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**Tom Carroll: Geometric Function Theory: A Second Course in Complex Analysis, Springer, 2024.**
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What is geometric function theory (or GFT for brevity)? Saying it is the "geometric bits of function theory" sounds like an unhelpful rearrangement of the words. A little more precise is "the parts of mathematics that utilize the theory of conformal mappings in the plane or their generalizations to higher dimensions and metric spaces". A broader attempt would be "the study of mappings that distort geometry by only a bounded amount". However, perhaps it is safest to paraphrase US Supreme Court Justice Stewart's description of a different topic: "I know it when I see it".

Assuming I know geometric function theory when I see it, I will list several topics I consider to be part of, or very close to, GFT (a highly personal and debatable list).

• **Complex dynamics:** The Fatou set of a polynomial $p$ is defined as the largest open set where the iterates of $p$ form a normal family, a concept at the center of geometric function theory. The Julia set is the complement; the set where the iterates are "chaotic". Any non-linear polynomial introduces some geometric distortion, which we might expect to accumulate as we iterate $p$, but because the iterates of small disks are conformal (as long as they avoid critical points), the distortion remains uniformly bounded by Koebe's 1/4-theorem, another central pillar of GFT. This explains the approximately self-similar character of Julia sets and the importance of understanding the critical orbits. Approximation results, such as Runge's theorem, are often used to create examples with novel properties, and the connectedness of the Mandelbrot set follows from constructing the Riemann map onto its complement. Thus holomorphic dynamics incorporates many tools of GFT.

• **Hyperbolic manifolds:** The disk has a natural hyperbolic metric and the Schwarz lemma says that holomorphic self-maps of the unit disk, $\mathbb{D}$, are contractions for this metric, binding GFT tightly to hyperbolic geometry. By the uniformization theorem, most Riemann surfaces are of the form $R = \mathbb{D}/G$ where $G$ is a discrete group of Möbius transformations acting on the disk (called a Fuchsian group). These are hyperbolic isometries of the disk, and they extend to a group of isometries on the hyperbolic upper half-plane (this is called a Kleinian group), giving a quotient that is a hyperbolic 3-manifold. Replacing the disk by others planar domains (usually with a fractal boundary) gives rise to other hyperbolic 3-manifolds. The study of such Kleinian groups and hyperbolic 3-manifolds is a rich mixture of GFT and topology, much of it inspired by the work of William Thurston.

• **Brownian motion:** A Brownian motion is a random continuous path in the plane (although it also makes sense in other dimensions). This seems firmly within real analysis and probability theory, but Brownian motion in a planar domain $\Omega$ is conformally invariant: it is mapped to another Brownian path by any conformal map on $\Omega$. This means that theorems about the the first hitting distribution of Brownian motion on the boundary of a domain (called harmonic measure) can, in the simply connected

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case, often be reduced to results about the boundary behavior of conformal mappings $f : \mathbb{D} \to \Omega$. This allows the tools of GFT to be applied, often with spectacular results. For example, Makarov used this approach to prove that harmonic measure on simply connected domains is always 1-dimensional in a precise sense, even if the boundary of $\Omega$ is not, e.g., it is a fractal. Extending such results to higher dimensions has been daunting, but remarkable recent progress has occurred because of advances in harmonic analysis, PDE and geometric measure theory.

• **SLE:** A Brownian motion is allowed to intersect itself, but defining random Jordan curves (non-self-intersecting paths) is much harder, and was not successfully done until Oded Schramm invented SLE [Sch00] using random conformal maps. He applied a differential equation of Loewner (very classic GFT) using Brownian motion as data. Schramm called these "stochastic Loewner Evolutions" but they are now known as "Schramm-Loewner evolutions", and for the last twenty years SLE has been one of the hottest topics in mathematics and physics (several Fields medals for related work).

• **Traveling salesman theorem:** The classical traveling salesman problem (TSP) in computer science is to find the shortest path that visits each point of a given finite set, but there is also an analytical version that asks which infinite sets $E$ can be visited by some finite length curve. Peter Jones [Jon90] defined an infinite series whose terms measure how close $E$ is to lying on a straight line at different points and scales and proved that the shortest curve containing $E$ has length bounded by a fixed multiple of this series. Jones's theorem has been extended to higher dimensions, Hilbert space and certain metric spaces, but his original proof was based on conformal maps and other basic tools of GFT, and has itself become a pillar of modern GFT with numerous applications to Julia sets, metric space analysis, Kleinian groups and Brownian motion.

Tom Carroll's book *Geometric Function Theory* does not deal directly with any of the applications of GFT mentioned above, but it does prepare a student for the study of all these topics by presenting many of the essential tools: spherical and hyperbolic geometry, normal families, the Riemann mapping theorem, Runge's approximation theorem, the distortion properties of conformal mappings (including Koebe's 1/4-theorem), Carathéodory convergence, and the uniformization theorem. Several of these topics can be found in other textbooks, though they are not generally covered in an undergraduate course. For example, the results on univalent functions and Carathéodory convergence are usually only found in more advanced books, such as [GM08] or [Pom92], and Carroll's book is an excellent preparation for reading these graduate level texts.

One non-standard example that caught my eye is Theorem 9.4, that says a Euclidean disk (which is obviously convex for the Euclidean metric) is also convex for the hyperbolic metric on any simply connected planar domain $\Omega$ containing $D$. This is a very pretty result of Jørgensen, and it is certainly a prototypical result of geometric function theory, but not one I have seen in a textbook before. Jørgensen's theorem follows a discussion of the differential equation $\Delta u = e^{2u}$ satisfied by the hyperbolic metric, another important topic not usually covered by a first course in complex analysis.

Another nice feature is the inclusion of Zalcman's lemma, along with more standard results about normal families, such as the theorems of Marty and Montel. Zalcman's lemma says that a family of holomorphic maps is not normal if we can extract a subsequence that has "blow-ups" that converge to a non-constant limit in a precise sense. The result is elementary, but extremely useful in holomorphic dynamics, but is not often included in introductory textbooks.

I do have a few minor quibbles. For example, conformal maps from the disk to a Jordan domain extend continuously to the boundary, but Carroll only proves this in a special case he calls "geometrically simple". A remark on page 175 gives the impression that this includes all Jordan domains, but this is not true; even a mild mannered fractal curve like the von Koch snowflake is not simple in the sense of Definition 6.4 (and much worse behavior is possible). Moreover, Carroll's proof takes eight pages, whereas the general case takes only two pages in [GM08] (assuming the Jordan curve theorem) and seven pages in [Mar19] (including a proof of the Jordan curve theorem).

There are 121 exercises, but I would have preferred even more. For example, the chapter on Runge's theorem has only three problems, and none utilize Runge's theorem itself. This is a pity, since Runge's theorem is a marvelous machine for generating unexpected examples, e.g., a sequence of polynomials converging pointwise on the plane to a discontinuous limit, a holomorphic function on the disk that has radial limits nowhere on the boundary, or a "universal" entire function whose translates can approximate any entire function. Learning Runge's theorem should include learning how to wield it.

A notable feature of Carroll's book is that it is carefully written, with plenty of discussion, motivation and extended explanations, in addition to the actual proofs; it has a conversational tone, and it is well suited for independent reading. This aspect is enhanced because it also contains introductions to many of the necessary parts of geometry and topology, as well as providing solutions to all the exercises. I recently taught complex analysis for first year PhD students, covering most of Marshall's more concise text [Mar19] in one semester, by assuming material from the parallel real analysis and topology classes. Several undergraduates attempted this class, but because of the rapid pace I set, only a few continued for the whole semester. In the future, I would recommend such students take our standard undergraduate complex variables course, followed by a reading course from Carroll's book. Such a plan would leave them well prepared to tackle books, papers or research projects in the areas mentioned earlier in this review. There are few (if any) books that fill the gap between the standard undergraduate material and more advanced texts as well as Carroll's book does. I am currently teaching a graduate course on quasiconformal mappings, and I recommended his book to any students needing to brush up on the essential prerequisites from classical GFT. It is a well organized, well written gateway to an enormous number of exciting topics in modern mathematics.

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