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Trivial Centre Group Orders

DES MACHALE

In memoriam Rex Dark

Abstract. We discuss the possible orders of finite groups that have trivial centre.

1. Introduction

The centre $Z(G)$ of a group G is defined to be $\{z \in G | zx = xz, \text{ for all } x \in G\}$. $Z(G)$ is a characteristic subgroup of G, which of course contains the identity element 1 of G. A group in which $Z(G) = \{1\}$ is said to have trivial centre. We discuss the question:

> For which natural numbers n does there exist a group G with $|G| = n$ and G has trivial centre?

This is sequence A060702 in Sloane [1] and begins 1, 6, 10, 12, 14, 18, 20, 21, 22, 24, 26, 30, 34, 36, 38, 39, 42, 46, 48, 50, 52, 54, 55, 56, 57, 58, 60, 62, 66, 68, 70, 72, 74, 75, 78, ...

We call these numbers the trivial centre group orders (TZ-numbers). The determination of all TZ-numbers seems to be a difficult problem, possibly out of reach at the moment, but we can list many classes of numbers which belong to this set.

(1) For each n, $4n + 2 = 2(2n + 1)$ is a TZ-number. This is because the dihedral group D_{2n+1} of order $2(2n + 1)$ has trivial centre.
(2) If p is an odd prime and $q > p$ is a prime such that p divides $q - 1$, then pq is a TZ-number. This is because under these conditions, there exists a unique group of order pq with trivial centre. This sequence begins 21, 39, 55, 57, 93, 111, 129, 183, 201, 203, 205, 219, ...
(3) Let p be a prime such that $p \equiv 1 \pmod 4$; then there are five isomorphism classes of groups of order $4p$. Two of those are abelian; there is D_{2p}, dihedral, and Q_p, dicyclic, given by $\langle a, b | a^{2p} = 1; b^2 = a^p, b^{-1}ab = a^{-1}\rangle$. All of these have non-trivial centre. But there is a fifth isomorphism class of groups, the semi-direct product of a cyclic group of order p by its unique cyclic subgroup of order 4 in its automorphism group. This group has trivial centre. Thus if $p \equiv 1 \pmod 4$ then $4p$ is a TZ-number. This sequence begins 20, 52, 68, 116, 148, 164, 212, ... ([1] A350115)
(4) Except for some small values of n, $n!$ and $n!/2$ are TZ-numbers because the symmetric group S_n and the alternating group A_n both have trivial centre.
(5) The simple non-abelian orders are clearly TZ-numbers. This sequence begins 60, 168, 360, 504, 660, 1092, 2448, 2520, 3420, 4080, ... ([1] A001034).
(6) We remark that the product of two TZ-numbers is also a TZ-number. This is because for direct products, $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$. But not every

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multiple of a TZ-number is a TZ-number. For example, 14 is a TZ-number but 28 is not.

(7) A perfect group G is a group which satisfies $G' = G$. Surprisingly, not all perfect groups have trivial centre. An example is $\mathrm{SL}(2,5)$. However, if G is perfect, it is known that $G/Z(G)$ has trivial centre (Grün's lemma).

(8) A complete group G is a group with trivial centre in which every automorphism is inner.
The sequence of complete orders begins 1, 6, 20, 24, 42, 54, 110, 120, 144, 156, 168, 216, 252, 272, 320, ... ([1] A341298).
In 1975, Rex Dark discovered a non-trivial complete group G which had odd order. It had order $33,209,467,522,096,377 = 3 \cdot 19 \cdot 17^{12}$ [3].
More recently, he showed that the smallest possible non-trivial complete group of odd order has order $352,947 = 3 \cdot 7^6$.

We can also list several classes of numbers which are *not* TZ-numbers.

(9) These include the cyclic orders, the abelian orders and more generally the nilpotent orders, which include the primes and the prime powers. We recall [2] that n is a nilpotent number, i.e. every group of order n is nilpotent, if n is of the form $p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$, p_i distinct and $p_i^k \not\equiv 1 \pmod{p_j}$ for all integers i, j and k, with $1 \le k \le a_i$. This sequence begins 1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 16, 17, 19, 23, 25, 27, 29, 31, 32, 33, 35, ... ([1] A056867).
Since a finite nilpotent group has non-trivial centre, none of the terms of this sequence, except the first, is a TZ-number.

(10) Let p be a prime such that $p \equiv 3 \pmod 4$, $p > 3$. Then there are precisely four isomorphism classes of groups of order $4p$ – two abelian, D_{2p}, dihedral, and Q_p, dicyclic. All of these groups have non-trivial centre. Thus $4p$ is never a TZ-number when $p \equiv 3 \pmod 4$. This sequence begins 28, 44, 76, 92, 124, 172, 188, ...

Some time ago, the author made the following

Conjecture. *If $n = 6t$, for some natural number t, then n is a TZ-number.*

I made some progress with this conjecture, for which the numerical evidence is overwhelming, but could not finish it. Then I discussed it over coffee with Rex Dark at a conference and this is the proof we came up with:

Theorem. *If n is of the form $6t$, for some natural number t, then n is a TZ-number.*

Proof. Let $n = 2^k \cdot 3 \cdot m$, with $k \ge 1$ and m odd. We show there is a group G with $|G| = n$ and $Z(G) = \{1\}$.

Case 1: Suppose first that k is odd, say $k = 2r+1$.

Take $H = S_3$, $K = K_1 \times K_2 \times \dots \times K_r$, with $K_i \simeq C_2 \times C_2$ and $L = C_m$.

Then $\mathrm{Aut}(C_2 \times C_2) \simeq S_3$, so each of the groups K_i can be regarded as a faithful H module. We can also make S_3 act on L by taking A_3 to centralise L and $S_3/A_3 \simeq C_2$ to invert L elementwise. Thus H acts on K_1, K_2, ... , K_r and on L, and we form the corresponding semidirect product $G = H \cdot (K \times L)$. Clearly, $|G| = 6 \cdot 4^r \cdot m = n$ and $Z(G) = \{1\}$. We note that this construction still works when $r = 0$ (so $K = \{1\}$) and/or when $m = 1$ (so $L = \{1\}$).

Case 2: Next suppose that k is even, say $k = 2r$, $r \ge 1$, and $m = 1$.

Take $H = C_3$, $K = K_1 \times K_2 \times \dots \times K_r$, with $K_i \simeq C_2 \times C_2$.

Then $C_3 \subseteq \mathrm{Aut}(C_2 \times C_2)$, so each of the groups K_i can be regarded as a faithful H module, and we form the corresponding semidirect product $G = HK$. Then $|G| = 3 \cdot 4^r = n$ and $Z(G) = \{1\}$.

Case 3: Finally, suppose that $k = 2r$ (with $r \ge 1$) and $m > 1$.

As in Case 1, we can construct a group G_1 with $|G_1| = 2^{2r-1} \cdot 3$ and $Z(G_1) = \{1\}$. We also take G_2 to be dihedral of order $2m$ and we form $G = G_1 \times G_2$. Then $|G| = 2^{2r-1} \cdot 3 \cdot 2m = n$ and $Z(G) = Z(G_1) \times Z(G_2) = \{1\}$, and we are done. □

2. Questions

Apart from a complete description of TZ-numbers, several other questions remain. Among these are:

Q1: What is the density of TZ-numbers? Our remarks (1) to (10) and our theorem could possibly throw some light on this question. Actual numbers in blocks of 100 less than 2000 appear to indicate that a figure hovering around 49.5% of natural numbers are TZ-numbers.
However, the preponderance of p-groups would seem to indicate that the proportion of *groups* with trivial centre is very small.

Q2: The consecutive numbers 20, 21, 22; 54, 55, 56, 57, 58; and 200, 201, 202, 203, 204, 205 are all TZ-numbers. We ask if there exist arbitrarily long sequences of this type.

Q3: Are there any other positive integers k (not a multiple of 6) for which kn is always a TZ-number? Clearly, k cannot be 10, 14, 20, 21 or 22.

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Des MacHale is Emeritus Professor of Mathematics at University College Cork where he taught for forty years. His main interests are in finite groups and rings, but he also dabbles in Number Theory, Euclidean Geometry, Trigonometric Inequalities, Combinatorial Geometry and Problem Posing and Solving. He has written several biographical books on George Boole but some would say his magnum opus is *Comic Sections Plus, the Book of Mathematical Jokes, Humour, Wit and Wisdom*, cf. `https://www.logicpress.ie/authors/machale`.

School of Mathematical Sciences, University College Cork
E-mail address: `d.machale@ucc.ie`