton Walk of 2015, Tony O'Farrell, the initiator of the annual walk, spoke about the turnpike not having been at Broome Bridge but at the next bridge, and about the "hackney stand". A hackney was "any carriage plying for hire" [britannica.com]. Because Hamilton wrote that he was sitting "on" a car, it may have been an open carriage, or a half-open one [en.wikipedia.org]. The turnpike can be seen on Historic Maps of Dublin, 1800-1849 [dublinhistoricmaps.ie], choose '1832 County of City,

Bridge, then the next bridge,[49] and because Hamilton later called the day of discovery "that to me memorable Monday"[2, 437], it can be imagined how delighted they were when walking the two kilometres between the bridges. It is not at all unlikely that during that walk they came up with the name 'quaternion' together; Hamilton had not named earlier discoveries, 'systems of rays' and 'conical refraction' were just what they are. Moreover, Hamilton never said much about what motivated him to call them 'quaternions', it seems that the only time he gave something of an explanation was a rather general remark in a footnote of his second book, *Elements of Quaternions*, "As to the mere word, Quaternion, it signifies primarily (as is well known), like its Latin original, "Quaternio," or the Greek noun *τετρακτύς*, a Set of Four: but it is obviously used here, and elsewhere in the present work, in a technical sense" [8, 111].

Yet they both were deeply religious, and knowing they read the Bible together, spoke with each other about religious beliefs, and prayed together [2, 311-312], it may have been more than a coincidence that the word 'quaternion' is in the Bible, The Acts 12:4, when the apostle Peter was delivered "to foure quaternions of souldiers."[50] And if perhaps, in happy excitement and seriousness not Hamilton, but Lady Hamilton, came up with the word 'quaternion', she may have asked him to keep that out of his public writings, knowing that he was wont to always give credit to whom credit was due [2, 464], [3, 76, 150]. Before accepting his marriage proposal she had asked him for a retired life, separate from his scientific, public life, and mentioning her as the name giver would certainly have gone too far. Of course, this is just an educated guess, but it would perfectly explain Hamilton's silence, both about the second part of the walk, and about why he chose the word 'quaternion'. If this was the case, by mentioning her in the *Letter to Tait* Hamilton came close to the boundaries of his wife's wishes, but he did it in the most general way.

It is not known if, or when, Lady Hamilton parted with her husband; depending on where she was going to, she may have preferred to continue walking when, at the Cross Guns Bridge, Hamilton took a car to the Academy at Grafton Street. While sitting on the car, he checked the law of the moduli.

5.5. **The law of the moduli.** In the *Letter to John Graves*, having given the considerations about i, j, k, Hamilton wrote that his "assumptions were now completed" [5, 492], and gave equations 5.1, albeit combined into four equations. Then he wrote about the requirement he had had for the triplets, and now for the quaternions,

> I considered it essential to try whether these equations were consistent with the law of moduli, namely,
>
> $$a''^2 + b''^2 + c''^2 + d''^2 = (a^2 + b^2 + c^2 + d^2)\,(a'^2 + b'^2 + c'^2 + d'^2),$$
>
> without which consistence being verified, I should have regarded the whole speculation as a failure. Judge then of my pleasure when, after a careful examination, I found that the whole twenty-four products not involving squares in the development of the sum of the squares of the four quadrinomials destroyed each other; and that the sixteen products involving squares were just those which arise otherwise from the multiplication of $a^2 + b^2 + c^2 + d^2$ and $a'^2 + b'^2 + c'^2 + d'^2$ [5, 492].

Boundary Commission', where it can be seen that the turnpike was indeed near the "Cross Guns or Westmoreland Bridge", due north of the city centre, nearly at the top of the map.

[49] The current railway bridge near Lock 7 was not there yet; it was built ca. 1864, see Buildings of Ireland [buildingsofireland.ie].

[50] *King James Bible, 1637-38* [archive.org]. It is also in John Milton's *Paradise Lost*, book 5 verse 180 [archive.org], and it is very likely that also Lady Hamilton had read the poem, because it is known that she had read her classics [2, 16]. For additional earlier uses of the word 'quaternion', even in a mathematical context, see the blog entry of 29 August 2021 [annevanweerden.nl].

This is in complete accord with *Book C., 1848* as given by Graves [2, 437-438], in which Hamilton wrote, see also the quote on p. 61,

> The pocket-book remains [...] in which [...] I did actually pencil at the time [...] the notes [equations 5.1]. In a jolted handwriting, the same pencilled page contains these other notes [visible in the middle of the lower pocket-book page, Figure 5.3], which were inserted while I was driving on that (to me memorable) Monday from the neighbourhood of the turnpike to the Academy, as the constituents of the quaternion product of the two quaternion factors,
>
> $$a + ib + jc + kd, \text{ and } \alpha + i\beta + j\gamma + k\delta.$$
>
> The dots referred to certain destructions of double products, by additions of positives to negatives, which I was examining on the car, in order to verify a conjecture which I instantly made, namely, that the law of the moduli would be found to hold good. [...] In connexion with such cancelling of terms, I pencilled also as follows [the equations on the upper pocket-book page, Figure 5.3]; which I think was done while I was sitting in the President's Chair [2, 437-438].

Explaining what the + and − signs meant, which can be seen on both pages of the pocket-book, Hamilton wrote that there were "24 (double) products, to be cancelled, 12 by 12 others" [2, 439], which would leave only "sixteen products involving squares," and that would show that quaternions obey the law of the moduli [5, 492]. Yet in the president's chair Hamilton wrote out only 9 cancelled products, as seen on the upper page of the pocket-book, and remarked, "But 3 were already known to be cancelled by 3, namely, those which did not involve d nor δ" [2, 439]. Setting $d = 0$, $\delta = 0$ [4, 105], would give a calculation similar to the multiplication of the triplets having their factor lines in one common plane with the unit line, $(a + ib + jc + 0)(x + ib + jc + 0)$, from which Hamilton had concluded that $ij = -ji$, see the quote on p. 62.

In the *Letter to Tait* Hamilton wrote, having "felt a problem to have been at that moment solved," see the quote on p. 61,

> Less than an hour elapsed before I had asked and obtained leave of the Council of the Royal Irish Academy, of which Society I was, at that time, the President – to read at the next General Meeting a Paper on Quaternions; which I accordingly did, on November 13, 1843 [2, 436].

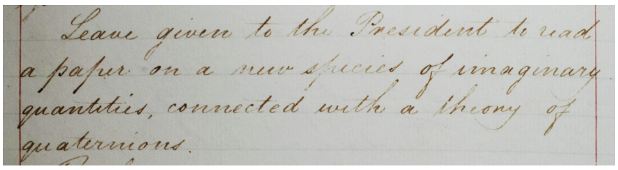
Leave given to the President to read a paper on a new species of imaginary quantities, connected with a theory of quaternions.

FIGURE 5.4. Minutes of the RIA council meeting of 16 October 1843.
"Leave given to the President to read a paper on a new species of imaginary quantities, connected with a theory of quaternions."
By permission of the Royal Irish Academy © RIA

And knowing that about discoveries, Hamilton's fellow mathematician James MacCullagh more often believed he actually had made or at least suggested them [2, 463],[51]

[51] In 1833, MacCullagh had claimed conical refraction [1, 685-692], and this tendency had been the reason that Hamilton wrote the *Letter to John Graves*, and sent it under cover to Robert Graves [2, 440]. In 1852, Hamilton wrote to De Morgan, "MacCullagh [...] was constantly fancying that people were plundering his stores, which certainly were worth the robbing" [3, 331].

Hamilton doubtless did not postpone discussing the equations with him until after the meeting; the sooner the better, and in company of a witness, William Sadleir.[52]

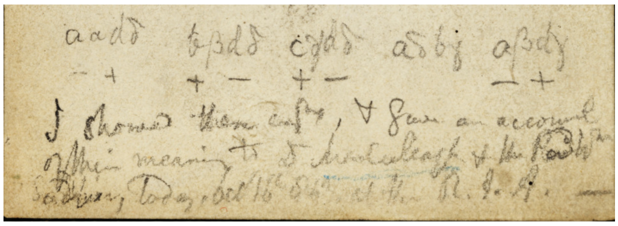

FIGURE 5.5. The lower part of the lower pocket-book page, on which Hamilton wrote, "I showed these equations, and gave an account of their meaning to Dr. MacCullagh and the Rev. William Sadleir today, October 16, 1843, at the R. I. A." With his blue pencil, Graves underlined MacCullagh's name. *Courtesy of the Board of Trinity College, the University of Dublin.*

5.6. **Chronologicality.** Hamilton's versions of the discovery containing moments in real life are found in *Book C., 1848*, the *Letter to Tait*, and the *Letter to Archibald*. There are some small differences, but as regards the sequence of events they are in complete agreement. Combining Hamilton's remark in the *Letter to Tait*, that "less than an hour elapsed" between the eureka moment and having asked and obtained leave at the Academy, and the known time of the start of the meeting, 15h30, a possible time line can be made of that afternoon. It can be used to scrutinise the credibility of Hamilton's account, and approximate the time of discovery.

What Hamilton described is that he was walking along the Royal Canal when his wife joined him. Just when they "came up" to Broome Bridge, he had his eureka moment; k was a third distinct imaginary, not to be confounded with i or j, and triplets were replaced by quadruplets. The "sparks" which fell from it were the "fundamental equations between i, j, k." He "pulled out on the spot a pocket-book" and entered the fundamental equations, and at that "very moment" felt the importance. "As they passed" Broome Bridge, he cut the "fundamental formula" on a stone of the bridge. They walked to the turnpike, he got a car to Dublin on which he started to check the law of the moduli, and he sat in the president's chair when he finished his calculations.

It further is assumed that Hamilton then asked leave to read a paper in November, and thereafter showed the equations to MacCullagh and Sadleir. Having been president, it may have sufficed to ask leave from them because both men were council members, nevertheless, asking leave first could be helpful in assuring his priority in case MacCullagh would again make a claim, which he indeed did in 1844 [2, 462-464]. That was also the importance of having this exact phrase in the minutes; it would be evidence of having then and there mentioned his discovery, including the word 'quaternion'.

5.7. **A most likely timeline.** To make a timeline of that afternoon, it is necessary to have distances and estimations of walking and driving speeds. The walk from the observatory to the Longford Bridge near Ashtown[53] is about 1.5 km. From the Longford Bridge along the Royal Canal to Broome Bridge is about 2.5 km, from Broome Bridge to the Cross Guns Bridge about 2 km, and from the Cross Guns Bridge to the Academy, then at Grafton Street [2, 192], about 3 km. Hamilton was a fervent walker, and doubtless so was his wife; she apparently did not have any problem with walking to

[52] As can be seen in Figure 5.5, Hamilton seems to have written 'Sadlier', but this was William Digby Sadleir, one of the members of the committee of science, see the *Dublin Almanac* for 1843, p. 165 [babel.hathitrust.org].

[53] For the Longford Bridge, near the 10th Lock, see Buildings of Ireland [buildingsofireland.ie].

Dublin.[54] And because it was not a walk for leisure because Hamilton had to preside, it can be assumed they walked about 6 km/h. An exact speed of the cars was not found, but by 1843 mail coaches could reach a speed of 16 km/h,[55] yet the Academy building being in Dublin city centre, the last part may have been somewhat slower, therefore the average speed can be taken as 12 km/h. Finally, the meeting would start at 15h30, which leads to the following timeline.

At 13h40, Hamilton left home, and walked to the south, 15 minutes. At 13h55, Lady Hamilton caught up with her husband at the Longford Bridge where he just turned left to start walking along the canal. They walked for 25 minutes until, near Broome Bridge, Hamilton had his eureka moment at 14h20. He experienced the moment, collected the "sparks", i.e., the fundamental equations for i, j, k, and wrote the equations in his pocket-book, 5 minutes, 14h25. He cut the fundamental formula on the bridge, 2 minutes, 14h27; it was an impulse, no time to dwell on it, he had to be in time for the meeting, it may have been Lady Hamilton who kept an eye on the time.[56] They walked to the cab stand near the Cross Guns Bridge, 20 minutes, 14h47, he got on a car, 2 minutes, 14h49, was driven to the Academy, 15 minutes, 15h04. He paid the carman and entered the Academy building, 5 minutes, 15h09, finished the calculations while sitting in the president's chair, 5 minutes, 15h14, asked and obtained leave, 4 minutes, 15h18. He showed the equations to MacCullagh and Sadleir, 8 minutes, 15h26, in time to return to his chair, make the short note in the pocket-book of having shown them his equations, and start the meeting at 15h30. This gives a total of 58 minutes between the eureka moment at 14h20 and having asked leave at 15h18, indeed "less than an hour" later, completely fitting in with his version of that day.

5.8. **Happiness.** For Hamilton, the combination of being close to his wife and working on mathematics was the happiest state of being [10, 133]. Having moreover been deeply religious, regarding marriage as a holy bond in which husband and wife became one,[57] he doubtless was very thankful that she was with him, precisely at that life-changing moment. And because symbolism was nothing strange for Hamilton [2, 256], it must have been gratifying that, as he mentioned in his *Book C., 1848*, she had given him the pocket-book in which he wrote the equations for the first time.

It is also known that Lady Hamilton was proud of her husband [9, 120]; in February 1833, two months before their marriage and having been highly praised for the discovery of conical refraction, Hamilton wrote to her, "All this, I know, will give you more pleasure than it does me, though I do not pretend that it gives me none; but I amuse myself sometimes imagining the delight with which you will open hereafter my presents from Europe and America, while I shall put on the Stoic, and tell you that I do not care for such things" [10, 156]. Yet because as a couple, they usually did not write letters to each other, there are no private letters about what she felt when her husband finally concluded the searches for triplets about which they had talked so often, leading to their sons asking about triplet multiplication during the breakfasts in October.[58] And although friends of Hamilton testified that he could not lie [10, 40], he was very well able to keep silent about something [10, 281], in this case about their happy walk between the bridges and, who knows, about how the quaternions received their name.

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[54]It was mentioned before that Lady Hamilton was frequently ill, implying a 'weak health', yet it is known from Graves' biography that she also rode a horse, see for instance [1, 26-27], [3, 162, 498].

[55]See for instance *Mail coach* [en.wikipedia.org].

[56]Hamilton more often just hurried after her when he was absorbed in something else [10, 354].

[57]*King James Bible, 1637-38*, Mark 10:8, "And they twain shall be one flesh : so then they are no more twain, but one flesh" [archive.org].

[58]It can easily be assumed that she joined in these conversations; there are indications that she had a quite technical mind, [10, 152, 182, 437], although due to their social circumstances undeveloped.

6. The creation of the quaternions; Hankins' reconstruction

In the 22nd section of his biography, called 'The Creation of Quaternions',[59] Hankins gave his own reconstruction of the discovery [9, 293-301],[60] in which his general doubts about Hamilton show very clearly. He changed quite some details of Hamilton's own version, of which the most unimportant one seems to be that Lady Hamilton had not "talked with him now and then," as Hamilton wrote, but "he had been talking to her" [9, 299]. But details make up a story, and comparing Hankins' reconstruction with that of Hamilton shows that Hankins also changed chronology, and even motivations.

To be able to compare and discuss where Hankins' reconstruction deviated from Hamilton's reports, quite some quotes from his 'quaternion section' will be given in the order in which he wrote them, each time followed by comments. Paraphrasing appeared to obscure too much of their essence, and although the collection of quotes is not meant to be a complete overview of the section, it will show that Hankins' confusion may have followed from the erroneous assumption that Hamilton had been walking in the morning instead of in the afternoon, and that he therefore took a large part of Hamilton's course of thought in the morning, as described in the *Evening entry*, as having happened during the walk. It led to his conclusion that Hamilton did not yet have a complete theory when he reached Broome Bridge and, consequently, in Hankins' reconstruction there was no eureka moment, just a sudden realisation as an intermediate moment.

6.1. **Quotes from the 'quaternion section', with comments.**

1.) Referring to the 1865 *Letter to Archibald* [2, 434], quoted on p. 59, Hankins wrote that Hamilton, having returned from the meeting of the British Association in Cork, had started a new search for the laws of multiplication of triplets.

> Every morning during the month of October, he tells us, he would come downstairs to breakfast and would be asked by his elder son, "Well, Papa, can you multiply triplets?" and every morning he had to confess that he could only add and subtract them. This episode has the characteristics of a charming, but unreliable, anecdote. Even if it is false in detail, however, it indicates that Hamilton was intensely engaged again in thinking about the triplets [9, 291].

Comments. Hamilton wrote the letter to his son Archibald, who had been eight years old at the time, which makes it perfectly possible that he remembered it well. It had been a memorable event; the discovery had such an impact on the family that thereafter the boys called Broome Bridge the 'Quaternion Bridge' [2, 435]. Knowing about Hamilton's astounding memory [1, 128], [2, 457-458], [3, 153] and that even until his last days he was working on his second book, *Elements of Quaternions*, calling Hamilton's memories as related to his son an "unreliable anecdote" should at least have been accompanied by arguments, but Hankins did not give any. An erroneous detail is that Hankins wrote that the "elder son", the then nine year old William Edwin, asked the question, although Hamilton explicitly mentioned both of his sons.[61]

2.) Having considered reasons for Hamilton to have returned to the problem of the triplets, Hankins wrote about the moment of discovery,

[59] Next to, perhaps, a philosophical stance, Hankins' choice of the word 'creation' may have been a reaction to John Graves, who had written to Hamilton, "I have not yet any clear view as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties" [9, 300].

[60] Hankins called other articles about Hamilton's discovery "modern reconstructions" [9, 441], and it therefore seemed appropriate to use this expression here.

[61] His daughter Helen had turned three in August.

> On the sixteenth of October, 1843, he and his wife were walking into Dublin along the Royal Canal to attend a council meeting of the Royal Irish Academy. Suddenly a possible resolution of the problem of triplets leapt into his mind [9, 293].

Comments. Hankins' use of the word "possible" is curious because, in the *Letter to Tait*, see p. 61, Hamilton had written that "the quaternions started into life, or light, full grown," which implies that he was immediately certain about the quaternions. That is in accord with having written, in *Book C., 1848* [2, 437], that after the discovery, he had "verified a conjecture which I instantly made, namely, that the law of the moduli would be found to hold good," see p. 66. Even if Hankins used the word "possible" in the sense of "acceptable," it conveys doubt, for which he did not give arguments.

3.) Hankins then gave a part of the quote from the *Letter to Archibald*, p. 60. He ended the quote with Hamilton's sentence,

> Nor could I resist the impulse – unphilosophical as it may have been – to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula [9, 293].

Hankins did not repeat the last part of the quote, in which Hamilton wrote out the "fundamental formula," but remarked in an endnote,

> Hamilton claimed to have written $i^2 = j^2 = k^2 = ijk = -1$ on the bridge, but it seems unlikely that the inspiration came to him in such a neat and condensed form [9, 441].

To be able to comment on this quote, it has to be combined with the following quote from Hankins. Having given a large part of the quote from the 1858 *Letter to Tait*, p. 61, in which Hamilton wrote that he had found the "fundamental equations between i, j, k; exactly such as I have used them ever since" [9, 293], Hankins commented,

> The pocket book remains to verify this story, which casts doubt on his later claim to have scratched the quaternion formulas on the bridge. Hamilton would scribble on any handy surface if he did not have paper along, but on this occasion he did have a notebook. Maybe he felt that the solemnity of the moment required an inscription in stone [9, 294].

Comments. The latter two quotes introduce Hankins' until then unexpressed supposition, that at the bridge Hamilton did not yet have a "full grown" system as he had written in the *Letter to Tait*, p. 61; Hankins did not repeat this part of the letter. But the part of the letter Hankins did give [9, 293], also contained Hamilton's remarks about having written the "fundamental equations between i, j, k" in his pocket-book, see equation 5.1, and Figure 5.3, and Hankins did not challenge that. Yet going from these equations to the "fundamental formula" $i^2 = j^2 = k^2 = ijk = -1$ is simple; if $ij = k$ and $k^2 = -1$, then $ijk = -1$. The born mathematical generalist that Hamilton was [3, 200-201, 286], and knowing how focused he then was, he could even have seen it while cutting the first part of the fundamental formula on the bridge; he would find new "tracks of discovery" while lecturing on a blackboard [10, 342]. Hankins' disbelief, which even led him to change Hamilton's word "formula" for what he wrote on the bridge into the plural "formulas", thus shows that he saw the writing on the bridge as an intermediate moment, not as an expression of a eureka moment. And suggesting that Hamilton's motive to cut formulae on the bridge was that he may have felt that the "solemnity" "required an inscription in stone," entirely passes over Hamilton's sentence, that he could not "resist the impulse."

4.) Hankins continued,

> After the discovery flashed across Hamilton's mind he quickly elaborated it while riding in a car the rest of the way to the council meeting. Once at the meeting he showed his discovery to [James] MacCullagh and to William Sadlier [9, 294].

Comments. This is a very clear statement; having to "elaborate" the discovery indicates again that Hamilton did not yet have a finished theory at the bridge. A small detail is that Hamilton wrote that he was sitting "on" a car, not "in" one [2, 437].

5.) About the sources for his section, Hankins wrote,

> [The] letter to Graves, a memorandum made the day of the discovery, and a later preface to his *Lectures on Quaternions* (1853) all contain attempts by Hamilton to reconstruct the train of thought that led him to his discovery. These documents make the moment of truth on the bridge in Dublin one of the best-documented discoveries in the history of mathematics [9, 295].

Comments. Concluding from the endnotes, Hankins gave references to Graves' biography, the *Evening entry*, and the *Letter to John Graves*, but not to the *Preface.* He also gave a reference to the paper for which Hamilton had asked leave, p. 66, [4, 111-116], but did not quote from it. His assumption that Hamilton made his discovery "on" the bridge, suggests that he did not know about the turnpike, nor about Seán Keating's charcoal drawing of the Hamiltons on the tow-path under the bridge.[62]

6.) Hankins then started to describe the walk anew, now embedded in mathematical contemplations.

> [Not finding a triplet algebra] was a problem that he had been turning over in his head and struggling with for some time prior to his discovery by the bank of the canal. What occurred to him there was the sudden realization that the "obvious" system of triplets that he had under consideration at the time would succeed if extended to a quaternion of four numbers [9, 295].

Comments. This time Hankins described what according to him Hamilton's "sudden realisation" was, in quote 2.), p. 69, only having mentioned that Hamilton "suddenly" had seen a possible resolution. So far, it does not contradict Hamilton's reports.

7.) Then Hankins gave Hamilton's "train of thought" about triplets which had led to his "sudden realization", and remarked that the square of a triplet

> should be another triplet, while the product that Hamilton obtained had four terms. He had to do something with the ij term – either set it equal to zero, or add it somehow to one of the other three terms. Hamilton thought he might find an answer in a geometrical analogy between triplets and complex numbers [9, 295].

[62] *Sir William Hamilton at The Royal Canal Dublin* by Seán Keating [sirwilliamorpen.com]. In accord with the state of the gossip then, Lady Hamilton is not happy, and Hamilton has to peek in his pocket-book in order to write the formula on the bridge. Incidentally, the initiators of the first plaque on Broome Bridge must have been right, as also Daniel Doyle acknowledged, choosing that side of the bridge for his sand sculpture, Figure 5.2; if the Hamiltons "came up" to the bridge, Hamilton doubtless cut the formula on the observatory side of the bridge [webhomes.maths.ed.ac.uk]. That first plaque was, in 1958, unveiled by the mathematician and politician Éamon de Valera, who at that time was Taoiseach. In 2009, the then damaged plaque was restored and relocated to a place higher up on the bridge [archive.maths.nuim.ie], and in 2021 it was accompanied by a sculpture [mckeanesculptor.ie].

Comments. That Hamilton "thought he might find an answer" in a geometrical analogy between triplets and complex numbers is curious; it had been Hamilton's basic assumption. In the *Letter to John Graves* he wrote, "You know that I have long wished [...] to possess a Theory of Triplets, analogous to my published Theory of Couplets, and also to Mr. Warren's geometrical representation of imaginary quantities" [5, 490].

8.) Next giving the equivalent laws of the moduli and of the norms, Hankins wrote,

> Hamilton was particularly anxious that the law of the norms hold for triplets as well as for number couples, because it would determine the length of the product line in the geometrical interpretation of triplets and would guarantee that division of one triplet by another would always give a unique quotient [9, 296-297].

Comments. Hankins chose to call the 'law of the moduli', the 'law of the norms' when Hamilton was calculating squares, even though Hamilton did not make such an explicit distinction [5, 10-11]; taking norms instead of moduli just makes it slightly easier to write down. Having called the length of a quaternion, μ, its "modulus" [5, 12], Hamilton's preference for the word 'modulus' points to his, what Graves called, "immediate prevision of the applicability of quaternions to physical phenomena" [2, 439].

9.) After discussing further steps made by Hamilton, Hankins continued that Hamilton's assumption of $ij = -ji$ did not remove the ij term from the general product of two triplets, and neither could the ij term be set to zero,

> because the product of the norms is the sum of four squares, not of three squares. But the sum of four squares cannot be the modulus of a triplet. The only thing to do was to set ij equal to some unknown k and see if it might somehow be removed [9, 298].

Comments. In both the *Evening entry* and the *Letter to John Graves*, Hamilton wrote that concluding that ij could not be equal to zero, that the $(bz - cy)^2$ term could not be neglected, led to k as a new imaginary, of which, as Hamilton wrote in the *Evening entry*, the square still could be equal to 1 or -1. That Hankins suggested that after Hamilton knew that the ij term could not be zero, he still thought he could remove it somehow, may have been connected to his remark in 7.), p. 71, that it could "somehow be added to one of the other three terms," but that consideration was not found in Hamilton's reports. Or, it comes from the first part of the *Evening entry*, in which Hamilton wrote about a new imaginary k, and then still contemplated that ij could be zero. But that was the part he thereafter corrected, as discussed on p. 62.

10.) Hankins continued,

> Hamilton had probably reached this far in his calculations before he set forth on his walk along the Royal Canal. He was musing over his triplets and their unfortunate inclination to make a quaternion of terms in the product, talking to his wife, but with the triplets in the back of his mind. Suddenly it occurred to him that if the product of two triplets had four terms and if the product of their norms also produced a norm with four terms, maybe he would have better luck multiplying expressions with four terms than he had had multiplying triplets. If k were taken to be a third imaginary in addition to i and j, his elements would become quaternions and maybe they would work better. He thought that possibly $k^2 = 1$, but could see immediately that this would not work, because some of the cross terms in the calculation of the norm would not cancel out. They would cancel, however, if $k^2 = -1$ [9, 298-299].

Comments. It is now clear why in Hankins' reconstruction there is no eureka moment; this "sudden occurrence" only yielded quaternions which might "work better." Hankins did not give references for his reasonings about three and four terms, yet Hamilton did mention it in his 1853 *Preface*, when he wrote that he saw, when multiplying triplets, "that although the product of the sums of squares of the constituents of the two factors could not in general be decomposed into three squares of rational functions of them, yet it could be generally presented as the sum of four such squares" [7, (45)-(46)], [4, 143-144], and that that was what had led him to quaternions. That is of course true, also in the *Preface* the fourth squared k term, $(bz - cy)^2$, was given as having been followed by the discovery, but this is the only time Hamilton mentioned the number of terms as a basic consideration. There are some more differences between the *Preface* and the other sources, such as giving the fundamental equations between i, j, k and the conclusion $k^2 = -1$ in reversed order [7, (46)], yet it was not contradictory, this was not a chronological report. In the *Preface* Hamilton was very clear however about having known the basic geometrical interpretations before he asked leave at the meeting [7, (47)-(48)]; in the *Letter to John Graves* he had also given geometrical contemplations, but there it was without a time indication. It is very well possible that this was one of the subjects Hamilton thought or even talked about during the second part of the walk; it may not have taken him much time to translate it from triplets to quaternions.

11.) Hankins continued,

> Although Hamilton does not tell us what in particular caused him suddenly to think of quaternions in place of triplets, I think it was probably the realization that the right members of equations 22.5 and 22.6 [the general product of two triplets, and the multiplication of their moduli, respectively, [9, 297]] contained four related terms. If the right member of 22.6 was the norm of the right member of 22.5, then the right member of 22.5 would have a conjugate quaternion analogous to the conjugate of a complex number. It is an easy matter to check, and Hamilton, who had great powers of mental calculation and who had been mulling over the calculation in his mind for years, probably did it in his head. Having already assumed $i^2 = j^2 = -1$ and $ij = -ji = k$, he got
>
> $$(a + ib + jc + kd)(a - ib - jc - kd)$$
> $$= a^2 + b^2 + c^2 + d^2(-k^2) - bd(ik + ki) - cd(jk + kj).$$
>
> The expected norm is $a^2 + b^2 + c^2 + d^2$, and it is obviously obtained by setting $k^2 = -1$ and $ik = -ki$, $jk = -kj$. These substitutions are entirely analogous to the values that Hamilton had already obtained for i^2, j^2, and ij. The symmetry of the equations for multiplication of imaginaries must have been a powerful impetus in favor of quaternions [9, 299].

Comments. Hankins thus assumed that Hamilton had found $k^2 = -1$ by calculating the norm, as he had stated in 10.), using a quaternion and its conjugate, as he argued here, and instead of that directly leading to the eureka moment and the "sparks" following it, the fundamental equations between i, j, k, it was just another intermediate step. But not in any of the reports did Hamilton refer to having found $k^2 = -1$ by calculating the norm using a conjugate, nor that he thus was led to $ik = -ki$, $jk = -kj$ as yet another intermediate step. Hamilton wrote that $k^2 = -1$ followed from the non-cancellation of the term $2ad\alpha\delta$ in the multiplication of two quadruplets, see p. 63, and when he did mention a conjugate, in the *Proceedings* of 13 November 1843, it was only as a statement [6, 429].

12.) Hankins continued,

> The next step was to check to see if the law of the norms held for quaternions. In order to do this he needed values for jk and ik. (He had already assumed $ij = k$.) He decided that $ik = -j$, because $ik = i(ij) = (ii)j = (-1)j$. And in like fashion probably $kj = -i$, because $kj = (ij)j = i(jj) = i(-1)$. Then summarizing his "multiplication assumptions," as he called them:
>
> $$\begin{gathered} i^2 = j^2 = k^2 = -1 \\ ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j \end{gathered} \tag{6.1}$$
>
> These are the equations that he scratched on the bridge in great excitement. They would preserve his priority at least until he could announce his discovery that afternoon at the academy. On the way into the council meeting he checked the law of the modulus, writing out the terms in his pocket notebook. He discovered that the law holds for quaternions [9, 299-300].

Comment That Hankins suggested that Hamilton needed values for jk and ik to be able to check the law of the moduli instead of therewith rounding off the discovery, again shows that he did not believe Hamilton, who had written that the quaternions came into life "full grown," that the relations between i, j, k were the "sparks" following the discovery, that he had instantly conjectured that the law of the moduli would hold, and that he had cut the fundamental formula, $i^2 = j^2 = k^2 = ijk = -1$, on the bridge. In Hankins' reconstruction there was no eureka moment, no "sparks", no fundamental formula on the bridge, just continued calculations.

And two inconsistencies emerge in this last quote. Contemplating Hamilton's motive to cut the "equations" (6.1) on the bridge, Hankins had suggested that "maybe he felt that the solemnity of the moment required an inscription in stone," see 3.), p. 70. But this time he claimed that Hamilton had scratched them on the bridge "in great excitement," that they would preserve his priority; these two interpretations suggest a different stage of discovery. The same holds for the sentence in quote 4.), p. 71; "after the discovery flashed across Hamilton's mind he quickly elaborated it while riding in a car" [9, 294], suggests a different stage of discovery than what Hankins wrote here, that "on the way into the council meeting he checked the law of the modulus."

6.2. **Conclusion.** Doubting Hamilton's claims so explicitly is in accord with the Hamilton who emerges from Hankins' biography; a man who would openly sacrifice truth for romance. Who only loved one woman, whom he had met in his youth, and who, having lost her, "remained on the edge of alcoholism" from 1848 until the end of his life [9, 224, 347]. Hankins should at least have given arguments for his doubts, and presented them in a logical and consistent manner; in accusing Hamilton of exaggeration about what he wrote on the bridge it is not enough to write in an endnote that it was "unlikely."

Yet also in general it was hardly ever found in Hankins' biography that, when he gave his opinions about Hamilton, he explained how he had come to his conclusions. Which in turn is in accord with the observation that ever since Graves' biography, anyone can claim anything about Hamilton, as long as it fits the depressed and alcoholic view of him which had already developed during the first half of the twentieth century, and is still worsening by the decade.

7. Metaphysics, and applying mathematics

In his reports of the discovery of the quaternions, Hamilton mentioned that the 'law of the moduli' was important, yet not a single remark was found about why he wanted his system to obey this law, or, for that matter, that he wanted it to be applicable for physics. Searching in the sources used here, only one sentence was found in which he said something about his motives; in 1852 he wrote to De Morgan, "I wished for such a modulus, as a child might wish for the moon" [3, 429-430].

But closing his 'quaternion section', Hankins wrote that Hamilton's "quest for the triplets" had been "extremely metaphysical [...]. Both he and [John] Graves combined metaphysical and mathematical concepts in an extraordinary way in their correspondence. Hamilton was convinced that the operations of the mind, including mathematical operations, are in perfect harmony with the operations of the physical world as God created it. To understand the mind is to understand the world, and vice versa" [9, 301].

Indeed, in the section before the 'quaternion section', Hankins had discussed how, following the philosopher and poet Samuel Coleridge in "the polar and triadic arrangement of nature," in 1831 Hamilton had been led to the metaphysics of polarity of power and existence [9, 285-286]. As a reference, Hankins gave a memorandum by Hamilton, about Coleridge and metaphysics, in which Hamilton remarked, "power can be manifested only by its effects, that is, by overcoming Resistance, which is Contrary Power. Existence is manifested by the struggle between two opposite tendencies" [1, 438-439].

This can be recognised in Graves' description of how Hamilton found the quaternions; he was very clear about how Hamilton connected his metaphysics to the real world. When introducing parts of the *Evening entry*, he wrote, "Thinking it of importance to put on record in this volume how immediate was his prevision of the applicability of quaternions to physical phenomena I transcribe the following passages" [2, 439].

"October 16th 1843. ... Every line of length unity is thus a geometrical mean between $+1$ and -1, and every line of length μ is a mean between $+\mu$ and $-\mu$, the line being here treated as a mere imaginary, but $\pm\mu$ as pure reals opposite to each other. And $\sqrt{-1}$ has in this view infinitely many possible positions." "Perhaps we might give a semi-metaphysical interpretation to this result by remarking that any quality itself unlocal, such as heat may be, is indifferently related to any one direction and to its opposite. This conception is very vague, but it seems to me not foreign from the subject." "In the quaternion (v, x, y, z), xyz may determine direction and intensity; while v may determine the quantity of some agent such as electricity. x, y, z are electrically polarised, v electrically unpolarised. ... The Calculus of Quaternions may turn out to be a Calculus of Polarities."

It can therefore be concluded that, for Hamilton, metaphysical reasonings were as straightforward as any physical reasoning could be. And although he thus did not explicitly mention that he wanted his triplet turned quaternion system to be applicable in physics, it is very obvious from what he did; already in the *Evening entry*, Hamilton wrote about points in space and about lines, angular distances, "coordinates of the end of the line," Warren's principles, and about "a desired agreement between the results of an algebraical process [...] and a geometrical interpretation" [4, 103-104].

8. An amazing memory, and humility

That Hamilton checked the law of the moduli while sitting on the car, and in the presidents' chair, by just placing dots and signs, is one of the many manifestations of Hamilton's astounding memory, which was very fit for, and highly trained by, his habit of calculating in his head. Next to acknowledging the psychological discovery of 1832, after which Hamilton fell in love with Helen Bayly, this amazing memory is of paramount importance in understanding who Hamilton was, because he used it for his private life as well.

How impressive Hamilton's memory was is already known from the first volume of Graves' biography; Graves wrote that when only five years old, Hamilton was "loving to recite Dryden, Collins, Milton, and Homer" [1, 47]. In 1813, eight years old and having met the "calculating boy" Zerah Colburn,[63] he "liked to perform long operations in Arithmetic in [his] mind; extracting the square and cube root, and everything that related to the properties of numbers" [1, 111]. Hamilton never lost this ability, as can be seen from the fact that even in his one-but-last year he learned a "new and ingenious" way of "contracted multiplication" from a Mr. Boyers, a self-taught arithmetician who was employed at Dunsinea; Hamilton mentioned that it "required rather more of mental calculation in passing from step to step, yet not more than would be found easy by a practised calculator" [3, 153].

That Hamilton also used his highly trained memory for his daily life was completely acknowledged by Graves, who remarked, "I may here note that to him throughout his life thoughts were events. He would remember, when he came to a particular spot in a road or field, the conversation which years before had passed there with a friend, and recall it to that friend's memory" [1, 128]. But that was overlooked by most authors of biographies and biographical sketches, and only when it became possible to search in Graves' scanned biography, hints of "where did I see that before" could be followed up. It became apparent that Hamilton more often connected events which happened many years apart, throwing new light on things he wrote or did.

For instance, in February 1825 Hamilton had written the Valentine poem for Catherine Disney, and in the poem he had promised "To lay my triumphs at thy feet" [1, 175]. In 1853, when Catherine felt that she might die soon and invited him for an interview, to finally tell him that she had been forced by her family to marry Barlow, he offered her, on his knees, his just published book *Lectures on Quaternions*, "which represented the scientific labours of my life" [9, 352], [12, 84]. That can be regarded as overly romantic, or even, as egotistic because she would never understand it, yet Hamilton simply kept his promise.

Or, when in 1832 Hamilton had written to Aubrey de Vere about having fallen in love with Helen Bayly, De Vere had answered "how intensely anxious" he would be "until I hear that you are as happy as you deserve to be" [2, 6]. More than two decades later, in 1855, when for some months opening up to De Vere about the terrible fate of Catherine Disney [10, 311], Hamilton wrote, "I have been as happy in my own marriage as I expected, and more than I deserved to be" [3, 37]. That can be read as if Hamilton felt he had not deserved to have been so happy, yet it was, foremost, a direct answer to De Vere's earlier question.

Next to the psychological discovery and Hamilton's amazing memory, Hamilton was also defined by a life-long battle against vanity. Having been extremely highly praised from his very first years, and having been surrounded by loving and supportive relatives from both sides of the family,[64] already in 1809, when Hamilton was four, uncle James in Trim was worried about his nephew becoming "spoiled by praise" [1, 34]. And when he was seventeen, his aunt Mary Hutton was critical about what she apparently saw as boasting, and "administered a lesson of humility" [1, 113].

But Graves did not recognise this ongoing battle in his adored friend [2, 284], and even contradicted what Hamilton wrote about it himself; that self-esteem and cautiousness were strong in him, but faith and veneration moderate [10, 190]. Graves argued that "competent friends" would judge that also faith and veneration were strong in Hamilton [2, 249], and he must have reacted to what he saw, a "simple, zealous, great man" [2, 286]. Yet Hamilton at times hinted at his battle against vanity, and he was even very clear about it when in 1848 he wrote to Catherine Disney, having alluded to the difficult

[63] See *Zerah Colburn, the 'mental calculator'* [annevanweerden.nl].

[64] For how large Hamilton's maternal family was, see *How Arabella Lawrence's sister Sarah introduced William Rowan Hamilton to Samuel Taylor Coleridge* [annevanweerden.nl].

times after having lost her, "Of some great sorrow I am sure that the discipline was necessary to tame, in some degree, a spirit of self-reliance, which has not yet, perhaps, been sufficiently subdued, but against which I am learning, at least, to be more and more on my guard" [2, 611]. And in a letter to a friend the next year, he wrote that he believed that he had needed that infliction to "feel and own a Master against whom I cannot defend myself" [2, 2nd fn. on 611]. That Graves saw him as "simple," may therefore have been a reason for silent happiness.

9. The need for a new biography

Hamilton's amazing memory, his psychological discovery, and his striving to remain humble, are crucial to understanding who he was. And although Graves knew about his friend's extraordinary memory, it is very unfortunate that neither he, nor Hankins, recognised the combination of these three subjects, and their influence on Hamilton's private life, because having missed it led to the most detailed and therewith thorough distortion of any reputation found in the history of science; to see who Hamilton was, all minutiae have to be scrutinised, because not any has been left untouched.

Only in our time, being able to search through scanned texts, it is possible to connect the myriads of details given in Graves' enormous biography. Hankins, writing his biography in the 1970s without the digital advantages, had to make do with the then prevalent view on Hamilton, at that time mainly due to Alexander Macfarlane, Eric Temple Bell, and Edmund Whittaker, who during the first half of the twentieth century, in an increasing series of evolving gossip, had been turning Hamilton into an unhappy, full-blown alcoholic, as was described in [11, 14-16]. Hankins did paint a more positive image of Hamilton than his contemporaries had done. Yet not explicitly contradicting these alcoholic views, not defending Helen Bayly, and introducing Catherine Disney as an absolute, all-consuming force in Hamilton's life while connecting his distress about her with alcohol, annihilated any positivity that was still left in the views on Hamilton's private life. And due to the internet, the gossip became more widespread than ever before. It appears there is only one thing that can change the distorted view on Hamilton, and that is a new biography.

Such a biography should be written by someone who is well acquainted with Ireland in the 19th century, has a thorough knowledge of mathematics and metaphysics, combined with physics and astronomy, theology, psychology and literature, and is prepared to look beyond the gossip and start anew. Such a biography would not change the way in which the mathematicians, physicists, computer scientists, engineers, etc.[65] carry on the work following from Hamilton's enormous legacy. Yet, of course depending on what will be found in still undiscussed original sources, it is expected to show the world that all that work was not done by a depressed, overly romantic, lonely and alcoholic man.

Hamilton was way ahead of his time, and in 1854 his son William Edwin wrote,

> Sir W. is surprised at his own apathy with respect to the mode in which the quaternions are regarded by the world: e.g. Mr. Carmichael told him that M. Terquem had reprinted in Paris an old investigation of Sir W.'s, and he had not yet even the curiosity to inquire what it was. So about the review of them in the Mechanics' Magazine, of which John Graves had told him. Sir W. considers this a mark of his own indifference to contemporary fame, and as arising from his conviction that his belonged to a future age entirely [2, 693].

[65]In 1850, quaternions were applied to the problem of skew bridges [2, 649]. And for instance for protein folding both Hamilton's quaternions and mechanics, or extensions thereof, can be used, see H. R. Frost, *A quaternion model for single cell transcriptomics*, 2023 [biorxiv.org], and A. Robert et al., *Resource-efficient quantum algorithm for protein folding*, 2021 [nature.com].

That age is now. Mainly due to our digital era valuing Hamilton's work differently than it was in Graves' time, or in that of Hankins, there still is very much unexplored original information about Hamilton. Letters of women, letters which were deemed too unimportant, letters which were not sent to Graves when he prepared the biography, letters which were too personal for Graves, and of which Hankins only gave sentences. Graves was ensnared by his social circumstances; Hankins did not have the digital advantages we now have. By any criterion, Hamilton is important enough to deserve a third original biography, and regain his fame in the "future age" he dreamt of.

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On splitting strategies for the numerical solution of stochastic delay differential equations with correlated noises

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Abstract. In this article we investigate the numerical solution of a scalar semilinear stochastic delay differential equation (SDDE) where the linear instantaneous feedback and nonlinear delayed feedback terms are perturbed by a pair of standard Brownian motions with correlation ρ. Such SDDEs may be naturally decomposed into two subsystems: a linear stochastic differential equation (SDE) without delay, and a nonlinear SDDE.

Splitting methods work by solving each subsystem separately and composing the results over a single step. Our main theoretical result provides a bound on the mean-square error of a particular strategy for doing this, known as Lie-Trotter splitting. This bound implies that the method is mean-square strongly convergent with order 1/2 when $\rho = 0$, so that the noises are uncorrelated, but assurances of convergence are lost when $\rho \neq 0$. Indeed we develop an upper bound on the global mean-square error with a term that depends linearly on the magnitude of the correlation, and is independent of the stepsize.

While our theoretical error bound is an estimate from above, we conduct numerical experiments that confirm the order of mean-square strong convergence of Lie-Trotter splitting in the $\rho = 0$ case, and demonstrate a rapid fall-off to effectively zero as $|\rho|$ increases. Similar numerical results are observed for an alternative commonly used strategy known as Strang splitting. Nonetheless, by carefully reorganising the subsystems into which we split the SDDE, we can improve the range of values of ρ over which a nonzero order of convergence is observed numerically.

1. Introduction

1.1. General background: stochastic differential equations (SDEs).

Differential equations are used to model systems that evolve dynamically according to one or more feedback mechanisms. Real-world systems such as financial markets are subject to uncertainty and it is necessary to incorporate the possibility of random effects disturbing the system. In this work, we model the random perturbations by Brownian motion processes, leading to an SDE of Itô-type. A one-dimensional linear SDE perturbed by a single scalar standard Brownian motion B may be written in the first instance as an integral equation

$$X(t) = X_0 + \int_0^t \mu X(s)ds + \int_0^t \sigma X(s)dB(s), \quad t \in [0,T], \tag{1}$$

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for some constants $\mu, \sigma \in \mathbb{R}$ and initial value $X(0) = X_0 \in \mathbb{R}$. The second integral on the RHS of (1) is taken with respect to the Brownian motion, and is known as an Itô integral. Equation (1) is said to have stochastic differential form when it is written

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t), \quad t \in [0, T]. \tag{2}$$

The SDE (2) is widely used to model asset prices in financial markets, and is a rare example of an SDE that admits an explicit solution (see Section 2). In the real-world form of this model, μ represents the constant underlying growth rate of an asset and σ its constant volatility. Nonetheless, most SDEs cannot be solved explicitly, and so it is necessary to develop effective and efficient numerical solution techniques. An excellent technical introduction to Itô integrals and SDEs may be found in Mao [14], with a more applied treatment available in [13, Chapter 10].

The SDE (2) has solutions that are Markovian. In many applications it is desirable to model non-Markovian dynamics. In finance, we may be motivated to model time-varying asset-price volatility with historical dependencies as a stochastic process. Alternatively we may be motivated by the fact that, despite classical assumptions about market efficiency, there is evidence that past prices of an asset may sometimes influence its future evolution: see the discussion in [1, 2], and [15]. Applications of SDEs to the modelling of population dynamics naturally prompt us to incorporate the effects of gestation delay (see [3]). An SDE which, like (2), incorporates only instantaneous feedback will not capture these properties.

1.2. **Numerical solution of stochastic delay differential equations (SDDEs).** Motivated by this discussion, let us extend (2) to include additional nonlinear feedback terms with delay in both the drift and diffusion coefficients. Consider the scalar semilinear SDDE of Itô-type

$$\begin{aligned} dX(t) &= [\mu X(t) + f(X(t-\tau))]\, dt + \sigma X(t) dW_1(t) + g(X(t-\tau)) dW_2(t), \quad t \in (0, T], \\ X(t) &= \psi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{3}$$

with delay length $\tau > 0$, and linear coefficients $\mu, \sigma \in \mathbb{R}$. An SDDE of the form (3) describes a system with two feedback channels, one of which is linear and instantaneous, the other nonlinear and subject to feedback delay of length τ. Both channels are noisy, and the noise perturbations, though not identical on each channel, may be interdependent. Therefore we suppose that $W = [W_1, W_2]^T$ is a correlated pair of standard Brownian motions such that

$$W(t) \sim \mathcal{N}(0, \Sigma), \quad \Sigma = t \cdot \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \rho \in (-1, 1).$$

Due to the presence of both noise and feedback delays, SDDE solutions can oscillate about an equilibrium point. We say that a nontrivial continuous function $y : [0, \infty) \to \mathbb{R}$ is oscillatory (about zero) if the set $Z_y = \{t \geq 0 : y(t) = 0\}$ satisfies $\sup Z_y = \infty$. This definition may be applied pathwise to solutions of (3), which are said to be almost surely (a.s.) oscillatory if and only if there exists an a.s. event on which all individual trajectories are oscillatory functions. The joint role of noise and feedback delays in oscillation of SDDE solutions is explored in [5, 6, 7, 8], building on the analysis of delay-induced oscillation in deterministic differential equations in Gopalsamy [11].

In this article, we are interested in the efficient numerical solution of solutions of (3) via splitting. Splitting methods work by decomposing an equation into two or more subequations, each of which may be solved separately over a single discretisation step in a manner most suited to its structure, before being recombined. Readers interested in a general introduction to numerical solution of SDEs may refer to Higham & Kloeden [12]. Those seeking an introductory reference focused only on numerical solution of SDDEs may consult Buckwar [9].

By Cholesky factorisation of the covariance matrix Σ and by the linear transformation property of the Gaussian distribution, it is equivalent to sample from the SDDE

$$\begin{aligned} dX(t) &= [\mu X(t) + f(X(t-\tau))]\,dt \\ &\quad + \sigma X(t)\left[\sqrt{1-\rho^2}\,dB_1(t) + \rho\,dB_2(t)\right] + g(X(t-\tau))dB_2(t), \quad t \in (0,T], \end{aligned} \tag{4}$$

where $B = [B_1, B_2]^T$ is a pair of independent standard Brownian motions, and therefore $B(t) \sim \mathcal{N}(0, \mathbb{I}_2 t)$. We are motivated by the observation that terms on the RHS of (3) (and (4)) may be naturally divided into linear terms without a feedback delay, and nonlinear terms with a feedback delay. This induces a natural decomposition of the SDDE into two subsystems: a linear SDE without delay that can be solved exactly, and a nonlinear SDDE, which must be solved by some suitably tailored SDDE approximation method. Our results indicate that care must be taken when following such a strategy, since the correlation of W_1 and W_2 can have significant and potentially detrimental effects on the rate at which a splitting method converges in the strong stochastic sense to the true solution of the SDDE.

In this article we will examine two specific discretisation strategies of splitting type that are distinguished by the manner in which they recombine these subsystems. They are referred to as Lie-Trotter and Strang splitting respectively. In the case of Lie-Trotter, we find that when W_1 and W_2 are independent sources of noise, so that $\rho = 0$, mean-square strong convergence can be confirmed theoretically as being of order 1/2. However, when $\rho \neq 0$ our theoretical bound of the strong RMS error, though uniform in the stepsize δt, is no longer guaranteed to converge to zero with δt. We do not present a theoretical error analysis for the Strang method in this article.

Numerically we explore the effect of varying ρ on the empirical order of strong RMS convergence for both Lie-Trotter and Strang splitting. For both methods, our observations are consistent with order-1/2 strong convergence when $\rho = 0$, and we observe a rapid drop-off to zero in the convergence order as $|\rho|$ increases. By modifying the split to ensure that individual independent Brownian motions are not replicated across subsystems, we can increase the effective range of ρ over which we observe a nonzero order of convergence.

1.3. **Structure of the article.** In Section 2, we demonstrate how a variation of constants form of the unique global solutions of (4) may be constructed via the method-of-steps. This is useful for motivating our proposed splitting strategies, characterising the pathwise error in the proof of our main theorems, and providing a reference solution for our numerical investigation.

In Section 3 we introduce the Lie-Trotter and Strang splitting strategies applied to the SDDE (4), and provide a rigorous definition of the strong convergence of a numerical method to the true solution of (4) via a continuous-time interpolant of the scheme. In Section 4 we derive moment bounds on important processes related to the SDDE (4) and its discretisation. In Section 5 we present our main strong convergence result for the Lie-Trotter splitting method.

In Section 6 we numerically verify the error bounds derived in Section 5 for test equations with oscillatory and nonoscillatory trajectories, and observe the relationship between the convergence bounds and the strength of the correlation coefficient ρ. To aid readability, a presentation of the proofs of our main results is deferred to Section 8.

2. Existence and uniqueness of SDDE solutions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[-\tau,T]}, \mathbb{P})$ be a complete probability space equipped with filtration $(\mathcal{F}_t)_{t\in[-\tau,T]}$. In (3) and (4) respectively we suppose that W and B are $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. We also suppose that the initial data ψ is an $\mathcal{F}_0$-measurable random function, right continuous in $\mathbb{R}$ and $\mathbb{E}[\|\psi\|_\infty^p] < \infty$ for any $p \geq 2$, where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$. Here we define the norm $\|\psi\|_\infty := \sup_{-\tau\leq t\leq 0}|\psi(t)|$.

Under the conditions of the following assumption, (4) has a unique strong solution over the interval $[-\tau, T]$ for any $T < \infty$, adapted to the filtration $(\mathcal{F}_t)_{t\in[-\tau,T]}$.

Assumption 2.1. *Suppose that f and each g_i satisfy a linear growth condition: there exists a positive constant K_T such that for all $x \in \mathbb{R}$*

$$\max\left\{|f(x)|^2, |g(x)|^2\right\} \leq K_T(1+|x|^2). \tag{5}$$

Note that f, g are pure delay functions and take no input from the present state $X(t)$. Therefore local Lipschitz conditions are not required to ensure existence and uniqueness; see for example [14, Chapter 5.3].

Proceed by the method of steps. First consider the inhomogeneous scalar linear SDE

$$dx(t) = [\mu x(t) + f(t)]\,dt + \sum_{i=1}^{2}[b_i x(t) + g_i(t)]\,dB_i(t), \quad t \in [t_0, T], \tag{6}$$

where the initial value $x(t_0) = x_0$ is an $\mathcal{F}_{t_0}$-measurable and $L^p(\Omega,\mathbb{R})$-valued random variable for some $p \geq 2$, where f and each g_i are Borel measurable and bounded $\mathbb{R}$-valued functions on $[t_0, T]$. The unique solution of (6) can be expressed for any $t \in [t_0, T]$ in variation of constants form as

$$x(t) = \Phi_{t_0}(t)\left(x_0 + \int_{t_0}^{t}\Phi_{t_0}^{-1}(s)\left[f(s) - \sum_{i=1}^{2}b_i g_i(s)\right]ds + \sum_{i=1}^{2}\int_{t_0}^{t}\Phi_{t_0}^{-1}(s)g_i(s)dB_i(s)\right), \tag{7}$$

where $\Phi_{t_0}(t)$ is the fundamental solution of the corresponding homogeneous equation, given by

$$\Phi_{t_0}(t) = \exp\left\{\left(\mu - \frac{1}{2}\sum_{i=1}^{2}b_i^2\right)(t-t_0) + \sum_{i=1}^{2}b_i(B_i(t) - B_i(t_0))\right\}. \tag{8}$$

See, for example, Mao [14, Chapter 3]. Note that $\Phi_s(t)$ defined in (8) admits the decomposition

$$\Phi_s(t) = \Phi_s(u)\Phi_u(t), \quad 0 \leq s \leq u \leq t \leq T, \tag{9}$$

and moreover, $\Phi_s(t)$ is independent of $\mathcal{F}_s$.

This gives us an avenue to construct a solution for (4) successively over each interval $((j-1)\tau, j\tau]$, $j = 0, \ldots, M$, where $M = T/\tau$ (suppose without loss of generality that $M \in \mathbb{N}$). Set

$$b_1 = \sqrt{1-\rho^2}\sigma; \quad b_2 = \rho\sigma; \quad g_1 \equiv 0; \quad g_2 \equiv g.$$

Denoting on each interval $(j\tau, (j+1)\tau]$ the $\mathcal{F}_{j\tau}$-measurable process $\psi_j(t) := X(t-\tau)$, an application of the SDE (6) over the interval $(j\tau, (j+1)\tau]$ with $t_0 = j\tau$ gives, for $j = 0, \ldots, M-1$,

$$\begin{aligned} X(t) = \Phi_{j\tau}(t)\Bigg(X(j\tau) &+ \int_{j\tau}^{t}\Phi_{j\tau}^{-1}(s)\left[f(\psi_j(s)) - \rho\,\sigma g(\psi_j(s))\right]ds \\ &+ \int_{j\tau}^{t}\Phi_{j\tau}^{-1}(s)g(\psi_j(s))dB_2(s)\Bigg), \quad t \in (j\tau, (j+1)\tau], \end{aligned} \tag{10}$$

where, specifying (8) for the SDDE (4),

$$\Phi_s(t) = \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(t-s) + \sqrt{1-\rho^2}\,\sigma(B_1(t) - B_1(s)) + \rho\,\sigma(B_2(t) - B_2(s))\right\}.$$

We can see that when $j = 0$, $\psi_0(t) = \psi(t)$ corresponds to the $\mathcal{F}_0$-measurable initial data, allowing us to use (10) to uniquely construct $\psi_1(t)$ over the first step $t \in (0, \tau]$.

We can then construct stepwise $\psi_2(t)$ over $t \in (\tau, 2\tau]$, $\psi_3(t)$ over $t \in (2\tau, 3\tau]$, and so on, until the solution on the full interval $[-\tau, T]$ has been constructed in its entirety.

3. Splitting Schemes for SDDEs

In this section, we define the Lie-Trotter and Strang splitting schemes, and provide a precise characterisation of the strong convergence of a numerical approximation to the true solution of (4). Fix $N \in \mathbb{N}$ and let $\{0 = t_0, t_1, \ldots, t_N = T\}$ be a uniform mesh over the interval $[0, T]$, so that each $t_n = n\delta t$, where $\delta t = T/N < \min\{\tau, 1\}$. We allow this notation to support half-steps, whereby $t_{n+\frac{1}{2}} = (n + 1/2)\delta t$.

3.1. **Lie-Trotter and Strang splitting strategies.** Decompose (4) into the following subequations:

$$\begin{aligned} dY^{[1]}(t) &= f\left(Y^{[1]}(t-\tau)\right) dt + g\left(Y^{[1]}(t-\tau)\right) dB_2(t), \qquad t \in (0, T], \\ Y^{[1]}(t) &= Y^{[1]}_{[-\tau,0]} = \psi(t), \qquad t \in [-\tau, 0]. \end{aligned} \tag{11}$$

and

$$\begin{aligned} dY^{[2]}(t) &= \mu Y^{[2]}(t)\, dt + \sqrt{1-\rho^2}\sigma Y^{[2]}(t)\, dB_1(t) + \rho\sigma Y^{[2]}(t)\, dB_2(t), \qquad t \in (0, T], \\ Y^{[2]}(0) &= Y^{[2]}_0 = 1. \end{aligned} \tag{12}$$

A splitting strategy proposes to solve each of subequations (11) and (12) separately and sequentially over each timestep, using the output of one as an initial condition for the other.

The first subequation may be written in integral form as

$$\begin{aligned} Y^{[1]}\left((n+1)\delta t\right) &= Y^{[1]}(n\delta t) \\ &\quad + \int_{n\delta t}^{(n+1)\delta t} f\left(Y^{[1]}(s-\tau)\right) ds + \int_{n\delta t}^{(n+1)\delta t} g\left(Y^{[1]}(s-\tau)\right) dB_2(s), \end{aligned} \tag{13}$$

and this may be solved over each step by a suitable approximation of the integrals on the right-hand side. For example if we use the Euler-Maruyama method to solve (13) we have

$$Y^{[1]}_{n+1} = \Psi^{[1]}(t_{n+1}; t_n, Y^{[1]}_n)$$

where

$$\Psi^{[1]}(t; s, r, r_\tau) = r + f(r_\tau)|t-s| + g(r_\tau)(B_2(t) - B_2(s)),$$

for $t_n \le s < t \le t_{n+1}$ and any $n = 0, \ldots, N-1$. The second subequation may be solved exactly as

$$Y^{[2]}_{n+1} = \Psi^{[2]}\left(t_{n+1}; t_n, Y^{[1]}_{n+1}\right)$$

where

$$\begin{aligned} &\Psi^{[2]}(t; s, r) \\ &= r\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)|t-s| + \sqrt{1-\rho^2}\sigma(B_1(t) - B_1(s)) + \rho\sigma(B_2(t) - B_2(s))\right\}, \end{aligned} \tag{14}$$

again for $t_n \le s < t \le t_{n+1}$ and any $n = 0, \ldots, N-1$. Given the maps $\Psi^{[1]}$ and $\Psi^{[2]}$, we can now characterise various splitting strategies according to the manner in which these maps are composed. Two common examples are as follows.

(1) The Lie–Trotter splitting scheme is given over each step of length δt by the composition

$$X_{n+1} = \Psi^{[2]}\left(t_{n+1}; t_n, \Psi^{[1]}\left(t_{n+1}; t_n, X_n, X_{n-\lfloor \tau/\delta t \rfloor}\right)\right). \tag{15}$$

(2) The Strang splitting scheme is characterised over each step of length δt by the composition

$$X_{n+1} = \Psi^{[1]}\left(t_{n+1}; t_{n+\frac{1}{2}}, \Psi^{[2]}\left(t_{n+1}; t_n, \Psi^{[1]}\left(t_{n+\frac{1}{2}}; t_n, X_n, X_{n-\lfloor\tau/\delta t\rfloor}\right)\right), X_{n-\lfloor\tau/\delta t\rfloor}\right). \tag{16}$$

3.2. **Continuous interpolants of splitting schemes.** The Lie-Trotter scheme (15), where solution map $\Psi^{[1]}$ is an approximation and based upon the Euler-Maruyama method, may be written in stochastic difference equation form as

$$X_{n+1} = \Phi_{t_n}(t_{n+1})\left(X_n + \delta t\, f\left(X_{n-\lfloor\tau/\delta t\rfloor}\right) + g\left(X_{n-\lfloor\tau/\delta t\rfloor}\right)\left(B_2(t_{n+1}) - B_2(t_n)\right)\right), \tag{17}$$

for $n = 0, \ldots, N-1$. This is appropriate for implementation in code. However, for our strong convergence analysis we will work with a continuous-time interpolant of (17), given over the step $[t_n, t_{n+1}]$ by

$$Y(t) = \Phi_{t_n}(t)\left(Y(t_n) + \int_{t_n}^{t} f\left(Y\left(t_{n-\lfloor\tau/\delta t\rfloor}\right)\right) ds + \int_{t_n}^{t} g\left(Y\left(t_{n-\lfloor\tau/\delta t\rfloor}\right)\right) dB_2(s)\right). \tag{18}$$

Notice that the discrete and continuous variants of the scheme coincide at the mesh-points, so that $Y(t_n) = X_n$ for all $n = 0, \ldots, N$.

The Strang scheme can be written as

$$\begin{aligned} X_{n+1} = &\left(X_n + \frac{\delta t}{2} f\left(X_{n-\lfloor\tau/\delta t\rfloor}\right) + g\left(X_{n-\lfloor\tau/\delta t\rfloor}\right)\left(B_2\left(t_{n+\frac{1}{2}}\right) - B_2(t_n)\right)\right)\Phi_{t_n}(t_{n+1}) \\ &+ \frac{\delta t}{2} f\left(X_{n-\lfloor\tau/\delta t\rfloor}\right) + g\left(X_{n-\lfloor\tau/\delta t\rfloor}\right)\left(B_2\left(t_{n+1}\right) - B_2\left(t_{n+\frac{1}{2}}\right)\right), \end{aligned} \tag{19}$$

for $n = 0, \ldots, N-1$. We do not present the continuous interpolant of (19), since the Strang scheme is investigated only numerically in this article, and (19) is sufficient for this. Nonetheless, we will define in the next subsection the notion of strong convergence of a numerical method in terms of its continuous interpolant.

3.3. **Strong convergence of splitting schemes.** Suppose that Y is the solution of the continuous interpolant of either a Lie-Trotter or a Strang splitting as described in the previous subsection, and let X be the solution of (4). We say that Y converges strongly to X in p^{th}-moment with order γ if and only if there exists $C > 0$ independent of δt such that

$$\max_{t\in[0,T]} \mathbb{E}\left[|X(t) - Y(t)|^p\right] \leq C\delta t^{\gamma p}. \tag{20}$$

If $p = 2$ we say that Y converges to X in mean-square with order γ, and we are concerned with this case here.

4. Moment bounds on key processes

4.1. **Moment bounds on the fundamental solution.** In our analysis we will require the following bounds which are special cases of moment bounds applying to matrix-valued variants of $\Phi_s(t)$ provided in Erdogan & Lord [10]. In what follows, let $L_p(\Omega, \mathbb{R})$ denote the space of scalar real-valued random variables U on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfy $\mathbb{E}\left[|U|^p\right] < \infty$.

Lemma 4.1. *Let $0 \leq s < t \leq T$, $0 < t - s < 1$, $p \geq 2$ and consider the process $\Phi_s(t)$ given in* (8)*. For each $\mathcal{F}_s$-measurable random variable $U \in L_p(\Omega, \mathbb{R})$, there exists a constant C_p such that*

$$\mathbb{E}\left[\,|U - \Phi_s(t)U|^p\,\right] \leq C_p\,|t - s|^{p/2}. \tag{21}$$

Lemma 4.2. *Suppose $p \geq 2$, $0 \leq s < t \leq T$ and consider the process $\Phi_s(t)$ given in* (8)*. Then*

(1) *for any $\mathcal{F}_s$-measurable random variable $V \in L_p(\Omega, \mathbb{R})$,*

$$\mathbb{E}\left[|\Phi_s(t)V|^p\right] \leq \exp\left[\left(p|\mu| + \frac{p(p-1)\sigma^2}{2}\right)(t-s)\right] \mathbb{E}\left[|V|^p\right]. \tag{22}$$

(2) *for any $\mathcal{F}_t$-measurable random variable $U \in L_p(\Omega, \mathbb{R})$,*

$$\mathbb{E}\left[|\Phi_s(t)U|^p\right] \leq K_r^{p/r}\, \mathbb{E}\left[|U|^q\right]^{p/q}, \tag{23}$$

where $K_r = \mathbb{E}\left[|\Phi_s(t)|^r\right] < \infty$ for $1/p = 1/r + 1/q$ with $r, q > 1$.

4.2. **Moment properties of the Lie-Trotter scheme.** For our main convergence analysis, we require the following mean-square bound on the continuous interpolant of the Lie-Trotter splitting given by the SDDE (18):

Lemma 4.3. *Let Y be the continuous interpolant of the Lie-Trotter splitting scheme given by the SDDE* (18)*, and assume that the linear growth condition* (5) *in Assumption 2.1 holds on f and g. Then, for any $p \geq 2$ there exists a constant $C > 0$ such that for all $t \in [0, T]$*

$$\mathbb{E}\left[\sup_{s \in [-\tau, t]} |Y(s)|^p\right] \leq C. \tag{24}$$

Next, we state a result on the path regularity of the scheme.

Lemma 4.4. *Let Y be the continuous interpolant of the Lie-Trotter splitting scheme given by the SDDE* (18)*, and assume that the linear growth condition* (5) *in Assumption 2.1 holds on f and g. Suppose $p \geq 1$ and fix a step $[t_n, t_{n+1}]$. For all $t_n \leq s < t \leq t_{n+1}$, there is a constant $C > 0$ such that*

$$\mathbb{E}\left[|Y(t) - Y(s)|^p\right] \leq C|t-s|^{p/2}.$$

The proofs of Lemmas 4.3 and 4.4 are deferred to Section 8.

5. Main theoretical result

In this section we will provide a bound on the strong error of the Lie-Trotter splitting method, where the nonlinear subsystem with delay has been discretised using the Euler-Maruyama method. This bound confirms order-1/2 strong convergence in mean-square when $\rho = 0$, and provides a uniform nonzero bound in the case where $\rho \neq 0$. The proof relies upon the following Gronwall inequality for delay-type equations.

Lemma 5.1 (Delay Gronwall Inequality [4])**.** *Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, and $c \geq 0$. Let $a, b : [t_0, T] \to \mathbb{R}_+$ be locally integrable. Assume $r \geq 0$ and $\tau : [t_0, T] \to \mathbb{R}_+$ is such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

Set $b(t) = 0$ for $t_0 - r \leq t < t_0$. If a Borel measurable, locally bounded function $x : [t_0 - r, T] \to \mathbb{R}_+$ satisfies

$$x(t) \leq c + \int_{t_0}^{t} b(u)\, x(u)\, du + \int_{t_0}^{t} a(u) x\left(u - \tau(u)\right) du, \qquad t_0 \leq t < T,$$

then

$$x(t) \leq K \exp\left(\int_{t_0}^{t} b(s)\, ds + \int_{t_0}^{t} a(s) \exp\left(-\int_{s-\tau(s)}^{s} b(u)\, du\right) ds\right), \qquad t_0 \leq t < T,$$

where

$$K := \max\left(c, \sup_{t_0 - r \leq s \leq t_0} x(s)\right).$$

We also require a technical condition on the drift coefficient of the SDDE governed by (4) beyond what is required for the existence and uniqueness of solutions.

Assumption 5.1. *Let f, g be the nonlinear coefficients of* (4). *Assume that for all $x, y \in \mathbb{R}$ there exists a positive constant $C > 0$ with*

$$\langle x - y,\ f(x-\tau) - f(y-\tau)\rangle \leq C\,|x-y|^2, \tag{25}$$

and

$$|g(x) - g(y)| \leq C|x-y|. \tag{26}$$

Note that our analysis also works if condition (25) in Assumption 5.1 is replaced with a global Lipschitz bound on f. Moreover, global Lipschitz bounds on f and g are sufficient to ensure the linear growth bounds captured in (5) in Assumption 2.1.

We now present our main convergence result.

Theorem 5.2. *Let X be a solution of* (4), *and let Y be the continuous interpolant of the Lie-Trotter splitting method for* (4), *given by* (18). *Then, for all $\rho \in (-1, 1)$ there exist positive constants $K_1, K_2 > 0$, independent of δt and ρ, such that*

$$\max_{t\in[0,T]} \mathbb{E}\left[|X(t) - Y(t)|^2\right] \leq K_1\delta t + K_2|\rho|. \tag{27}$$

In the special case that $\rho = 0$, the Lie-Trotter splitting method is strongly mean-square-convergent with order $1/2$.

The proof of Theorem 5.2 relies directly upon Lemmas 4.1–4.4, and 5.1. A detailed presentation is deferred to Section 8.

Notice from (27) in the statement of Theorem 5.2 that if $\rho \neq 0$ the global strong error is bounded above uniformly in δt, and we can neither confirm nor rule out strong convergence. We investigate this numerically in the next section.

6. Numerical Experiments

In this section, we present numerical experiments to explore the theoretical result provided by Theorem 5.2 in Section 5. We will do this for both the Lie-Trotter and Strang splitting schemes for two example SDDEs.

6.1. Numerical estimation of the strong convergence order γ.

Motivated by the definition of strong convergence in (20), we assume that the root-mean-square error satisfies for some $C > 0$

$$\mathcal{E}_{\mathrm{RMS}} := \sqrt{\max_{k=0,\ldots,N} \mathbb{E}\left[|X(t_k) - Y(t_k)|^2\right]} \approx C(\delta t)^{\gamma}. \tag{28}$$

We then take logarithms on both sides of (28):

$$\log \mathcal{E}_{\mathrm{RMS}} \approx \log(C) + \gamma \log(\delta t).$$

In the numerical experiments, the expectation in $\mathcal{E}_{\mathrm{RMS}}$ is approximated simulating the pathwise error between the reference solution and numerical approximation over a sample of $M' = 500$ simulated trajectories. As reference solution, we use the variation of constants form of the exact solution of the SDDE given by (7), where over a very fine mesh of size 2^{-18}, the integral for the drift has been approximated using the trapezoid rule, and the integral for the drift has been approximated using an explicit rectangular approximation. The value of γ is then estimated as the slope of the regression line of $\log(\mathcal{E}_{\mathrm{RMS}})$ on $\log(\delta t)$.

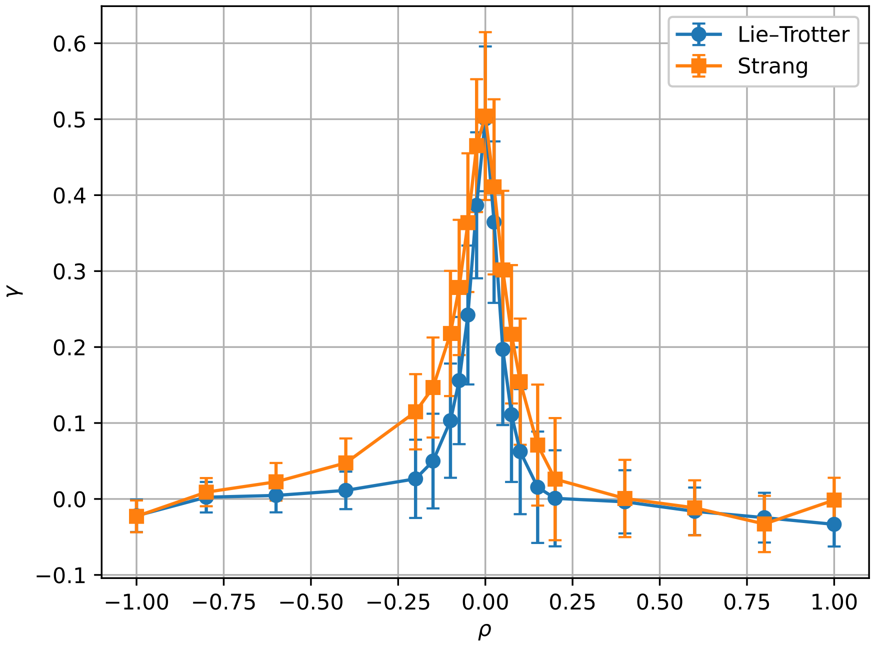

FIGURE 1. Strong convergence order γ against ρ (Example 6.1).

6.2. **Investigating the effect of changing ρ on numerical convergence order.** We will numerically investigate the strong convergence order of the Lie-Trotter and Stang splitting schemes given by (17) and (19) respectively.

Example 6.1. First consider consider the SDDE (4) with

$$f(x) = -x, \qquad g(x) = x,$$

and parameters

$$\mu = 0, \qquad \sigma = 1.2, \qquad \tau = 1, \qquad T = 8, \qquad \rho \in [-1, 1].$$

The initial condition is $X(t) = 1$ for $t \in [-\tau, 0]$. The reference solution is computed with time step 2^{-18}, whereas the numerical schemes are tested with

$$\delta t = \{\, 2^{-10},\, 2^{-11},\, 2^{-12},\, 2^{-13},\, 2^{-14}\}. \tag{29}$$

Figure 1 illustrates the observed numerical order of strong convergence with respect to the correlation parameter ρ for both the Lie–Trotter and Strang splitting schemes. We see that the numerical order of strong convergence is consistent with $\gamma = 1/2$ when $\rho = 0$, and declines rapidly but smoothly to zero as the size of $|\rho|$ increases. This result is consistent with the statement of Theorem 5.2 in the case of the Lie-Trotter scheme, and shows that the Strang scheme behaves similarly for the same numerical example.

Example 6.2. For comparison, we consider the case of (4) with

$$f(x) = 0.6x, \qquad g(x) = x,$$

and

$$\mu = -0.4, \qquad \sigma = 1.2, \qquad \tau = 1, \qquad T = 8, \qquad \rho \in [-1, 1].$$

We use the same initial condition, reference time step and δt values as in Example 6.1.

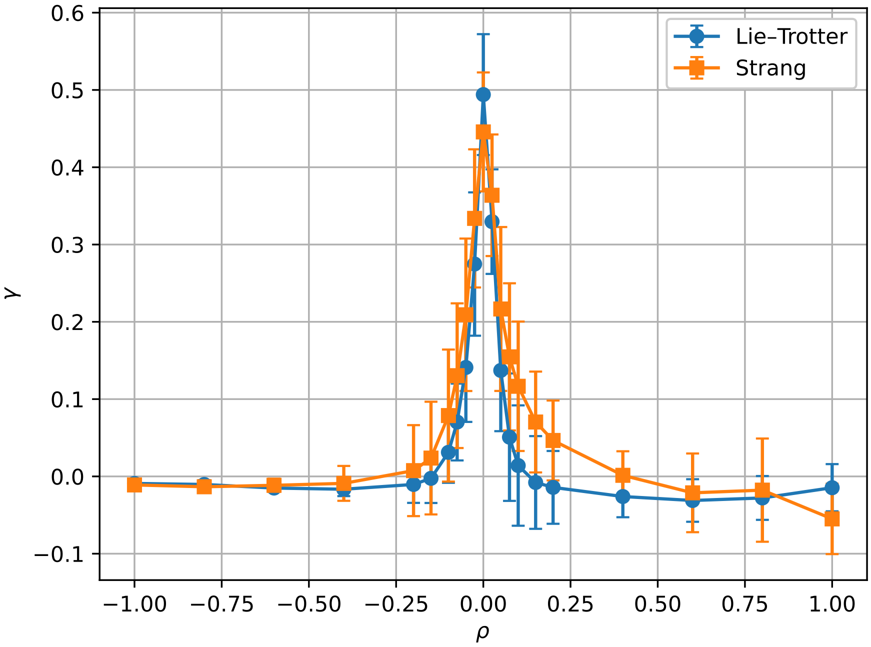

FIGURE 2. Strong convergence order γ against ρ (Example 6.2).

Figure 2 presents a plot of numerical order of strong convergence against ρ. In particular, when $\rho = 0$, the numerical convergence order is close to $\gamma = 1/2$, in agreement with the theoretical results established in Subsection 5, and we observe that the numerical order of strong convergences drops to zero as $|\rho|$ increases.

We finish by noting that, in both Figure 1 and 2, we take 20 groups of 25 trajectories to estimate the standard deviation in the error bars. Once the error bar associated with an observed order of convergence includes zero, we may conclude that the method is not converging numerically over the stepsize range (29) for this value of ρ.

6.3. **An alternative method: varying the split.** The proof of Theorem 5.2 in Section 8 relies on an application of the stochastic product rule to solutions (exact or approximate) of 11 and 12. Since these subsystems share the standard Brownian motion B_2 in common, this creates a square variation term that contributes to the overall error bound in the analysis. This term can be eliminated by moving the diffusion term with B_2 in 12 to (11), suggesting an alternative splitting based upon the new decomposed SDE subsystems

$$\begin{aligned} dY^{[1]}(t) &= f(Y^{[1]}(t-\tau))\,dt + \left[g(Y^{[1]}(t-\tau)) + \rho\sigma Y^{[1]}(t)\right] dB_2(t), \quad t \in (0,T]; \\ Y^{[1]}(t) &= Y^{[1]}_{[-\tau,0]} = \psi(t), \quad t \in [-\tau, 0], \end{aligned}$$

and

$$\begin{aligned} dY^{[2]}(t) &= \mu Y^{[2]}(t)\,dt + \sqrt{1-\rho^2}\,\sigma Y^{[2]}(t)\,dB_1(t), \quad t \in (0,T]; \\ Y^{[2]}(0) &= Y^{[2]}_0 = 1. \end{aligned}$$

with the construction and composition of associated maps as described according to the Lie-Trotter approach in Section 3. For this new variant, we have confirmed that

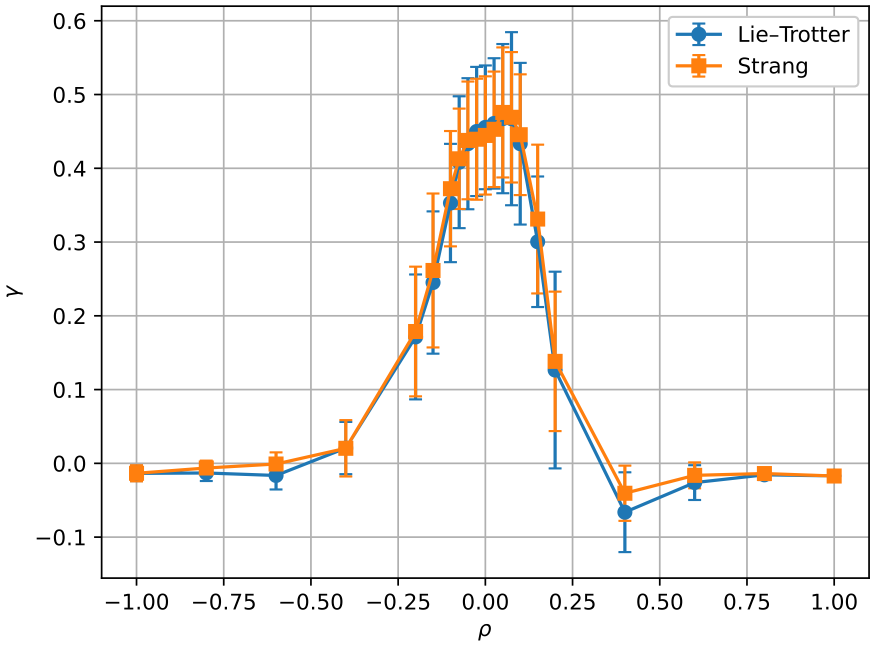

FIGURE 3. Strong convergence for Lie-Trotter variant showing order γ against ρ (Example 6.1).

an error bound of the form (27) in the statement of Theorem 5.2 still holds; the proof is similar to that of Theorem 5.2 and omitted for brevity. As such, we do not see an immediate theoretical advantage to this variant. However, in practice, we observe in Figures 3-4 that the range of values of ρ over which nonzero order of strong convergence is observed has increased, improving the useful domain of the method.

7. Conclusions and further work

We have shown that the Lie-Trotter splitting scheme is strongly convergent of order $1/2$ for a class of semilinear SDDEs when the noise perturbations on the delay and non-delay parts are independent. In the case where the SDDE contains correlated noises, we can ensure an upper bound on the global error of the Lie-Trotter scheme, however this bound is independent of the stepsize δt and does not guarantee convergence. Numerically we see that the observed convergence order reduces to zero rapidly as $|\rho|$ increases, and similar behaviour is seen for Strang splitting. By selecting a different decomposition of the SDE into subsystems that avoids repeating instances of the same standard Brownian motions across different subsystems, we can improve the range of ρ over which numerical strong convergence is observed, though the form of the obtainable theoretical error bound is the same.

We conclude that these specific splitting strategies, though intuitively natural, are only suitable for the numerical solution of such SDDEs in the case where the perturbing noises are independent or (at best) very weakly correlated. A variant that ensures the independence of noise perturbations across subsystems is seen to be more numerically effective.

There are many possible future directions for this work. These include developing a lower bound on the order of strong convergence, and extending the theoretical results

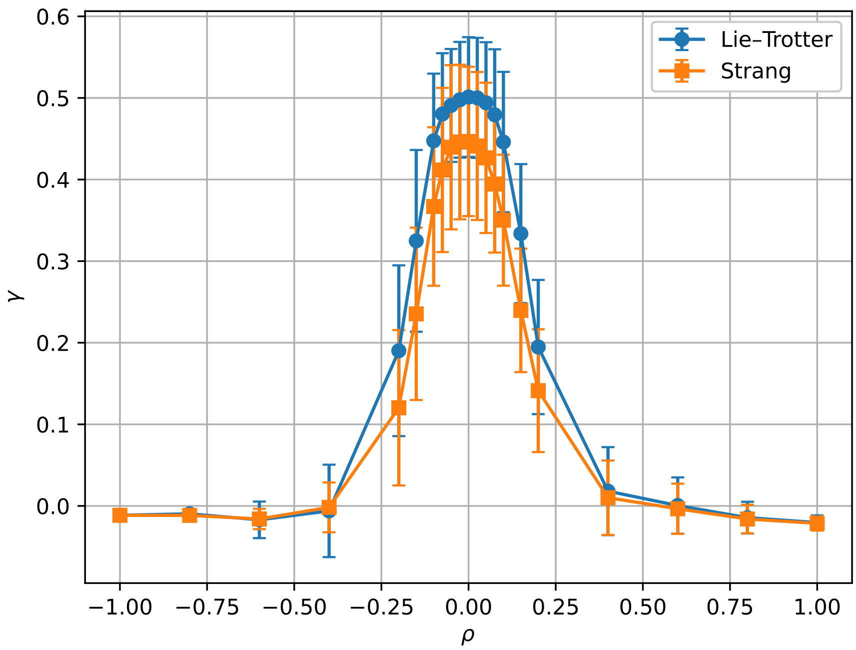

FIGURE 4. Strong convergence for Lie-Trotter variant showing order γ against ρ (Example 6.2).

in Sections 4 and 5 to finite-dimensional systems and to splitting schemes beyond Lie-Trotter; identifying further splitting strategies that are numerically effective when $\rho \neq 0$; investigating the dynamic consistency of these schemes, including their ability to capture oscillatory behaviour, and to empirically reproduce stationary distributions.

8. Proofs

Proof of Lemma 4.3. Fix any $n = 0, \ldots, N-1$. For $t \in [t_n, t_{n+1}]$ the solution of (18) may be written $Y(t) = Z_1(t)Z_2(t)$, where

$$\begin{aligned} dZ_1(t) &= \mu Z_1(t)dt + \sqrt{1-\rho^2}\sigma Z_1(t)dB_1(t) + \rho\sigma Z_1(t)dB_2(t); \\ dZ_2(t) &= f(\bar{Y}(t_n))dt + g(\bar{Y}(t_n))dB_2(t), \end{aligned}$$

with initial values $Z_1(t) = 1$, $Z_2(t_n) = Y(t_n)$, and using the notation $\bar{Y}_n := Y(t_{n-\lfloor \tau/\delta t \rfloor})$. An application of the stochastic product rule followed by the transformation $V(y) = |y|^p$ leads to the semimartingale representation

$$\begin{aligned} |Y(t)|^p &= |Y(t_n)|^p \\ &\quad + p\int_{t_n}^t \left[\Phi_{t_n}(s)f(\bar{Y}_n)|Y(s)|^{p-1} + \rho\sigma\Phi_{t_n}(s)g(\bar{Y}_n)|Y(s)|^{p-1} + \mu|Y(s)|^p\right] ds \\ &+ \frac{p(p-1)}{2}\int_{t_n}^t \left[\sigma^2|Y(s)|^p + \Phi_{t_n}(s)^2 g(\bar{Y}_n)^2|Y(s)|^{p-2} + 2\rho\sigma\Phi_{t_n}(s)g(\bar{Y}_n)|Y(s)|^{p-1}\right] ds \\ &+ p\int_{t_n}^t \sqrt{1-\rho^2}\sigma|Y(s)|^p dB_1(s) + p\int_{t_n}^t \left[\rho\sigma|Y(s)|^p + \Phi_{t_n}(s)g(\bar{Y}_n)|Y(s)|^{p-1}\right] dB_2(s), \end{aligned}$$

for $t \in [t_n, t_{n+1}]$. For brevity, we outline the proof under the assumption that both f and g are globally Lipschitz continuous and $f(0) = g(0) = 0$, in which case there exists $C > 0$ such that

$$|f(y)| \vee |g(y)| \leq C|y| \text{ for all } y \in \mathbb{R}. \tag{30}$$

A proof that requires only the linear growth bound (5) on f and g is more complex, and may be achieved using the alternative transformation $V(y) = (1 + |y|^2)^{p/2}$ (see for example Theorem 5.4.1 in Mao [14]).

Iterate both sides back to the initial data and make use of (30) and the fact that $|Y(0)| \leq \|\psi\|$ to get

$$\begin{aligned}
|Y(t)|^p \leq \|\psi\|^p + C\int_0^t \Bigg[& |Y(r)|^p + |Y(r)|^{p-1}\sum_{j=0}^{N-1}\left(|\bar{Y}_j|\Phi_{t_j}(r)\mathcal{I}_{\{r\in[t_j,t_{j+1}]\}}\right) \\
& + |Y(r)|^{p-2}\sum_{j=0}^{N-1}\left(|\bar{Y}_j|^2\Phi_{t_j}(r)^2\mathcal{I}_{\{r\in[t_j,t_{j+1}]\}}\right)\Bigg]\,dr \\
& + C\int_0^t |Y(r)|^p dB_1(r) + C\int_0^t |Y(r)|^p dB_2(r) \\
& + C\int_0^t \sum_{j=0}^{N-1}\left(|\bar{Y}_j|\Phi_{t_j}(r)\mathcal{I}_{\{r\in[t_j,t_{j+1}]\}}\right)|Y(r)|^{p-1}dB_2(r),
\end{aligned}$$

where $C > 0$ is a generic constant and $\mathcal{I}_A$ is the indicator of the set A. Take the supremum over the time set on both sides

$$\begin{aligned}
\sup_{s\in[-\tau,t]}|Y(s)|^p \leq{} & \|\psi\|^p \\
& + C\int_0^t \left[\sup_{u\in[-\tau,s]}|Y(u)|^p\left(1 + \sum_{j=0}^{N-1}\left(\Phi_{t_j}(s) + \Phi_{t_j}(s)^2\right)\mathcal{I}_{\{s\in[t_j,t_{j+1}]\}}\right)\right]ds \\
& + C\sup_{s\in[0,t]}\int_0^s |Y(r)|^p dB_1(r) + C\sup_{s\in[0,t]}\int_0^s |Y(r)|^p dB_2(r) \\
& + C\sup_{s\in[0,t]}\int_0^s \sum_{j=0}^{N-1}\left(|\bar{Y}_j|\Phi_{t_j}(r)\mathcal{I}_{\{r\in[t_j,t_{j+1}]\}}\right)|Y(r)|^{p-1}dB_2(r)
\end{aligned}$$

We may now take expectations on both sides, invoke the Burkholder-Davis-Gundy inequality to control the diffusion terms, and apply the bounds (22) in Lemma 4.2 to control the contributions from $\Phi_{t_n}(s)$. The proof may then be completed via an application of the continuous form of the Gronwall inequality found for example in [14, Theorem 1.8.1]. □

Proof of Lemma 4.4. On an individual step, Y can be written as:

$$Y(u) = \Phi_{t_n}(u)\left(Y(t_n) + \int_{t_n}^u f\left(\bar{Y}_n\right)\,ds + \int_{t_n}^u g\left(\bar{Y}_n\right)\,dB_2(s)\right), \quad u \in [t_n, t_{n+1}].$$

Using the decomposition $\Phi_s(t) = \Phi_s(u)\,\Phi_u(t)$, we get

$$\begin{aligned} &Y(t) - Y(s) \\ &\quad = \Phi_{t_n}(s)\,(\Phi_s(t) - 1)\,Y(t_n) + \Phi_{t_n}(s)\,(\Phi_s(t) - 1)\left(\int_{t_n}^{s} f\left(\bar{Y}_n\right)\,dr + \int_{t_n}^{s} g\left(\bar{Y}_n\right)\,dB_2(r)\right) \\ &\qquad + \Phi_{t_n}(t)\int_s^t f\left(\bar{Y}_n\right)\,dr + \Phi_{t_n}(t)\int_s^t g\left(\bar{Y}_n\right)\,dB_2(r). \end{aligned}$$

By the elementary inequality $|a_1 + a_2 + \cdots + a_5|^p \le 5^{\,p-1}\,(\,|a_1|^p + |a_2|^p + \cdots + |a_5|^p)$, we can estimate

$$\begin{aligned} &\mathbb{E}|Y(t) - Y(s)|^p \\ &\quad \le 5^{p-1}\,\underbrace{\mathbb{E}\,|\Phi_{t_n}(s)\,(\Phi_s(t) - 1)\,Y(t_n)|^p}_{=:\tilde{A}} + 5^{p-1}\,\underbrace{\mathbb{E}\left|\Phi_{t_n}(s)\,(\Phi_s(t) - 1)\int_{t_n}^{s} f\left(\bar{Y}_n\right)\,dr\right|^p}_{=:\tilde{B}} \\ &\qquad + 5^{p-1}\,\underbrace{\mathbb{E}\left|\Phi_{t_n}(s)\,(\Phi_s(t) - 1)\int_{t_n}^{s} g\left(\bar{Y}_n\right)\,dB_2(r)\right|^p}_{=:\tilde{C}} \\ &\qquad + 5^{p-1}\,\underbrace{\mathbb{E}\left|\Phi_{t_n}(t)\int_s^t f\left(\bar{Y}_n\right)\,dr\right|^p}_{=:\tilde{D}} + 5^{p-1}\,\underbrace{\mathbb{E}\left|\Phi_{t_n}(t)\int_s^t g\left(\bar{Y}_n\right)\,dB_2(r)\right|^p}_{=:\tilde{E}}. \end{aligned} \tag{31}$$

We will estimate each of the terms $\tilde{A}$-$\tilde{E}$ separately. In what follows, let $C > 0$ denote a generic positive constant that may change over the course of the proof, but which is always independent of the difference $t - s$.

<u>Term $\tilde{A}$:</u> Observe that since $t_n \le s \le t$, $\Phi_{t_n}(s)$, $\Phi_s(t)$, and $Y(t_n)$ are pairwise mutually independent. Therefore, by (21) in Lemma 4.1 with $U = Y(t_n)$, (22) in Lemma 4.2 with $V = 1$, and Lemma 4.3, there exists a constant $\bar{C} > 0$ such that

$$\begin{aligned} \tilde{A} = \mathbb{E}\,|\Phi_{t_n}(s)\,(\Phi_s(t) - 1)\,Y(t_n)|^p &\le \mathbb{E}\,|\Phi_{t_n}(s)|^p\,\mathbb{E}\,|(\Phi_s(t) - 1)Y(t_n)|^p \\ &\le C\,(t - s)^{p/2}. \end{aligned}$$

<u>Term $\tilde{B}$:</u> Recall that $\bar{Y}_n = Y(t_{n-\lfloor \tau/\delta t\rfloor})$ and $\delta t < \tau$. Then $\Phi_{t_n}(s)$, $\Phi_s(t)$, and $\int_{t_n}^{s} f\left(\bar{Y}(r)\right)\,dr$ are pairwise mutually independent, and we obtain in a similar way that

$$\tilde{B} = \mathbb{E}\left|\Phi_{t_n}(s)\,(\Phi_s(t) - 1)\int_{t_n}^{s} f\left(\bar{Y}_n\right)\,dr\right|^p \le C\,(t - s)^{p/2}\mathbb{E}\left|\int_{t_n}^{s} f\left(\bar{Y}_n\right)\,dr\right|^p.$$

Since $p \ge 1$, the function $|\cdot|^p$ is convex. Apply Jensen's inequality to the last factor on the RHS and bring the expectation inside the integral to get

$$\mathbb{E}\left|\int_{t_n}^{s} f\left(\bar{Y}_n\right)\,dr\right|^p \le (s - t_n)^{p-1}\int_{t_n}^{s} \mathbb{E}\left|f\left(\bar{Y}_n\right)\right|^p dr.$$

Applying (5) in Assumption 2.1, we get

$$\begin{aligned} \tilde{B} &\le C\,(t - s)^{p/2}\int_{t_n}^{s} \mathbb{E}\left|f\left(\bar{Y}_n\right)\right|^p\,dr \\ &\le C(t - s)^{p/2}(s - t_n)^{p-1}\int_{t_n}^{s} \mathbb{E}\left(1 + \bar{Y}_n^2\right)^{p/2} dr \\ &\le C(t - s)^{p/2}. \end{aligned}$$

At the last step we invoke the bound (24) in the statement of Lemma 4.3, and the fact that $s - t_n \le T$.

Term $\tilde{C}$: A similar argument, making additional use of Burkholder-Davis-Gundy inequality (see, for example [14, Theorem 1.7.1]) to bound the absolute moment of the Itô integral, gives

$$\begin{aligned}
\tilde{C} &= \mathbb{E}\left|\Phi_{t_n}(s)\,(\Phi_s(t)-1)\int_{t_n}^{s} g(\bar{Y}_n)\,dB_2(r)\right|^p \\
&\le C(t-s)^{p/2}\mathbb{E}\left[|\Phi_{t_n}(s)|^p\left|\int_{t_n}^{s} g(\bar{Y}_n)\,dB_2(r)\right|^p\right] \\
&\le C(t-s)^{p/2}\left(\mathbb{E}\left[|\Phi_{t_n}(s)|^{2p}\right]\right)^{1/2}\left(\mathbb{E}\left[\left|\int_{t_n}^{s} g(\bar{Y}_n)\,dB_2(r)\right|^{2p}\right]\right)^{1/2} \\
&\le C(t-s)^{p/2}(s-t_n)^{(p-1)/2}\left(\int_{t_n}^{s}\mathbb{E}\left|g\left(\bar{Y}_n\right)\right|^{2p}dr\right)^{1/2} \\
&\le C(t-s)^{p/2}(s-t_n)^{(p-1)/2}\left(\int_{t_n}^{s}\mathbb{E}\left(1+\bar{Y}_n^2\right)^p dr\right)^{1/2} \\
&\le C(t-s)^{p/2}.
\end{aligned}$$

Note the use of the Cauchy-Schwarz inequality at the second estimation.

Term $\tilde{D}$: Again, $\Phi_{t_n}(t)$ is independent of $\int_s^t f(\bar{Y}(r))\,dr$, and we may apply Lemma 4.2 and Hölder's inequality to show that

$$\begin{aligned}
\tilde{D} = \mathbb{E}\left|\Phi_{t_n}(t)\left(\int_s^t f(\bar{Y}_n)\,dr\right)\right|^p &= \mathbb{E}|\Phi_{t_n}(t)|^p\,\mathbb{E}\left|\int_s^t f(\bar{Y}_n)\,dr\right|^p \\
&\le C\,\mathbb{E}\left|\left(\int_s^t \left|f\left(\bar{Y}_n\right)\right|^p\,dr\right)^{1/p}(t-s)^{(p-1)/p}\right|^p \\
&\le C(t-s)^{p-1}\mathbb{E}\int_s^t\left|f\left(\bar{Y}_n\right)\right|^p dr.
\end{aligned}$$

Next, bringing the expectation inside the integral and applying (5), we get

$$\begin{aligned}
\tilde{D} &\le C(t-s)^{p-1}\int_s^t \mathbb{E}\left(1+\bar{Y}_n^2\right)^{p/2}dr \\
&\le C(t-s)^p \le C(t-s)^{p/2},
\end{aligned}$$

since $t - s < 1$.

Term $\tilde{E}$: By Part 2 of Lemma 4.2 we have for $1/p = 1/r + 1/2$ with $r > 1$,

$$\tilde{E} = \mathbb{E}\left|\Phi_{t_n}(t)\int_s^t g(\bar{Y}_n)\,dB_2(r)\right|^p \le K_r^{p/r}\left(\mathbb{E}\left|\int_s^t g(\bar{Y}_n)\,dB_2(r)\right|^q\right)^{p/q}. \tag{32}$$

An application of the Burkholder-Davis-Gundy inequality yields

$$\mathbb{E}\left|\int_s^t g(\bar{Y}_n)\,dB_2(r)\right|^q \le C(t-s)^{q/2-1}\int_s^t \mathbb{E}|g(\bar{Y}_n)|^q dr \le C(t-s)^{q/2}.$$

Apply the linear growth bound (5) to g, and the moment bound (24) in Lemma 4.3. Then (22) in Part 2 of Lemma 4.2 yields

$$\tilde{E} \le C(t-s)^{p/2}.$$

Substituting the estimates for terms $\tilde{A}$-$\tilde{E}$ back into (31) yields the statement of the Lemma. □

Proof of Theorem 5.2. Suppose that the interval $[t_n, t_{n+1}]$ lies on the mth delay interval $[m\tau, (m+1)\tau]$. The exact solution for $t \in [t_n, t_{n+1}]$ is given by

$$\begin{aligned}
X(t) =& \Phi_{m\tau}(t) \left[X(m\tau) + \int_{m\tau}^{t} \Phi_{m\tau}^{-1}(s)[f(X(s-\tau)) - \rho\sigma g\left(X(s-\tau)\right)]ds \right. \\
&\left. + \int_{m\tau}^{t} \Phi_{m\tau}^{-1}(s) g(X(s-\tau)) dB_2(s) \right] \\
=& \Phi_{m\tau}(t_n)\Phi_{t_n}(t) \left[X(m\tau) + \int_{m\tau}^{t_n} \Phi_{m\tau}^{-1}(s)[f(X(s-\tau)) - \rho\sigma g\left(X(s-\tau)\right)]ds \right. \\
&\left. + \int_{m\tau}^{t_n} \Phi_{m\tau}^{-1}(s) g(X(s-\tau)) dB_2(s) \right] \\
&+ \Phi_{m\tau}(t_n)\Phi_{t_n}(t) \left[\int_{t_n}^{t} \Phi_{m\tau}^{-1}(s) f(X(s-\tau)) ds + \int_{t_n}^{t} \Phi_{m\tau}^{-1}(s) g(X(s-\tau)) dB_2(s) \right] \\
=& \Phi_{t_n}(t) \left(X(t_n) + \Phi_{m\tau}(t_n) \left[\int_{t_n}^{t} \Phi_{m\tau}^{-1}(s)[f(X(s-\tau)) - \rho\sigma g\left(X(s-\tau)\right)]ds \right. \right. \\
&\left. \left. + \int_{t_n}^{t} \Phi_{m\tau}^{-1}(s) g(X(s-\tau)) dB_2(s) \right] \right).
\end{aligned}$$

Now define the (signed) error function:

$$\begin{aligned}
E(t) :=& X(t) - Y(t) \\
=& \Phi_{t_n}(t) \left(X(t_n) + \Phi_{m\tau}(t_n) \left[\int_{t_n}^{t} \Phi_{m\tau}^{-1}(s)[f(X(s-\tau)) - \rho\sigma g\left(X(s-\tau)\right)]ds \right. \right. \\
&\left. + \int_{t_n}^{t} \Phi_{m\tau}^{-1}(s) g(X(s-\tau)) dB_2(s) \right] \\
&\left. - Y(t_n) - \int_{t_n}^{t} f\left(Y\left(t_{n-\lfloor \tau/\delta t \rfloor}\right)\right) ds - \int_{t_n}^{t} g\left(Y\left(t_{n-\lfloor \tau/\delta t \rfloor}\right)\right) dB_2(s) \right) \\
=& \Phi_{t_n}(t) \left(E(t_n) + \int_{t_n}^{t} [\Phi_{t_n}^{-1}(s)[f(X(s-\tau)) - \rho\sigma g\left(X(s-\tau)\right)] - f\left(Y\left(t_{n-\lfloor \tau/\delta t \rfloor}\right)\right)]ds \right. \\
&\left. + \int_{t_n}^{t} [\Phi_{t_n}^{-1}(s) g(X(s-\tau)) - g\left(Y\left(t_{n-\lfloor \tau/\delta t \rfloor}\right)\right)] dB_2(s) \right) \\
=& Z_1(t) Z_2(t),
\end{aligned}$$

where Z_1 and Z_2 can be written as components of a system of Itô-type SDEs with

$$dZ_1(t) = \mu Z_1(t)dt + \sqrt{1-\rho^2}\sigma Z_1(t) dB_1(t) + \rho\sigma Z_1(t) dB_2(t),$$

and

$$\begin{aligned}
dZ_2(t) = & \left(\frac{f(X(t-\tau)) - \rho\sigma g\left(X(t-\tau)\right)}{Z_1(t)} - f\left(Y\left(t_{n-\lfloor \tau/\delta t \rfloor}\right)\right) \right) dt \\
& + \left(\frac{g(X(t-\tau))}{Z_1(t)} - g\left(Y\left(t_{n-\lfloor \tau/\delta t \rfloor}\right)\right) \right) dB_2(t),
\end{aligned}$$

with initial conditions on the interval $[t_n, t_{n+1}]$ given by $Z_1(t_n) = 1$ and $Z_2(t_n) = E(t_n)$. Since the exact solution $X(t)$ admits a finite second moment and, by Lemma 4.3, $Y(t)$ also admits a finite second moment, it follows that

$$\mathbb{E}\left[E(t_n)^2\right] = \mathbb{E}\left[|X(t_n) - Y(t_n)|^2\right] \leq 2\,\mathbb{E}\left[X(t_n)^2\right] + 2\,\mathbb{E}\left[Y(t_n)^2\right] < \infty.$$

We can use the stochastic product rule to also write $E(t)$ as an Itô-type SDE:

$$\begin{aligned}
dE(t) =& Z_1(t)dZ_2(t) + Z_2(t)dZ_1(t) + dZ_1(t)dZ_2(t)\\
=& \left[f\left(X(t-\tau)\right) - \Phi_{t_n}(t)\, f\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right) - \rho\sigma\Phi_{t_n}(t) g\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right) + \mu E(t)\right] dt\\
&+ \sqrt{1-\rho^2}\sigma E(t) dB_1(t)\\
&+ \left[g\left(X(t-\tau)\right) - \Phi_{t_n}(t)\, g\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right) + \rho\sigma E(t)\right] dB_2(t).
\end{aligned}$$

We may then apply Itô's formula to compute the SDE governing $E(t)^2$, write in integral form for any $t \in [t_n, t_{n+1}]$, and take expectations on both sides, making use of the martingale property to eliminate the Itô integrals:

$$\begin{aligned}
\mathbb{E}[E(t)^2] =& \mathbb{E}\left[E(t_n)^2\right] + (2\mu + \sigma^2)\int_{t_n}^{t} \mathbb{E}\left[E(s)^2\right] ds\\
&+ 2\int_{t_n}^{t} \mathbb{E}\left[\underbrace{\left(f(X(s-\tau)) - \Phi_{t_n}(s) f\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right)\right) E(s)}_{=:\bar{A}}\right] ds\\
&+ \int_{t_n}^{t} \mathbb{E}\left[\underbrace{\left(g(X(s-\tau)) - \Phi_{t_n}(s) g\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right)\right)^2}_{=:\bar{B}}\right] ds\\
&- 2\rho\sigma\int_{t_n}^{t} \mathbb{E}\left[\underbrace{\Phi_{t_n}(s) g\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right) E(s)}_{=:\bar{C}}\right] ds. \qquad (33)
\end{aligned}$$

We can rewrite the term $\bar{A}$ as follows:

$$\begin{aligned}
\bar{A} =& \left(f(X(s-\tau)) - \Phi_{t_n}(s) f\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right)\right) E(s)\\
=& \underbrace{E(s)\left(f(X(s-\tau)) - f(Y(s-\tau))\right)}_{=:\bar{A}_1} + \underbrace{E(s)\left(f(Y(s-\tau)) - \Phi_{t_n}(s)\, f\left(Y(s-\tau)\right)\right)}_{=:\bar{A}_2}\\
&+ \underbrace{E(s)\Phi_{t_n}(s)\left(f(Y(s-\tau)) - f\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right)\right)}_{=:\bar{A}_3}.
\end{aligned}$$

We can rewrite the term $\bar{B}$ as follows:

$$\begin{aligned}
\bar{B} =& \left[g(X(s-\tau)) - \Phi_{t_n}(s) g\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right)\right]^2\\
=& \left[\underbrace{\left(g(X(s-\tau)) - g(Y(s-\tau))\right)}_{=:\bar{B}_1} + \underbrace{\left(g(Y(s-\tau)) - \Phi_{t_n}(s)\, g\left(Y(s-\tau)\right)\right)}_{=:\bar{B}_2}\right.\\
&\left. + \underbrace{\Phi_{t_n}(s)\left(g(Y(s-\tau)) - g\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right)\right)}_{=:\bar{B}_3}\right]^2.
\end{aligned}$$

We bound each of the terms $\bar{A}_i$, $\bar{B}_i$, $i = 1, 2, 3$ separately. As in the proof of Lemma 4.4, C will represent a generic constant that may change in value over the course of the proof, but which is always independent of δt.

Term $\bar{A}_1$: By (25) in Assumption 5.1, there exists a constant $C > 0$ such that

$$\bar{A}_1 = E(s)\left(f(X(s-\tau)) - f(Y(s-\tau))\right) \le C E(s)^2, \quad \text{a.s.}$$

This bound applies pathwise and therefore

$$\mathbb{E}\left[\bar{A}_1\right] \leq C\mathbb{E}\left[E(s)^2\right]. \tag{34}$$

Term $\bar{A}_2$: Apply the elementary inequality $ab \leq a^2/2 + b^2/2$ to get

$$\bar{A}_2 = E(s)f(Y(s-\tau))(1-\Phi_{t_n}(s)) \leq \frac{1}{2}E(s)^2 + \frac{1}{2}f(Y(s-\tau))^2(1-\Phi_{t_n}(s))^2, \quad a.s.$$

Take expectations on both sides:

$$\begin{aligned}\mathbb{E}\left[\bar{A}_2\right] &\leq \frac{1}{2}\mathbb{E}\left[E(s)^2\right] + \frac{1}{2}\mathbb{E}\left[f(Y(s-\tau))^2(1-\Phi_{t_n}(s))^2\right] \\ &\leq \frac{1}{2}\mathbb{E}\left[E(s)^2\right] + \frac{1}{2}\mathbb{E}\left[f(Y(s-\tau))^2\right]\mathbb{E}\left[(1-\Phi_{t_n}(s))^2\right] \\ &\leq \frac{1}{2}\mathbb{E}\left[E(s)^2\right] + C\delta t. \end{aligned}\tag{35}$$

At the second step we use the fact that since $s-\tau < t_n$, the terms $f\left(Y(s-\tau)\right)$ and $1-\Phi_{t_n}(s)$ are mutually independent. At the third step we apply the moment bounds in Lemmas 4.1 and 4.3.

Term $\bar{A}_3$: Apply the same elementary inequality to get

$$\bar{A}_3 \leq \frac{1}{2}E(s)^2 + \frac{1}{2}\Phi_{t_n}(s)^2\left(f(Y(s-\tau)) - f\left(Y\left(t_{n-\lfloor\tau/\delta t\rfloor}\right)\right)\right)^2 \quad a.s.$$

By the independence of $\Phi_{t_n}(s)^2$ and $\left(f(Y(s-\tau)) - f\left(Y\left(t_{n-\lfloor\tau/\delta t\rfloor}\right)\right)\right)^2$, and taking expectations on both sides,

$$\begin{aligned}\mathbb{E}\left[\bar{A}_3\right] &\leq \frac{1}{2}\mathbb{E}\left[E(s)^2\right] + \frac{1}{2}\mathbb{E}\left[\Phi_{t_n}(s)^2\right]\mathbb{E}\left[\left(f(Y(s-\tau)) - f\left(Y\left(t_{n-\lfloor\tau/\delta t\rfloor}\right)\right)\right)^2\right] \\ &\leq \frac{1}{2}\mathbb{E}\left[E(s)^2\right] + \frac{1}{2}C\mathbb{E}\left[\left(f(Y(s-\tau)) - f\left(Y\left(t_{n-\lfloor\tau/\delta t\rfloor}\right)\right)\right)^2\right] \\ &\leq \frac{1}{2}\mathbb{E}\left[E(s)^2\right] + \frac{1}{2}C\mathbb{E}\left[\left(Y(s-\tau) - Y\left(t_{n-\lfloor\tau/\delta t\rfloor}\right)\right)^2\right] \\ &\leq \frac{1}{2}\mathbb{E}\left[E(s)^2\right] + \frac{1}{2}C\delta t, \end{aligned}\tag{36}$$

where we use Lemma 4.2 at the second step, (5) in Assumption 2.1 at the third step, and Lemma 4.4 at the last step.

By combining the estimates (34), (35), and (36), we arrive at the overall estimate

$$\mathbb{E}\left[\bar{A}\right] \leq C(\mathbb{E}\left[E(s)^2\right] + \delta t). \tag{37}$$

Next, we have $(\bar{B}_1 + \bar{B}_2 + \bar{B}_3)^2 \leq 3\bar{B}_1^2 + 3\bar{B}_2^2 + 3\bar{B}_3^2$. We estimate term-by-term.

Term $\bar{B}_1^2$: By (26) in Assumption 5.1, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\bar{B}_1^2\right] \leq C\mathbb{E}\left[E(s-\tau)^2\right]. \tag{38}$$

Term $\bar{B}_2^2$: By the independence of $(1-\Phi_{t_n}(s))^2$ and $g\left(Y(s-\tau)\right)^2$, Lemma 4.1 and Lemma 4.2, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\bar{B}_2^2\right] = \mathbb{E}\left[g(Y(s-\tau))^2(1-\Phi_{t_n}(s))^2\right] \leq C\delta t. \tag{39}$$

Term $\bar{B}_3^2$: By the independence of $\Phi_{t_n}(s)^2$ and $\left(g(Y(s-\tau)) - g\left(Y\left(t_{n-\lfloor\tau/\delta t\rfloor}\right)\right)\right)^2$, applying Lemma 4.2, the bound (26) in Assumption 5.1, and Lemma 4.4, there exists a constant $C > 0$ such that

$$\begin{aligned}\mathbb{E}\left[\bar{B}_3^2\right] =&\mathbb{E}\left[\Phi_{t_n}(s)^2\left(g(Y(s-\tau)) - g\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right)^2\right]\right.\\ =&\mathbb{E}\left[\Phi_{t_n}(s)^2\right]\mathbb{E}\left[\left(g(Y(s-\tau)) - g\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right)^2\right]\right.\\ \leq& C\delta t. \end{aligned} \tag{40}$$

By combining the estimates (38), (39), and (40), we arrive at the overall estimate

$$\mathbb{E}\left[\bar{B}\right] \leq C\left(\mathbb{E}\left[E(s-\tau)^2\right] + \delta t\right), \tag{41}$$

for some $C > 0$.

Term $\bar{C}$:

By the independence of $\Phi_{t_n}(s)^2$ and $g\left(Y\left(t_{n-\lfloor \tau/\delta t\rfloor}\right)\right)^2$, as well as the bounds given in Lemmas 4.2 and 4.3, there exists a constant $C_{12} > 0$ such that

$$-2\rho\sigma\mathbb{E}\left[\bar{C}\right] \leq |\rho| C\left(\mathbb{E}\left[E(s)^2\right] + 1\right). \tag{42}$$

Note that we have kept the linear dependence on $|\rho|$ explicit in this bound, rather than absorbing it into the constant C.

Substituting the bounds (37), (41), and (42), back into (33) and subtracting $\mathbb{E}\left[E(t_n)^2\right]$ from both sides gives

$$\begin{aligned}&\mathbb{E}\left[E(t)^2\right] - \mathbb{E}\left[E(t_n)^2\right]\\ &\leq (2\mu+\sigma^2)\int_{t_n}^{t}\mathbb{E}\left[E(s)^2\right]ds + C\int_{t_n}^{t}\left(\mathbb{E}\left[E(s)^2\right] + \mathbb{E}\left[E(s-\tau)^2\right] + \delta t + C|\rho|\right)ds\\ &\leq C\int_{t_n}^{t}\mathbb{E}\left[E(s)^2\right]ds + C\int_{t_n}^{t}\mathbb{E}\left[E(s-\tau)^2\right]ds + C\delta t^2 + C|\rho|\delta t. \end{aligned} \tag{43}$$

Let $\bar{N} = \max\{n : t_n \leq t\}$ be the index of the meshpoint immediately preceding the time t in (43). Now set $t = t_{\bar{N}}$ and sum both sides over $n = 0, \ldots, \bar{N}$. Finally, since (43) also holds if we replace t_n with $t_{\bar{N}}$ we can add it again to both sides yielding the inequality

$$\mathbb{E}\left[E(t)^2\right] \leq C\int_0^t \mathbb{E}\left[E(s)^2\right]\,ds + C\int_0^t \mathbb{E}\left[E(s-\tau)^2\right]\,ds + CT\delta t + C|\rho|T.$$

We have used the fact that $E(0) = 0$. Since this inequality holds for all $t \in [0, T]$, an application of the delay form of the Gronwall inequality (see Lemma 5.1) yields the statement of the Theorem. □

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Factors influencing productive teaching experiences of early career mathematics tutors in higher education: A case study of a professional development programme

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ABSTRACT. Graduate teaching assistants are often tasked with teaching large groups of students, commonly in workshops or tutorials, a number of times per week in each semester of the academic year. This paper discusses a case study on the challenges faced by early career mathematics tutors while learning-to-teach and the factors that shape their teaching experiences in higher education. These factors were identified from the analysis of data that was collected during a professional development (PD) programme centred around a community of practice, whose participants were a group of mathematics/statistics PhD students ($n = 7$). The factors that emerged from the sessions were 'Teaching', 'Concurrent and Post Teaching Reflection', the 'Affective Domain', 'Students', 'Past Experiences', and 'External Factors'. Each of these factors have multiple subcategories which are described in this paper. The findings also indicate that the PD programme played a meaningful role in alleviating the challenges that the tutors in this study faced by increasing their confidence and supporting their pedagogical development and perseverance. Considerations for future PD programmes are also discussed.

1. INTRODUCTION

Much research has been conducted into the learning and teaching of mathematics at primary- and post-primary levels of education. By comparison, there is less research on the same issues in higher education (HE) [14, 31]. However, it has been found that many students experience difficulty in transitioning from post-primary education to HE, including those who were high achievers in mathematics at post-primary level. Factors that negatively affect students' transition from post-primary education to HE include: changes in instructional practices in HE, shifts in the nature of the curriculum, changing assessment practices, changes in students' emotional disposition towards mathematics, increased speed of the introduction of new topics, a greater diversity of tasks, increased autonomy in the solving process, a new balance in the 'tool' and 'object' dimensions of mathematical concepts, and the increased role of definitions [1, 4, 10].

Compounding these issues is the so-called 'mathematics problem', i.e. the finding that the mathematical standards of beginning undergraduates have been in decline for many years in various countries around the world [16, 33, 34]. It is apparent that mathematics educators in HE need to possess teaching skills and knowledge of a very high level to help them to facilitate a positive and successful HE mathematics experience for their students, particularly during the transition phase when students' issues have been found to be most intensive. In contrast to this finding, new academic staff (including lecturers at various levels) in HE have been found to experience difficulty in

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terms of their self-confidence and preparation for their teaching roles [2, 39]. These difficulties are also evident amongst graduate teaching assistants (GTAs) in mathematics [11, 36, 39] and research suggests that the issues persist should GTAs become teaching staff in HE institutions [2]. It has been stated that the "impact of [teacher] experience is strongest during the first few years of teaching; after that, marginal returns diminish" [29, p.1]. Rice [29] further suggests that the quality of a teacher's training and preparation programme may have a greater influence on their effectiveness than teaching experience itself. Based on these findings, providing HE mathematics instructors with a teaching preparation programme during the early phases of their careers may be critical in terms of positively influencing their future teaching. However, some GTAs' research supervisors deem that professional development (PD) of GTAs' teaching skills does not warrant the time that it takes away from their research, and in some cases the supervisors do not permit, or do not buy-in to, GTAs' engagement with training [8, 14]. This is despite the finding that GTAs' participation in teaching PD does not change the length of time it takes them to complete their studies [5]. Our case study investigates the research questions:

- What challenges do GTAs encounter in their early teaching experiences?
- Does participating in a PD programme centred around community of practice (CoP) help to alleviate these challenges?

2. Literature Review

2.1. **Students' Cognitive and Emotional Difficulties Related to HE Mathematics.** The transition from post-primary education to HE can present significant cognitive and emotional difficulties for students in their mathematical studies, particularly in first year [4]. Difficulties have been reported for students undertaking degrees in mathematics [10], as well as for students taking service mathematics modules (i.e. students who take mathematics modules as part of their degrees, but whose degrees are not in mathematics, such as science/engineering students) [34]. The cognitive challenges stated in the literature include difficulties with the increased use of symbolism and generalisation, the role of definitions, formal reasoning and proof, abstraction, modelling, and the faster pace and changed expectations of HE mathematics [10, 13, 24, 32]. These challenges are compounded by students' prior exposure to more routine or procedural approaches in post-primary education and lecturers' assumptions about students' mathematical preparation [13, 24]. Furthermore, the 'mathematics problem' (see Section 1), has long been recognised as a detrimental factor for students transitioning to HE and its effects have been observed in Ireland, the United Kingdom, and internationally [12, 16, 18, 34]. Alongside these cognitive challenges, students may experience affective/emotional difficulties such as anxiety, insecurity, low confidence, fear, frustration, and negative self-perceptions of their mathematical ability [4, 10]. High levels of mathematical anxiety and low mathematical self-efficacy have been reported among specific groups of HE students, including engineering students, psychology students, and mature students [17, 23, 25, 35].

2.2. **The Role and Challenges of GTAs.** GTAs form the majority of future mathematics lecturing staff and their preparation for teaching roles does, and will, impact mathematics education in HE both now and in the future [6, 9, 11, 21]. Nearly all mathematics PhD students teach at some point during their postgraduate studies [8] with nearly two-thirds of this teaching in the USA being in first-year modules such as calculus or below that level [9]. They have important roles in module delivery, including having more contact with students than the module lecturers [27], and impacting on students' mindset and decisions about which major to pursue [8, 28, 37]. Despite this, they typically have very little teaching experience and/or education or training on teaching [11]. While they have been found to have strong subject matter knowledge, they have also been found to teach in a didactic, lecture-type, way with limited opportunities for student engagement [9, 36]. It has also been found that fundamental teaching skills such as voice projection, board work, checking for understanding, responding to different student learning styles and fostering participation and engagement among students all required improvement in GTAs' teaching skills [9, 36]. This highlights the fact that even though a teacher's knowledge of the subject matter is clearly vital and impacts on their acquisition of all other teaching skills [30], they do need guidance on teaching. Wood et al. [39] investigated the teaching and learning challenges that early career teachers (lecturers and tutors) in HE in Australia faced when teaching mathematical sciences. Respondents ($n = 111$) to a survey in Wood et al. [39], listed management of large classes, gaps and diversity in student knowledge and abilities, negative attitude of students, and lack of knowledge of effective and contemporary teaching methods as challenges that they faced in teaching. They also reported that the use of technology and presentation skills were areas in which they would like training.

2.3. **Professional Development for GTAs.** Despite the fact that many GTAs only teach in departments for a relatively short period of time, their participation in PD on teaching can "pay out-sized dividends in undergraduate teaching" [8, p.687]. In relation to previous PD for GTAs on teaching mathematics, most of the research is descriptive (i.e. the types of training/PD undertaken) [7]. Ellis [11] describes three models for GTA PD on teaching: (i) The Apprenticeship Model, which emphasises mentoring, observation, and progressive responsibility as tutors transition into their new role; (ii) The Coordinated Innovation Model, which combines pre-semester preparation, ongoing coordination around student-centred teaching, teaching observations, and weekly meetings within the teaching team; and (iii) The Peer-Mentor Model, which utilises experienced GTAs to support newer tutors and foster communities of practice. Murdoch et al. [20], in their project with GTAs in a school of criminology in Canada, also state four recommendations to enhance GTAs' training for teaching. They recommend that GTA training should be mandatory, discipline-specific, varied, ongoing, and that participants should be compensated.

A study with 253 STEM GTAs in the USA showed that more time engaged in PD on teaching and learning was correlated with higher teaching self-efficacy [6]. A similar finding is reported in [5]. Musgrave and Carlson [21] studied the effects of an intervention targeted at 19 precalculus and calculus GTAs in the USA. The intervention was designed to incorporate pre-semester workshops and weekly 90-minute seminars during the academic term where they prepared upcoming classes, discussed what was involved in understanding the upcoming concepts and practiced delivering mini-presentations on how they would support their students' learning in the class. The intervention focused on what was needed to understand and learn the key ideas of each lesson and encouraged the tutors to focus on mathematical meaning and precise communication. They found that while the GTAs had deficiencies in their understanding of average rate of change pre-intervention, mostly focusing on it in terms of computational or geometric interpretations, after the intervention 15 of the 19 GTAs attempted to give an explanation of it with a more conceptual focus. Although research on the experiences of GTAs has been carried out in other countries, there is a shortage of research in the Irish setting.

Deshler et al. [9] state that changes in knowledge, changes in beliefs about learning and teaching, increases in student success, and the analysis of student surveys, interviews, teaching observations and portfolios are some of the factors that can be analysed to determine how effective a PD programme is for GTAs.

The literature above indicates that GTAs occupy a significant position in undergraduate mathematics education, often teaching a large number of students, some of whom are experiencing substantial cognitive and emotional challenges related to mathematics in HE. Also, GTAs may begin teaching with limited preparation for teaching, low confidence, and uncertainty about how to support student engagement and understanding. These findings suggest a need for discipline-specific, ongoing professional development that supports GTAs at the outset of their teaching roles.

3. Methodology

3.1. **Sample.** The participants in this case study consisted of 7 PhD students (denoted P1 - P7) with teaching responsibilities (preparing and delivering tutorials and marking coursework and examinations) in the Department of Mathematics and Statistics in an Irish HE institution. The attendees at a tutor training day (Section 3.2.1) were asked to complete consent forms, as well as pre and post tutor training questionnaires which were completed immediately prior to and at the end of the training day (Appendix A). Participation in this case study was voluntary. Our sample consists of the participants, hereafter referred to as 'tutor(s)', who completed all of these forms during the tutor training day. The sample of 7 PhD students represented approximately 30% of the population of PhD students with teaching responsibilities in the department. As the case study took place prior to the second semester of the academic year, all of the tutors had some prior teaching experience (Table 1). Ethical approval was granted from the university's research ethics committee. The previous teaching experience of the tutors, in addition to any prior teaching training they received, is shown in Table 1.

Tutor	**Length of time teaching prior to this research**	**Previous tutor training**
P1	6 semesters	Yes. 1 tutor training day*
P2	1 semester	No
P3	2 years	No
P4	3.5 years	Yes. 1 tutor training day*
P5	1 semester	Yes, 2 years initial mathematics teacher training course
P6	7 years on an irregular basis	No
P7	1 semester	Tutor training (1 week) for teaching English to children. Sport coaching course (1 week)

Table 1. Teaching background of tutors in the sample (*The tutor training day these tutors previously completed is as described in Section 3.2.1, excluding Workshop 4).

The average length of time that six of the tutors had been teaching was 3.33 semesters (one tutor who stated a length of time, stated they only taught irregularly so was omitted in the calculation of this average) and so can be deemed to be in the early stages of their teaching career. Three of the tutors had received tutor training in the past in the form of a tutor training day, one tutor had completed two years of initial post-primary mathematics teacher education before changing their course of study to mathematical sciences, whilst the last three tutors had not received any training prior to this research. The average amount of time tutors spent teaching at the time of this case study was 3.6 hours per week (tutors met with students between 3 and 6 times a week for an hour to two hours at a time, with class sizes of approximately 20 - 50 students). The tutors stated that, on average, they felt that students would rate their teaching as 6 out of 10 (range 2.5 - 7.5), they rated their own confidence in teaching as 6 out of 10 (range 3 - 8) and rated their positive feelings towards teaching before the semester began as 8 out of 10 (range 3 - 9). The tutorials that the tutors were teaching were timetabled from Weeks 2 - 12 or Weeks 3 - 12 (inclusive, starting in Week 2 or 3 depending on the module they taught) of the semester.

3.2. **Professional Development Programme for Tutors.** The PD programme developed for the tutors in the sample was motivated by the National Forum for the Enhancement of Teaching and Learning in Higher Education's Professional Development Framework for all staff who teach in HE [22]. It consisted of a tutor training day, reflective practice and community of practice (CoP) sessions.

3.2.1. *Tutor training day.* Prior to the semester in which the case study commenced, tutors participated in a five-hour training day comprising four workshops whose activities addressed all five domains in the National Forum PD Framework [22].[1]

The workshops focused on several key areas of tutor development. Workshop 1 introduced tutors to their role, the characteristics of the students they would teach, and challenges associated with supporting mathematics learning. Workshop 2 developed communication skills, including explaining concepts, questioning techniques, listening skills, and the effective use of technology to support student understanding. Workshop 3 encouraged tutors to critically reflect on teaching practice through the analysis of recorded teaching examples and discussion of tutorial planning. Workshop 4 provided opportunities for tutors to apply their learning by planning and delivering short teaching sessions, engaging in self-reflection, and receiving feedback from peers and facilitators.

Collectively, the workshops were designed to support tutors in developing the knowledge, skills, and reflective practices required for effective mathematics tutoring, while aligning with the domains of the National Forum PD Framework.

3.2.2. *Written reflections.* In alignment with elements of domains 1 and 2 of the National Forum PD framework, the tutors were requested to write reflections on their tutorials during the semester on a prescribed reflection document (Appendix B). The decision to complete each reflection point in the document was left to the discretion of the tutors. The primary aim of the written reflections was to provide a platform for tutors to reflect on their teaching: to identify and analyse aspects of their teaching that went well, aspects that could be improved, and to formulate and implement methods to alleviate any difficulties they experienced. The reflections were also used to provide material for the facilitated CoP sessions (Section 3.2.3).

3.2.3. *Community of Practice (CoP) sessions.* Following the tutor training day, the authors held CoP sessions with the tutors twice over the course of a twelve-week semester (in the fourth and seventh weeks). The objective of the CoP sessions was to facilitate the sharing of common concerns and best-practice experiences among tutors, with the aim of co-creating new knowledge to advance their collective teaching practices. These CoP sessions were guided by a list of open questions (Appendix C) in order to identify those experiences the tutors were having during their teaching, and to provide a space for them to discuss teaching and their written reflections with each other if they were comfortable doing so. During the CoP sessions, concurrent feedback and advice, including teaching

[1] The five domains are (edited for brevity): 1. Personal development; 2. Professional identity, values and development; 3. Professional communication and dialogue; 4. Professional knowledge and skills; 5. Digital capacity.

demonstrations, were given by the authors who acted as mentors[2] during these CoP sessions. The tutors also shared best practice and created new knowledge with their peers and with the authors. They did this through discussing problems they experienced in their teaching, and questioning and reaching consensus on potential solutions raised.

3.3. **Data Analysis.** The data sources as described above were (i) pre and post tutor training day questionnaires (ii) transcriptions of CoP sessions and (iii) tutors' written reflections.

(i) Pre and post tutor training day questionnaires. The tutors' rating of their confidence, positive feeling towards teaching and any comments made were compared from the pre to post tutor training day questionnaires. The findings and comments made by the tutors on these questionnaires were listed under the domains of the National Forum's PD framework that they corresponded to.

(ii) Transcriptions of CoP Sessions. The CoP sessions were audio recorded and transcribed verbatim. Six tutors attended the Week 4 session and five attended the Week 7 session. Inductive thematic analysis, using line-by-line open coding [3, 19], was conducted in NVivo 11. The authors and a colleague independently analysed the transcripts without pre-defined codes, identifying factors that influenced tutors' early teaching experiences. Codes and themes were developed inductively from the data using the six-step process of Braun and Clarke [3, p.87]:

(1) Familiarizing yourself with your data
(2) Generating initial codes
(3) Searching for themes
(4) Reviewing themes
(5) Defining and naming themes
(6) Producing the report

Following their individual analyses, a meeting was held to discuss and agree upon the final themes relating to factors affecting tutors' early teaching experiences. After coding the data into primary categories, a second round of analysis was undertaken to identify secondary categories. Comparisons of tutors' verbal interactions within each primary category informed the development of these secondary categories, which also emerged inductively from the data.

(iii) Tutors' Written Reflections. Similar to the CoP sessions, inductive thematic analysis was used to examine the tutor's reflections. Also, the reflections completed by the tutors were examined as follows:

- Data from the tutors' reflections after their first tutorial were examined to determine whether the tutor training day impacted on their teaching and if yes, then how.
- Data from the tutors' reflections after subsequent tutorials were examined to view the tutors' teaching and learning experience, to determine whether the CoP sessions affected anything in their teaching, and to determine if there were any developments in their reflective approaches.

The data provided in the pre and post tutor training day questionnaires, the tutors' written reflections and the CoP sessions were combined as shown in Table 2 to address each of our research questions.

3.4. **Limitations.** This study has several limitations. As a small-scale qualitative case study involving a voluntary sample of seven PhD tutors from one Irish university, the findings are not generalisable to the population. Participants may also have

[2]The terms authors/mentors are used interchangeably depending on their appropriateness.

Research question	Data sources
What challenges do GTAs encounter in their early teaching experiences?	Reflections + post tutor training questionnaire + transcriptions of CoP sessions.
Does participating in a PD programme centred around CoP help to alleviate these challenges?	Reflections + pre and post tutor training questionnaires + transcriptions of CoP sessions.

Table 2. The data sources used to address each of the research questions

had more positive dispositions towards teaching and professional development than non-participants, introducing a degree of self-selection bias. The collected data also relied largely on self-reported surveys and reflections. In addition, variation in tutors' prior teaching experience and inconsistent completion of reflective journals limited the extent to which the participants' learning to teach trajectories could be analysed and/or compared. Finally, while the study provides insight into tutors' perceptions of factors which influenced their teaching experience and some insights into their perceived changes in their teaching, it does not include any classroom observations or student outcome data, which constrains claims about the impact of the programme on the participants' teaching practice or on their students' learning.

4. Results

This section presents the findings from the data analysis. First, we compare the pre and post tutor training questionnaires, examining tutors' self-reported changes in confidence and positive feelings towards teaching, what they learned from the training day and intended to implement, and any remaining questions or concerns. Second, we present themes from the post tutor training questionnaires and tutors' reflections after their first tutorial to explore perceived impacts of the tutor training on practice. Finally, we present the themes that emerged from the CoP sessions.

4.1. **Pre and Post Tutor Training Day questionnaires.** The tutors' rating of their confidence on a scale of 1-10 (1-not confident at all; 10-very confident) increased from a median of 6 prior to the tutor training day to a median of 7 afterwards. The median rating of the tutors' level of positive feelings towards teaching on a scale of 1-10 (1-I am dreading it; 10-I can't wait) decreased from a median of 8 to a median of 7 from pre to post tutor training questionnaires. Three tutors (P2, P3, and P5) reported an increase in their positive feelings towards teaching, two tutors (P4 and P7) reported no change, and two tutors (P1 and P5) reported a decrease.

No responses were given by the tutors to the question "Any additional comments" on the pre tutor training day questionnaire. In response to the question "state three things that you have learned from the tutor training day that you plan on implementing into your teaching?" on the post tutor training day questionnaire, the tutors listed the elements shown in Table 3. The domains of the National Forum's PD framework that these elements correspond to are also shown in Table 3.

In response to the question "state three issues/questions that you still have, that you feel were not addressed in the tutor training course?" on the post tutor training questionnaire, the tutors listed the issues in Table 4. The domains of the National Forum's PD framework that they correspond to are also shown in Table 4.

4.2. **Themes from Reflections and CoP sessions.** A total of 30 reflections were completed by the tutors. Given the difference in the number of reflections completed

Domain	Findings and Comments
Personal Development: the 'self' in teaching and learning (Domain 1).	Increased confidence and increased level of positive feelings towards teaching as detailed above.
Professional communication and dialogue in teaching and learning (Domain 3).	Speak slower; more awareness of my speed of writing and talking, in addition to organising the writing on the board better (e.g. divide the space on the board into three equal parts by drawing two vertical lines). Summarise important points at the end of the tutorial.
Professional knowledge and skills in teaching and learning (Domain 4).	Ask more guiding questions. Try to facilitate students' completion of questions during tutorial (as opposed to just writing up the solution). Try to cover less material to allow for increased student participation. Walk around the room while students are attempting tutorial problems. Show students why their suggested and/or completed solution may not be correct. Planning – spend more time thinking about different ways to explain.

Table 3. Learning experienced by the tutors and their links to the National Forum's PD framework

Domain(s)	Findings and Comments
Professional communication and dialogue in teaching and learning (Domain 3). Professional knowledge and skills in teaching and learning (Domain 4).	Maybe more practical advice on how to engage 1 to 1 with students in a very large class. How to phrase and pick precise English statements to describe each step in a solution.
Personal and professional digital capacity in teaching and learning (Domain 5).	How to use technology – projector, acetate sheet etc.

Table 4. Issues raised by the tutors and their links to the National Forum's PD framework

across participants, analysis of the tutors' first tutorial experiences was based on reflections from tutors P1–P5, while longitudinal analysis was based on tutors P1, P2, and P5, who provided sufficient data across multiple weeks (Section 4.2.2).

4.2.1. *Reflections completed after the tutors' first tutorial (P1-P5) and common themes noted in the post tutor training day questionnaires.* In the post tutor training questionnaire (Q1), tutors were asked to list three things that they have learned from the tutor training day that they plan on implementing in their teaching, while the reflection document (Q3) asked if the tutor training course had impacted on their teaching and (Q2) to list three things that they felt could have been improved and/or did not go well in their first tutorial. Each of the tutors who completed a reflection after their first tutorial mentioned at least one common item in their responses to these questions (Table 5).

Tutor	**Post-training day questionnaire (Plan on implementing in teaching)**	**Reflection after first tutorial (Impact of tutor training on teaching)**	**Reflection after first tutorial (Things that could have been improved/did not go well)**
P1	Speak slower.	Slowed down – students not getting too much information too quickly.	Was too quick to respond and said 'no' to incorrect responses to questions.
	If someone gives an approach to a question that is incorrect, go through it to help students to see why it is incorrect	Made sure students had time to take things down from board.	
P2	Ask more guiding questions.	Students were interactive, students answered questions posed and asked questions.	
	Summarise important points at the end of the tutorial.		Because I ran out of time at the end, I didn't get the chance to summarise what we went through in the tutorial
P3	Try to cover less material to allow for increased student participation.	The tutor training day emphasised the importance of clarity and understanding over quantity of material covered. As such I spent considerably more time on each question than I would have previously. Had multiple questions from students, questions were tackled slowly with students and motivation outlined at start of each question. Encouraged those who asked questions, a point that was raised at the training.	Too much time dedicated to one question.
	Try to get students to tackle questions in tutorial.	The tutor training day highlighted that ideally students should be attempting questions themselves in the tutorial, this is something I hope to implement, I indicated this to students in the first tutorial.	Students did not attempt questions themselves today.
P4	Explain when you've stopped writing (i.e. give students time).	Conscious of not blocking board as I write and explaining when students aren't writing.	May cover the board when I write. Explain when I've stopped writing.
P5	Students should be attempting questions	Asked more probing questions, tried to get students more involved. Answered students' questions and ensured they understood the answer.	Silence when I asked questions, blank faces and no answers. No answers to questions posed.

TABLE 5. Common items listed by tutors in the post tutor training questionnaires and in their reflections after their first tutorial.

Comparison of the post tutor training questionnaires and the reflections after the tutors' first tutorial indicated that tutors attempted to implement many of the practices emphasised during the training day, including questioning, encouraging student participation, greater lesson preparation, covering less material in greater depth, and allowing students time to process information. Difficulties reported included limited student participation, timing, and balancing the quality of the content taught with the quantity of content to be covered.

4.2.2. *Overview of learning journey of P1, P2 and P5.* Tutors P1, P2 and P5, who completed reflections across multiple weeks, showed evidence of developing reflective practice over time, moving towards a more proactive and adaptive approach to teaching. Common themes included increased use of questioning, greater attention to student participation and understanding, adaptation of teaching approaches, and an increasing emphasis on quality of learning over quantity of content covered. Their reflections also indicate increased confidence, such as taking time to think before "jumping into an explanation I haven't thought through", and a greater willingness to implement ideas raised in the CoP sessions in their teaching and classroom management.

4.2.3. *Transcriptions of CoP sessions.* Six factors (hereafter referred to in this subsection as 'primary categories' for the purpose of analysis) that shaped the tutors' experiences of teaching mathematics in HE were identified from the thematic analysis of the transcriptions of the CoP sessions.

(1) Teaching - skills in teaching and teaching approaches.
(2) Concurrent and post teaching reflection – the tutor documents what they do in their own teaching and reflects on the means and purpose of what they do. Identifying and analysing this information may lead to changes and improvements in their teaching practice.
(3) The affective domain – the tutor's own confidence, motivation, and patience, and the concern they have for their students.
(4) Students – the knowledge and behaviour of the tutor's students.
(5) Past experiences – the prior teaching experience of the tutor, the tutor's knowledge of the module they are teaching and the tutor's preconceptions of students in general.
(6) External factors – factors to do with resources (size of room, technology etc.) and knowledge of university procedures and management.

These six primary categories, were subdivided into secondary categories (Table 6). For example, the primary category of teaching is subdivided into two secondary categories: teaching skills (planning, explanation, and adaptive practice), and teaching approaches and pedagogical orientation. Themes within the secondary categories are included in (Table 6) alongside the corresponding secondary category.

4.2.4. *Themes within the primary categories from the transcriptions of the CoP sessions and reflections.* This section outlines the challenges experienced by tutors, viewed in relation to how frequently they were referenced in the CoP sessions and reflections. It was observed that tutors did not always mention all challenges in their written reflections; some were identified only through discussion during the CoP sessions.

Themes within the primary category of 'teaching'.

In the CoP sessions, the primary category of 'teaching' accounted for the highest percentage of all coded statements (48.9% by 5/6 tutors, and 34.4% by all 5 tutors, in the first and second CoP sessions respectively), with most of the statements falling in the secondary category of 'teaching skills (planning, explanation, and adaptive practice)'. The dominant themes within this secondary category based on both frequency and the proportion of tutors who mentioned them, are shown in Table 7.

Primary Category	**Secondary Category**	**Secondary Category Themes**	**Number of coded Statements**
Teaching	Teaching skills (planning, explanation, and adaptive practice)	Teaching Abstract Concepts. Quality versus Quantity. Difficulty with changing own behaviour. Motivating Students. Knowledge of Students. Facilitating Students' Input. Planning. Questioning. Mixed-Ability Teaching. Conscious Awareness.	$n = 37(41.1\%)$ CoP1 $n = 26(27.1\%)$ CoP2
	Teaching approaches and pedagogical orientation	Teaching Style and associated opinion on benefit to students. Outlining expectations. Contextualising. Perseverance in new approaches.	$n = 7(7.8\%)$ CoP1 $n = 7(7.3\%)$ CoP2
Concurrent and Post teaching Reflection	Reflective practice on teaching and student learning	Ability to acknowledge what went well. Perception of student learning. Learning from reflection.	$n = 14(15.6\%)$ CoP1 $n = 29(30.2\%)$ CoP2
Affective Domain	Affective dispositions towards teaching and students	Confidence. Motivation. Patience. Concern for Students.	$n = 4(4.4\%)$ CoP1 $n = 11(11.5\%)$ CoP2
Students	Students' engagement and behaviour	Engagement. Misbehaviour.	$n = 4(4.4\%)$ CoP1 $n = 5(5.2\%)$ CoP2
	Students' knowledge	Prior knowledge and preparedness	$n = 3(3.3\%)$ CoP1 $n = 10(10.4\%)$ CoP2
Past Experiences	Prior experience and assumptions	Prior teaching experience. Knowledge of module. Preconceptions of students.	$n = 7(7.8\%)$ CoP1 $n = 5(5.2\%)$ CoP2
External Factors	Institutional and resource factors	Classroom. Technology. Tutorial Sheets content and amount to complete. Class size. Knowledge of university procedures and management.	$n = 14(15.6\%)$ CoP1 $n = 3(3.1\%)$ CoP2

Table 6. Primary and Secondary Categories Identified as Factors that Shape the Teaching Experiences of Early Career Mathematics Tutors in Higher Education (with n = number of statements coded, and % = percentage of overall statements coded. CoP1 and CoP2 = Community of Practice Session 1 and Session 2 respectively.)

Across reflections following their first tutorial and during the first CoP session, tutors consistently identified three key areas of development: 'questioning', 'conscious awareness'[3], and 'facilitating student input'. The themes of 'questioning' and 'conscious

[3]Conscious awareness refers to tutors becoming more reflective about their teaching behaviours (e.g. pacing, clarity, repetition, and interaction with the class).

First CoP session	Second CoP session
Conscious awareness (*1)	Questioning (1)
Questioning (2)	Quality versus quantity (1)
Facilitating students' input (2)	Conscious awareness (2)
Planning (3)	Mixed-ability teaching (2)
	Motivating Students (2)

Table 7. Themes coded within the secondary category of 'teaching skills (planning, explanation, and adaptive practice)' at each CoP session. *Denotes the order of frequency of codings, note some themes tied in terms of order of frequency.

awareness' also emerged in the second CoP session. These were recognised as important teaching practices that tutors aimed to incorporate into their teaching practices but found difficult to implement effectively.

Questioning emerged as the most prominent challenge. Tutors found it difficult to get students to engage with questions, either because students were reluctant to respond or because tutors felt unsure how to pose effective, probing questions. Practical constraints, such as large class sizes, limited time due to large amount of content coverage requested by the lecturer, and uncertainty about handling unexpected answers further hindered their use of questioning as a teaching tool. In the second CoP session, the theme of 'questioning' was the most frequently coded theme. The tutors continued to experience some difficulties in developing this skill.

The theme of 'conscious awareness' was mentioned much less in the second CoP session than in the first CoP session. An item that was listed in the tutors' reflections after their first tutorial as something that could have been improved on was tutors being too quick to respond 'no' to incorrect answers (i.e. being too blunt and not encouraging students' attempts to contribute). Tutors found it challenging to avoid discouraging students while still guiding them effectively, and to engage quieter students when a few dominated discussions. A discussion on how best to respond to students' incorrect answers (theme of 'facilitating students' input') and on potential ways to engage quieter students occurred in the first CoP session. Additionally, time management emerged as an ongoing issue, affecting tutors' ability to implement planned practices (such as summarising at the end of tutorials). Overall, the findings indicate that while tutors were aware of key pedagogical strategies, translating them into practice proved complex due to both skill-related and contextual factors.

Between Weeks 4 and 7, tutors' focus shifted from their own teaching performance to student-centred concerns, particularly balancing 'quality versus quantity' , 'motivating students', and addressing 'mixed-ability' groups. These themes, absent in earlier discussions, became more prominent in the second CoP session, indicating developing pedagogical awareness. Tutors increasingly reflected on how their teaching impacted student engagement and learning, while also highlighting ongoing challenges, especially in managing mixed levels of student abilities and deciding how to allocate time effectively across learners with varied needs. Examples of some of the statements made by the tutors in the second CoP session are shown in Figure 1.

In the reflections that they completed after their first tutorial, tutors identified an ongoing tension between 'quantity versus quality' in their teaching. While the tutor training emphasised prioritising student understanding, tutors found it difficult to balance this with covering all required material. Slowing down improved student comprehension and satisfaction, but often meant less content was completed. This trade-off left tutors uncertain about the 'right' approach, highlighting a persistent challenge in

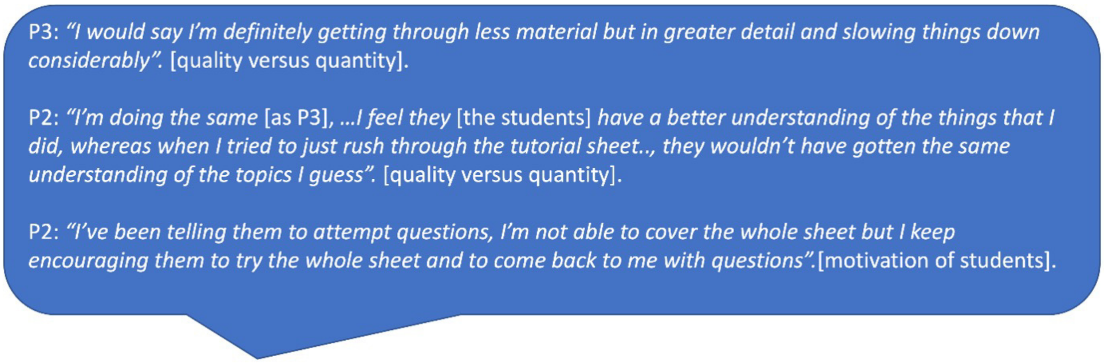

FIGURE 1. Examples of statements made in relation to the secondary category of 'teaching skills (planning, explanation, and adaptive practice)' in the second CoP session (Week 7).

managing instructional priorities.

Themes within the primary categories of 'concurrent and post teaching reflection' and 'external factors'.

In the first CoP session, the primary categories of 'concurrent and post teaching reflection' and 'external factors' each accounted for the second-highest percentage of coded statements (15.6%; 5/6 tutors). In the second CoP session, 'concurrent and post teaching reflection' and 'students' accounted for the second and third highest percentage of coded statements respectively (30.2%; all 5 tutors, and 15.6%; 4/5 tutors). Examples from the first session are shown in Figure 2, illustrating difficulties in relation to the themes of 'learning from reflection', 'perceptions of students' learning', 'tutorial sheet content', 'classroom' and 'technology'. Tutors found it difficult to gauge student progress; for example, P6 asked, "is there a way to measure it?". In the second CoP session, external factors such as pressure to complete lengthy tutorial sheets were seen as limiting opportunities for deeper student engagement through questioning. When expectations around content coverage were relaxed, tutors felt better able to prioritise interaction and understanding. In the second CoP session, reflection itself was viewed as valuable, helping tutors become more aware of their teaching in the moment and more intentional about improving their practice in future sessions.

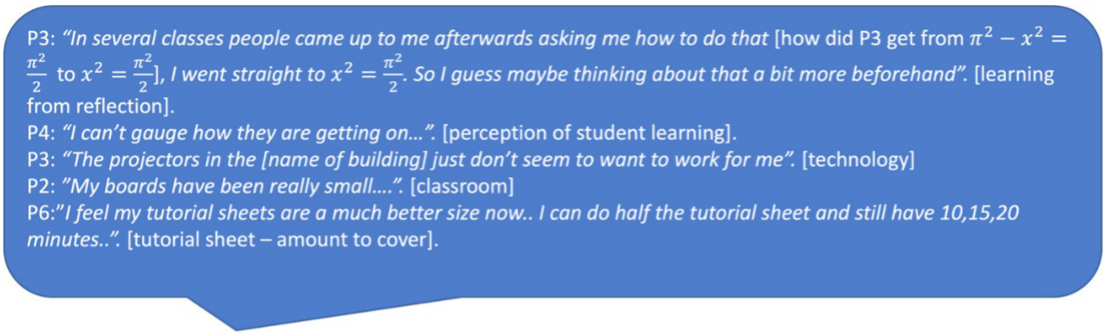

FIGURE 2. Examples of statements made in relation to the primary categories of 'concurrent and post teaching reflection' and 'external factors' in the first CoP session (Week 4).

Themes within the primary categories of 'students', 'past experiences' and 'affective domain'.

Tutors showed an increased focus on 'students', driven mainly by the secondary category 'students' knowledge', which rose from 3.3% of coded statements (2/6 tutors) in the first CoP session to 10.4% (3/5 tutors) in the second. Tutors expressed concern about students' knowledge and their ability to remain engaged when encountering difficulties. They were surprised by some of the challenges students faced and felt that gaps in knowledge and understanding slowed progress through tutorial sheet questions. Examples of statements relating to students' knowledge from the second CoP session are shown in Figure 3.

> P1:..*"I guess what surprised me about this was like we did partial fractions,.., and they couldn't even start the question. It wasn't like here's a question, solve it, it was like here's a question, here's how you should solve it, now do the partial fractions and they couldn't do that".*
>
> P3:....*"it's sort of worrying to see at the end of the tutorial like, ask a question like that and you're like hmmm".*

FIGURE 3. Examples of statements made in relation to the secondary category of 'students' knowledge' in the second CoP session (Week 7).

In the first CoP session, tutors' discussions focused on their own 'past experiences' and on their 'students'. Comments about students centred mainly on 'students' knowledge', and their 'engagement and behaviour', while references to 'past experiences' drew on tutors' 'preconceptions of students' and 'prior teaching experience'. Examples of statements made in relation to these themes in the first CoP session are shown in Figure 4.

> P3: *"..just trying not to skip over too much of the basics in case.."* [students - knowledge]
> P1: *"..I was getting like a lot of responses..".* [students – engagement]
> P4: *"I was surprised I didn't have any chatters".* [preconception of students].
> P4: *"...this is the fourth time I've taught it* [particular module]*, so I'm really quite confident in how to go about the questions..".* [prior teaching experience].

FIGURE 4. Examples of statements made in relation to the primary categories of 'students' and 'past experiences' in the first CoP session (Week 7).

In the second CoP session, tutors' discussions included themes related to the 'affective domain' and 'past experiences' (fourth and fifth highest percentage of coded statements respectively: 11.5% by all 5 tutors and 5.2% by 4/5 tutors respectively). Most affective domain comments reflected 'concern for students', while references to 'past experiences' centred on 'preconceptions of students' and 'prior teaching experience'. Compared with the first CoP session, affective domain themes became substantially more prominent, particularly 'concern for students', 'motivation', and 'patience'. Some examples of these statements made in relation to the primary category of 'affective domain' in the second CoP session are shown in Figure 5.

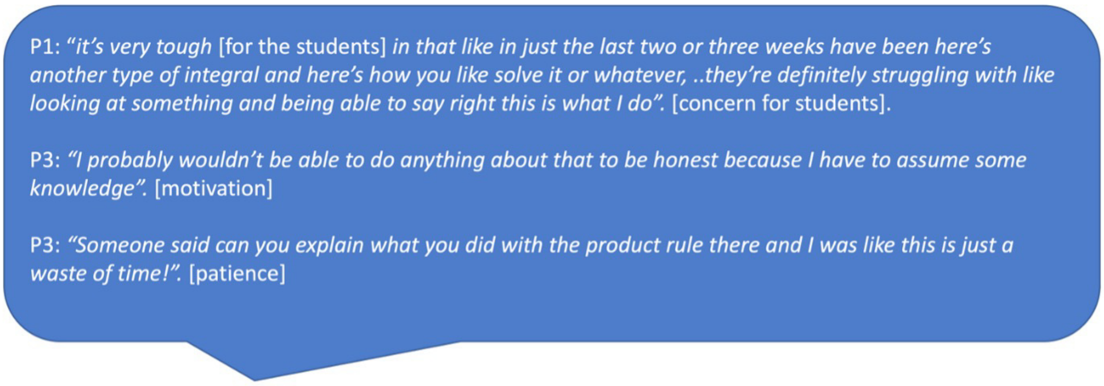

FIGURE 5. Examples of statements made in relation to the primary category of 'affective domain' in the second CoP session (Week 7).

In the second CoP session, the primary category of 'external factors' accounted for the least percentage of all coded statements (3.1% by 2/6 tutors) with the theme of 'tutorial sheet - content and amount to complete' coded highest. There was decreased focus by the tutors in categories 'external factors' and 'past experiences' from the first to the second CoP session.

The challenges that the tutors experienced in their early teaching experiences occurred in the six primary categories as described in Section 4.2.3. The factors that shaped the teaching experiences of the tutors in this case study were seen to remain largely the same from Week 4 to Week 7. The order of frequency of the factor 'external factors' changed from the second highest coded category to the lowest coded category from Week 4 to Week 7, with the factor 'past experiences' changing from the third highest coded category to the fifth highest coded (or second lowest) category.

5. Discussion

Based on findings from the review of literature, the provision of a teaching preparation programme to HE mathematics instructors, particularly during the early phases of their careers, may be critical in terms of positively influencing their future teaching [29]. Both a teacher's experience, and the training and preparation programme they engage with, impact considerably on their teaching practice [8, 20, 21], with teacher training most likely having the greater influence [29]. In our study, the tutors' voices highlighted the importance, necessity and benefit of early PD initiatives on their initial teaching experiences. The authors reiterate that the small sample size limits the extent to which conclusions can be drawn from these findings, and we cannot generalise the results beyond our sample. We structure the discussion around the two research questions guiding the study: (RQ1) the challenges GTAs encounter in their early teaching experiences, and (RQ2) whether participation in a PD programme centred around community of practice (CoP) helps to alleviate these challenges.

5.1. **RQ1: What challenges do GTAs encounter in their early teaching experiences?** Our analysis revealed difficulties related to questioning, facilitating students' input, identifying the source of students' misunderstandings, managing mixed-ability classrooms, and balancing quality of learning with coverage of content. The facilitation of students' input and mixed-ability teaching have previously been highlighted as challenges for tutors [9, 39]. Although the tutors in our study were aware of the benefits of implementing particular teaching skills from the tutor training day in their teaching, and did try to incorporate these, they found themselves sometimes doing the opposite of what they meant to implement and then becoming very aware of this during their teaching. From the comments made by the tutors in relation to questioning, specifically figuring out what probing questions might help the student to gain understanding, the findings suggest that strong content knowledge does not necessarily translate into pedagogical questioning skills (similar to [30]), reinforcing similar research findings on GTA pedagogical gaps [9, 36]. This may be a result of a combination of never having thought about the need for probing questions, a lack of practice, and being somewhat conditioned to the lecture style of teaching from their own university experience. The lecture style of teaching has previously been found to be favoured by GTAs [9].

Tutors also found it challenging to determine precisely where students were experiencing difficulty. Questions that revealed substantial gaps in prior knowledge were described as "worrying" and tested tutors' confidence and patience. For example, P3 found it concerning that a student asked about how to implement the product rule while they were completing questions which required its use with partial derivatives. As reported in Section 2.1, these experiences align with well-documented weaknesses in undergraduate students' mathematical abilities and the cognitive challenges they face, for both mathematics and service mathematics students [10, 12, 13, 16, 18, 24, 32, 34]. Although discussion of the 'mathematics problem' and common areas of difficulty for students formed part of our Workshops 1 and 2 during the tutor training day, more emphasis on this is required in future PD programmes.

Simultaneously, with the discussion in the CoP sessions suggesting a positive shift in the tutors' focus from self to student (Section 4.2.1), items mentioned also suggests that the tutors were experiencing difficulty in the secondary category of teaching approaches and pedagogical orientation, specifically in the area of 'persevering with new approaches'. The difficulty of persevering with unfamiliar teaching approaches when confronted with challenges related to weaknesses in students' mathematical abilities further compounded tutors' frustrations. The tutors in Wood et al. [39] stated that they lacked knowledge of contemporary teaching approaches and desired training in them. However, based on our study it is noteworthy that the implementation of new teaching methods may come with a degree of difficulty. Tutors also had trouble maintaining patience when faced with "jarring questions" from students, that is questions that indicate severe deficiencies in knowledge of material met earlier in their course/module. In general, the tutors' increased concern for students' knowledge and understanding, coupled with seeing first-hand the benefit of the new teaching approaches to students' understanding, resulted in the tutors wanting to persevere with these approaches in order to benefit students, but they found it difficult. It was also noticed by the authors that, at times, a very literal interpretation of a teaching approach resulted in more difficulty being experienced with respect to levels of frustration and perseverance. For example, tutors were advised that they could complete the initial part of a student's incorrect suggestion for a solution to a problem to allow the student to understand why it was incorrect/not a useful suggestion. P1 interpreted this to mean that they had to proceed with an incorrect suggestion to the point of exhaustion and mentioned that when a student in their class suggested an incorrect substitution for an integrand, following this teaching approach led to a "mess" on the board. P1 stated "I'm not going to keep going and have some sort of mess on the board that I'm going to get flustered with because I don't know how this is supposed to work out". Demonstrations in addition to discussions around teaching approaches will be included in future iterations of the PD programme to alleviate this.

The tutors embraced the idea of providing a quality learning experience for their students and were keen to provide this. However, at times they experienced challenges in trying to achieve a balance of providing a quality experience while also covering all of the content. This was further exasperated if the tutorial sheets provided to them were particularly long, the lecturer's request was that all material on the sheet should be completed, and/or the students found the content very difficult. Contextual factors, including unfamiliarity with technology and with university procedures and management, added to these challenges and were also found to potentially divert the tutors' attention away from their pedagogical development at times.

5.2. **RQ2: Does participating in a PD programme centred around community of practice (CoP) help to alleviate these challenges?** The findings indicate that participation in the PD programme supported tutors in addressing many of the challenges identified in RQ1. The increase in confidence towards teaching that was self-reported by the tutors was an anticipated outcome of participating in the tutor training day, as a focus on personal development formed part of the design. This is similar to the findings presented in [5] and [6]. Developing confidence, in addition to recognising the role that one's emotions play in any human interaction, forms part of the domain of 'personal development: the self in teaching and learning' of the National Forum's PD framework [22]. The decrease in positive feelings toward teaching reported by the tutors was unexpected. One possible explanation is that participation in the training day increased tutors' awareness of areas in which they may need to improve their teaching practice. At the same time, the increase in self-reported confidence may indicate that they felt they had acquired the knowledge and skills necessary to implement these improvements. The increase in confidence was also of practical importance in reducing anxiety among tutors in this sample and increasing their willingness to implement new teaching approaches (Section 4.2.2).

The CoP sessions provided valuable insights and further elaboration on the difficulties experienced with questioning skills that the tutors mentioned in their reflections. Using the P1 example from Section 5.1, it seems they may not have engaged in reflection to a depth where they was able to gain understanding of potential reasons for an issue occurring and to consider some potential ways to alleviate the issue. In response to the question posed by one of the mentors "after you reflected [...] did you think [...] if something like that comes up again, did you think about what you were going to do?". P1 stated that they "didn't really think, I mean my thought was maybe just not to ask that kind of question but at the same time I don't want to just do out the answers". The CoP session was helpful in providing additional time for tutors to reflect on any difficulties that they were experiencing and to reach an understanding of a possible solution/strategy to use to alleviate these difficulties.

By prompting tutors to consider students' thinking more explicitly, the CoP sessions supported professional growth and heightened awareness of students' difficulties. For example, through mentor questioning, P3 examined whether a student's difficulty stemmed from gaps in prior knowledge or from applying familiar rules in a new context. Such discussions also helped to resolve difficulties arising from literal (and unintended) interpretations of new teaching approaches.

The impact of the tutor training day on the tutors' teaching was discussed in Sections 4.1 and 4.2. A comparison of the post tutor training day questionnaires and the tutors' reflections after their first tutorial revealed that, in general, the tutors did try to implement the items they mentioned in their questionnaires (Table 5). This comparison also revealed several other items that the tutors learned from the tutor training day (not mentioned in the tutors' questionnaires) that they took with them into their teaching (Section 4.2.1). These included questioning to gauge understanding, planning, clearer presentation, encouraging student participation, and prioritising quality over quantity, and monitoring students' progress during class (Tables 3 and 5). The tutors focused on developing various skills and approaches based on what they considered important from what they learned on the tutor training day, and from their own experience as a learner. They worked on implementing these skills and approaches at times that were appropriate in their learning to teach journey, for example, those with the least teaching experience initially tended to focus on more foundational practices (such as introducing themselves to a group). In the tutors' reflections after their first tutorial, students asking/answering questions was noted as a sign of the class going well by all tutors. This shows that the PD programme had a positive effect on the issues that tutors experience regarding facilitating students' input as found by [9]. Also, tutors asking more probing questions, giving students opportunities to ask questions at the end of problems and/or at several points during more difficult problems, and trying to get their students more involved were all noted as things that went well in the tutors' first tutorial. While 'mixed-ability teaching' and 'motivating students' were not mentioned in the first CoP session, the tutors in the second CoP placed emphasis on motivating students' engagement with the material with continued focus on questioning and managing diverse student needs.

Similar to [39], responses to the post tutor training day questionnaire indicated that some tutors felt the training on technology in Workshop 2 did not address their needs sufficiently, and suggested that more training in this area was necessary. This arose again in the first CoP session. Training on how to use technology and training on any other specific technologies that are available to tutors within an institution should be included within early PD initiatives, specifically before tutors begin their teaching (e.g. checking the classrooms and equipment with each tutor before they begin teaching in order to address any potential issues with classroom resources prior to teaching). In light of the fact that online education is arguably now more prevalent than ever, this training should also include the use of online education platforms. The provision of information on university procedures and management should also be included (e.g. how to change the classroom assigned to a more suitable classroom). By including training on 'external factors' in early PD initiatives, future cohorts of tutors could avoid experiencing difficulty with them and therefore be able to focus more on their teaching without these additional distractions, which may lead to a more productive learning-to-teach experience. The authors would also recommend increased allocation of time where tutors engage in practising their teaching of mathematics, with observation, discussion and feedback by peers and mentors in early PD initiatives – this was highlighted by the tutors as an aspect of PD with which they felt they needed more engagement.

5.3. **Other Outcomes from the PD Programme.** A notable outcome of the PD programme was a shift in tutors' focus from themselves to their students. This included greater recognition of the difficulties that many students, particularly first years [4], experience in their mathematics studies, increased approachability, and adaptive teaching practices aimed at supporting understanding and encouraging engagement. Using the results obtained from this sample of tutors, focusing on external factors in advance of the tutors beginning to teach may have facilitated discussion of mixed-ability teaching at an earlier stage in the CoP sessions. For example, the provision of a session (with discussion) on strategies for mixed-ability teaching could have been included within the first CoP session, or between Weeks 4 and 7. The tutors would then have had the knowledge of some approaches for mixed-ability teaching in advance of them requiring it, providing a more productive learning to teach experience.

The recurring quality versus quantity theme suggests that the tutors were beginning to evaluate teaching success less by content coverage and more by students' understanding of the material. Evidence of the tutors emphasising this was shown by their engagement in practices such as the use of questioning and explaining to promote understanding, and by their inclusion of examples of where the mathematics being learned is applied within the students' chosen field of study. This contrasts with the findings in the literature which state that students find that classes lack applications [26] and students also have trouble with just accepting content delivered to them in HE and not questioning why it works, along with the findings that mathematics in HE can be very procedural [15, 26, 38].

The increase in references (in the CoP sessions) to concurrent and post reflection and the nature of those statements suggest that the tutors have positive opinions of engaging in reflection. In general, the tutors saw the benefits of engaging in reflective practice in terms of increased awareness of their own teaching and their perception of their students' learning and understanding. In addition, the tutors also acknowledged the benefit of engaging in reflection in terms of improvements in their planning and subsequent improvement in the flow of their lessons and on their students' learning experience. Completing the reflection document after each tutorial was noted as being helpful to the tutors in our study in spotting patterns in their teaching, having time to think about why certain things may be occurring in their teaching and classrooms, and thinking about possible ways in which they could overcome any difficulties. The tutors who completed the reflection document after each of their tutorials over several weeks, showed progression in their reflective practices. As mentioned in Section 4.2.4, in several instances, the information provided by the tutors in their reflections facilitated further insight into a reference to one of the six factors made by that tutor in the CoP sessions.

The tutors reported improvements in their personal development, professional communication and dialogue, and increased knowledge and further development of their skills in teaching and learning after participating in the tutor training day. The CoP sessions were instrumental in supporting the tutors with their engagement of new approaches particularly when they encountered difficulty. The tutors noted that the provision of a safe space to discuss any aspect of their teaching among their peers and mentors, was very helpful in terms of chatting about their teaching experiences, gaining clarification and reassurance on certain aspects of their teaching approaches, reflecting on their teaching, gaining advice on possible ways aspects of their teaching could be improved, discussing new ideas and classroom management issues, acting out teaching approaches, and learning about ways technology could be used to enhance students' understanding. This finding provides further supporting evidence of the benefits of one of the core components in each of the models described by Ellis [11] - Apprenticeship Model (Teaching mentor); Coordinated Innovation Model (Weekly meetings); Peer-Mentor Model (Experienced peer mentor delivering ongoing seminars). The CoP sessions were also helpful in providing another opportunity to engage in reflection with the added benefit of collaborative discussion regarding any issues with peers and mentors.

Overall, while early teaching challenges are unavoidable, the findings from the cohort of tutors in our study suggest that a PD programme centred around a CoP can play a meaningful role in alleviating these challenges by supporting tutors' confidence, pedagogical development, and perseverance during the formative stages of their teaching careers.

6. Conclusion

The tutors in this study highlighted the concerns they had with respect to beginning to teach, in addition to revealing the factors that impacted on their early teaching experiences. It was seen that the prominence of each of the six factors on the tutors' early teaching experience increased/decreased or remained relatively the same between the CoP sessions. This information is useful for individuals who are providing PD initiatives, in terms of factors to consider for training, discussion and exploration in PD.

The tutors in our study encountered and engaged with many of the challenges identified by past research [39] including management of large classes, gaps and diversity in student knowledge and abilities, lack of knowledge of effective and contemporary teaching methods, and use of technology in teaching within a structured PD context. By engaging early with challenges such as these, the tutors in our study may be better prepared to address these issues should they choose to progress into academic teaching roles.

The tutors in this study reported the importance and benefit of PD initiatives on their early teaching experiences. The authors would recommend the provision of PD programmes for all GTA's who teach mathematics in HE, ensuring that those who teach in HE are given PD from the earliest stage of their teaching career. The authors would also recommend that the duration of such PD programmes be of a sufficient length to cater for the GTAs' PD needs while also taking into consideration both the research and teaching commitments of the GTAs. Following feedback received from the tutors in this study, the authors have extended the duration of the tutor training day in this study to two days. The first day will consist of the four workshops as described in this paper, with further information, training and discussion on the mathematical abilities of students, technology in teaching and external factors. At the end of the first day, the tutors will be asked to plan a lesson, noting that they will be asked to teach part of that planned lesson to their peers and the mentors on the second day. The second day of the tutor training course will consist of the tutors practising their teaching of mathematics with observation, completing a reflection document after their teaching, and then engaging in discussion and receiving feedback by peers and mentors. Following the teaching practice and discussion, tutors will be asked to discuss their completed reflections and to engage in a discussion on the benefits of engaging in planning for teaching and reflective practice. The authors have also refined the PD programme to include two individual meetings with tutors to discuss their reflective practice, in particular areas of growth, implementation of any new approaches, areas of continued difficulty and possible ideas for alleviating any difficulties. These meetings would be held between the first and second CoP session and following the second CoP session. Mathematics education specialists and mathematics support staff are well positioned to offer PD initiatives such as that outlined in this paper. The combination of their mathematics knowledge, pedagogical knowledge, and their experience and knowledge of university students are invaluable in providing early PD initiatives to those who teach mathematics in HE. The provision of such PD initiatives is in line with the National Forum's recommendations for all staff who teach in HE.

The tutors in this study had a strong desire to provide high-quality instruction in mathematics but they required support to aid them in their endeavours. The PD programme outlined in this paper was noted by the tutors as being beneficial and helpful to them in facilitating the provision of such high-quality instruction, as well as leading to increased confidence and reduced levels of stress and anxiety. The CoP sessions as part of the PD programme facilitated meaningful peer discussions on teaching and it is hoped that engaging in PD initiatives as early mathematics tutors may lead to a continuation of peer dialogue on teaching in the future and a continued improvement in the teaching in universities.

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Appendix A. Pre and Post tutor training questionnaires (Note: Format of the questionnaires shortened for publishing purposes)

Pre tutor training course questionnaire

Q1. How long have you been teaching mathematics/statistics? If you have not taught before please just state 'n/a'.

Q2. Have you ever received any teacher training before this (please circle)? Yes / No. If 'yes', please explain below.

Q3. How many hours per week (on average) do you teach/will you be teaching this semester?

Q4. If you have taught before what do you think a student who has attended all of your tutorials would say about your tutorials?

Q5. On a scale of 1 to 10 (please circle) how do you think a student would rate your teaching if you have taught before? (1 = not good; 10 = excellent)

1 2 3 4 5 6 7 8 9 10

Q6. My peers and I discuss teaching and teaching approaches (please circle):

Never Very rarely Rarely Occasionally Frequently Very frequently

Q7. The lecturers and I discuss teaching and teaching approaches (please circle):

Never Very rarely Rarely Occasionally Frequently Very frequently

Q8. On a scale of 1 to 10 (please circle) how would you rate the amount of support available to you with your teaching? (1 = there's no help; 10 = excellent. Everyone is very helpful and help is often offered from staff members.)

1 2 3 4 5 6 7 8 9 10

Q9. On a scale of 1 to 10 (please circle) how confident do you feel about teaching? (1 = not confident at all; 10 = very confident.)

1 2 3 4 5 6 7 8 9 10

Q10. On a scale of 1 to 10 (please circle) how positive are you feeling towards teaching this semester? (1 = I am dreading it; 10 = I can't wait).

1 2 3 4 5 6 7 8 9 10

Q11. Any other comments (worries about teaching/ questions you have etc.)

Post tutor training course questionnaire

Q1. State three things that you have learned from the tutor training course that you plan on implementing/taking with you into your teaching.

Q2. State three issues/questions you still have that you feel were not addressed in the tutor training course.

Q3. On a scale of 1 to 10 how confident do you feel about teaching? (1 = not confident at all; 10 = very confident.)

1 2 3 4 5 6 7 8 9 10

Q4. On a scale of 1 to 10 how positive are you feeling towards teaching this semester? (1 = I am dreading it; 10 = I can't wait.)

1 2 3 4 5 6 7 8 9 10

Any other comments?

Appendix B. Reflection Documents

Reflection after first tutorial

(1) List three things that you felt were good and/or went well in your first tutorial.

(2) List three things that you felt could have been improved and/or did not go well in your first tutorial.
(3) Do you feel that the tutor training course impacted on your teaching in any way (please circle)?

Yes No

If 'yes', please explain in the space provided below.
(4) Any other comments on your first tutorial?

Post tutorial reflection (following each tutorial)

(1) What content did your tutorial cover?
(2) List three things that you felt were good and/or went well in your tutorial.
(3) List three things that you felt could have been improved and/or did not go well in your tutorial.
(4) What will you do to help alleviate the issues you have stated in Q3?
(5) If you feel the past mentoring session(s) has affected anything in your teaching please explain.
(6) Any other comments?

Appendix C. Open ended questions for CoP sessions (semi-structured and subject to change).

(1) How is your teaching going this semester?
(2) Have you experienced any successes? If so, what are they?
(3) Have you encountered any difficulties? If so, what are they?
(4) How have you tried to alleviate the difficulties you have experienced?
(5) Have you noticed any difference in how your tutorials have started this semester compared to in previous semesters?
(6) What do you think a student who has attended your classes might say about them?
(7) Have you had any classroom management issues? If so, what have you done to combat them?
(8) Have you found any of the mathematics you are teaching particularly difficult to explain? Have you resorted to procedural teaching (i.e. teaching the students how to do a problem step by step without effort to teach for understanding)?
(9) If yes to 7, did you change anything in the weeks following the class where you had that experience?
(10) Have you reflected on your classes after they were over? *(NOTE this will be advised in the tutor training day).* If so, what were some points you reflected on? Did it change anything about your teaching afterwards?

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A celebration of the 25th anniversary of the Mathematics Learning Centre at the University of Limerick

JASON CURRAN

Dedicated to Prof. John O'Donoghue and all of the staff, past and present, who work and have worked in the Mathematics Learning Centre at the University of Limerick.

Abstract. Mathematics Support Centres (MSCs) are centres located in third-level institutions that are aimed at supporting students with their mathematics modules. There has been an increased demand for MSCs in the last few decades to combat the so-called "*Mathematics Problem*". While the first MSC was established in Queensland, Australia in 1986, now MSCs are much more wide-spread throughout the world, particularly in Europe. The first MSC established in Ireland was the Mathematics Learning Centre (MLC) at the University of Limerick. The MLC was established in September 2001 and in 2026, the MLC celebrates its 25th birthday. The author of this paper wishes to pay homage to the centre by providing the history of the centre from its founding to 2026. In writing this article, the author had correspondence with all previous managers of the centre and also with key people in the history of the centre.

1. Brief history of mathematics and statistics support

Mathematics and Statistics Support (MSS) refers to support offered to students at third-level institutions (not necessarily mathematics students), that is not a part of their regular programmes of teaching. MSS is usually delivered via a Mathematics Support Centre (MSC), and many MSCs have been established throughout the world, with the first examples being in Australia, the UK and Ireland. At first, MSS was considered "nice to have but not essential" [16, p. 6], however the perception of MSS has morphed into an essential part of university provision [23]. MSCs were established in universities in reaction to the so-called "*Mathematics Problem*" (MP) [23]. The MP refers to the declining mathematical readiness of incoming students into universities. The problem was first observed in the UK [34], where it caused concern mostly in engineering and physical science departments. Diagnostic testing provided research-based evidence that the mathematical abilities of incoming students into UK universities were indeed declining [21]. Subsequent studies showed that the problem was also present in Ireland [27, 8]. See Section 4 for a discussion on the diagnostic testing carried out at the University of Limerick. There are many issues that contribute to the MP, but what appears to be clear is that secondary-school mathematics does not seem to adequately prepare students for the mathematics they encounter at third level [23].

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In the UK, contributing factors to the MP include that second-level students only study mathematics up to the age of 16 [18] and that there is a lack of qualified teachers, at least in England [32]. In Ireland, there was a shortage of fully qualified mathematics teachers in secondary schools, with out-of-field teachers (OOFTs) (i.e., teachers who are qualified to teach other subjects but not mathematics) carrying 48% of the responsibility of teaching mathematics nationwide in 2010 [31]. This problem has been partly addressed by the Professional Diploma in Mathematics for Teaching (PDMT) qualification that upskills OOFTs to mathematics teachers [14]. The rate of out-of-field teaching in mathematics in Ireland is now 25%, a clear reduction from 2010. However, despite reducing the number of OOFTs and having 98% of students studying mathematics up to the age of 18 [40], there is still evidence of the MP in Ireland [2], indicating that additional factors, other than these, contribute to the MP.

Due to the myriad of issues faced by incoming students, universities established MSCs throughout the 1990s and early 2000s. These early MSCs catered for first-year university students, enrolled in engineering and science degrees, who were deemed 'at risk' of failing their mathematics modules [23]. However, since the early 2000s, the number of disciplines catered to by MSCs has increased dramatically, partly in response to the increased demand for analytical and quantitative methods used in humanities disciplines [23]. Engagement by students in mathematics courses also increased since the early years of MSS support [22]. Moreover, not all students who attend MSCs are struggling as there is evidence that MSCs in Ireland were used by mathematically-strong students to further improve their grades [24], particularly in second and third year. Second- and third-year students now engage more with MSS compared to the earlier years of MSS delivery when first years comprised the majority of student contacts.

The way in which MSS is delivered has evolved over the past few decades. In the early years, academic staff were mostly responsible for delivering support. However, as MSCs became more established and respected as an integral part of their respective universities, dedicated, full-time staff became more common [23]. Now many centres have a director or manager, with the actual support being delivered by postgraduate students or part-time lecturers. Postgraduate students, particularly those not coming from an education background, can lack various pedagogical skills that are needed for high-quality learning amongst service-mathematics students [38] (i.e., students enrolled in engineering, business, science, technology and education courses) and therefore, tutor training has been identified as central to the continued success of MSS delivery. There have been many efforts to provide training to tutors, including at the University of Limerick [7]. Tutor training has been shown to have benefits beyond teaching, with tutors developing a more professional attitude, growing in confidence and becoming better not only at teaching but also at mathematics more generally [15].

There is substantial evidence that students recognise the value of MSS [29], including mature students (i.e., students who are at least 23 years of age on the 1st of January of the year in which they start their course) [5]. In Ireland, a nationwide survey was carried out regarding student engagement with MSS and the impact of MSS on students. The report found that, "over 56% of the students felt the [MSS] was helpful to their confidence in mathematics" [29, p. 37]. Moreover, "over 56% of respondents felt that [MSS] had an impact on their mathematics performance" [29, p. 39]. MSS has also assisted with drop-out rates at universities. For example, in a study at Maynooth University in Ireland, "26 Maynooth University students said that they had considered dropping out of university due to mathematical difficulties but cited the MSC (at Maynooth University) as influencing their decision to remain" [25, p. 35]. Additionally, MSS has been shown to improve the mathematical standards of students in universities [26]. Despite these positive outcomes, there is evidence that some weaker students tend not to engage for a variety of reasons including fear, embarrassment, feeling intimidated or a lack of awareness that they even needed help [35, Table 1].

The rest of the paper is structured as follows: In Section 2, we provide the methodology employed in collecting the data for this article. In Section 3, we provide the history of the Mathematics Learning Centre (MLC) at the University of Limerick. In Section 4, we provide details on the research and outreach activities carried out by staff in the MLC. In Section 5, we discuss the future of the MLC by considering the impact of Generative AI (GenAI) on the centre and on the MSS community in general.

2. Methodology

Several sources of information were used when writing this article, including the MLC's annual reports, interviews conducted with previous managers of the MLC, quotes collected from previous tutors who worked in the MLC, a statement from **sigma** (a network for excellence in mathematics and statistics support based in the UK) and a statement from the Irish Mathematics Learning Support Network (IMLSN). Much of the quantitative data supplied in this article was made available to the author through MLC annual reports. These reports were graciously provided by the current managers of the MLC and provide information on student engagement with each aspect of the MLC, the modules for which students sought help and the courses to which students belonged. In more recent reports, aims and goals for the upcoming year were also included. These reports proved invaluable for providing context and background information on the MLC, particularly in the case of the earlier years.

The author also conducted interviews with the person credited with founding the centre, six of the seven previous managers of the MLC and had email correspondence with one of the previous managers. The questions used for these interviews are given in the Appendix of this article. The interviews were carried out on Microsoft Teams in December 2025 and were recorded. Each interview lasted between 30 and 90 minutes and the recordings of the interviews were stored in a secure OneDrive folder. The author also reached out to others who were less directly involved with the MLC, but who nonetheless had an impact on the history of the MLC. Such people include Prof. Sarah Moore, Prof. Ailish Hannigan and many previous tutors of the centre. The author also collected a statement from the IMLSN committee, and from the chair of the **sigma**-network (Dr Mark Hodds) and the director of **sigma** at Coventry University (Prof. Duncan Lawson) regarding their outlook on how GenAI will impact MSS in the future.

3. Chronological history of the MLC at the University of Limerick

3.1. **Foundation of the centre: 1990 - 2001.** The history of the MLC at the University of Limerick began with Prof. John O'Donoghue who, in 1991, transferred to the Department of Mathematics and Statistics at the University of Limerick from Thomond College. At this time, the Department of Mathematics and Statistics was responsible for delivering many service-mathematics modules which can have student numbers that reach well into the hundreds. It was identified, albeit anecdotally, that many students enrolled in such courses were struggling with the level of mathematics required to successfully complete their respective courses. In response, the Head of Department at the time, Prof. Michael Wallace, had introduced support tutorials to aid students. These tutorials proved popular with students, however they did not address the underlying reasons why students were struggling. Prof. O'Donoghue, who had a PhD in mathematics education, was responsible for teaching mathematics and mathematics-education modules. Therefore he found himself at the forefront of the problem and through discussions and encouragement from Prof. Wallace, he was well positioned to address the problem.

In the 1997/98 academic term, Prof. O'Donoghue administered diagnostic tests to students of Technological Mathematics 1, a large service-mathematics module. The test consisted of 40 questions and each test was administered in the students' first mathematics lecture and was hand corrected with an all-or-nothing, 1 point per question marking scheme. Students who received 20 points or less were deemed 'at risk' and advised to attend once-per-week support tutorials for the module as well as a front-end skills package of four tutorials on arithmetic and algebra. Of the 257 students that participated, 69 (or approximately 27%) were deemed 'at risk'. Moreover, it was found that students who were deemed 'at risk' but who availed of the supports provided were significantly more likely to pass their mathematics modules compared to those who had been deemed 'at risk', but who did not engage with the available supports.

The results of this pilot study indicated that many incoming students at the University of Limerick were underprepared for the level of mathematics in their mathematics modules. At the time, the underpreparedness of students had been observed in the UK [34] and elsewhere in Ireland [19]. However, the reasons for this were not well understood nor well researched. The results of the tests caused concern among the staff of the Department of Mathematics and Statistics, including the Head of the Department. It was identified that while the supports put in place were certainly helping students, this was a permanent problem and not a transient one. Hence, the problem would require a more permanent solution.

In the mid-to-late 1990s, the position of Dean of Teaching and Learning was established in the University of Limerick and was filled by Prof. Evan Petty. In conjunction with the position, an award for Teaching Excellence was organised and, importantly for the development of the MLC, a Teaching Fellowship position was created. The Fellowship position involved conducting research over one year on how to improve the teaching within the university. Prof. O'Donoghue made a successful application for the position and his project had several aims, with the first being the design and administration of a bespoke diagnostic test. This test would be a key indicator of student preparedness and would also help identify the areas in which students were struggling. The second aim of his project was to provide on-going, customised support (including support tutorials) to students deemed 'at risk' with the aim of addressing their specific needs. The third aim was to have qualified/trained tutors administer the support tutorials. This project was completed during the academic year 1998/99 and the results largely corroborated with those of the 1997/98 pilot project. 30% of first-year students in Technological Mathematics 1 and Science Mathematics 1 enrolled in the 1998/99 cohorts were deemed 'at risk'. However, the research also showed that support tutorials were helpful in addressing this problem. Another conclusion of the study was that continuous intervention was required. Hence, a recommendation of the project was the foundation of a mathematics support centre. This project was significant as it provided research-based evidence for the need for a support centre for mathematics in the university.

Following this recommendation, Prof. O'Donoghue and Prof. Wallace applied to the Vice President of Academic Affairs (VPAA) for funding to found the support centre. However, this proved ultimately unsuccessful. The VPAA recommended applying for the Higher Education Authority (HEA) Targeted Initiative funding with the university providing administrative support for the application. The HEA Targeted Initiative funding was a government scheme aimed at "support[ing] institutional activities in a range of areas including access" [17, p. 52]. This funding application was successful, however additional help was required to successfully get the MLC set up. The Department of Mathematics and Statistics proved a huge ally in this regard by providing the physical space for the MLC and assisted in providing the equipment necessary for the centre. Another key ally was Prof. Sarah Moore who was the Dean of Teaching at the time and a strong advocate for support centres in general. In September 2001, the MLC at the University of Limerick was opened and, while Prof. O'Donoghue is credited with founding the centre, in his own words "*without that kind of support [from the Dept. of Math. and Statistics, Prof. Michael Wallace and Prof. Sarah Moore], we would have had a much harder ride and maybe much more difficulty in getting anything started*".

3.2. **Initial phase of the centre: 2001 to 2010.** The first manager of the centre was Dr Helen Purtill (née MacMullen) who started as manager of the MLC in September 2001. During her time as manager, the MLC was properly established and many of the customs of the centre that were instituted at this time are still in place in 2026. For example, the centre was open to drop-ins from 10am to 12pm and from 2pm to 4pm, Monday through Friday and support tutorials for specific modules were organised. Dr Purtill attended the lectures for large service-mathematics modules to advertise the centre to the students who were most likely to derive benefit from the service. Furthermore, Dr Purtill conducted a visit to the MSC at Loughborough University in England to exchange ideas on the running of a support centre. Loughborough University (and Coventry University) have two of the longest established MSCs in the UK (dating back to the 1990s) and in 2005 they jointly established **sigma** which was designated by the Higher Education Funding Council for England as a Centre for Excellence in Teaching and Learning (CETL), specialising in university-wide mathematics and statistics support. Dr Purtill's visit began a long collaboration between the University of Limerick and Loughborough and Coventry Universities regarding MSS and research.

During the 2001/02 academic year, 431 individual students attended the centre, which caused problems regarding space. Dr Purtill remarks that, "*at times the centre would become very busy and some students would attend multiple times during the week - it became difficult to manage space and student expectations.*" Evening support tutorials were also provided for the large service-mathematics and statistics modules. These were aimed at providing extra support for mature students and those who may have had gaps in their mathematics knowledge, such as those transferring in from different courses. The courses which comprised the highest percentage of students attending the drop-in were the Bachelors in Business Studies (20.1% in semester 1 and 21.9% in semester 2), Bachelors of Science in Applied Physics (10.6% in semester 1) and Bachelors of Science in Computer Systems (21.6% in semester 2). Applied Physics and Computer Systems may be a surprise as one may assume that physics and computer-science students would be relatively strong in mathematics. However, even students enrolled in mathematics courses attended the centre (4.5% in semester 1 and 7.1% in semester 2). This was early evidence of the secondary-tertiary transition, a now well-documented phenomenon where even mathematically-strong students struggle with the transition between secondary school and university [4]. A positive impact of the mathematics students' attendance was that other students appreciated that even mathematically-strong students find mathematics difficult at university. Dr Purtill remarks that, "*initially we were concerned that students may feel embarrassed to seek help from the centre. But having engineers and mathematics students also attend gave all students the confidence to come and ask questions.*" One clear indication of the success of the MLC was that the failure rates (grades: D1, D2 and Fail) in several service-mathematics modules had declined markedly. In Science Mathematics 1 the failure rate dropped from 29.4% in the 2000/01 academic year to 10.9% in the 2003/04 academic year. Engineering Mathematics 1 (37.7% to 9.8%), Technological Mathematics 2 (29.8% to 16.3%) and Business Mathematics 2 (4.3% to 3.3%) also saw drops in their failure rates. While it is not certain, there is at least an indication that the MLC was partly responsible for the improvement. The statistics above can be found in [10, Table 4].

Dr Purtill was manager until 2003, when Dr Olivia Fitzmaurice (née Gill), who was previously a tutor in the MLC, became the manager. Dr Fitzmaurice was acquainted with Prof. John O'Donoghue from her time studying in the University of Limerick as a Physical Education and Mathematics secondary-school teacher. After talking to Prof. O'Donoghue on a visit to the University of Limerick and completing a higher diploma in Maynooth University, Dr Fitzmaurice completed a PhD under Prof. O'Donoghue's supervision. During her PhD, she tutored in the MLC and she remained as manager of the MLC until 2010. Between 2003 and 2010, the MLC saw a huge increase in the number of students attending. For example, in the 2003/04 academic year, the MLC recorded 1,154 individual students attending the drop-in centre and 511 individual students were recorded attending support tutorials. In the 2009/10 academic year the MLC recorded 2,203 individual students attending the drop-in centre and 1,126 individual students were recorded attending support tutorials. This sharp rise occurred in parallel with developments in how the MLC was funded. The university had included the MLC in its strategic plan in 2001, which helped solidify the centre as an important part of the university, but funding was still precarious as the Targeted Initiative funding was due to expire in 2005. However, in 2004, the Government of Ireland insisted on the participation of all departments in universities around the country in a quality review. The review would consist of a panel of international specialists and the results of this review would be reported back to the government and would impact on government policy and funding decisions. The Mathematics and Science Quality Review panel commended the MLC in the University of Limerick highly. This was seminal for the MLC as it established the reputation, credibility and need for the MLC. One recommendation of the review panel was the maintaining of the MLC as a permanent feature of the university. The executive committee within the university agreed and guaranteed the MLC would continue to be funded internally for at least three years past the end of the Targeted Initiative funding. The Targeted Initiative funding formally ended in 2005, but, through negotiation, it was extended for one year on a reduced budget, after which the university then funded the MLC until 2009. Between 2009 and 2012, the President of the University of Limerick, Prof. Don Barry (who was a former member of the Department of Mathematics and Statistics) funded the MLC through the President's Fund, which was a fund that the president of the university could use for internal university projects.

A major factor in the continued securing of funding after 2009 was that there was a undoubted economic benefit bestowed upon the university by the MLC. Tutors in the MLC noticed many students remarking that mathematics at university was too difficult and that they were considering dropping out of university as a result. Aside from the reputational cost to the university, this would also incur an economic cost. Through assisting a student with their mathematics and keeping the student in the university, especially if they were in first year, the university would retain thousands of euros in fees, which would otherwise have been lost. The number of students required for the MLC to pay for itself in this indirect way was estimated to be only eight students. In 2004, the Dean of Teaching and Learning had completed an independent review of the MLC with results showing that for 20% of students (i.e., certainly more than eight), the MLC was instrumental in retaining them in university. After 2012, the MLC was placed within the Centre for Teaching and Learning (which today is named the Centre for Transformative Learning (CTL)). This was done with four other support centres in the University of Limerick: the Regional Writing Centre, the Science Learning Centre, the Peer Supported Learning Centre and the ICT Learning Centre. Being in the CTL solidified the funding for the MLC and was a big step in securing the long-term survival of the centre.

3.3. **Developments in the MLC in the 2010s.** Between 2010 and 2015, the MLC had three separate managers. The first was Dr Lisa O'Keeffe, who obtained her PhD from the University of Limerick in 2011. Dr O'Keeffe, like most of the managers, worked in the MLC during her PhD. The second manager was Dr Mark Prendergast who took over from Dr O'Keeffe in 2012. The third manager in this period was Dr Páraic Treacy. Dr Treacy started his PhD in 2009 and became manager of the centre in 2012 after Dr Prendergast. Dr Treacy remained in the role until December 2014. By this time, the MLC was a "*well-oiled machine*" as described by Dr Treacy and the attendance numbers were increasing, both in the drop-in centre and in the support tutorials. The Leaving Certificate helpdesk was a series of outreach classes given to secondary-school students who wanted help preparing for their end of school examinations. This project was started in the 2000s and was running smoothly but was further developed by Dr Treacy. There was also more collaboration with the other learning centres in the university, particularly the Science Learning Centre. The managers of both centres regularly met and compared strategies in order to improve the supports offered by both centres. While the MLC appeared to be going from strength-to-strength, that is not to say that there were no problems. Dr O'Keeffe, Dr Prendergast and Dr Treacy all faced similar issues during their respective times as manager. Firstly, staffing became a major issue. The managers reported finding it more difficult to find enough staff, particularly those with the necessary expertise to work in the centre. There was a large staff turn-over at this time which compounded the issue. Secondly, physical space was also becoming an issue. By this point, the MLC now composed of two separate rooms located beside each other. However, each room is quite small and space would prove a major problem for the next few years.

Following Dr Treacy, the next manager of the MLC was Dr Richard Walsh. Dr Walsh completed a Bachelors degree in Physical Education and Mathematics in the University of Limerick in 2011. He then completed a PhD under the supervision of Dr Olivia Fitzmaurice and Prof. John O'Donoghue in 2015. He became manager of the MLC in the same year. During the initial few years in the role, the number of student contact hours with the MLC sharply increased once again. In the 2014/15 academic year, the total number of contacts recorded by the MLC (this includes drop-in, support tutorial and examination revision) was 6,169. This number increased to 7,448 in 2015/16 and then to 10,023 in 2016/17. This was by far the largest number of contacts recorded by the MLC up to this date, with the previous maximum being in 2009/10 with 8,953. While this development may seem positive, it put severe pressure on the MLC's resources, particularly staffing. For context, only 11 tutors worked in the MLC in the 2016/17 academic year compared to 22 in the 2009/10 academic year. The reason for the low staff engagement was in part due to the fact that the some PhD students' funding providers did not allow them to teach during their first year or, if they could teach, the hours in which they could were restricted. In the 2016/17 MLC report, it was suggested that more staff with "*knowledge of higher-order mathematics*" and "*who are mathematics-education focused*" be hired. Although, the report also admits that, "*this goal was also put forward for the past 2 years, however, it is proving very difficult to achieve.*"

Another development regarding staffing in the MLC around this time was the introduction of tutor training. The MLC report from 2014/15 provides a clear description:

> *"The MLC ran its first tutor training day in August 2015. This training day was targeted at incoming tutors (mostly first-year postgraduates) who had not taught before. The manager of the MLC and two PhD students in mathematics education delivered the course. The main focus was at improving tutoring in mathematics support but was also extended to mathematics teaching in general. 17 tutors took part in the training day with*

11 of these coming from the University of Limerick. The others came from other tertiary institutions WIT (1) (Waterford Institute of Technology - now called South-East Technological University) and LIT (5)(Limerick Institute of Technology - now Technological University of the Shannon). This training day was also run in other institutions nationwide in conjunction with other MSC managers involved in the IMLSN. In total, 42 tutors took part nationwide."

The general sentiment of participants was that the training was worthwhile, although some improvements could be made such as more time being dedicated to developing tutors' questioning and assessment skills. An account of the training days, including an analysis of participant interviews was published in [7]. Tutor training remains an important component of working in the MLC to this day. In fact, the author of this paper recalls the tutor training he received when he started in 2020. He recalls that the training allowed him to acquire pedagogical knowledge prior to beginning tutoring. Moreover, this pedagogical knowledge can be developed further and it helps tutors develop skills for their future careers.

"The MLC played a pivotal role in shaping my approach to teaching. It opened my eyes to the diverse challenges students face when learning mathematics and reinforced for me that effective teaching isn't about one 'right' method – it's about communication, flexibility, and meeting learners where they are. My time in the MLC also strengthened my own understanding of mathematics, as working with students challenged me to explain concepts in clear, accessible, and accurate ways." – Dr Patrick Johnson: Tutor of the MLC from January 2002 to May 2006.

"I found working the MLC to be a challenging, rewarding, and deeply enriching learning experience that has shaped my professional career as a secondary-school mathematics teacher and mathematics-education lecturer. Working with students in a one-to-one capacity and in small groups, I acquired knowledge of common misconceptions, developed my teaching skills, and gained a deeper understanding of the processes that students engage in when learning mathematics. Significantly, for my professional career, I gathered insights on the common misconceptions that student hold, irrespective of the nature of mathematics being studied, that pertain to fundamental concepts in mathematics introduced in primary school. I learnt that mathematics is way of thinking than spans varying branches and requires a deeper understanding of fundamental concepts. This perspective on mathematics, emergent from my experience in the MLC, informs my current work in initial teacher education." - Dr Stephen Quirke: Tutor of the MLC from September 2015 to May 2019.

Due to the increase in student numbers and the increased responsibilities of the MLC manager, the case was made for a second full-time position within the MLC. In 2017, Dr Aoife Guerin, who had completed her PhD in the University of Limerick in the same year under the supervision of Dr Fitzmaurice and Prof. O'Donoghue, was appointed to this position. Dr Guerin's year-long contract was renewed in 2018 and she was appointed permanent co-manager in 2019.

Another significant development for the MLC was the securing, in 2019, of €46,573 of funding alongside the other four learning centres in the University of Limerick from the National Forum for the Enhancement of Teaching and Learning in Higher Education to support the development of a Digital Learning Support Hub (DLSH). This initiative, in part, led to the development of 150 mathematics support videos across 3 service-mathematics modules (Science Mathematics 1, Science Mathematics 2, and Engineering Mathematics 2) which were traditionally popular in the MLC for in-person support. The videos, in principle, ranged up to 10 minutes in length (max.) and were aligned with each of the modules' tutorial sheets (i.e., weekly worksheets given to the students by the lecturer). For example, a collection of these videos would be released to students on concepts relating to Science Mathematics 1's first tutorial sheet at a time that corresponded to when they were studying that material and were expected to attempt that first tutorial sheet. Then another set would be released corresponding to the second tutorial sheet at an appropriate time, etc. In addition, as well as these videos, the MLC developed online questions that students could attempt alongside solutions that they could access after attempting the problems. All of these resources were housed on the university's Virtual Learning Environment (VLE) - Sulis at the time - on a site that was established by the MLC. This work dramatically improved the MLC's online resource support capability.

3.4. **Developments in the MLC in the 2020s.** The timing of the National Forum funding was serendipitous as the shift to online teaching due to the COVID-19 pandemic in March 2020 occurred in the middle of this funding. In Ireland, all higher-education institutions were instructed to move to online teaching on the 12th of March 2020. We point to [28] for further information regarding the impact of COVID-19 on the teaching and learning of mathematics at third-level in Ireland. For the University of Limerick, the instruction to move to online teaching fell in the middle of the university's second semester. The expertise gained by Dr Walsh and Dr Guerin from working on the DLSH project resulted in a relatively seamless transition between the in-person and online delivery of services in the MLC. The MLC was also in a position to provide workshops for colleagues in the Department of Mathematics and Statistics on the online tools that were available to them for teaching. This assisted the staff in the department with the transition to teaching lectures and tutorials online. This is good example of the continued close relationship between the MLC and the Department of Mathematics and Statistics, even after the MLC moved to the Centre for Transformative Learning. In fact, Prof. O'Donoghue expressed the sentiment that all successive Heads of the Department of Mathematics and Statistics (beginning with Prof. Wallace) were very supportive of the MLC and always offered assistance when asked. This relationship is acknowledged by the current head of the department, Prof. Ailish Hannigan.

"The MLC began as an initiative of the Department of Mathematics and Statistics and we continue to have a strong relationship with the centre. It plays a vital role in supporting and encouraging all our students to achieve their full potential in mathematics and statistics." - Prof. Ailish Hannigan.

The MLC built upon the DLSH initiative by continuing to develop online resources, aimed at other service-mathematics modules, even after the funding had ceased in 2021. The DLSH service proved very popular with 7,902 visits being made to the site in the 2020/21 academic year. This wealth of online resources led to a major shift in how the MLC operated. In-person contacts with the MLC after 2020 decreased markedly, while online engagement soared. This was particularly true for the service-mathematics modules, which now had significant online support available for them. The MLC has its own site on the current VLE for the university, Brightspace. Students can access this site via a link and can then navigate to the specific page for the module they want help with. Each page contains videos in which Dr Walsh or Dr Guerin go through solutions to problems, provide explanations of topics using Geogebra and a collection of other resources. The pages are updated when a new lecturer takes over a module, although this is usually an uncomplicated procedure. Dr Walsh explains, "*The only thing that might happen is that from year-to-year, a lecturer might put something that we have done under tutorial three, they might include it in tutorial two. So, we might just need to rejig it from a student facing perspective*".

As mentioned, there was a notable decrease in the number of students attending the physical centre, particularly from service-mathematics modules, once the online resources were developed. This coincided with an increase in the proportion of students from mathematics modules attending. With these modules came more advanced topics as explained by Dr Walsh, "*It's more common for me [now] to help students with concepts related to more advanced topics as opposed to more basic maths*". He clarifies further, "*Laplace transforms and metric spaces are super common, whereas I probably never helped anybody with those in 2015*". This shift to mathematics-focused modules has been documented in the literature [33, 23]. Dr Guerin agrees with Dr Walsh that there are more students seeking support with advanced topics, however, she notes that students have always presented to the MLC with more advanced topics, they were just fewer in number and also presented less often. For her, during her time as a part-time tutor in the MLC (2013-2017), she recalls "*students would have come into the MLC with those more advanced modules but they'd have their notes with them and we would work on the problem together*".

Dr Guerin notes that the MLC is based on an integrated, supportive and collaborative self-help model and through their teaching and learning approaches, everyone in the MLC endeavours to increase the students' understanding, confidence and enjoyment of mathematics while promoting perseverance. She notes that the creation of a mutually respectful learning environment in the MLC is as important as the mathematics being learned and taught. Indeed students have commented on 'the lovely atmosphere', the 'non-judgmental support', 'the ability of tutors to explain concepts' and the 'increased understanding and decreased stress-levels' as being substantial contributors to their increased understanding, confidence and successful completion of their mathematics modules.

We conclude this section by looking specifically at how the work of a tutor changed between 2020 and 2026, as major changes were made in the delivery of MLC support services during this time. The author of this paper started working in the MLC in January of 2020 and hence is familiar with the operation of the MLC during the lock-downs. The drop-in centre ceased operation in March 2020 with an online appointment system being used in its place. Tutors would be assigned a number of hours per week that students could then book, with a maximum of two hours per week per student. Students could attend appointments alone or in groups, however from the MLC report in 2020/21, most students attended alone. Students were expected to provide the exact questions/problems that they were struggling with, which allowed the tutors to prepare in advance for the session. This made the sessions resemble an appointment service, which while common in other MSCs, was not common in the MLC at the time. Such a service had advantages and disadvantages, as elucidated by a previous tutor.

"The online booking system worked well, and there was rarely an issue with the meeting links. As students had to be (semi-)prepared, this meant I wasn't going in blind which I personally appreciated. As an individual, I would rather spend time preparing to properly help a student in need than trying to remember 'on-the-fly'. However, it was often a challenge to gauge how well a student was grasping the information and their own personal level of confidence with the problems and the subject at hand." - Dr Aoife Hurley: tutor in the MLC from September 2021 to August 2022.

In order to facilitate teaching online, tutors were provided with writing tablets, which they could use to write on Microsoft Word or Microsoft PowerPoint while sharing their screen with the student. Since the return of in-person teaching in 2022, the role of the tutor has shifted back to an in-person one. The sentiment of the current tutors mirrors those of many of the previous tutors especially around the support they receive and the benefit they get.

"Working in the MLC has been great experience for me. It is rewarding to help a student who is struggling with a topic and guide them to a place of understanding. My time in the MLC has given me a perspective on the mathematics that students do across the whole university and I look forward to learning more in my time at the MLC." - Mr Brendan Lowry: Current tutor of the MLC (as of February 2026).

4. Outreach and Research

As demonstrated throughout this article, staff in the MLC have conducted a significant amount of research (see [2, 3] for recent publications) and been involved in several outreach programmes (for example the Leaving Certificate help-desk or the DLSH project). In this section, we briefly highlight some additional projects not mentioned previously and we elaborate further on the extensive diagnostic testing carried out at the University of Limerick, which was mentioned in Section 3. We begin with the latter, as this forms a major piece of the research conducted by the MLC, particularly in the early years.

To recall, each diagnostic test consisted of a 40-question test administered to science and technology students in their first mathematics lecture at university. Students were not made aware of the test prior to taking it and completion of the test was done voluntarily by the students without the use of a formula and tables book or a calculator. The questions were open ended and were a mixture of Junior Certificate and ordinary-level Leaving Certificate questions[1]. Students were given 45 minutes to complete the test and a rough work column was provided on each page. A 'Don't Know' box was included after each question and students were encouraged to use this rather than guess. One point was assigned to each question with an all-or-nothing marking scheme. Since calculators were not permitted, students had to rely on their fundamental knowledge of mathematical procedures. Students who scored less than 20 on the test were deemed 'at risk' and were recommended to attend the MLC support tutorials for their respective modules. What made the diagnostic tests such an important research tool is the database that was created over the course of the twenty years the test was administered. Several journal articles were published (see [12, 36, 37, 8] for example) using the data from the diagnostic tests and two researchers had the data from the diagnostic tests as a key part of their PhD theses [6, 9].

In addition to the diagnostic testing, Dr Fitzmaurice, who was manager at the time, adapted the front-end tutorials mentioned in Section 3 into a bridging course called "Head Start Maths"; a voluntary mathematics revision course that runs in August, before students start in the university. Dr Fitzmaurice created the course by using her lecture notes from the "Access Course", which was a one-year course offered to mature students who, upon completion, usually started on a degree programme in the university. The funding for the project was acquired through a secondment from **sigma** in the UK, with Prof. Duncan Lawson supervising the project. This is evidence of the close ties the MLC has, and continues to have, with **sigma**. Furthermore, in 2008, Dr Fitzmaurice contributed an article on the "Head Start Maths" programme to the proceedings of the Centres of Excellence in Teaching and Learning - Mathematics, Statistics and Operations Research conference (referred to as the CETL-MSOR conference) [11]. The CETL-MSOR conference is a major annual mathematics-education conference and, in 2024, it was hosted in the University of Limerick with staff in the MLC

[1] The Junior Certificate is a state examination completed around the age of 15. The Leaving Certificate is a state examination completed at the end of secondary school.

constituting a large part of the organising committee. "Head Start Maths" began as a one-week revision course, but was extended to a two-week course following feedback. This service, which is administered by the MLC, is still running in 2026.

Another project administered by Dr Fitzmaurice was "Maths Casts". These were short, recorded screen casts developed to explain specific topics to students. Dr Fitzmaurice acquired funding from the Quality Department in the University of Limerick to complete this project and she conducted the project with Prof. Tony Croft from Loughborough University in England and Prof. Birgit Loch from Swinburne University of Technology in Melbourne, Australia. Dr Fitzmaurice stated that the rationale behind this project was that help for the same topics was being requested consistently in the MLC. She states, "*domain and range used to make me want to quit my job, I knew I was going to have the first two weeks explaining this over and over.*" Hundreds of videos were created as part of the "Maths Casts" project and these were amalgamated with other similar videos across Europe on the European Virtual Laboratory of Mathematics. Dr Guerin was also part of the project before working at the MLC.

As the first mathematics support centre in Ireland, the MLC has close ties with many of the other centres in Ireland. In particular, the support centres in University College Dublin, Maynooth University and the University of Galway were all established following consultation by their managers with the manager of the MLC. The close ties between support centres on the island of Ireland culminated in 2009 with the founding of the Irish Mathematics Learning Support Network (IMLSN). Several managers and tutors from the MLC in the University of Limerick were/are on the IMLSN committee. To conclude this section, the author points to [1] for a comprehensive account of the development of MSS on the island of Ireland and, in particular, to [1, Figure 2], which shows the founding timeline for each MSC on the island up to 2016.

5. Future of the MLC and Mathematics and Statistics Support

In this section, we speculate on the future of the MLC in the University of Limerick. For this, the author relied on the interviews with Dr Walsh and Dr Guerin, the current directors of the centre. However, much of what Dr Walsh and Dr Guerin said was echoed by many in the MSS community.

The main theme that surfaced with regard to the future of the MLC and indeed MSS support in general was GenAI. Both Dr Walsh and Dr Guerin noticed a significant drop in attendance numbers in the second semester of the 2024/25 academic year (i.e., January to May 2025). The centre had seen a drop in attendance numbers after COVID-19, however this was attributed to the increase in the online support resources developed between 2019 and 2021. By the first semester of the 2024/25 academic year, (i.e., September to December 2024), "*things were starting to get back to normal*", as commented by Dr Guerin. Both Dr Walsh and Dr Guerin speculate, as the author suspects many in the MSS community do, that the cause of the attendance drop is due, at least in part, to the proliferation of GenAI in education. Dr Walsh comments that "*AI is inevitably already a large part of how students study.*" However, an interesting question is, can students rely wholly on the online resources and GenAI in order to get them through their degrees? For some students, Dr Guerin believes the answer is affirmative: "*For some students the online [MLC support] and AI is enough and thus we are not seeing them in the drop-in any more*". However, she points out that: "*For some modules, [students] have always had fully worked solutions [for their tutorials], but they still came in [to the drop-in centre]*". Indeed, this point is evidenced by the experiences of Dr Walsh and Dr Guerin. Dr Walsh commented that "*I have students saying [to me in the drop-in centre] that AI taught me this, but they have no idea what the output is*". Dr Guerin concurs saying, "*We still get students in [the MLC] saying I looked it* *up on AI and I couldn't understand it. So, they still do come in [to the centre]*". This indicates that while GenAI may be having a major impact on MSS services, there is still space for MSCs. Dr Walsh comments that "*AI tools can be a fantastic learning resource, but I do think that students, particularly service-mathematics students who are struggling, need to speak to someone in person about the difficulties that they are having*", while Dr Guerin commented that she believes "*MLC support and AI as being integrated into the provision of MSS in the future*". This sentiment is echoed in a recent study at a university in Norway which indicates that while students generally find explanations provided by AI easy to understand and helpful, a majority still prefer human explanations [13]. This study shows that MSS is still valued by students. The author received the following statement from the **sigma** network.

It is inevitable that GenAI will impact on both the provision of maths and stats support and the way that students engage with it. Maths and stats support providers should seek to harness the power of GenAI and the support it can give students on a 24/7 basis and enable them to use it effectively. We firmly believe that maths and stats support is fundamentally a human endeavour and that, good though GenAI is and will become, there will always be many students who will respond best to human to human interaction. Credit: Mark Hodds (Chair, **sigma** network) and Duncan Lawson (Director, **sigma**, Coventry University). [Received: January 2026].

The author also received the following statement from the committee of the IMLSN.

GenAI is already influencing how students seek MSS, with some students possibly using it instead of, or alongside, visiting support centres. As GenAI becomes an everyday tool for students, its use in study will inevitably feel normal. As usage increases, MSS may need to place greater emphasis on reasoning and critical thinking, and shift towards supporting students to critically evaluate AI-generated work. It is important that it is used to enhance learning rather than become a substitute for it. Credit: Committee of the Irish Mathematics Learning Support Network. [Received: February 2026].

6. Conclusion

It is clear that the MLC at the University of Limerick has had a significant impact on both the students and staff of the University of Limerick over the past 25 years.

"*[Working in the MLC] was a valuable experience in terms of helping students along on their mathematics journey. The impact was (and still is) huge since these students rely on the MLC as an essential support within their [University of Limerick] degree.*" - Dr Kevin Burke: tutor in the MLC from January 2011 to December 2014.

It is also clear that the MLC has changed a lot over its 25 years. The foundation of the centre was a delicate procedure that relied on support from many people and the centre's survival was not a certainty until years after its foundation. It was only through the dedication and hard work of several people that the centre became what it is today.

"*The story of the Maths Learning Centre at the University of Limerick is an example of the best kind of innovation in higher education: evidence-based, discipline-led. I was lucky to have had the chance to work with Prof. John O'Donoghue to help realise the vision, but all the credit goes to him and his colleagues at the centre and in the department. What a legacy.*" - Prof. Sarah Moore: previous Dean of Teaching and Learning and Associate Vice President Academic at the University of Limerick.

Funding was the main struggle of the centre in the 2000s, with the centre not securing a reliable source of funding until it came under the Centre for Transformative Learning in 2012. In the 2010s, staffing and space proved a major issue. Student engagement with the centre saw a dramatic increase and this coincided with a reduced number of available tutors. The end of the 2010s also saw a shift towards online support, which would be followed by a further online shift in the 2020s. While this shift from in-person to online led to a decrease in the in-person student engagement, the development of GenAI has had perhaps an even greater effect. On the question of GenAI and what lies ahead for the MLC, no one can be absolutely certain. However, hopefully this article has shown that the MLC in the University of Limerick, and MSS in general, has proven itself to be an adaptive service that has already faced and overcome many challenges, and there is much hope and optimism within the community that MSS will continue to do so into the future.

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Appendix

Here, we provide the questions that were provided to the previous managers of the MLC ahead of their respective interviews.

1. How did you initially get involved in the MLC?
2. How did the MLC develop between when you took over as manager and when you left the position?
3. What caused you to pursue these developments?
4. What challenges did the MLC face during your time as manager and how did you overcome them?
5. How did MLC student engagement change during your time as manger (more/fewer students, the module cohorts that looked for help, online vs in-person, the number of walk-in hour, etc.)?
6. How did you handle the funding of the MLC during your time as manager?
7. Were there any particular metrics that were influential in securing additional funding/support?
8. How did/do you measure impact and effectiveness of the MLC?
9. Did/Do you conduct tutor training with staff, and how did/do you do this?
10. How did/do you recruit tutors?
11. How did/do you collaborate with other departments (e.g. Maths and Stats etc.)?
12. Is there anything you would change about the MLC if you could? And why?
13. How did technological developments impact on the MLC?
14. What is some of the best feedback you have received about the MLC, and some of the negatives?

Prof. O'Donoghue received a slightly amended list of questions that included questions 7 - 14 from the list sent to previous managers and the following:

1. Why did you decide to set up the MLC/what motivated you to do so?
2. Did you look at similar centres in other universities when setting up/developing the MLC?
3. Who was the MLC originally aimed at, as in what kind of students (year/course/etc.)?
4. What were the challenges in setting up the MLC?
5. Who assisted in setting up the MLC - such as the maths department, government grants, etc?
6. How was the MLC received when it opened and how did you justify keeping the MLC open after the original funding ended?

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A Combinatorial Approach to Wealth Distribution in Multi–Aggregate Systems

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Abstract. We study a closed monetary system in which agents are organised into interacting aggregates. Starting from a combinatorial description of the accessible configurations of the system, we derive analytically the joint equilibrium distribution of aggregate wealth and aggregate size. The derivation follows from a counting argument combined with an entropy maximisation principle. In a continuous formulation the allocation of wealth among agents naturally leads to Dirichlet-type structures, which provide a convenient framework for describing aggregation effects. Numerical simulations of the stochastic dynamics confirm the analytical predictions and reproduce the theoretical distributions.

1. Introduction

The statistical characterization of wealth distribution has long attracted the attention of economists, mathematicians, and statisticians. Beyond its relevance for economic policy and social welfare, it raises a more fundamental scientific question: are there universal statistical laws governing the allocation of economic resources in large populations?

The first systematic empirical investigation is dated 1897 and it is commonly attributed to Vilfredo Pareto, who observed that the upper tail of income distributions follows a power law [14]. This empirical regularity, now known as Pareto's law [4, 5], suggests that the probability density of income x behaves asymptotically as

$$p(x) \propto x^{-(1+\alpha)},$$

for sufficiently large values of x.

More generally, the attempt to understand social and economic phenomena through statistical regularities dates back to the nineteenth century. Adolphe Quetelet, for instance, proposed the idea of a "social physics", arguing that collective properties of human societies may exhibit stable statistical patterns [17].

During the twentieth century several researchers attempted to explain wealth distributions using stochastic models. Robert Gibrat proposed that proportional random growth of incomes leads naturally to lognormal distributions, an idea known as Gibrat's Law [9]. Later contributions by Kalecki and Champernowne showed that stochastic processes can also generate Pareto-type tails, providing theoretical mechanisms capable of reproducing the empirical regularities observed by Pareto [11, 3].

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Most of these classical approaches adopt a "one-body" perspective in which the wealth of each individual evolves independently according to a stochastic growth process. While such models capture some empirical features, they do not explicitly account for the interactions among economic agents.

A different viewpoint emerged with the development of *econophysics*, which applies methods from statistical physics to economic systems [13, 21, 19]. In this framework the economy is viewed as a many-body system composed of interacting agents. A particularly influential example is the model proposed by Dragulescu and Yakovenko, in which agents randomly exchange money while the total amount of money in the system is conserved [7].

Several extensions of this kinetic exchange framework have been proposed, including models with saving propensity and heterogeneous interaction rules [2, 4, 1].

By analogy with the Boltzmann–Gibbs distribution of energy, this model predicts an exponential stationary distribution of money, a result that has received considerable empirical support for the bulk of income distributions in many countries [21].

Although kinetic exchange models [5, 16, 18] successfully reproduce several stylized facts of wealth distributions, they typically treat agents as identical units interacting only at the individual level. Real economic systems, however, display a clear hierarchical structure, where individuals belong to firms, firms to sectors, and sectors to larger economic systems.

In this work we extend the statistical-mechanics approach to wealth dynamics by introducing an additional level of organization. Instead of considering only individual agents, we study a closed monetary system composed of interacting *aggregates*, each representing a group of agents sharing common characteristics. This intermediate level allows us to capture both the microscopic dynamics of monetary exchanges and the mesoscopic evolution of groups within the system.

Our presentation complements our earlier study [12]. There, the emphasis was primarily on the statistical–mechanics viewpoint, the resulting equilibrium laws, and their validation against numerical simulations. In the present article we focus instead on the mathematical structure of the model and provide a more detailed derivation of the main results, with the aim of making the underlying arguments transparent to a mathematical audience.

Within this framework we investigate the joint statistical distribution of wealth and aggregate size and study how interactions across different levels of organization shape the macroscopic patterns of wealth allocation.

The novelty of the present work lies in the introduction of an intermediate aggregate level together with an explicit combinatorial derivation of the joint equilibrium distribution of wealth and aggregate size.

Our goal is not to reproduce the full complexity of real economies, but rather to identify the minimal statistical structure capable of generating the main empirical regularities of wealth distributions. In this sense the model can be viewed as an idealized laboratory system, in the spirit of statistical mechanics, for studying how patterns of economic inequality emerge from simple stochastic interactions.

2. The multi–aggregate monetary model

We now introduce the mathematical structure of the monetary system that will be studied in the remainder of the paper. The model can be viewed as an extension of the kinetic exchange framework introduced by Yakovenko [7], in which money is randomly exchanged between agents while the total amount of money in the system remains conserved.

We consider a closed monetary economy composed of D agents. Each agent i possesses a non–negative quantity of money m_i. The total amount of money in the system is fixed and equal to M, so that the conservation law

$$\sum_{i=1}^{D} m_i = M$$

holds at all times.

Agents are organised into N_a aggregates, denoted by A_k, for $k = 1, \ldots, N_a$. An aggregate represents a group of agents sharing common characteristics, such as belonging to the same firm, economic sector, or institutional community.

For each aggregate A_k, we denote by d_k the number of agents belonging to that aggregate. The quantity d_k therefore represents the *size* of the aggregate A_k. The conservation of the total number of agents implies that

$$\sum_{k=1}^{N_a} d_k = D.$$

The total monetary wealth of the k-th aggregate is defined as

$$M_k = \sum_{i \in A_k} m_i,$$

so that summing over all aggregates we recover the total monetary mass

$$\sum_{k=1}^{N_a} M_k = M.$$

The state of the system at a given time can therefore be described by the set of pairs

$$(M_k, d_k), \qquad k = 1, \ldots, N_a,$$

which represent respectively the wealth and the size of each aggregate A_k.

The dynamics of the system is driven by two classes of stochastic interactions.

Monetary exchange between agents.
Two agents i and j are randomly selected and exchange a quantity Δm of money according to the rule

$$(m_i, m_j) \to (m_i - \Delta m,\, m_j + \Delta m), \qquad m_i + m_j = (m_i - \Delta m) + (m_j + \Delta m).$$

This mechanism conserves the total wealth of the system.

Aggregate interaction.
Aggregates themselves may interact through the exchange of agents. If Δn agents move from aggregate A_k to aggregate A_ℓ, the sizes of the two aggregates change according to

$$(d_k, d_\ell) \to (d_k - \Delta n,\, d_\ell + \Delta n), \qquad d_k + d_\ell = (d_k - \Delta n) + (d_\ell + \Delta n).$$

This mechanism conserves the total number of agents.

Both interaction mechanisms preserve the global quantities M and D. As a consequence, the system evolves within the set of configurations satisfying the constraints

$$\sum_{k=1}^{N_a} M_k = M, \qquad \sum_{k=1}^{N_a} d_k = D.$$

In analogy with statistical mechanics we assume that, in equilibrium, all microscopic configurations compatible with these conservation laws are equiprobable. Under this hypothesis the equilibrium properties of the system can be derived through a combinatorial counting argument.

In particular, we will determine the joint probability distribution of aggregate wealth and aggregate size. This distribution characterizes the statistical structure of the system at equilibrium and will be derived explicitly in the next section.

3. Derivation of the equilibrium distribution

In this section we derive the joint equilibrium distribution of aggregate wealth and aggregate size within the framework introduced in the previous section. We prove the following:

Theorem 3.1. *In the context of the framework set out in Section 2 and in terms of the number of aggregates N_a, the total wealth M, and the total number of agents D, the joint equilibrium distribution $p(m,d)$ of aggregates wealth m and aggregate size d is given by*

$$p(m,d) = \binom{m+d-1}{d-1}\frac{N_a M^m (D-N_a)^{d-1}}{(D+M)^{m+d}}. \tag{1}$$

The corresponding marginal distributions of aggregate size $p(d)$ and of aggregate wealth $p(m)$ are given by

$$p(d) = \frac{N_a}{D^d}(D-N_a)^{d-1} \tag{2}$$

and

$$p(m) = \frac{N_a}{M}\left(\frac{M}{M+N_a}\right)^{m+1} \tag{3}$$

Proof. Let $n_{m,d}$ denote the number of aggregates having monetary wealth m and size d. The quantities $n_{m,d}$ therefore describe the macroscopic configuration of the system.

Because the total number of aggregates, agents and money are fixed, the following conservation relations must hold:

$$\sum_{m,d} n_{m,d} = N_a, \qquad \sum_{m,d} d\, n_{m,d} = D, \qquad \sum_{m,d} m\, n_{m,d} = M.$$

These relations represent the macroscopic constraints of the system.

We now compute the number of microscopic configurations compatible with a given macrostate $\{n_{m,d}\}$. First, the N_a aggregates must be distributed among the possible states (m,d), which can be done in

$$\frac{N_a!}{\prod_{m,d} n_{m,d}!}$$

different ways. In addition, for each aggregate characterised by size d and wealth m, the quantity m must be distributed among d agents. The number of such configurations is

$$\binom{m+d-1}{d-1}.$$

Taking these contributions together, the total number of microscopic configurations associated with the macrostate $\{n_{m,d}\}$ is

$$W(n) = \frac{N_a!}{\prod_{m,d} n_{m,d}!}\prod_{m,d}\binom{m+d-1}{d-1}^{n_{m,d}}.$$

The quantity $W(n)$ counts the number of microscopic configurations of the system corresponding to the macroscopic state $\{n_{m,d}\}$. Under the assumption that all microscopic configurations compatible with conservation constraints are equally likely, the probability of observing a given macrostate is proportional to $W(n)$. Consequently, the equilibrium configuration corresponds to the maximum of $W(n)$, or equivalently to the entropy

$$S = \ln W(n).$$

Using Stirling's approximation

$$\ln(n!) \approx n \ln n - n,$$

which is valid for large n, we obtain

$$S = N_a \ln N_a - N_a - \sum_{m,d} n_{m,d} \ln n_{m,d} + \sum_{m,d} n_{m,d} + \sum_{m,d} n_{m,d} \ln \binom{m+d-1}{d-1}.$$

The equilibrium configuration corresponds to the maximum of the entropy subject to the conservation constraints. Introducing Lagrange multipliers α, β and γ we impose

$$\delta \left(S - \alpha \sum d n_{m,d} - \beta \sum m n_{m,d} - \gamma \sum n_{m,d} \right) = 0.$$

Taking the derivative with respect to $n_{m,d}$ and setting it equal to zero gives the equilibrium condition and leads to

$$n_{m,d} = \binom{m+d-1}{d-1} e^{-\alpha d} e^{-\beta m} e^{-\gamma}.$$

Dividing by the total number of aggregates we obtain the joint equilibrium distribution

$$p(m,d) = C \binom{m+d-1}{d-1} e^{-\beta m} e^{-\alpha(d-1)} \tag{4}$$

where C is the normalization constant into which the term involving γ has been absorbed.

We now determine the parameters C, α and β by imposing the normalization of the probability distribution together with the macroscopic constraints. The normalization condition reads

$$\sum_{m=0}^{\infty} \sum_{d=1}^{\infty} C \binom{m+d-1}{d-1} e^{-\beta m} e^{-\alpha(d-1)} = 1.$$

To simplify this expression we compute the marginal distribution of aggregate size,

$$p(d) = \sum_{m=0}^{\infty} C \binom{m+d-1}{d-1} e^{-\beta m} e^{-\alpha(d-1)}.$$

Using the generating function identity

$$\sum_{m=0}^{\infty} \binom{m+d-1}{d-1} x^m = \frac{1}{(1-x)^d},$$

valid for $|x| < 1$, and setting $x = e^{-\beta}$ we obtain

$$p(d) = C e^{\alpha} \frac{e^{-\alpha d}}{(1 - e^{-\beta})^d}.$$

Similarly, summing over d yields the marginal distribution of aggregate wealth

$$p(m) = C e^{\beta} \frac{e^{-\beta(m+1)}}{(1 - e^{-\alpha})^{m+1}}.$$

Returning to the normalization condition we obtain

$$C \sum_{d=1}^{\infty} e^{\alpha} \frac{e^{-\alpha d}}{(1 - e^{-\beta})^d} = 1.$$

This geometric series converges provided that

$$0 < \frac{e^{-\alpha}}{1-e^{-\beta}} < 1.$$

Evaluating the series gives

$$C\,\frac{1}{1-e^{-\alpha}-e^{-\beta}} = 1. \tag{5}$$

We next impose the constraint on the average aggregate size

$$\sum_{d=1}^{\infty} d\,p(d) = \frac{D}{N_a}.$$

Using the identity

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

we obtain

$$C\frac{1-e^{-\beta}}{(1-e^{-\alpha}-e^{-\beta})^2} = \frac{D}{N_a}. \tag{6}$$

Finally we impose the conservation of the average wealth

$$\sum_{m=0}^{\infty} m\,p(m) = \frac{M}{N_a}.$$

Repeating the same steps we obtain

$$C\frac{e^{-\beta}}{(1-e^{-\alpha}-e^{-\beta})^2} = \frac{M}{N_a}.$$

Introducing the variables $x = e^{-\alpha}$, $y = e^{-\beta}$ and $z = C$ the problem reduces, by Eqs 4, 5, 6, to the system

$$\begin{cases} \dfrac{z}{1-x-y} = 1 \\ \dfrac{(1-y)z}{(1-x-y)^2} = \dfrac{D}{N_a} \\ \dfrac{yz}{(1-x-y)^2} = \dfrac{M}{N_a} \end{cases}$$

whose solution is

$$x = \frac{D-N_a}{D+M}, \qquad y = \frac{M}{D+M}, \qquad z = \frac{N_a}{D+M}.$$

Substituting these expressions we finally obtain

$$C = \frac{N_a}{D+M}, \qquad \alpha = -\ln\left(\frac{D-N_a}{D+M}\right), \qquad \beta = -\ln\left(\frac{M}{D+M}\right).$$

If we substitute these values in $p(m,d)$, $p(d)$ and $p(m)$, we get 1, 2 and 3.
Note that

$$e^{-\alpha} + e^{-\beta} = \frac{D-N_a}{D+M} + \frac{M}{D+M} = 1 - \frac{N_a}{D+M} < 1,$$

thus the convergence condition is always satisfied for systems satisfying $N_a < D$. $\square$

4. Numerical validation

In this section we provide a numerical validation of the analytical results derived in the previous sections. The purpose of the simulations is not to study the full dynamical behaviour of the system, but rather to verify that the stochastic evolution of the model leads to the equilibrium distributions predicted by the theoretical analysis.

4.1. **Simulation setup.**

We simulate a closed monetary system composed of D agents organised into N_a aggregates. Each agent initially possesses the same amount of money, while aggregates are assigned an initial size distribution consistent with the constraint

$$\sum_{k=1}^{N_a} d_k = D.$$

The total monetary mass of the system is fixed and equal to M,

$$\sum_{i=1}^{D} M_i = M,$$

so that the system evolves under the same conservation laws assumed in the theoretical derivation.

The dynamics of the simulation follows the stochastic rules introduced in Section 2. At each step, we randomly select: two agents from the whole population of aggregates, and an amount of money Δm to be exchanged. If the amount of money Δm exceeds the wealth of the randomly selected agents, a new value for Δm is sampled. In addition, aggregates may interact through the exchange of a random number of agents Δn. Both mechanisms preserve the total number of agents and the total amount of money.

The simulation is iterated for a sufficiently large number of steps so that the system reaches a stationary statistical regime. Once this regime is reached we sample the quantities (m_k, d_k) associated with each aggregate in order to reconstruct the empirical distributions of aggregate wealth and aggregate size.

In the numerical experiments we consider systems with $D = 10^4$ agents and $N_a = 500$ aggregates and evolve the system for approximately 10^7 interaction steps before sampling the stationary distributions.

4.2. **Comparison with theoretical predictions.**

The theoretical analysis developed in Section 3 predicts that the joint equilibrium distribution of aggregate wealth and aggregate size is given by Eq. (4), where the parameters C, α and β depend only on the macroscopic quantities M, D and N_a.

From this joint distribution, we obtain the marginal distributions of aggregate wealth and aggregate size. These theoretical predictions can then be compared with the distributions obtained from the simulated data.

Figure 1 shows the empirical distributions obtained from the numerical simulation together with the corresponding theoretical curves. The agreement between the two is very strong for both quantities. In particular, the distribution of aggregate wealth exhibits the exponential decay predicted by the analytical model, while the aggregate size distribution follows the theoretical law derived in Section 3.

This comparison confirms that the stochastic dynamics of the system, despite being defined at the level of individual interactions, converges towards the equilibrium distribution predicted by the statistical–mechanical analysis.

As an additional consistency check we consider the limiting case in which the system contains a single aggregate. In this situation the model reduces to the original kinetic exchange framework studied by Yakovenko [7]. The simulations reproduce the exponential wealth distribution predicted in that setting, providing a further validation of the numerical implementation.

Overall, the numerical experiments support the analytical results derived in the previous sections and confirm that the proposed multi–aggregate framework preserves the main statistical properties of the original kinetic exchange model while introducing an additional level of organisation in the system.

5. Conclusions

In this work we have studied the statistical properties of a closed monetary system in which agents are organised into interacting aggregates. Starting from simple conservation principles for the total number of agents and the total monetary mass, we derived the joint equilibrium distribution of aggregate wealth and aggregate size.

The derivation is based on a combinatorial description of the accessible configurations of the system. Assuming that all microscopic configurations compatible with the macroscopic constraints are equiprobable, the equilibrium distribution follows from a standard entropy maximisation argument [10, 15]. The resulting probability law, derived in Section 3 (see Eq. (4)), depends only on the macroscopic quantities that characterise the system, namely the total wealth, the number of agents, and the number of aggregates.

A central aspect of the analysis is the combinatorial factor appearing in the equilibrium distribution, which counts the number of possible ways in which wealth can be distributed among the agents belonging to a given aggregate. This term provides the mathematical bridge between the statistical mechanics formulation of the model and classical results from combinatorics and probability theory, making explicit how macroscopic statistical laws emerge from the enumeration of admissible microscopic configurations.

When the allocation of wealth is considered in a continuous framework, the same combinatorial structure naturally leads to Dirichlet-type distributions [6, 8]. The aggregation properties of the Dirichlet family offer a convenient mathematical framework for describing how individual wealth variables combine when agents are grouped into larger units, providing a direct connection between microscopic allocation mechanisms and the statistical behaviour of aggregates.

The analytical results derived in the paper have been validated through numerical simulations of the stochastic dynamics. The simulated distributions of aggregate wealth and aggregate size are in strong agreement with the theoretical predictions obtained from the combinatorial analysis, confirming that the stochastic evolution of the system reproduces the equilibrium laws derived analytically. This comparison is illustrated in Figure 1.

From a mathematical perspective, the model illustrates how relatively simple conservation constraints, when combined with elementary counting arguments, can generate non–trivial statistical laws in systems composed of many interacting agents.

Several mathematical extensions of the present framework appear natural. One may consider interaction mechanisms in which exchange rates depend explicitly on aggregate characteristics, or investigate alternative conservation constraints. Another promising direction concerns continuous formulations of the model, where Dirichlet structures and related families of probability distributions may play an even more central role in describing aggregation phenomena.

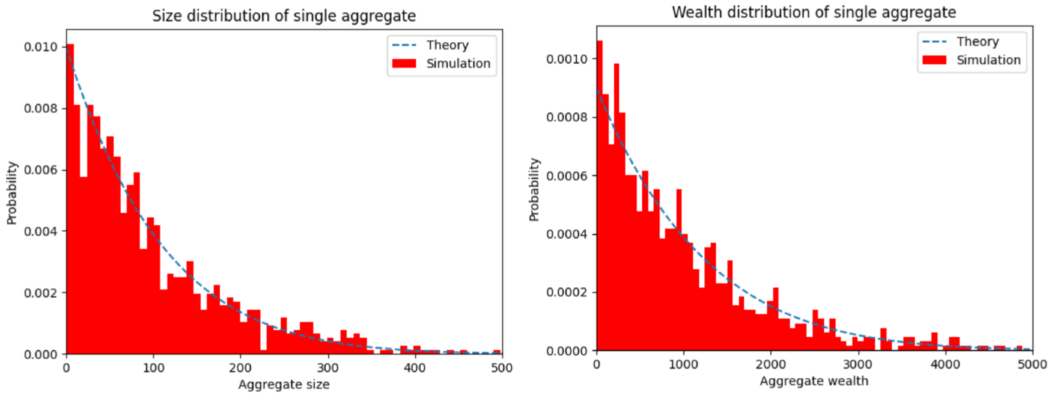

Figure 1. Numerical validation of the equilibrium distributions derived in Section 3. The figure compares the empirical distributions obtained from the stochastic simulations with the theoretical predictions of the combinatorial model (see Eq. (4)). The left panel shows the distribution of aggregate size, while the right panel shows the distribution of aggregate wealth. In both cases the simulated data are in strong agreement with the analytical distributions, confirming that the stochastic dynamics of the system converges to the equilibrium law derived from the combinatorial analysis.

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Bilel Krichen: Spectral Theory for Linear Operators: Demicompactness and Perturbation Theory, CRC Press, 2025. ISBN: 9781032936352, EUR 210.00 (hardback) or EUR 189.00 (eBook), 320+xxiii pp.

REVIEWED BY EDWARD L. BACH

This book attempts to present the spectral theory of demicompact operators. Demicompact operators were introduced by Petryshyn [4], who was interested in constructing fixed points for operators that are not necessarily norm-one contractions. In the last ten years, a number of papers have been published investigating spectral and perturbation theory for demicompact operators and related classes. Several of these papers are referenced in the book's extensive bibliography.

Some examples of demicompact operators are given in Chapter 5 of the book:

(1) Every (not necessarily bounded) compact linear operator T with domain and range in a Banach space X;
(2) Every (not necessarily bounded) strongly continuous linear operator T with domain and range in a Banach space X;
(3) Every (not necessarily bounded) linear operator T with domain and range in a Banach space X, for which 1 is in the resolvent of T.

The book begins with an outline of the basic tools and results of functional analysis needed to develop and prove results in the spectral theory of demicompact operators (Chapters 1-4). Chapter 5 contains the definition of demicompact operators and related classes (weakly and relatively weakly Riesz and demicompact operators). Chapter 6 is a self-contained chapter dealing with nonlinear operators. The remaining chapters investigate aspects of the Fredholm and spectral theory of demicompact operators and related classes.

A researcher who is interested in the spectral theory of linear Banach-space operators or elements in Banach algebras will find a lot of familiar results and considerations here. Because the simple topological characterisation of demicompact operators apparently does not lead to a simple spectral characterisation (the examples above illustrate why there might be none), and because there is no operator ideal playing the role that the socle plays in Riesz and Fredholm theory (see [1]) for demicompact operators (as shown in, for example, [2, Remark 2.3], the demicompact operators are not themselves an ideal), it is likely there will be difficulties in developing the corresponding theory for demicompact elements of a Banach algebra. The present book does not attempt to develop such a theory, although [2] is clearly proceeding along these lines for relatively decompact elements [2, Definition 2.7] of a Banach algebra.

This book could be used by researchers in functional analysis or operator theory who wish to familiarise themselves with the theory of demicompact operators and related classes. Alternatively it could be used, together with [1] and [3], in a class for postgraduates with a title such as "Spectral and Perturbation Theory in Banach Algebras".

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Pierre Py: Lectures on Kähler Groups, Princeton University Press, 2025.

ISBN:978-0-691-24715-1, USD 65.00, 382+xiii pp.

REVIEWED BY BENJAMIN MCKAY

Py's book is the first survey in almost thirty years on this exciting area of research at the intersection of algebraic geometry, Hodge theory, topology, analysis of harmonic maps, complex analysis, and geometric group theory. The heart of the book is a focus on relating morphisms from Kähler groups to holomorphic factorizations of maps of Kähler manifolds, often after covering, so perhaps maps of noncompact Kähler manifolds. This book would already be indispensable for its many appendices, which provide exceptionally clear introductions to the many strands of geometry that the book winds together, including ends of spaces, ends of groups, groups acting on trees, and harmonic maps. The reader is assumed fluent in differential, Riemannian and complex geometry, Hodge theory and Kähler geometry. This book would be difficult for a doctoral student, but is accessible to researchers in Kähler geometry.

Recall that a *Kähler manifold* is a complex manifold with a Riemannian metric which is compatible in the sense that (a) the metric is Hermitian for the complex structure, and (b) the parallel transport of the metric is complex linear. A compact complex manifold is a *smooth projective variety* if it admits a holomorphic embedding into a complex projective space. Kähler manifolds first arose in algebraic geometry: every smooth projective variety is a compact Kähler manifold. Compact Kähler manifolds satisfy various well known relations on their cohomology groups which are not satisfied by some compact complex manifolds. The original motivation to introduce Kähler manifolds was to clarify these relations, for application to smooth projective varieties. However in classical algebraic geometry, compact Kähler manifolds naturally arise which are not smooth projective varieties. Moreover, there are compact Kähler manifolds which cannot even be deformed to smooth projective varieties.

A group is finitely presented precisely when it is the fundamental group of a compact complex manifold [4], [6]. A *Kähler group* is a group arising as the fundamental group of a compact Kähler manifold. Serre proved that every finite group is a Kähler group, and asked for the classification of Kähler groups. The known constructions of Kähler groups are quite diverse, as the book explains.

A *projective group* is a group arising as the fundamental group of a smooth projective variety. Clearly every projective group is Kähler, but it is not known if there are Kähler groups which are not projective, perhaps the most important open question about Kähler groups. Surprisingly, this book does not survey the results on this question. As examples:

(1) Compact oriented surfaces arise as Riemann surfaces, so surface groups are Kähler.
(2) Finite groups are Kähler, after Serre.
(3) Products of Kähler groups are Kähler.

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(4) Any torus of even dimension admits the structure of an abelian variety, so free abelian groups of even rank are Kähler.
(5) Take any bounded symmetric domain in complex Euclidean space, for example the unit ball, and quotient by any torsion-free discrete cocompact group π of biholomorphisms. The quotient is a smooth projective variety with fundamental group π.

An obstruction: if π is the fundamental group of a compact Kähler manifold M then in group cohomology $H^1(\pi, \mathbb{R}) = H^1(M, \mathbb{R})$. This vector space has even dimension. To see this: each element of $H^1(M, \mathbb{R})$ is a harmonic 1-form. Hodge theory shows that each harmonic 1-form on a Kähler manifold is the real part of a holomorphic 1-form. Holomorphic 1-forms constitute a complex vector space, so $H^1(M, \mathbb{R})$ has even dimension. On the other hand, the Hopf surface $M = (\mathbb{C}^2 - \{0\})/(z \sim 2z)$ is a compact complex manifold with $\pi_1(M) = \mathbb{Z}$, so $H^1(M, \mathbb{R}) = \mathbb{R}$, so M is not Kähler.

The hard Lefschetz theorem yields more stringent restrictions on Kähler groups. The rational homotopy type of a compact Kähler manifold is a formal consequence of its cohomology ring, which restricts Massey products on the group cohomology of Kähler groups. Harmonic map theory restricts the actions of Kähler groups on symmetric spaces and trees [3]. One of the most important steps in the book: Py simplifies [3] and generalizes the result, simplifying the theory of linear fractional representations of Kähler groups. Every morphism from a Kähler group to a surface group arises from a holomorphic map of the Kähler manifold to a Riemann surface. A Kähler group acting on a tree factors through a holomorphic map to a Riemann surface, so from a surface group acting on the tree.

The book proves these results, with complete proofs, making use of extensive appendices on essential related material. Prior to this important and comprehensive book, the study of Kähler groups had been without a guide book for nearly 30 years [1], while crucial steps were made. For example, Delzant's 2010 work on BNS invariants for Kähler groups is explained clearly here.

In addition, the author proves a number of new results. For example, the Castelnuovo–de Franchis theorem which says that, on any compact Kähler manifold, any two linearly independent holomorphic 1-forms which wedge to zero are both pulled back by the same holomorphic map to a compact Riemann surface. The author proves a powerful L^2 generalization on noncompact Kähler manifolds, which plays a central role in the subsequent theory.

The author gives a clear introduction to the theory of ends of covering spaces of compact Kähler manifolds. He outlines nonabelian Hodge theory, and explains how Corlette and Simpson apply it to study linear fractional representations of Kähler groups, generalizing their approach to apply to compact Kähler manifolds.

Unfortunately, the author does not discuss the Shafarevich conjecture (that the universal covering space of a smooth projective variety is holomorphically convex). Nonetheless, the methods that are explained here are closely related to the progress that has been made on that conjecture.

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Cónall Kelly: Computation and Simulation for Finance: An Introduction with Python, Springer, 2024. ISBN:978-3-031-60574-1, EUR 53.49, 330+xvi pp.

REVIEWED BY NIALL MADDEN

The book introduces mathematical modelling in finance, and is aimed at undergraduate and beginning graduate students. The origins of the book are in undergraduate and postgraduate courses delivered by the author in UCC and the African Institute for Mathematical Sciences (AIMS). This shows throughout: there is a natural flow in the narrative where theoretical concepts are followed by specific examples from finance, which are then investigated computationally. Thus this book serves as an introduction to asset pricing, to stochastic methods, and to associated numerical methods.

Previous experience of coding in Python, though of course useful, is not assumed: the opening chapter provides a summary of Python fundamentals and the NumPy package, as well as the web-based Jupyter environment. Other elements of Python are introduced as needed, such as random number generation (§1.3.1), root-finding with SciPy's `fsolve` (§2.3.2), and Pandas DataFrame objects (Chap. 6). However, in general there is very little reliance on Python packages, and readers are encouraged to develop their own functions and reuse these later as needed. When that occurs, there are helpful tables summarising which earlier presented functions are being reused, and in which sections they were first developed.

Some basic understanding of probability is required: for example, it is assumed that the reader is familiar with the terminology of σ-algebras. However, these opening sections deftly weave such formalities with concrete discussion of financial assets to help develop intuition. Consequently, stochastic differential equations and Itô calculus are introduced early, and in a natural way. An understanding of fundamental linear algebra, calculus, and differential equations is also expected.

The book is organised into three sections: *Modelling Assets and Markets*, *Computational Pricing Methods in the Black-Scholes Framework*, and *Simulation Methods Beyond the Black-Scholes Framework*.

The first of these provides the necessary understanding to, for example, derive the Black-Scholes-Merton PDE via the Feynman-Kac theorem linking partial differential equations and stochastic processes. Related problems, such as option price sensitivities (the Greeks), are also presented. That section nicely encapsulates the general approach of this book: the concepts are introduced and then explored through a practical investigation of dynamic hedging in Python. The reader who carefully reproduces the computations (and completes the related exercises) is bound to develop a good understanding of this topic.

The second section, on numerical methods, deals with binomial methods, Monte Carlo methods, and then finite difference methods. The first of these is introduced first for European options, and then those with early exercise facilities. The Monte Carlo Method chapter includes study of exotic options, and the topic of variance reduction.

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I dwell a little more on the lengthy chapter on finite difference schemes for the Black-Scholes-Merton PDE, since these methods are closer to my own interests. This chapter includes some relatively standard material on transforming it to the heat equation, and derivation of the standard explicit central finite difference scheme from Taylor series. Less standard, I think, for books on mathematical finance (with the possible honourable exception of Higham [1]) is the careful numerical analysis of the method, covering the stability and consistency of the scheme.

This is followed by a presentation of the unconditionally stable Implicit Euler and Crank-Nicolson schemes. Since implementing implicit schemes requires the solution of a linear system, a detailed presentation of a Python program for the Implicit Euler scheme is given. (This leads to one of my very few quibbles with this book, and it is a very minor one. The author is correct to mention that there is a large body of research on the topic of inverting large sparse matrices. However, much of that is concerned with avoiding explicitly inverting large sparse matrices at all costs. In this example, SciPy's `solve_banded` could be used with minimal changes to the code).

The third and final section of the book concerns more advanced topics, many quite modern and based on recent publications. Its three chapters are on modelling multivariate data, stochastic models for interest rates and, of most interest to this reviewer, numerical approximation of stochastic differential equations. Here the presentation manages to be both succinct yet complete. I particularly appreciated the discussion of the theory and computation of the weak and strong orders of convergence of the Euler-Maruyama scheme. I originally picked up this volume in the hope it would help me compute these for a problem in computational fluids; I was not disappointed.

Overall, this book is a very welcome addition to the literature. There are well-established texts introducing undergraduate students to the mathematics of option pricing and related topics (e.g., [1, 2, 3]). However, the present volume distinguishes itself by complementing a rigorous treatment of both stochastics and numerical analysis with practical computing.

The preface of this book suggests that a one-semester course could be based on Part 1, and a selection of topics from Part 2. For more advanced students, Parts 2 and 3 could each be the basis for a one-semester course. Each chapter concludes with a section on "further reading", and a set of exercises. This again reflects how the book would be a very suitable basis for a course on computational financial mathematics. The preface also promises an "up-to-date treatment" of numerical techniques for computational finance. It certainly delivers that and, in particular, summarises some of the most recent advances, making it a suitable reference for anyone with research interests in stochastic differential equations.

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PROBLEM PAGE

J.P. MCCARTHY

Problems

The first of this issue's problems comes courtesy of Des MacHale of University College Cork:

Problem 97.1. A prime number p is called *isolated* if $p-2$ and $p+2$ are both composite. For example, 37 is an isolated prime.

Show that there are infinitely many isolated primes.

The second problem was sent in by Seán M. Stewart of the King Abdullah University of Science and Technology, Saudi Arabia:

Problem 97.2. Evaluate

$$\int_0^1 \frac{x\,\tan^{-1}(x)}{\sqrt{1-x^2}\,(1+\sqrt{1-x^2})}\,dx.$$

Finally a problem from Finbarr Holland of University College Cork.

Problem 97.3. Let n be a positive integer, and denote by $H_n = [h_{ij}]$ the square matrix of order $n \times n$ where

$$h_{ij} = \frac{1}{\max(i,j)}, \qquad (i,j = 1,2\ldots,n).$$

Prove that

$$\det H_n = (n!)^{-2},$$

and determine H_n^{-1}.

I would like to thank Anthony G. O'Farrell for pointing out that Problem 71.3 has never been solved in these pages. Problems 80.2 and 94.3 share a similar status; however, unlike Problem 71.3, solutions of these were provided by the proposers. We invite readers to attempt all these problems: any solution to 71.3 will be published in Issue 99, while solutions of 80.2 and 94.3 will be published (one way or another) in Issue 98. These older problems may be accessed via `https://www.irishmathsoc.org/bulletin/index.php?file=byvol`.

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Solutions

Here are solutions to two problems from *Bulletin* Issue 94, all of which were solved by the North Kildare Problem Club (and the proposers). We provide the solutions of the NKPC. Problem 94.3 remains at large.

*Problem 94.1***.** Let X, Y be independent and identically distributed random variables with values in a finite group G. Let $H < G$ be a subgroup such that $\mathbb{P}[X \in H] \in (1/2, 1)$. Prove that

$$\mathbb{P}[XY \in H] < \mathbb{P}[X \in H].$$

Solution 94.1. If we fix $x \in G$, then the map $y \mapsto xy$ is bijective. Because H is a subgroup, if x happens to be in H, then the image of H under this map is H, and because H is a subgroup we see that $xy \in H$ iff $y \in H$. Likewise for multiplication on the other side. So, we end up with a table:

	$x \in H$	$x \in G \setminus H$
$y \in H$	$xy \in H$	$xy \in G \setminus H$
$y \in G \setminus H$	$xy \in G \setminus H$	undetermined

If we let $p = \mathbb{P}[X \in H]$, then we see that the maximum probability for $\mathbb{P}[XY \in H]$ is $p^2 + (1-p)^2$, but $p > p^2 + (1-p)^2$ when $p \in (1/2, 1)$. □

Problem 94.2 was solved by the proposer Des MacHale, and North Kildare Problem Club. Again, we provide the solution of the NKPC:

*Problem 94.2***.** If G is a group with centre Z and $|G/Z| = n!$, for some integer $n > 1$, show that G/Z is non-abelian.

Solution 94.2. Let's say that p is a *solo prime factor* of an integer $n \in \mathbb{N}$ if p is prime, p divides n, but p^2 does not divide n. Note that $2!$ has a solo prime factor $p = 2$. If $n > 2$ is an integer, then by Bertrand's postulate there is a prime p such that $n/2 < p < n$. Hence p divides $n!$, but no multiple $2p, 3p, \ldots$ occurs among $1, 2, \ldots, n$. Thus $p^2 \nmid n!$. Therefore $n!$ has a solo prime factor when $n > 1$.

Suppose G is a group with centre Z, and the order $|G/Z|$ is finite, and has a solo prime factor. We claim that G/Z is non-abelian.

Suppose, on the contrary, that G/Z is abelian. Then the commutator $[g, h] = g^{-1}h^{-1}gh$ belongs to Z whenever $g, h \in G$. Thus, for any $g_1, g_2, h \in G$, we have

$$\begin{aligned}
[g_1, h][g_2, h] &= g_1^{-1}h^{-1}g_1h[g_2, h] \\
&= g_1^{-1}[g_2, h]h^{-1}g_1h \\
&= g_1^{-1}g_2^{-1}h^{-1}g_2hh^{-1}g_1h \\
&= (g_2g_1)^{-1}h^{-1}(g_2g_1)h \\
&= [g_2g_1, h].
\end{aligned}$$

Hence, by induction,

$$[g, h]^m = [g^m, h],$$

whenever $g, h \in G$ and $m \in \mathbb{N}$. It follows that if $g^m \in Z$, then $[g, h]^m = 1$. Interchanging the rôles of g and h, we conclude that if $g^m \in Z$ and $h^r \in Z$, then

$$[g, h]^m = [g, h]^r = 1.$$

This tells us that if the order of gZ in G/Z is relatively prime to the order of hZ, then g commutes with h.

Choose a solo prime factor p of $|G/Z|$. Since G/Z is abelian, it is a product of cyclic groups whose orders have product $|G/Z|$, so it contains a cyclic group of order p. Choose $g \in G$ such that gZ has order p, i.e. $g \notin Z$ and $g^p \in Z$. Since G/Z is finite abelian and p occurs only once in $|G/Z|$, we may write

$$G/Z = \langle gZ \rangle \times K,$$

where gZ has order p and $|K| = |G/Z|/p$. Hence every $hZ \in G/Z$ has the form

$$hZ = (gZ)^s (kZ)$$

for some integer s and some $kZ \in K$. In particular, the order of kZ is relatively prime to p. Thus $h = g^s kz$ for some $z \in Z$, and it follows that h commutes with g. But this means that $g \in Z$, a contradiction. The claim is proven.

If we assume that G is finite, then the claim becomes an almost obvious consequence of some general facts: The assumption that G/Z is abelian implies that G is nilpotent, and each nilpotent finite group is the direct product of its Sylow subgroups. Then the centre Z of G is the internal direct product of the centres of the Sylow subgroups of G, and the quotient G/Z is the product of the corresponding quotients. But a p-group cannot have a centre of index p, so the order of G/Z cannot have a solo prime factor.

Example: let D be a dihedral group of order eight and let E be the group of 3×3 upper-triangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathrm{GF}(3)$, the field with three elements. Then with $G = D \times E$, we get $|Z| = 6$ and $|G/Z| = 2^2 \cdot 3^2$ and G/Z is abelian. This shows that the condition that $|G/Z|$ have no solo prime factor is essential. □

Here are solutions to the problems from *Bulletin* Issue 95. Again, all solved by the North Kildare Problem Club (and the proposers).

Problem 95.1 was also solved by Yagub N. Aliyev, ADA University, Baku, Azerbaijan; Riccardo Della Martera; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; and Kee-Wai Lau, Hong Kong, China. We present the solution submitted by Kee-Wai Lau:

Problem 95.1. Suppose $p \geq 2$ is an integer, and

$$u_n = \frac{p^{pn}\,(n!)^p}{(pn)!\,n^{(p-1)/2}}, \qquad (n = 1, 2, \ldots).$$

Prove that $(u_n)_{n \geq 1}$ is a strictly decreasing sequence, and determine its limit.

Solution 95.1. For any real $x > 0$, let

$$f(x) = \sum_{k=1}^{p-1} \ln(px + k) - \frac{(p-1)(\ln(x(x+1)) + 2\ln p)}{2}.$$

Then

$$f'(x) = p\sum_{k=1}^{p-1} \frac{1}{px+k} - \frac{p-1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right), \text{ and}$$

$$f''(x) = \frac{p-1}{2}\left(\frac{1}{x^2} + \frac{1}{(x+1)^2}\right) - p^2\sum_{k=1}^{p-1} \frac{1}{(px+k)^2}.$$

Since
$$\frac{1}{(px+k)^2} < \frac{1}{(px+k)^2-\frac{1}{4}} = \frac{1}{px+k-\frac{1}{2}} - \frac{1}{px+k+\frac{1}{2}},$$
so, telescoping:
$$\sum_{k=1}^{p-1} \frac{1}{(px+k)^2} < \frac{1}{px+\frac{1}{2}} - \frac{1}{px+p-\frac{1}{2}} = \frac{4(p-1)}{(2px+1)(2px+2p-1)}.$$
Therefore
$$\begin{aligned} f''(x) &> \frac{p-1}{2}\left(\frac{1}{x^2}+\frac{1}{(x+1)^2}-\frac{8p^2}{(2px+1)(2px+2p-1)}\right) \\ &= \frac{(p-1)\left[(4p^2+4p-2)x^2+(4p^2+4p-2)x+2p-1\right]}{2x^2(x+1)^2(2px+1)(2px+2p-1)} > 0. \end{aligned}$$
Since $\lim_{x\to\infty} f'(x) = 0$, we have $f'(x) < 0$. Note that $e^{f(n)} = \frac{u_n}{u_{n+1}}$ and
$$\lim_{x\to\infty} e^{f(x)} = \lim_{x\to\infty} \frac{\prod_{k=1}^{p-1}(px+k)}{(x(x+1))^{(p-1)/2}p^{p-1}} = 1,$$
so $\lim_{x\to\infty} f(x) = 0$, and hence $f(x) > 0$.

It follows that $\frac{u_n}{u_{n+1}} = e^{f(n)}$ is greater than one, so that $(u_n)_{n\geq 1}$ is a strictly decreasing sequence. By using Stirling's approximation that $N! \sim \sqrt{2\pi}\, N^{N+1/2}e^{-N}$ as N tends to infinity, we readily obtain that
$$\lim_{n\to\infty} u_n = \frac{(2\pi)^{(p-1)/2}}{\sqrt{p}}. \qquad \square$$

Here is a solution of the NKPC:

Problem 95.2. Let $\mathcal{A}$ be a regular polytope in n-dimensional Euclidean space $\mathbb{E}^n$ with $n \geq 2$. Let O be the centroid of $\mathcal{A}$, and $\mathcal{S}$ a hypersphere in $\mathbb{E}^n$ with O in its interior. Let $\{A_i\}_{i\in I}$ be the set of vertices of $\mathcal{A}$. For all $i \in I$, say ray OA_i meets $\mathcal{S}$ at B_i, and the opposite ray of OA_i meets $\mathcal{S}$ at C_i.

Prove that
$$\sum_{i\in I} |OB_i| = \sum_{i\in I} |OC_i|.$$

Solution 95.2. Let $\mathcal{A}$ be any regular polytope in $\mathbb{E}^n$, having vertex set $\{A_i\}_{i\in I}$, and let O be the centroid of its vertices. Let $\mathcal{S}$ be the hypersphere with centre P and radius r and suppose O lies inside this sphere. For all $i \in I$, say ray OA_i meets $\mathcal{S}$ at B_i, and the opposite ray of OA_i meets $\mathcal{S}$ at C_i. Without loss of generality, we assume $|OA_i| = 1$ for each $i \in I$.

Consider the triangle PB_iC_i and its plane. The point O lies on the line $a_i := B_iC_i$. Let M_i be the midpoint of a_i.

Since B_iC_i is a chord of the hypersphere $\mathcal{S}$, the line PM_i is perpendicular to a_i. Hence OM_i is the projection of OP onto the line a_i. Taking directed lengths along a_i in the direction $\overrightarrow{OA_i}$, we therefore have
$$|OB_i| - |OC_i| = 2\,\overrightarrow{OP}\cdot\overrightarrow{OA_i}.$$
Equivalently,
$$|OC_i| - |OB_i| = 2\,\overrightarrow{PO}\cdot\overrightarrow{OA_i}.$$

Adding, we get

$$\sum_{i\in I}|OC_i| - \sum_{i\in I}|OB_i| = 2\,\overrightarrow{PO}\cdot\sum_{i\in I}\overrightarrow{OA_i}.$$

However this equals zero because O is the centroid of the vertices, so

$$\sum_{i\in I}\overrightarrow{OA_i} = \overrightarrow{0}.$$

Therefore

$$\sum_{i\in I}|OB_i| = \sum_{i\in I}|OC_i|. \qquad \square$$

Problem 95.3 was also solved by Kee-Wai Lau, Hong Kong, China. We give Kee-Wai Lau's solution:

*Problem 95.3***.** In ΔABC, show that:

$$\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}+\left(\frac{a}{c}+\frac{c}{a}\right)\cos^2\frac{B}{2}+\left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2}\le\frac{3}{2}\left(\frac{R}{r}+1\right)$$

Solution 95.3. Since

$$\begin{aligned}\cos^2\frac{A}{2} &= \frac{1+\cos A}{2} = \frac{(a+b+c)(b+c-a)}{4bc},\\ \cos^2\frac{B}{2} &= \frac{(a+b+c)(c+a-b)}{4ca},\\ \cos^2\frac{C}{2} &= \frac{(a+b+c)(a+b-c)}{4ab},\end{aligned}$$

we obtain, after simplification,

$$\begin{aligned}&\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}+\left(\frac{a}{c}+\frac{c}{a}\right)\cos^2\frac{B}{2}+\left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2}\\ &=\frac{a+b+c}{4}\left[\frac{(b^2+c^2)(b+c-a)}{b^2c^2}+\frac{(c^2+a^2)(c+a-b)}{c^2a^2}+\frac{(a^2+b^2)(a+b-c)}{a^2b^2}\right]\\ &=\frac{(a+b+c)(ab+bc+ca)}{2abc}. \qquad (1)\end{aligned}$$

Now let

$$s=\frac{a+b+c}{2}.$$

We need the following known results:

$$\begin{aligned}ab+bc+ca &= s^2+4rR+r^2, &(2)\\ abc &= 4srR, &(3)\\ s^2 &\le 4R^2+4rR+3r^2, &(4)\\ R &\ge 2r. &(5)\end{aligned}$$

Identities (2) and (3) are well known. Inequality (4) appears in item 5.8 on p. 50 of the book by O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969. Inequality (5) is Euler's inequality.

By (1), (2), and (3), the left-hand side equals

$$\begin{aligned}\frac{(a+b+c)(ab+bc+ca)}{2abc} &= \frac{2s(ab+bc+ca)}{2abc} \\ &= \frac{s(ab+bc+ca)}{abc} \\ &= \frac{ab+bc+ca}{4rR} \\ &= \frac{s^2+4rR+r^2}{4rR}.\end{aligned}$$

Hence the required inequality is equivalent to

$$\frac{s^2+4rR+r^2}{4rR} \leq \frac{3}{2}\left(\frac{R}{r}+1\right),$$

which is equivalent to

$$s^2 \leq 6R^2+2rR-r^2. \tag{6}$$

Now the right-hand side of (6) can be written as

$$6R^2+2rR-r^2 = 4R^2+4rR+3r^2+2(R-2r)(R+r).$$

By Euler's inequality (5), we have $R-2r \geq 0$, and hence

$$2(R-2r)(R+r) \geq 0.$$

Therefore, by (4),

$$\begin{aligned}s^2 &\leq 4R^2+4rR+3r^2 \\ &\leq 4R^2+4rR+3r^2+2(R-2r)(R+r) \\ &= 6R^2+2rR-r^2.\end{aligned}$$

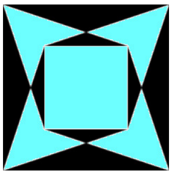
Thus (6) holds, and consequently

$$\begin{aligned}&\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}+\left(\frac{a}{c}+\frac{c}{a}\right)\cos^2\frac{B}{2}+\left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2} \\ &\quad\leq \frac{3}{2}\left(\frac{R}{r}+1\right).\end{aligned}$$

This completes the proof. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (preferably raw LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by the end of October. The solution to a problem is published two issues after the issue in which the problem first appeared. If possible, please include solutions with your submissions.

Department of Mathematics, Munster Technological University



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