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Wiring Switches to More Light Bulbs

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Abstract. Given n buttons and n bulbs so that the ith button toggles the ith bulb and perhaps some other bulbs, we compute the sharp lower bound on the number of bulbs that can be lit regardless of the action of the buttons. In the previous article we dealt with the case where each button affects at most 2 or 3 bulbs. In the present article we give sharp lower bounds for up to 4 or 5 wires per switch, and we show that the sharp asymptotic bound for an arbitrary number of wires is $\frac{1}{2}$. (Even if you've found their buttons, you can please no more than half the people all the time!)

1. Introduction

1.1. **The function $\mu(m, n)$.** This article is a continuation of [2], and we refer to that article for motivation and context. The focus of our attention is the function $\mu(n, m)$, which counts the minimum number of bulbs that can always be lit by some switching choice when each of n bulbs has a dedicated button (=switch) that switches it and up to $m - 1$ other bulbs on or off. The problem is rephrased in precise terms using vectors and matrices over $\mathbb{F}_2$, the field with two elements, as follows:

Each conceivable wiring from n buttons to r bulbs may be represented by an element of the set $\mathcal{M}(n, r, \mathbb{F}_2)$ of all $n \times r$ matrices over $\mathbb{F}_2$, by letting column i represent the effect of button i. Replacing n and r by their maximum, and filling in with zeros, we might as well use square matrices, so for us a wiring corresponds to a directed graph G on n vertices, represented by an $n \times n$ matrix W over $\mathbb{F}_2$. A column vector in $\mathbb{F}_2^n$ may represent either the state (lit or unlit) of the n bulbs, or a choice (press or don't press) for n buttons. The effect of switch choice x on state c gives state $Wx + c$.

We are focussed on wirings with 1 on the diagonal, and we call these *admissible wirings*, but we shall have occasional use for inadmissible wirings.

The *Hamming norm* $|\cdot| : \mathbb{F}_2^n \to \mathbb{Z}_{\geq 0}$ is defined by letting $|u|$ be the the number of 1 entries in u. We define $M(W, c) := \max\{\, |Wx + c| : x \in Z_2^n \,\}$; it represents the maximal number of bulbs that can be lit by a choice of switches, given initial state c.

Given a wiring W, the *associated degree of vertex i* is the Hamming norm of the i-th column of W (the out-degree of node i in the graph G, the number of bulbs affected by button i). The degree of W is the maximum associated degree.

For any $n \in \mathbb{N}$, and any set A of $n \times n$ matrices over $\mathbb{F}_2$, we define

$$\mu_A = \min\{M(W, 0) \mid W \in A\}\,,$$
$$\nu_A = \min\{M(W, c) \mid W \in A,\ c \in \mathbb{F}_2^n\}.$$

For $n, m \geq 1$, let $A(n, m)$ be the set of $n \times n$ matrices over $\mathbb{F}_2$ that have 1s all along the diagonal and satisfy $\deg(W) \leq m$. For $n \geq m \geq 1$, let $A^*(n, m)$ be the set of matrices

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in $A(n,m)$ for which $\deg(i) = m$, for all $i \in S$. The class of all admissible wirings on n vertices is $A(n) := A(n,n)$.

The functions we study are:

$$\mu(n,m) := \mu_{A(n,m)}\,, \qquad \mu^*(n,m) := \mu_{A^*(n,m)}\,, \qquad \mu(n) := \mu(n,n)\,,$$
$$\nu(n,m) := \nu_{A(n,m)}\,, \qquad \nu^*(n,m) := \nu_{A^*(n,m)}\,, \qquad \nu(n) := \nu(n,n)\,,$$

It is convenient to define $\mu(0,m) = 0$ for all $m \in \mathbb{N}$. Given $n \ge m$, we have the following trivial inequalities:

$$\nu(n,m) \le \nu^*(n,m) \le \mu^*(n,m) \tag{1.1.1}$$

$$\nu(n,m) \le \mu(n,m) \le \mu^*(n,m) \tag{1.1.2}$$

1.2. **Results.** General formulae for ν and ν^*, and formulae for $\mu(\cdot,m)$ and $\mu^*(\cdot,m)$ for $m = 2,3$ were determined in [2]. We'll summarise these in Section 2 below, but right now we mention only that if $m = 2,3$, then $\mu(n,m)$ and $\mu^*(n,m)$ are asymptotic to $2n/3$ as $n \to \infty$. By contrast, we will see that for $m = 4,5$, both functions $\mu(n,m)$ and $\mu^*(n,m)$ are asymptotic to $4n/7$. In fact we have the following result:

Theorem 1.1. *Let $n \in \mathbb{N}$.*

(a) For $j = 4,5$, $\mu(n,j)$ is given by the equation

$$\mu(n,j) = \begin{cases} \left\lceil \dfrac{4n}{7} \right\rceil, & n \ne 7k-2 \text{ for some } k \in \mathbb{N}, \\ \left\lceil \dfrac{4n}{7} \right\rceil + 1 = 4k, & n = 7k-2 \text{ for some } k \in \mathbb{N}. \end{cases}$$

(b) If $n \ge 3$, then $\mu^*(n,4) = 2\left\lceil \dfrac{2n}{7} \right\rceil$, the least even integer not less than $\mu(n,4)$.

It is not hard to show that $\mu(n,m) \ge n/2$ for all $n,m \in \mathbb{N}$. This is asymptotically sharp according to the following result.

Theorem 1.2. $\lim_{n\to\infty} \mu(n)/n = 1/2$.

In fact, this shows that $\dfrac{\mu(n)}{\nu(n)} \to 1$ (cf. Theorem C below).

1.3. **Outline.** The article is organized as follows. After some introductory material in Section 2, we consider $\mu(n,m)$ and $\mu^*(n,m)$ for numbers of the form $(n,m) = (2^{k+1}-1, 2^k)$ in Section 3. This special case involves a wiring related to Hadamard matrices, and allows us to deduce Theorem 1.2.

In Section 4, we give an explicit upper bound $U(n,m)$ for $\mu(n,m)$. This upper bound has the appearance of being rather sharp: indeed, we know of no pair (n,m) such that $\mu(n,m) < U(n,m)$. Whether $\mu(n,m) = U(n,m)$ for all n,m is an interesting open question. The upper bound $U(n,m)$ sheds light on the formulae for $\mu(n,m)$ given above and in Section 2 which, although convenient for understanding the asymptotics of $\mu(n,m)$ as $n \to \infty$, do not seem to follow any clear pattern as m changes. The sequence $U(n,n)$ is connected to OEIS sequence A046699, which is of meta-Fibonacci type, and has a number of combinatorial descriptions in terms of trees.

In Section 5, we prove that if $\mu(\cdot,m) = U(\cdot,m)$ for $m = 2^k - 2$, then this equation also holds for $m = 2^k + i$, $i \in \{-1,0,1\}$. Theorem 1.1(a) will follow immediately from this result but Theorem 1.1(b) still requires a proof, which can be found in Section 6.

2. A recap of previous results and ideas

For ease of reference, we state and label some results from [2]. We need them either for proofs or for comparison purposes.

2.1. **Theorems from** [2]**.** We begin by listing the three main results in [2]: in the order listed below, these were Theorems 1.1, 1.2, and 3.2 in that article.

Theorem A. *Let* $n \in \mathbb{N}$.

(a) $\mu(n,2) = \left\lceil \frac{2n}{3} \right\rceil$.

(b) *If* $n \geq 2$, *then* $\mu^*(n,2) = 2\left\lceil \frac{n}{3} \right\rceil$, *the least even integer not less than* $\mu(n,2)$.

Theorem B. *Let* $n \in \mathbb{N}$.

(a) $\mu(n,3) = \mu(n,2)$.

(b) *If* $n \geq 3$, *then*

$$\mu^*(n,3) = \begin{cases} 4k-1, & n = 6k-3 \text{ for some } k \in \mathbb{N}, \\ \mu(n,3), & \text{otherwise.} \end{cases}$$

Note that $\mu^*(n,3) = \mu(n,3) + 1$ in the exceptional case $n = 6k - 3$.

Theorem C. *Let* $n, m \in \mathbb{N}$, $m > 1$.

(a) $\nu(n) = \nu(n,m) = \left\lceil \frac{n}{2} \right\rceil$.

(b) *If* $n \geq m$, *then*

$$\nu^*(n,m) = \begin{cases} \nu(n,m)+1, & \text{if } n \text{ is even and } m \text{ odd,} \\ \nu(n,m), & \text{otherwise.} \end{cases}$$

In particular, $\nu^*(n,2) = \nu^*(n) = \nu(n)$ *for all* $n > 1$.

2.2. **Lemmas from** [2]**.** The next four results were, in the order listed below, Lemmas 3.1, 5.1, and 5.2, and Corollary 3.3 in [2].

Lemma D. *Let* $n \in \mathbb{N}$. *For all* $W \in A(n)$ *and* $c \in \mathbb{F}_2^n$, *the mean value of* $|Mx + c|$ *over all* $x \in \mathbb{F}_2^n$ *is* $n/2$. *In particular,* $M(W,c) \geq n/2$ *and* $M(W,c) > n/2$ *if the cardinality of* $\{i \in [1,n] \cap \mathbb{N} \mid c_i = 1\}$ *is not* $n/2$.

Lemma E. *Let* $m \geq 2$ *and* $n \geq 1$. *Then either* $\mu(n+m,m) = \mu(n+m,m-1)$, *or*

$$\mu(n+m,m) \geq \mu(n,m) + \nu(m) = \mu(n,m) + \left\lceil \frac{m}{2} \right\rceil .$$

Lemma F. *Let* $n, m, n' \in \mathbb{N}$, *with* $n \geq m$. *Then*

$$\mu^*(n+n', m+1) \leq \mu^*(n,m) + n' .$$

Corollary G. *If* λ *is any one of the four functions* μ, μ^*, ν, *or* ν^*, *then* $\lambda(\cdot, m)$ *is sublinear for all* m:

$$\lambda(n_1+n_2, m) \leq \lambda(n_1,m) + \lambda(n_2,m) , \tag{2.2.1}$$

as long as this equation makes sense (i.e. we need $n_1, n_2 \geq m$ *if* $\lambda = \mu^*$ *or* $\lambda = \nu^*$*).*

2.3. **Edge functions.** Associated with the graph G is its vertex set S (which we treat as an initial segment $S(n) := \{1, \dots, n\}$ of the set $\mathbb{N}$ of natural numbers) and the *edge function* $F : S \to 2^S$, where $j \in F(i)$ if there is an edge from i to j, and the *backward edge function* $F^{-1} : S \to 2^S$, where $j \in F^{-1}(i)$ if there is an edge from j to i. We extend the definitions of F and F^{-1} to 2^S in the usual way: $F(T)$ and $F^{-1}(T)$ are the unions of $F(i)$ or $F^{-1}(i)$, respectively, over all $i \in T \subset S$. We say that $T \subset S$ is *forward invariant* if $F(T) \subset T$, or *backward invariant* if $F^{-1}(T) \subset T$. Given a wiring W, associated graph G, and $T \subset S$, we denote by W_T and G_T the subwiring and subgraph, respectively, associated with the vertices in T: more precisely, W_T is the matrix obtained by deleting all rows and columns of W other than those with index in T, and G_T is obtained by retaining only the vertices in T and those edges in G between vertices in T.

2.4. **Pivoting.** We now recall the concept of *pivoting*, as introduced in [2, Section 5]. Pivoting about a vertex i, $1 \le i \le n$, is a way of changing the given wiring W to a special wiring W^i such that $M(W^i, c) \le M(W, c)$. Additionally, pivoting preserves the classes $A(n, m)$ and $A^*(n, m)$.

Let us fix a wiring $W = (w_{i,j})$ on n vertices, and let $F : S \to 2^S$ denote the edge function associated to W, where $S = S(n)$. Given $T \subset S$, and $i \in S$, we define $W^{i,T}$ by replacing the jth column of W by its ith column whenever $j \in F(i) \setminus T$. We refer to the wiring $W^{i,T}$ as the *pivot of W about i relative to T*. If T is nonempty, we refer to this process as *partial pivoting*, while if T is empty we call it *(full) pivoting* and write G^i, W^i, and F^i for the resulting graph, matrix, and edge function, respectively.

As in [2], we use the notation $\hat{K}_r$ to denote an augmented complete graph on r vertices, i.e. a complete graph augmented by a loop at each vertex. Full pivoting about vertex i just rewires $F(i)$ so that it becomes a $\hat{K}_{\deg(i)}$, which is thus a forward-invariant subgraph of W^i.

We refer to a forward-invariant $\hat{K}_r$ subgraph of a wiring graph W as an F_r (relative to W).

For $t \in \{0, 1\}$, we denote by $t_{p\times q}$ the $p \times q$ matrix all of whose entries equal t, and let $t_p = t_{p\times p}$. The matrix of a $\hat{K}_r$, is $1_{r\times r}$. This is (of course) different from the $r \times r$ identity matrix I_r, except when $r = 1$.

Pivoting relative to any T is a process with several nice properties: it has the non-increasing property $M(W^{i,T}, c) \le M(W, c)$, it preserves membership of the classes $A(n, m)$ and $A^*(n, m)$, and if $F^{i,T}$ is the edge function of $W^{i,T}$, then $F^{i,T}(i) = F(i)$ is an augmented complete subgraph of the associated graph $G^{i,T}$, but might not be forward invariant in $G^{i,T}$.

2.5. **Graphical conventions.** We continue the graphical conventions introduced in [2]. Thus, we do not show loops or the internal edges in a $\hat{K}_r$, and a single arrow issuing from $\hat{K}_r$ represents r edges, one from each vertex in the $\hat{K}_r$, all sharing the same target. If several arrows from a $\hat{K}_r$ point to some $\hat{K}_s$, then distinct arrows have distinct targets (so the number of arrows will not exceed s). For instance, Figure 1 shows three views of a $\hat{K}_6$. Notice how the 36 directed edges of the $\hat{K}_6$ are hidden to varying degrees in this figure, and how the arrows represent multiple edges — two each in the version with $\hat{K}_2$s, and three each in the version with $\hat{K}_3$s. To reduce clutter further, we introduce the additional convention that an two-headed arc stands for a pair of arrows, one in each direction. This gives the two more views of $\hat{K}_6$ shown in Figure 2 in which individual two-headed arcs stand for up to six directed edges in the $\hat{K}_6$.

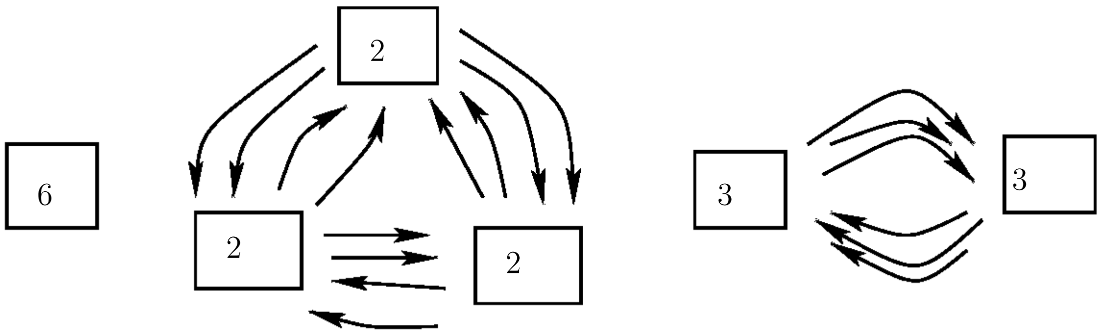
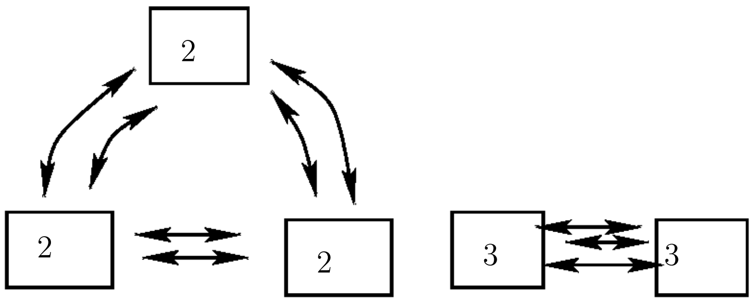


FIGURE 1. Views of $\hat{K}_6$



[h]

FIGURE 2. More views of $\hat{K}_6$

3. THE CASE $(n, m) = (2^{k+1} - 1, 2^k)$

3.1. We begin with some observations for general n, m that will be useful here or in later sections. Trivially, $\mu(n, m)$ is nonincreasing as a function of m, but it is also easy to see that it is also nondecreasing as a function of n: given a wiring $W \in A(n, m)$ such that $|Wx| \leq \mu(n, m)$ for all $x \in \mathbb{F}_2^n$, it may be that vertex n has degree 1, in which case it is clear that if W' is obtained by eliminating the last row and column of W, then $|W'x'| \leq \mu(n, m)$ for all $x' \in \mathbb{F}_2^{n-1}$.

If instead vertex n has degree larger than 1 then, by pivoting if necessary, we may assume that vertex n forms a part of a forward invariant $\hat{K}_j$ for some $j > 1$. Because the effect of pressing vertex n is the same as the effect of pressing any other vertex in the $\hat{K}_j$, the set of vectors Wx, as $x = (x_1, \ldots, x_n)^t$ ranges over all vectors in $\mathbb{F}_2^n$ for which $x_n = 0$, coincides with the set of vectors Wx as x ranges over all of $\mathbb{F}_2^n$. It follows that if we define W' as in the previous case, then $|W'x'| \leq \mu(n, m)$ for all $x' \in \mathbb{F}_2^{n-1}$.

In contrast, we do not know whether or not $\mu^*(n, m)$ is an nondecreasing function of n.

3.2. Another easily proven inequality is:

$$\mu(n+1, m) \leq \mu(n, m) + 1\,. \tag{3.2.1}$$

To see this, we need only consider the matrix $W \in A(n+1, m)$ which has block diagonal form $\mathrm{diag}(W', I_1)$, where $W' \in A(n, m)$ satisfies $M(W', 0) = \mu(n, m)$.

3.3. We now prove a pair of closely related lemmas. We will only use the second one in this section, but we will need the first one later.

Lemma 3.1. *Let* $m, m', n \in \mathbb{N}$ *and* $m \leq n$. *Then*

$$\mu(nm', mm') \leq m'\mu(n, m)$$
$$\mu^*(nm', mm') \leq m'\mu^*(n, m)$$

Proof. Essentially the same proof works for μ and μ^*, so we write down only the one for μ. Let $W \in A(n, m)$ be such that $M(W, 0) = \mu(n, m)$. We construct a new matrix W' by replacing each entry $w_{i,j}$ in W by an $m' \times m'$ block, each of whose entries is $w_{i,j}$, i.e. W' is the Kronecker product $W \bigotimes 1_{m' \times m'}$. It is readily verified that $W \in A(nm', mm')$.

The graph of W is obtained by replacing each vertex j in the original graph G by m' new vertices which we will label (j, j'), $1 \leq j' \leq m'$. Pressing vertex (j, j') changes the status of some other vertex (i, i') if and only if pressing j changes the status of vertex i in the original graph. In the new wiring W', each bulb of W has been replaced by a bank of m' bulbs, all of which are switched synchronously by any of their associated switches and it is clear that $M(W', 0) = m'M(W, 0)$. □

Figure 3 illustrates the proof that $\mu^*(18, 9) \leq 3\mu^*(6, 3)(= 12)$, i.e. the case $n = 6$, $m = 3$, $m' = 3$. The graph W is the graph from Figure 12 in [2], the wiring example

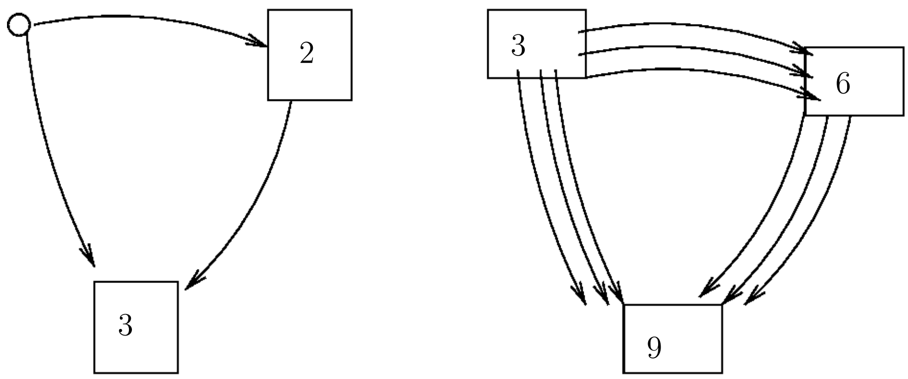


FIGURE 3. W and W'

which concludes the proof that $\mu^*(6, 3) = 4$. To construct W', each vertex of W has been replaced by a $\hat{K}_3$ and each edge by three edges, one to each vertex of the $\hat{K}_3$ that replaces the original target. Thus, the original $\hat{K}_2$ and $\hat{K}_3$ become a $\hat{K}_6$ and a $\hat{K}_9$, respectively. In terms of bulbs and switches, each bulb becomes a bank of 3 bulbs, and each switch a bank of 3 switches, all having the same effect.

It is also possible to view the new wiring W' as a row of m' copies of W, suitably wired together, and when we think of it in this way we refer to the copies as *clones* of W. Figure 4 shows this view of the above example. The view in Figure 4 is comparatively cluttered, but it is still substantially less messy than the full wiring graph, which has 162 directed edges.

Lemma 3.2. *Let* $n, m, m' \in \mathbb{N}$ *with* $m'm \geq n+1$. *Then* $\mu(m'n+1, m'm) \leq m'\mu(n, m)$.

Proof. Let $W \in A(n, m)$ be such that $M(W, 0) = \mu(n, m)$. As in the previous lemma, we construct a new matrix $W' = W \bigotimes 1_{m' \times m'}$. The wiring W' is a wiring for $m'n$ vertices which can be split into n banks of m' vertices that are always in sync (either all on or all off). We add one last vertex v and get a new wiring by connecting v to itself and to one vertex from each of the m' sets of clones. In terms of matrices, this can be achieved by defining a matrix with block form

$$W'' = \begin{pmatrix} W' & V \\ 0_{1 \times m'n} & I_1 \end{pmatrix} \tag{3.3.1}$$

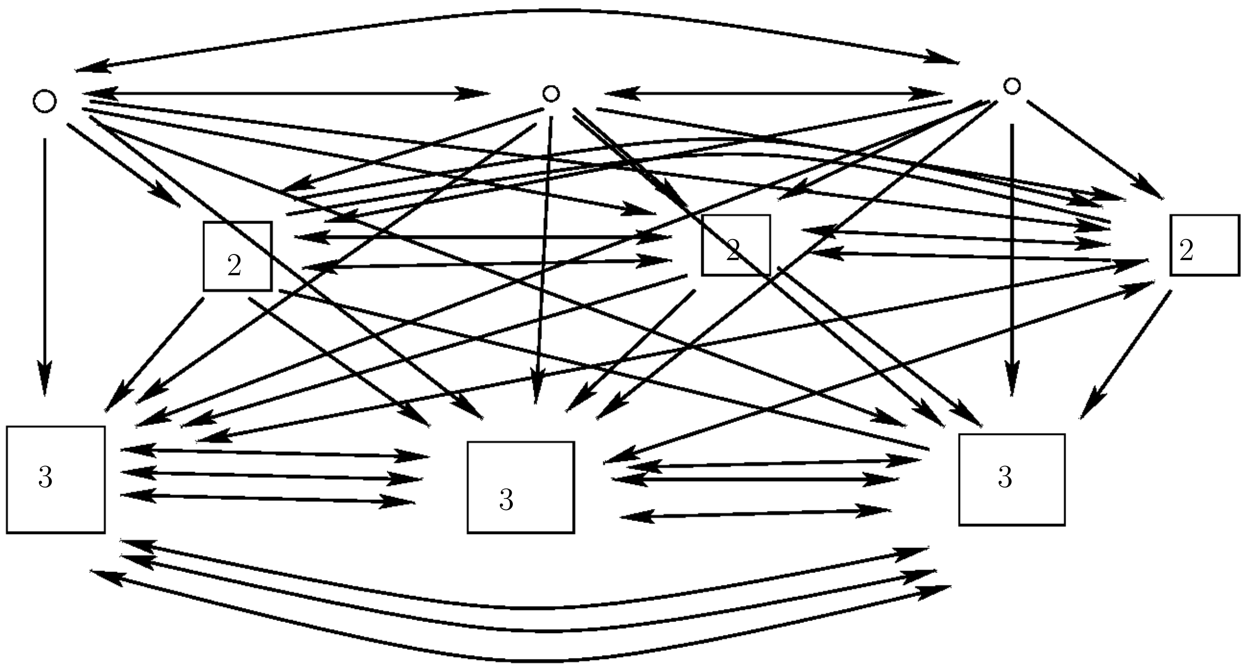


Figure 4. W'

where $V = (v_i)$ is a $m'n \times 1$ column vector with $v_i = 1$ if i is a multiple of m', and $v_i = 0$ otherwise. Using the inequality $m'm \geq n + 1$, it is readily verified that $W \in A(m'n + 1, m'm)$.

If we do not press v, then it is clear (as in the previous proof) that we can light at most $m'M(W, 0) = m'\mu(n, m)$. Suppose therefore that we press v (together with some combination of other vertices). Partitioning each set of m' clones into two subsets S' and S'', where S'' has cardinality 2 and includes the vertex which is toggled by v, it is clear that all vertices in each of the S' sets remain in sync, that precisely one vertex in each S'' is lit, and that v itself is lit. Thus, we can light at most $(m' - 2)\mu(n, m) + n + 1$ if v is pressed. Since we know from Lemma D that $\mu(n, m) = M(W, 0) > n/2$, we have $n + 1 \leq 2\mu(n, m)$, and so $(m' - 2)\mu(n, m) + n + 1 \leq m'\mu(n, m)$, and we are done. □

3.4. We now state our first main result for m close to a power of 2.

Theorem 3.3. *For all $k \in \mathbb{N}$, and $m \geq 2^k$,*

$$\mu(2^{k+1} - 1, m) = \mu^*(2^{k+1} - 1, 2^k) = 2^k .$$

3.5. Using this theorem, it is easy to deduce Theorem 1.2, i.e. $\lim_{n\to\infty} \mu(n)/n = 1/2$:

Proof of Theorem 1.2. Lemma D implies that $\liminf_{n\to\infty} \mu(n)/n \geq 1/2$, so it suffices to show that $\limsup_{n\to\infty} \mu(n)/n \leq 1/2$. Fixing $k \in \mathbb{N}$, let us assume that $n > p := 2^{k+1}-1$. We write $n = ap+r$, where $a \in \mathbb{N}$ and $0 \leq r \leq p-1$. By inequality (2.2.1), Theorem 3.3, and the fact that $\mu(\cdot, \cdot)$ is nondecreasing in its first argument and nonincreasing in its second, we see that

$$\mu(n) \leq \mu(n, 2^k) \leq a\mu(p, 2^k) + \mu(r, 2^k) \leq (a + 1)2^k .$$

Letting $n \to \infty$, it follows easily that $\limsup_{n\to\infty} \mu(n)/n \leq 2^k/(2^{k+1} - 1)$. Since k can be chosen to be arbitrarily large, it follows that $\limsup_{n\to\infty} \mu(n)/n \leq 1/2$, as required. □

3.6. Sylvester-Hadamard matrices. Before proving Theorem 3.3, we need to discuss the Sylvester-Hadamard matrices, which are defined as follows:

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and inductively H_{2^k} is given in block form by

$$H_{2^k} = \begin{pmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{pmatrix} .$$

Equivalently, H_{2^k} is the Kronecker product $H_2 \bigotimes H_{2^{k-1}}$.

Let h be the rescaled Haar function given by $h(t) = 1$ if $\lfloor t \rfloor$ is even and $h(t) = -1$ otherwise. Let $h_p(t) = h(2^{-p}t)$ for all $p \in \mathbb{N}$, so that each function h_p is periodic. It is straightforward to verify that for fixed $k \in \mathbb{N}$ and $1 \leq j \leq 2^k$, the jth column $(a_{i,j})_{i=1}^{2^k}$ of H_{2^k} is always given by a pointwise product of one or more of the column vectors $(h_p(i-1))_{i=1}^{2^k}$, $1 \leq p \leq k$, and that any such product gives some column of H_{2^k}. It follows that a pointwise product of any number of the columns of H_{2^k} is another column of H_{2^k}, a fact that will be useful in the following proof.

3.7. Proof of Theorem 3.3.

Proof. By Lemma D and the fact that $\mu(\cdot,\cdot)$ is nonincreasing in its second argument, we have that $\mu^*(2^{k+1}-1, 2^k) \geq \mu(2^{k+1}-1, m) \geq 2^k$. Conversely, by taking $n = 2^{j+1}-1$, $m = 2^j$, and $m' = 2$ in Lemma 3.2, we deduce inductively $\mu(2^{k+1}-1, 2^k) \leq 2^k$, and so $\mu(2^{k+1}-1, m) \leq 2^k$.

It remains to get the same upper bound for $\mu^*(2^{k+1}-1, 2^k)$. For this, we need to work a little harder. Fix k and let $n = 2^{k+1}-1$. We claim that if we delete the first row and column of the Sylvester-Hadamard matrix $H_{2^{k+1}}$, and change each 1 entry to a 0 and each -1 to a 1, then we get an $n \times n$ matrix $W = W_k$ over $\mathbb{F}_2$ such that each column of W has exactly 2^k ones, and such that the pointwise sum of any two columns of W is another column of W or is a column of zeros.

The fact that any pointwise product of columns of $H_{2^{k+1}}$ is another column of that same matrix means that any pointwise product of columns of $H_{2^{k+1}}$ has either zero or 2^k entries equal to -1. Pointwise products for $H_{2^{k+1}}$ correspond to pointwise sums mod 2 for W, so the claim is established.

It follows from the claim that for each $x \in \mathbb{F}_2^n$, the vector Wx is some column of W, so $|Wx| = 2^k$ or 0.

For $k \in \mathbb{N}$, the graph with matrix W_k does not have a loop at each vertex, i.e. it corresponds to an inadmissible wiring. But if V is any matrix all of whose columns are columns of W_k, and which has only 1's on the diagonal, then $V \in A^*(n, 2^k)$ and for each $x \in \mathbb{F}_2^n$ we have $|Vx| = 2^k$ or 0 (because $Vx = WPx$ for some projection P), so we deduce that $M(V, 0) = 2^k$. The simplest way to construct such a matrix V from W is to repeat columns 1,2,4 and so on, respectively, once, twice, four times, etc. In other words, take column i of V equal to column 2^j of V whenever $2^j \leq i < 2^{j+1}$. This concludes the proof. □

3.8. Towers V_k. The matrix $V = V_k$ in the foregoing proof has the property that the nonzero entries occur in blocks that are of the form $1_{r \times r}$, where r runs through powers of 2. Graphically, this wiring V corresponds to a tower of $k+1$ augmented complete graphs, one of degree equal to each power of 2, as illustrated (sideways on) in Figure 5. The corresponding matrix is

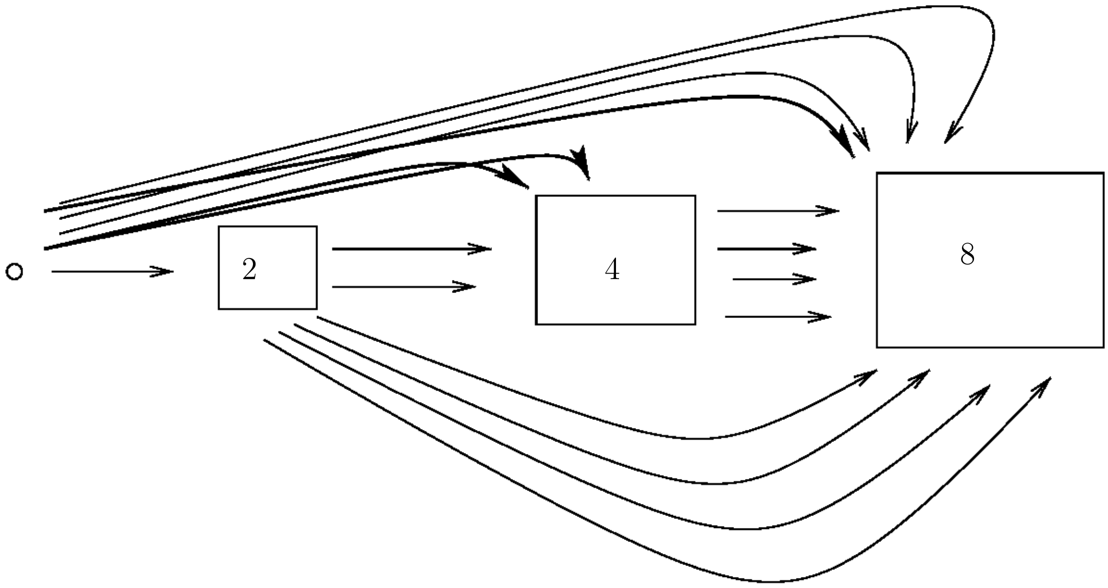


FIGURE 5. V_3

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
1&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
1&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&0&0&0&0&1&1&1&1&1&1&1&1\\
1&1&1&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}$$

The transition from W_k to V_k in the proof can be described by a sequence of pivots: A first pivot produces a forward-invariant $\hat{K}_{2^k}$, then a partial pivot with respect to the $\hat{K}_{2^k}$ produces a $\hat{H}_{2^{k-1}}$, and so on. The process converts a rather symmetrical inadmissible graph into an asymmetric admissible tower. Alternative constructions that amount to multiplying W_k by a permutation matrix convert the inadmissible graph to a symmetric admissible graph without a proper forward-invariant subgraph. Figure 6 shows an example, obtained by permuting the columns of V_2 to the order $(1, 2, 5, 6, 3, 4, 7)$ (As usual, the loops at the vertices are not shown.) This could be illustrated rather prettily on a regular tetrahedron by placing 1 at the apex, the $2, 4, 6$ as the vertices of the base triangle, and placing the remaining three points on the edges halfway up, with 5 on the edge $1-2$, 7 on $1-4$, and 3 on $1-6$. All the arrows can then be drawn on faces of the tetrahedron.

3.9. Remark. Note that there are Hadamard matrices H_{4n} of dimension $4n$ for many $n \in \mathbb{N}$, not just powers of 2; in fact, they are conjectured to exist for all dimensions $4n$ [10]. Since by definition the rows of an Hadamard matrix are pairwise orthogonal,

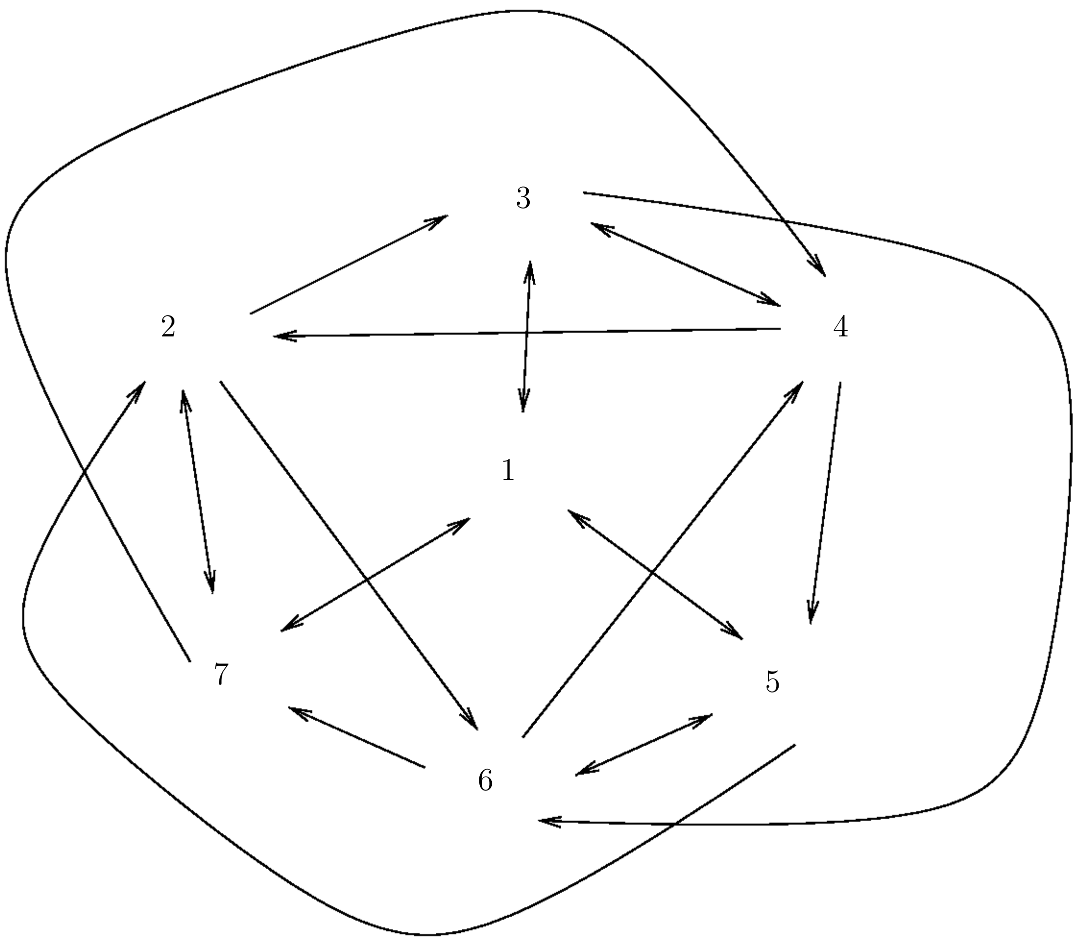


FIGURE 6. An alternative V for $k = 2$

one might wish to use H_{4n}^t as we used the symmetric Sylvester-Hadamard matrices. However, this is not possible for several reasons: we do not in general have a complete row and column of 1s suitable for deleting (although there is always an equivalent Hadamard matrix with this property), there may not be -1s along the diagonal of an associated minor, and some pointwise products of more than two columns of H_{4n} may have more than $2n$ entries equal to -1 (even if n is a power of 2). For instance, in the Paley-Hadamard matrix

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \end{pmatrix},$$

the pointwise product of columns 2, 3, and 5 contains all -1s, except from the first entry. For all these reasons, the method for Sylvester-Hadamard matrices does not in other cases produce a $W \in A(4n-1, 2n)$, let alone W such that $M(W, 0) = 2n$.

3.10. Codes. Hadamard matrices generate Hadamard codes, which have a certain optimality property. Recall that the code associated with H^{2^k} has 2^{k+1} codewords that make up a group $G < (\mathbb{F}_2^{2^k}, +)$. The above wiring $W \in A^*(2^k - 1, 2^{k-1})$ can by constructed from the code as follows. First, let $H < G$ be the order 2 subgroup generated by $(1, \dots, 1)^t \in G$, and select the element in each coset of H, other than H itself, that has a 0 in the first coordinate. Discarding the first coordinate of each selected codeword yields a set of projected codewords that give the columns of W.

It would be interesting to know if there are any further connections between optimal codes and optimal wirings. There is a reason to expect that (near-)optimal linear codes may be associated with (near-)optimal wirings: a near-optimal linear code is one in which the minimum over all codewords w of the Hamming distance $|w|$ is about as large as possible, so if we take many of these codewords as the columns of the wiring matrix (perhaps after discarding one or more coordinates, as we did for Hadamard codes), we get a matrix for which $|Wx|$ is fairly large, except for the relatively few times when $Wx = 0$. A relatively large minimum nonzero value for $|Wx|$ should therefore be associated with a relatively small maximum value for $|Wx|$, since Lemma D says that the average of $|Wx|$ over all $x \in \mathbb{F}_2^n$ is $n/2$.

4. An upper bound for $\mu(n, m)$

In this section, we establish an upper bound $U(n, m)$ for $\mu(n, m)$ in all cases. This upper bound seems rather sharp, insofar as we know of no values n, m for which $\mu(n, m)$ and $U(n, m)$ differ. We also investigate $U(n, m)$ and a related nondecreasing sequence $(a(n))_{n=1}^{\infty}$ which we use to define U.

4.1. The sequence $a(n)$. We first define $(a(n))$ by the following inductive process:

$$\begin{aligned} a(1) &= 1\,, \\ a(2^k - 1 + i) &= 2^{k-1} + a(i)\,, && 1 \le i \le 2^k - 1,\ k \in \mathbb{N}\,, \\ a(2^{k+1} - 1) &= 2^k\,, && k \in \mathbb{N}\,. \end{aligned}$$

Thus $(a(n))$ begins:

1, 2, 2, 3, 4, 4, 4, 5, 6, 6, 7, 8, 8, 8, 8, 9, 10, 10, 11, 12, 12, 12, 13, 14, 14, 14, 15, 16, 16, 16, 16, 16, 17, ...

It is not hard to verify that the above sequence has the following alternative description: it is the nondecreasing sequence consisting of all positive integers, where the frequency of each integer n is the 2-adic norm of $2n$.

Note that $a(n) \le 2^k$ whenever $n \le 2^{k+1} - 1$.

If we add an extra 1 term to the beginning of the sequence $(a(n))$, we get a sequence $(b(n))$ listed in the OEIS (Online Encyclopedia of Integer Sequences) as A046699 [1]. The sequence (b_n) is defined by the initial conditions $b(1) = b(2) = 1$, and the following recurrence relation:

$$b(n) = b(n - b(n-1)) + b(n - 1 - b(n-2))\,, \qquad n > 2\,.$$

It can be deduced from this that $(a(n))$ satisfies the same recurrence relation as $(b(n))$: we just need to modify the initial conditions. We leave the verification of this to the reader, with the hint that it is straightforward to deduce it from the two inequalities $a(n) > n/2$ and $a(n+1) \le a(n) + 1$.

Such so-called *meta-Fibonacci sequences* go back to D. Hofstadter [7, p. 137], and are generally considered to be rather mysterious. Indeed, one of them was the subject of a \$10 000 prize offered by the late J. Conway [4]. However $(a(n))$ and $(b(n))$ are clearly rather tame members of this family, and one or other has appeared elsewhere in the context of binary trees; see [9], [5], [3], and [6].

4.2. The function U. Having defined $(a(n))$, we are now ready to define our upper bound function U. Given positive integers n and m, we choose the nonnegative integer k for which $2^k \le m < 2^{k+1}$, and we let q and r be the integers with $n = (2^{k+1}-1)q + r$, with $q \ge 0$ and $1 \le r < 2^{k+1}$. Thus, q and r are the usual integral quotient and remainder when n is divided by $2^{k+1}-1$, except when the remainder is zero, and in that case $r = 2^{k+1}-1$ and $q = (n-r)/(2^{k+1}-1)$. We then define $U(n,m) = q2^k + a(r)$.

Note that given n and m, the integers k, r, q so defined are unique, and that $U(n,m) \ge a(n)$. Following our usual notation, we write $U(n) = U(n,n)$, so that $U(n)$ is just an alternative notation for $a(n)$.

Proposition 4.1. *We have $\mu(n,m) \le U(n,m)$ for all $n, m \in \mathbb{N}$.*

Proof. The result is trivially true when $m = 1$. We prove the result for $2^k \le m < 2^{k+1}$ by induction on k. Assuming $\mu(\cdot,m) \le U(\cdot,m)$ for $2^{k-1} \le m < 2^k$, we need to prove that this estimate also holds for $2^k \le m < 2^{k+1}$. From now on, we assume that $2^k \le m < 2^{k+1} - 1$.

Sublinearity of $\mu(\cdot,m)$ and the inductive hypothesis gives $\mu(n, 2^k - 1) \le 2^k$ for all $n < 2^{k+1} - 1$. Since any wiring with a vertex of degree at least 2^k allows us to light at least 2^k vertices, we must have

$$\mu(n,m) = \mu(n, 2^k - 1) \le U(n, 2^k - 1) = U(n,m)$$

for $n < 2^{k+1} - 1$. If $n = 2^{k+1} - 1$, then Theorem 3.3 yields

$$\mu(n,m) = \mu(n, 2^k) = 2^k = U(n,m)\,.$$

Finally, the required inequality follows readily for all $n \ge 2^{k+1}$ by using the case $n < 2^{k+1}$ and sublinearity of $\mu(\cdot,m)$. Thus, we have proven the inductive step, and we are done. ☐

4.3. Sublinearity. Equation (2.2.1) says that $\mu(\cdot,m)$ is sublinear for all m. We now prove the same for $U(\cdot,m)$

Theorem 4.2. *For all $n_1, n_2, m \in \mathbb{N}$, we have $U(n_1+n_2,m) \le U(n_1,m) + U(n_2,m)$.*

Proof. Since $U(\cdot,m)$ is unchanged as m varies over a dyadic block, it suffices to assume that $m = 2^k$ for some $k \ge 0$. We will show that $U(n,2^k) = \mu'(n,2^k)$, where $\mu'(n,2^k) = \mu_{A'(n,2^k)}$ and $A'(n,2^k)$ is an appropriate set of wiring matrices $W \in M(n,n;\mathbb{F}_2)$ that has the following closure property: if $W_i \in A'(n_i, 2^k)$ for $i = 1,2$, then the block diagonal matrix

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}$$

lies in $A'(n_1+n_2, 2^k)$. In terms of wirings, this just says that a disjoint union of two wirings in this class for given $m = 2^k$ also lies in this class for the same value of m. Sublinearity then follows easily from the definition of μ' and this closure property.

To define $A'(n,2^k)$, we first define W_j for each $j \ge 0$ to be some wiring in $A^*(2^{j+1} - 1, 2^j)$ such that $M(W,0) = \mu^*(2^{j+1}-1, 2^j) = 2^j$; this exists by Theorem 3.3. We now define $A'(n,2^k)$ to be the collection of matrices $W \in M(n,n;\mathbb{F}_2)$ that are of block diagonal form with $n_j \ge 0$ diagonal blocks of type W_j for some $0 \le j \le k$. Thus, $n = \sum_{j=0}^{k} n_j(2^{j+1}-1)$.

We first show that $\mu'(n,2^k) \le U(n,2^k)$. Taking $k = 0$, it is clear that $U(n,1) = \mu'(n,1) = n$; note that the identity matrix is the only element of $A'(n,1)$. We therefore assume that $k > 0$.

If $n < 2^{k+1}-1$, then $U(n,2^k) = U(n,2^{k-1})$, so by choosing $W \in A'(n,2^{k-1})$ satisfying $M(W,0) = U(n,2^{k-1})$, we get

$$\mu'(n,2^k) \le \mu'(n,2^{k-1}) \le U(n,2^{k-1}) = U(n,2^k)\,.$$

If $n = 2^{k+1} - 1$, then $\mu'(n, 2^k) \le M(W_k, 0) = 2^k = U(n, 2^k)$.

Finally if $n > 2^{k+1} - 1$, then we simply write $n = q(2^{k+1} - 1) + r$ in the usual way and select a wiring W that decomposes into q copies of W_k plus one copy of a wiring $W' \in A'(r, 2^k)$ satisfying $M(W', 0) = U(r, 2^k)$ to deduce that $\mu'(n, 2^k) \le q2^k + U(r, 2^k) = U(n, 2^k)$, as required.

We now prove the opposite inequality by induction. As mentioned above, the case $k = 0$ is clear. Suppose that $\mu'(\cdot, 2^j) = U(\cdot, 2^j)$ for all $0 \le j < k$, and suppose that $\mu'(n', 2^k) = U(n', 2^k)$ for all $n' < n$, $n' \in \mathbb{N}$. Let $W \in A'(n, 2^k)$ be such that $M(W, 0) < U(n, 2^k)$. Since $U(n, 2^k) \le U(n, 2^{k-1}) = \mu'(n, 2^{k-1})$, W must have a vertex of degree 2^k which is part of a W_k. Let W' be the subwiring obtained from W by removing this W_k, and so $W' \in A'(n - 2^k, 2^k)$. Now $M(W_k, 0) = 2^k$ and by minimality of n, we have $M(W', 0) \ge U(n - 2^k, 2^k)$, so $M(W, 0) \ge U(n - 2^k, 2^k) + 2^k \ge U(n, 2^k)$, as required. □

Since $U(n, m) = a(n)$ whenever there exists $k \in \mathbb{N}$ such that $n/2 < 2^k \le m$, it follows that $(a(n))$ is also sublinear, a fact we now record.

Corollary 4.3. *For all $n_1, n_2 \in \mathbb{N}$, we have $a(n_1 + n_2) \le a(n_1) + a(n_2)$.*

5. Results for m near a power of 2

5.1. In this section, we give some results for $2^k - 2 \le m \le 2^k + 1$. Our first result follows rather easily from Theorem 3.3.

Proposition 5.1. *For all $2 \le k \in \mathbb{N}$, we have $\mu(\cdot, 2^k - 1) = \mu(\cdot, 2^k - 2)$.*

Proof. Let $m := 2^k - 1$. By Proposition 4.1, we have $\mu(n, m - 1) \le 2^{k-1} + 1 \le m$ for all $n \le 2^k$. It follows that $\mu(n, m) = \mu(n, m - 1)$ for $n \le 2^k$, since any wiring W with a degree m vertex satisfies $M(W, 0) \ge m$.

Suppose inductively that $\mu(n', m) = \mu(n', m-1)$ for all $1 \le n' < n$, where $n > 2^k$, and we wish to extend this equation to $n' = n$. By Lemma E, either $\mu(n, m) = \mu(n, m - 1)$ and we have established the inductive step, or

$$\mu(n, m) \ge \mu(n - m, m) + 2^{k-1} = \mu(n - m, m - 1) + 2^{k-1}$$

and so

$$\begin{aligned}
& \mu(n-m, m-1) + 2^{k-1} \le \mu(n, m) \\
\text{(trivial estimate)} \quad & \le \mu(n, m-1) \\
\text{(sublinearity)} \quad & \le \mu(n-m, m-1) + \mu(m, m-1) \\
\text{(trivial estimate)} \quad & \le \mu(n-m, m-1) + \mu(m, 2^{k-1}) \\
\text{(by Theorem 3.3)} \quad & = \mu(n-m, m-1) + 2^{k-1}\,.
\end{aligned}$$

The inductive step, and so the lemma, follows from equality of the first and last lines. □

Theorem 5.2. *Let $m = 2^k$ for some $k \in \mathbb{N}$, and suppose that $\mu(\cdot, m-1) = U(\cdot, m-1)$. Then $\mu(\cdot, p) = U(\cdot, p)$ also holds for $p = m$ and $p = m + 1$. In particular, $\mu(\cdot, m) = \mu(\cdot, m + 1)$.*

Proof. Proposition 4.1 tells us that $\mu(\cdot, \cdot) \le U(\cdot, \cdot)$, so we must prove inequalities in the opposite direction. For $k = 1$, the desired conclusion follows from Theorems A and B, so we assume that $k > 1$.

We first consider the case $p = m$. Suppose for the sake of contradiction that $m = 2^k > 2$ is such that $\mu(\cdot, m) \ne U(\cdot, m)$, even though $\mu(\cdot, m - 1) = U(\cdot, m - 1)$. Also for the sake of contradiction, assume that $n \in \mathbb{N}$ is the smallest number such that $\mu(n, m) < U(n, m)$, and that $W \in A(n, m)$ is such that $M(W, 0) < U(n, m)$. Since

$\mu(n,m-1) = U(n,m-1) \geq U(n,m)$, it follows that W must contain a vertex of degree m. This certainly implies that $M(W,0) \geq m = U(2m-1,m)$, so $n \geq 2m$.

We write $n = q(2m-1)+r$, where $q,r \in \mathbb{N}$ and $r < 2m$. By induction we have $M(W,0) < qm + \mu(r,m)$. We may also assume that we cannot increase the number of F_m subgraphs in W by any amount of pivoting; recall that an F_m is a forward invariant augmented complete subgraph on m vertices.

We now carry out what for later reference we call a *Partition by Degree argument*: we partition the set of n vertices into subsets A and B, where A consists of all vertices that lie in an F_m, and B consists of all other vertices. Let us write n_A, n_B for the cardinalities of A and B, respectively.

Since we cannot increase the number of F_m subgraphs by pivoting, we have $W_B \in A(n_B, m-1)$. Since we can light all vertices in A by pressing one vertex in every F_m, we must have

$$n_A < U(n,m) = qm + \mu(r,m) \leq (q+1)m\,.$$

But n_A is a multiple of m, so $n_A \leq qm$. Alternatively, we can first light at least $\mu(n_B, m-1)$ of the B-vertices followed by at least $\nu(n_B,m) \geq n_A/2$ of the A-vertices, and so

$$\mu(n_B, m-1) + n_A/2 < qm + \mu(r,m)\,. \tag{5.1.1}$$

Suppose $n_A = qm$, and so $n_B = n - qm = q(m-1)+r$. By assumption, $\mu(n_B, m-1) = qm/2 + \mu(r, m-1)$, and so

$$\mu(n_B, m-1) + n_A/2 = qm + \mu(r,m-1) \geq qm + \mu(r,m)\,,$$

contradicting (5.1.1). If n_A is smaller than qm, it must be smaller by $q'm$ for some $q' \in \mathbb{N}$, thus increasing $\mu(n_B, m-1)$ by at least $q'm/2$:

$$\mu(n-qm-q'm, m-1) \geq \mu(n-qm-q'(m-1), m-1) = \mu(n-qm, m-1) + \frac{q'm}{2}\,.$$

Thus, $\mu(n_B, m-1) + n_A/2$ is at least as large as in the case $n_A = qm$, and we still get a contradiction.

We next prove that $\mu(n, m+1) = \mu(n,m)$. Again for the sake of contradiction, we suppose that $m = 2^k > 2$ is such that $\mu(\cdot, m+1) \neq U(\cdot, m+1)$, even though $\mu(\cdot, p) = U(\cdot, p)$ when $p = m-1$. This last equation holds also for $p = m$ by the first part of the proof. Note that $U(n,m+1) = U(n,m) = \mu(n,m)$.

Suppose also for the sake of contradiction that n is minimal for the inequality

$$\mu(n, m+1) < U(n, m+1) = \mu(n,m)\,.$$

Now, $U(n, m+1) \leq m+1$ for $n \leq 2m$, so as in the first part of the proof, we must have $n > 2m$. We again write $n = q(2m-1)+r$, where $q,r \in \mathbb{N}$ and $r < 2m$. Let $W \in A(n, m+1)$ be such that $M(W,0) = \mu(n, m+1)$, and we assume that the number of F_{m+1}s cannot be increased by pivoting, and that the only possible pivoting operations that may increase the number of $\hat{K}_m$ subgraphs are those that decrease the number of F_{m+1} subgraphs; recall that a $\hat{K}_m$ is an augmented complete subgraph on m vertices (which is not necessarily forward invariant).

We carry out another Partition by Degree argument, with A consisting of all vertices that lie in a $\hat{K}_m$ or an F_{m+1}, and $W_B \in A(n_B, m-1)$. Now, W must be a vertex of degree $m+1$, since $M(W,0) < \mu(n,m)$, and so W contains at least one F_{m+1}. Suppose that there are at least two F_{m+1}s. We can light at least $\mu(n-2m-2, m+1)$ of the other vertices, followed by at least $2\nu(m+1) = m+2$ of the vertices in the pair of

F_{m+1}s. Now

$$
\begin{aligned}
\text{(minimality of } n) \quad \mu(n-2m-2,m+1)+m+2 &= U(n-2m-2,m+1)+m+2 \\
&= U(n-3,m)+2 \\
&= \mu(n-3,m)+\mu(3,m) \\
\text{(sublinearity)} \quad &\geq \mu(n,m)\,,
\end{aligned}
$$

contradicting the fact that $\mu(n,m+1) < \mu(n,m)$.

Thus, there is precisely one F_{m+1}, and n_A is equivalent to 1 mod m. We distinguish between those $\hat{K}_m$s that are forward invariant, which we denote as usual by F_m, and those that are not, which we denote by N_m. The one external link of each N_m is to the F_{m+1}, since otherwise we could pivot to get a second F_{m+1}. Furthermore, any two N_ms must link to the same vertex in the F_{m+1}, since if this were not the case, we could pivot about a vertex in one N_m to get a wiring with one N_m linked to a second N_m, which in turn links to a F_{m+1}, and such a configuration would allow us to get a second F_{m+1} by pivoting about the vertex in the first N_m.

It follows that we can light all except possibly one of the vertices in A, and so $n_A-1 < qm+\mu(r,m-1) \leq (q+1)m$, which self-improves to $n_A \leq qm+1$. Alternatively, as in the first part of the proof, we get

$$\mu(n_B,m-1)+(n_A+1)/2 < qm+\mu(r,m-1)\,. \tag{5.1.2}$$

Suppose $n_A = qm+1$, and so $n_B = q(m-1)+r-1$. By the inductive hypothesis, $\mu(n_B,m-1) = qm/2+\mu(r-1,m-1)$, and so by Lemma F,

$$\mu(n_B,m-1)+(n_A+1)/2 = qm+\mu(r-1,m-1)+1 \geq qm+\mu(r,m) = qm+\mu(r,m-1),$$

contradicting (5.1.2). The case where n_A is smaller than qm is ruled out as in the first part of the proof. □

5.2. Partition by degree arguments. Since we will be seeing other variations of the above *Partition by Degree arguments*, let us describe the common features of these arguments, so that we can be sketchy in all subsequent uses of it. Given a wiring W on n vertices, we partition the set of vertices into two subsets, typically called A and B, and we denote the cardinality of A and B by n_A and n_B, respectively. The wiring will be initially pivoted so that A is forward invariant and W_A will consist only of $\hat{K}_j$s for various $j \geq 2^k$. There will be very few links between different $\hat{K}_j$s in A, allowing us to light almost all except at most n_0 of the vertices in A by pressing one vertex in each $\hat{K}_j$; for instance, n_0 was either 0 or 1 in the two Partition by Degree arguments in the above proof. This gives the bound $n_A \leq K-n_0$, where K equals either $\mu(n,m)$ or an assumed value of $\mu(n,m)$ from which we wish to derive a contradiction. By the structure of A, we often know that n_A has a certain value mod 2^k, allowing us to improve the estimate $n_A \leq K-n_0$ to $n_A \leq n_1$ for some $n_1 \leq K-n_0$.

By somehow maximizing the number of $\hat{K}_j$s in A, we arrange for the restricted wiring W_B to lie in $A(n_B,m')$ for some $m' \leq 2^k-1$, so we may light at least $\mu(n_B,m')$ of these vertices followed by at least $\nu(n_A)$ vertices in A. This gives the inequality

$$\mu(n_B,m')+\nu(n_A) \leq K \tag{5.2.1}$$

The aim of the Partition by Degree argument is now either to derive a contradiction, or to show that $n_A = n_1$. To do this, we first consider the possibility that $n_A = n_1$, and we typically deduce that $\mu(n-n_1,m')+\nu(n_1)$ either equals or exceeds K. If instead we allow n_A to decrease below n_1, then n_A typically must be decreased by a multiple of 2^k, and $\mu(n_B,m')$ increases by at least as much as $\mu(n_A)$ decreases. Thus, if $\mu(n-n_1,m')+\nu(K_1) > K$, we get a contradiction to (5.2.1) also for any value of n_A less than n_1, and we are done. In other instances of this argument, $\mu(n-n_1,m')+\nu(K_1) =$

K, but taking a value of n_A smaller than K_1 increases $\mu(n_B, m')$ strictly more than $\mu(n_A)$ decreases, so we conclude that n_A must equal n_1 and $n_B = n - n_1$, as we are seeking to prove in such instances.

5.3. A technical lemma. We now give a lemma which makes no mentions of wirings and vertices but which we will need later. In this lemma, $|u|$ denotes the Hamming norm of a vector $u \in \mathbb{F}_2^N$, as defined in Section 1.

Lemma 5.3. *Let n, N and M be positive integers. Then the following are equivalent:*
(1) There exist vectors $a_j = (a_{i,j})_{i=1}^N \in \mathbb{F}_2^N$, $1 \le j \le n$ such that

$$\left|\sum_{j=1}^{n} \lambda_j a_j\right| = M \in \mathbb{N}, \qquad \text{for all } \lambda = (\lambda_j) \in F_n := \mathbb{F}_2^n \setminus \{0\}. \tag{5.3.1}$$

(2) $M = 2^{n-1}q$ for some $q \in \mathbb{N}$, and $N \ge 2M - 2^{1-n}M$.
Assuming these conditions are fulfilled, all solutions $(a_{i,j})$ to (5.3.1) are equivalent modulo permutations of the i and j indices.

Proof. Assuming the conditions (2) are fulfilled, with $M = 2^{n-1}q$, we see that $N \ge (2^n - 1)q$, so we can allocate $(2^n - 1)q$ vertices into $2^n - 1$ pairwise disjoint sets of q vertices each. We label these sets S_k for $1 \le k \le 2^n - 1$, and write $S = \bigcup_{i=1}^{2^n-1} S_k$. Writing $d_{n-1;k} \dots d_{1;k} d_{0;k}$ for the binary expansion of $1 \le k \le 2^n - 1$, we let $a_{i,j} := d_{j-1;k}$ for all $i \in S_k$, and $a_{i,j} = 0$ of $i \notin S$. It is readily verified that (5.3.1) holds with this choice of $(a_{i,j})$.

Conversely, suppose that $A := (a_{i,j})$ satisfy (5.3.1). Note that this condition implies the same condition with n replaced by any number $1 \le n' \le n$, and if we take $n' = n-1$, we can replace a_{n-1} by either a_n or $a_{n-1} + a_n$ and the condition remains true. For each $u = (u_j) \in F_n$, we write $S_n(u; A)$ for the set of indices $1 \le i \le n$ such that $a_{i,j} = u_j$ for all j. Trivially, such sets $S_n(u; A)$ are pairwise disjoint. Writing $\#(\cdot)$ for set cardinality, we claim that $\#(S_n(u; A)) = 2^{1-n}M$. Since $\#(F_n) = 2^n - 1$, it follows from the claim that $N \ge 2M - 2^{1-n}M$. Also, the fact that $\#(S_n(u; A))$ is independent of $u \in F_n$ means that there is essentially only one such solution, modulo permutations of the indices, so the result follows from the claim. We prove this by induction on n.

If $n = 1$, the claim is trivial. For $n = 2$, note that $|a_1 + a_2| = |a_1| + |a_2| - 2K = 2M - 2K$, where K is the number of indices i for which $a_{i,1} = a_{i,2} = 1$. Since $2M - 2K = M$, we must have $K = M/2$. This readily implies the result for $n = 2$.

Suppose inductively that the result is true for $n < m$, where $m > 2$, and we want to prove it for $n = m$. Let us define the following matrices

$$A_1 = (a_{i,j})_{\substack{1 \le i \le N \\ 1 \le j \le m-2}}, \quad A_2 = (a_{i,j})_{\substack{1 \le i \le N \\ 1 \le j \le m-1}}, \quad A_3 = (b_{i,j})_{\substack{1 \le i \le N \\ 1 \le j \le m-1}}, \quad A_4 = (c_{i,j})_{\substack{1 \le i \le N \\ 1 \le j \le m-1}},$$

where

$$b_{i,j} = \begin{cases} a_{i,j}, & j \le m-2, \\ a_{i,m}, & j = m-1, \end{cases}$$

$$c_{i,j} = \begin{cases} a_{i,j}, & j \le m-2, \\ a_{i,m-1} + a_{i,m}, & j = m-1, \end{cases}$$

We assume that $A := (a_{i,j})_{\substack{1 \le i \le N \\ 1 \le j \le m}}$ satisfies (5.3.1) for $n = m$, so certainly A_s satisfies (5.3.1) for $1 \le s \le 4$ (for $n = m-2$ or $n = m-1$).

By the inductive assumption $\#(S_{m-1}(u; A_s)) = 2^{2-m}M$ for each $u \in F_{m-1}$ and $2 \le s \le 4$. Considering separately those $u = (u', u_{m-1}, u_m) \in F_{m-2} \times F_1 \times F_1 = F_m$

such that $u' \neq 0$ and $u' = 0$, we can in both cases argue as for $n = 2$ above that $\#(S_m(u)) = 2^{1-n}M$, as required. □

5.4. Towers again. We return to the special wirings V_k discussed in Subsection 3.8. For the rest of this section, a $\overline{K}_i$ will mean a $\hat{K}_{2^i}$, i.e. an augmented complete graph on 2^i vertices. The wiring $V_k \in A^*(2^{k+1}-1, 2^k)$ consists of augmented complete subgraphs $\overline{K}_0$, $\overline{K}_1$, $\overline{K}_2$,. . .,$\overline{K}_k$, such that each vertex of each $\overline{K}_i$ toggles zero vertices in $\overline{K}_j$ for $j < i$ and toggles 2^{j-1} vertices in each $\overline{K}_j$, $i < j \le k$. In addition the set of vertices toggled in $\overline{K}_j$ by any $\overline{K}_i$ vertex for $i < j$ is independent of which vertex in $\overline{K}_i$ is chosen, so we may as well restrict ourselves to considering vertex press sets where we are allowed to press only one vertex, which we call the *designated vertex*, in each $\overline{K}_i$. We say that $\overline{K}_i$ is *activated* if its designated vertex is pressed. Thus, each $\overline{K}_i$ can be viewed as a single switch which toggles 2^{j-1} indices in $\overline{K}_j$ for each $j > i$. We assume that this wiring is arranged so that activating one or more of the $\overline{K}_i$, $i < j$, always lights exactly 2^{j-1} of the vertices in $\overline{K}_j$. This is possible by Lemma 5.3. In view of the uniqueness in Lemma 5.3, this defines the wiring V_k uniquely up to relabeling of the vertices within each $\overline{K}_j$, $1 \le j \le k$.

The next theorem generalises our remark that the Sylvester-Hadamard wiring W_k can be pivoted to obtain V_k. In this theorem, we push further with the ideas in the proof of Theorem 5.2 to show that if $\mu(\cdot, p) = U(\cdot, p)$ for $p < 2^k$ then, modulo (full and partial) pivoting, there really is only one optimal wiring in $A(n, 2^k)$ for each $n = q(2^{k+1} - 1)$, $q \in \mathbb{N}$, namely q disjoint copies of V_k.

Theorem 5.4. *Suppose that* $\mu(\cdot, p) = U(\cdot, p)$ *for all* $p < m := 2^k$*, for some* $k \in \mathbb{N}$*. If* $n = q(2m - 1)$ *for some* $q \in \mathbb{N}$*, and if* $W \in A(n, m)$ *is such that* $M(W, 0) = \mu(n, m)$*, then* W *can be pivoted to the block diagonal wiring* $\operatorname{diag}(V_k, \ldots, V_k)$*, where* $V_k \in A^*(2m-1, m)$ *is as above; both full and partial pivoting operations may be required.*

Proof. We will construct a chain of restricted wirings, so let us write W_k in place of W for our initial wiring, and we also write N_k in place of n. We assume without loss of generality that W_k has the property that no additional $\overline{K}_k$ subgraphs can be obtained by pivoting. We denote by B_k the set of all N_k vertices.

We do a Partition by Degree argument, partitioning the N_k vertices into two sets: A_k, of cardinality n_k, contains all vertices in any $\overline{K}_k$, and B_{k-1}, of cardinality N_{k-1}, contains all the other vertices. We denote by W_{k-1} the wiring W_k restricted to B_{k-1}. Then, $W_{k-1} \in A(N_{k-1}, 2^k-1)$, since otherwise we could create an extra $\overline{K}_k$ by pivoting.

As usual, we have $n_k \le \mu(n, 2^k) = q2^k$ and

$$\mu(N_{k-1}, 2^k - 1) + \frac{n_k}{2} \le \mu(n, 2^k) = q2^k\,. \tag{5.4.1}$$

The first inequality forces $n_k \le q2^k$, so $N_{k-1} \ge q(2^k - 1)$. If $N_{k-1} = q(2^k - 1)$, we get equality in (5.4.1), but this inequality cannot hold if $n_k < q2^k$, since it would force the inequality $U(i2^k, 2^{k-1}) \le i2^{k-1}$ for some $i \in \mathbb{N}$, which itself can be reduced to $U(i, 2^{k-1}) \le 0$, $i \in \mathbb{N}$, which we know to be false. Thus, the only possible value for (n_k, N_{k-1}) is $(q2^k, q(2^k - 1))$.

The fact that this choice of (n_k, N_{k-1}) only satisfies (5.4.1) with equality means that we can analyze the wiring more closely and rule out any wiring that creates any slippage in the left-hand side bounds. In particular, if there were a vertex in W_{k-1} of degree $j > 2^{k-1}$, we could pivot about it relative to A_k to get a $\hat{K}_j$. Vertices in the $\hat{K}_j$ have at most $2^k - j$ links outside the $\hat{K}_j$, which must all be in A_k because of the pivoting process). By pressing a $\hat{K}_j$ vertex and then one vertex in every $\overline{K}_k$, we light all the vertices in the $\hat{K}_j$ and all except at most $2^k - j$ vertices in A_k, thus giving a contradiction since $q2^k - (2^k - j) + j > q2^k$. A $\overline{K}_{k-1}$ also leads to a contradiction unless its vertices link to exactly 2^{k-1} vertices in A_k.

Thus, $W_{k-1} \in A(q(2^k-1), 2^{k-1})$ and (5.4.1) forces $M(W_{k-1}, 0) = \mu(q(2^k-1), 2^{k-1})$. Thus, W_{k-1} satisfies assumptions similar to those of W_k, but with k replaced by $k-1$. We can continue this process, creating a chain of restricted wirings W_j and associated partition sets A_j consisting of the vertices in q copies of $\overline{K}_j$ and B_j of cardinality $N_{j-1} = q(2^j - 1)$ such that $W_{j-1} := W_{B_{j_1}} \in A(N_{j-1}, 2^{j-1})$, for $j = 0, \dots, k$.

Since all $\overline{K}_j$s are obtained by (partial or full) pivoting, all vertices in any one $\overline{K}_j$ link to the same set of vertices. As in the discussion of V_k before this theorem, we may as well restrict to vertex press sets where we are only allowed to press a single *designated vertex* in each $\overline{K}_j$, and we say that $\overline{K}_j$ is *activated* if its designated vertex is pressed. We also talk about a $\overline{K}_j$ being *switched* if its designated vertex is one of the vertices given by a perturbation y of an existing vertex press set x, thus yielding a vertex press set $x + y$.

For $1 \le j \le k$, we know that the designated vertex in any one $\overline{K}_{j-1}$ links to 2^{j-1} vertices in A_j. By activating every $\overline{K}_{j-1}$, and then activating any $\overline{K}_j$s in which fewer than 2^{j-1} vertices are lit, we could light strictly more than $q2^j = \mu(N_j, 2^j)$ vertices in A_j if there were at least one $\overline{K}_j$ that had either strictly more, or strictly less, than 2^{j-1} lit vertices after every $\overline{K}_{j-1}$ had been activated. It follows that the links from any two different K_{j-1}s must be to distinct sets of vertices in A_j, and that these links must be evenly distributed, in the sense that there must be 2^{j-1} of them in each $\overline{K}_j$.

Suppose now that $1 < j \le k$. By activating every $\overline{K}_{j-2}$ and every $\overline{K}_j$, we light $q2^{j-2}$ vertices in A_{j-2} and the same number in A_{j-1}, and arguing as above we see that there must be exactly 2^{j-1} vertices lit in each $\overline{K}_j$. We can continue this argument to deduce inductively that any one vertex in $A_{j'}$ is linked to exactly 2^{j-1} vertices in A_j if $j \ge j'$, and to no vertices in A_j if $j < j'$. Furthermore there are links to 2^{j-1} vertices in any given $\overline{K}_j$ from designated $\overline{K}_{j'}$ vertices whenever $j' < j$.

We next prove that all of these $\overline{K}_j$s are arranged in V_ks. This is trivial if $k = 1$, since each $\overline{K}_0$ has only one link to A_1, and each $\overline{K}_1$ has a link from one of the $\overline{K}_0$s. Suppose inductively that all the A_js for $j \le k-1$ are arranged into q copies of V_{k-1}. We wish to prove the same with $k-1$ replaced by k.

We fix one particular $\overline{K}_{k-1}$, which we call L_{k-1} and, for each $0 \le j < k-1$, denote by L_j the copy of $\overline{K}_j$ that is linked to L_{k-1}. The sets L_j, $0 \le j \le k-1$ lie in a particular copy of V_{k-1} that we will call U_{k-1}. For each $0 \le j < k-1$, let x_j be the vertex press sets where we activate every $\overline{K}_{k-1}$ other than L_{k-1}, and we also activate L_j. By the properties of the V_{k-1}, this results in having 2^{k-1} vertices lit in each V_{k-1}, and some vertices in A_k are also lit as a result of the 2^{k-1} links from each $\overline{K}_{k-1}$ and from L_j into A_k. By then activating any $\overline{K}_k$ where fewer than half of the vertices are lit, we get at least $q(2^{k-1} + 2^{k-1}) = q2^k$ vertices lit in B_k, the maximum amount allowed.

But we would get strictly more than this if the vertex press set x_j resulted in any number of lit vertices other than 2^{k-1} in any A_k. Thus, x_j must result in $q2^{k-1}$ lit vertices in A_k, with exactly 2^{k-1} of these in each $\overline{K}_k$. But there are only 2^{k-1} links from each $\overline{K}_k$ or from L_j to A_k, so it must be that no two of these links are to the same vertex in A_k, since otherwise there would be fewer than $q2^{k-1}$ vertices lit in A_k as a result of x_j.

Consider more generally a vertex press set x where we press the designated vertex in $\overline{K}_{k-1}$ in all cases except L_{k-1}, and we also press the designated vertex in one or more of the sets L_j, $0 \le j \le k-1$. For all such x, we get 2^{k-1} lit vertices in each V_{k-1}, so again we must have $q2^{k-1}$ lit vertices in A_k, with exactly 2^{k-1} of these in each $\overline{K}_k$. Since we have seen that the links to A_k from U_{k-1} are disjoint from the links to A_k from every

$\overline{K}_{k-1}$ other than L_{k-1}, it follows that any nontrivial combination of activations of the sets L_j, $0 \le j \le k-1$ toggles the same number of vertices in each $\overline{K}_k$ and 2^{k-1} such vertices across the union all all $\overline{K}_k$s.

Denoting by L_k some particular $\overline{K}_k$ where nontrivial combination of activations of the sets L_j toggle at least one vertex, we assume the number of such toggles is M. Viewing our designated vertices in L_j as switches for $0 \le j \le k-1$, we now apply Lemma 5.3 and the fact that $1 \le M \le 2^{k-1}$ to deduce that $M = 2^{k-1}$. This uses up all the available links from U_{k-1} to A_k. Now U_{k-1} is a fixed but arbitrary V_{k-1}, so it follows that each V_{k-1} is linked only to a single $\overline{K}_k$, and so our full wiring consists of q copies of V_k, as required. □

6. The cases $m = 4, 5$

6.1. In this section, we prove Theorem 1.1. Throughout, an n-optimal wiring is a wiring $W \in A(n,m)$ for which $M(W,0) = \mu(n,m)$; the parameter m is in all such cases understood.

Proof of Theorem 1.1. Part (a) can be restated as $\mu(\cdot,p) = U(\cdot,p)$ for $p = 4,5$. It is readily verified from Theorem B(a) that $\mu(\cdot,p) = U(\cdot,p)$ when $p = 3$, so it extends to $p = 4,5$ by Theorem 5.2.

We now prove Part (b). The desired formula for $\mu^*(n,4) - 4k$, $n = 7k + i \ge 4$, is given by a_i in the following table:

i	1	2	3	4	5	6	7
a_i	2	2	2	4	4	4	4

It is readily verified that a_i equals the least even integer not less than $\mu(7k+i,4)-4k$. Since pressing any vertex for a wiring in $A^*(n,4)$ preserves the parity of the number of lit vertices, $\mu(7k+i,4)$ must be even. Thus $\mu^*(n,4) \ge 4k + a_i$.

We now prove the converse by induction. The nontrivial part is to prove it for $4 \le n \le 10$. Once this is proved, it follows inductively for all $n = 7k + i > 10$ using (2.2.1):

$$\mu^*(7k+i,4) \le \mu^*(7(k-1)+i,4) + \mu^*(7,4) \le (4(k-1)+a_i) + 4 = 4k + a_i\,.$$

It remains to prove that $\mu^*(n,4) \le 4k + a_i$ when $4 \le n \le 10$. Trivially $\mu^*(4,4) = 4$ and

$$\begin{aligned}
&\mu^*(5,4) \le \mu^*(4,3) + 1 = 4 && \text{(Lemma F)}\,,\\
&\mu^*(6,4) \le 2\mu^*(3,2) = 4 && \text{(Lemma 3.1)}\,,\\
&\mu^*(7,4) = 4 && \text{(Theorem 3.3)}\,,\\
&\mu^*(8,4) \le \mu^*(7,3) + 1 = 6 && \text{(Lemma F)}\,,\\
&\mu^*(9,4) \le \mu^*(8,3) + 1 = 7 && \text{(Lemma F)}\,.
\end{aligned}$$

All except the last of these is sharp, and parity considerations allow us to improve the last one to the sharp $\mu^*(9,4) \le 6$.

Finally, $\mu^*(10,4) \le 6$ follows by consideration of the wiring

$$W_{10} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 1&0&0&1&1&1&0&0&0&0\\ 0&1&1&1&1&1&0&0&0&0\\ 0&0&0&1&1&1&0&0&0&0\\ 1&0&0&0&0&0&1&1&1&1\\ 0&1&1&0&0&0&1&1&1&1\\ 0&0&0&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1 \end{pmatrix}$$

(cf. Figure 7 All columns except columns 1, 2, 4, and 7 are duplicates of these columns,

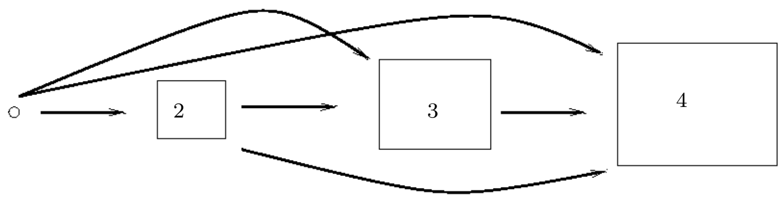


FIGURE 7. W_{10}

so we can restrict ourselves to sets of vertex presses involving only these four vertices. With this restriction, we can proceed to list all sixteen possible values of x, and deduce that $M(W_{10}, 0) = 6$. □

6.2. Let us mention an alternative, more instructive, way of proving that $M(W_{10}, 0) \le 6$. Again, we may restrict ourselves to pressing only some combination of vertices 1,2,4, and 7. Note first that W_{10} consists of one copy each of a $\hat{K}_1$, $\hat{K}_2$, $\hat{K}_3$, and $\hat{K}_4$ (vertices 1, 2–3, 4–6, and 7–10, respectively), and $\hat{K}_i$ is connected to $\hat{K}_j$ only if $i < j$. The subwiring for vertices 1–3 is such that we can never light all three vertices (by parity, since all vertices have degree 2), and to get only one unlit vertex, we must press vertex 1 and/or vertex 2. But all three of these possibilities throws both the $\hat{K}_3$ and $\hat{K}_4$ out of sync since the links from vertices 1 and 2 into the $\hat{K}_3$ are different from each other, and similarly for the links into the $\hat{K}_4$. Furthermore, the $\hat{K}_3$ and $\hat{K}_4$ vertices remain out of sync regardless of whether we press vertices 4, 7, or both. Thus, the unlit vertices always include either all of 1–3, or at least one vertex each from 1–3, 4–6, and 7–10. Thus, $M(W_{10}, 0) \le 7$ and parity considerations improve this to $M(W_{10}, 0) \le 6$.

6.3. Questions. So far, we know this:

Theorem 6.1. *For $m \in \mathbb{N}$, $m \le 5$, we have $\mu(\cdot, m) = U(\cdot, m)$.*

This naturally prompts the following question, which we cannot answer.

Question 6.2. *Is it true that $\mu(\cdot, m) = U(\cdot, m)$ when $m > 5$?*

Theorem 3.3 states that if m is a power of 2, then there exists wirings for $(n, m) = (2m - 1, m)$ that are optimal in the sense that $\mu^*(n, m)$ and $\mu(n, m)$ both equal to $(n + 1)/2$ (the smallest possible value for $\mu(n, m)$ according to Lemma D). This is evidence that powers of 2 are significant boundaries for the behavior of $m \mapsto \mu(\cdot, m)$ and $m \mapsto \mu^*(\cdot, m)$. This fact motivates the following pair of open questions with which we close the article. Note that the first one is simply a weaker version of Question 6.2.

Question 6.3. *Is it true that $\mu(n,m)$ is independent of m for all $2^k \le m \le 2^{k+1}-1$, $k \in \mathbb{N}$?*

The answer to the above question is affirmative if we restrict to $m \le 5$.

Question 6.4. *Is it true that $\mu(n,m_1) - \mu^*(n,m_2)$ is bounded independent of $n,k \in \mathbb{N}$ for all $2^k \le m_1, m_2 \le 2^{k+1}-1$, $k \in \mathbb{N}$?*

The answer to this last question is affirmative if we restrict to $m_1, m_2 \le 4$.

References

[1] A046699, The Online Encyclopedia of Integer Sequences, http://oeis.org/A046699.

[2] S.M. Buckley and A.G. O'Farrell, *Wiring switches to light bulbs.* Bulletin IMS 94 (2025) 69–88.

[3] B.W. Conolly, *Meta-Fibonacci sequences*, in S. Vajda, ed., Fibonacci and Lucas Numbers and the Golden Section, Wiley, New York, 1986, pp. 127–137.

[4] J. Conway, *Some Crazy Sequences*, Lecture at AT&T Bell Labs, July 15, 1988.

[5] C. Deugau and F. Ruskey, *Complete k-ary trees and generalized meta-Fibonacci sequences*, in Fourth Colloquium on Mathematics and Computer Science: Algorithms, Trees, Combinatorics and Probabilities, Discrete Math. Theor. Comput. Sci. Proc., Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2006, pp. 203–213.

[6] N.D. Emerson, *A family of meta-Fibonacci sequences defined by variable-order recursions*, J. Integer Seq. **9** (2006), Article 06.1.8.

[7] D.R. Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid*, Basic Books, 1979.

[8] A. Isgur, D. Reiss, and S.Tanny, *Trees and meta-Fibonacci sequences*, Electron. J. Combin., **16** (2009), #R129.

[9] B. Jackson and F. Ruskey, *Meta-Fibonacci sequences, binary trees, and extremal compact codes*, Electron. J. Combin., **13** (2006), #R26.

[10] N.J. Sloane. Library of Hadamard Matrices. http://neilsloane.com/hadamard/. Accessed 4 October 2024.

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