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PROBLEM PAGE

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Problems

Let us start with a thank you to all contributors, and then a call for more problems; at the moment the problem bank is running low.

The first of this edition's problems comes courtesy of Finbarr Holland of University College Cork.

Problem 95.1. Suppose p is a positive integer, and

$$u_n = \frac{p^{pn}\,(n!)^p}{(pn)!n^{(p-1)/2}}, \qquad (n = 1, 2, \ldots).$$

Prove that $(u_n)_{n\geq 1}$ is a strictly decreasing sequence, and determine its limit.

The second problem was sent in by Tran Quang Hung of the Vietnam National University at Hanoi, Vietnam. Given points X, Y in Euclidean space, the *ray* XY is the set of points Z satisfying $\overrightarrow{XZ} = t\,\overrightarrow{XY}$ for some $t \geq 0$. Similarly the *opposite ray* of XY is the set of points W satisfying $\overrightarrow{XW} = -t\overrightarrow{XY}$.

Problem 95.2. Let $\mathcal{A}$ be a regular polytope in n-dimensional Euclidean space $\mathbb{E}^n$ with $n \geq 2$. Let O be the centroid of $\mathcal{A}$, and $\mathcal{S}$ a hypersphere in $\mathbb{E}^n$ with O in its interior. Let $\{A_i\}_{i\in I}$ be the set of vertices of $\mathcal{A}$. For all $i \in I$, say ray OA_i meets $\mathcal{S}$ at B_i, and the opposite ray of OA_i meets $\mathcal{S}$ at C_i.

Prove that

$$\sum_{i\in I} |OB_i| = \sum_{i\in I} |OC_i|.$$

Finally a problem from Marian Uršarescu, "Roman–Vodă", National College, Roman, Romania. As standard, r and R denote the inradius and circumradius, respectively.

Problem 95.3. In ΔABC, show that:

$$\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2} + \left(\frac{a}{c}+\frac{c}{a}\right)\cos^2\frac{B}{2} + \left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2} \leq \frac{3}{2}\left(\frac{R}{r}+1\right)$$

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Solutions

Here are solutions to the problems from *Bulletin* Number 93. In taking over from Ian Short, I may have inadvertently lost some solutions, my apologies if this is the case.

The first problem was solved by the North Kildare Problem Club.

*Problem 93.1***.** Find a simple closed curve in the plane that does not have an inscribed regular pentagon.

Solution 93.1. Take the boundary T of a regular triangle. Suppose the regular pentagon P is inscribed in T. Since no three vertices of P are collinear, there must be at least two vertices of P on each of two sides of T, and hence the angle between two lines joining pairs of vertices of P must be a multiple of $\pi/3$. But in fact all such angles are multiples of $\pi/10$. □

The second problem was solved by Yagub N. Aliyev, of ADA University, Baku, Azerbaijan; and the North Kildare Problem Club. We provide the solution of Yagub:

*Problem 93.2***.** Determine the least positive integer n for which a continued fraction

$$\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{b_3+\cdots+\cfrac{1}{b_n}}}}$$

has value ∞, where b_i are Gaussian integers each of modulus greater than 1.

Solution 93.2. $n = 1$ is impossible, because

$$\frac{1}{b_1} = \infty \iff b_1 = 0.$$

$n = 2$ is also impossible, because

$$\frac{1}{b_1 + \frac{1}{b_2}} = \infty \iff b_1 + \frac{1}{b_2} = 0 \iff b_1 b_2 = -1,$$

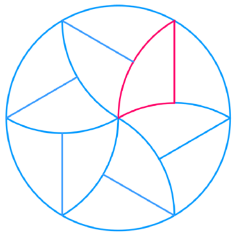
which implies that $|b_1 b_2| = 1$ but $|b_1 b_2| = |b_1| \cdot |b_2| > 1$, a contradiction.

$n = 3$ is possible, with, e.g. $b_1 = 1 + i$, $b_2 = -1 + i$, $b_3 = 1 + i$. □

The third problem was solved by Yagub N. Aliyev, the North Kildare Problem Club, and the proposer Andrei Zabolotskii of the Open University. Here is the solution of the Problem Club:

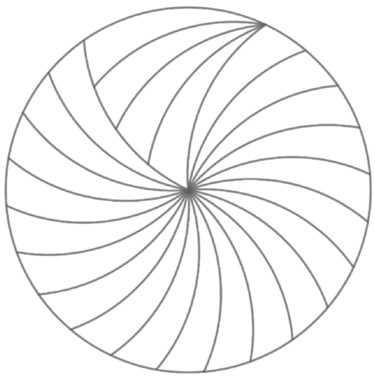
*Problem 93.3***.** Dissect a disc into a finite number of congruent connected pieces (reflections allowed) in such a way that at least one piece does not contain the centre of the disc inside it or on its boundary.

Solution 93.3. If we represent the disc as the unit disc in the complex plane, the three edges of the piece NE of 0 (coloured red in the below figure) can be parametrized.



The long arc has length $\pi/3$ and is given by
$$1 + e^{it}, \text{ for } 2\pi/3 \leq t \leq \pi.$$
The short arc has length $\pi/6$ and is given by
$$e^{5\pi i/3} + e^{it}, \text{ for } \pi/2 \leq t \leq 2\pi/3.$$
The straight edge has length $\sqrt{3} - 1$ and is given by
$$e^{i\pi/3} - t, \text{ for } 0 \leq t \leq \sqrt{3} - 1.$$

For good measure, here is the construction of Yagub:



□

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (preferably LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. If possible, please include solutions with your submissions.

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