Irish Math. Soc. Bulletin
Number 90, Winter 2022, 42–45
ISSN 0791-5578

PROBLEMS

IAN SHORT

Problems

The first problem this issue is an elementary observation I stumbled upon recently.

Problem 90.1. Let M be any 3-by-3 matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

over a field, where $e \neq 0$, and let A, B, C, D be the four submatrices of M given by

$$A = \begin{pmatrix} a & b \\ d & e \end{pmatrix}, \quad B = \begin{pmatrix} b & c \\ e & f \end{pmatrix}, \quad C = \begin{pmatrix} d & e \\ g & h \end{pmatrix}, \quad D = \begin{pmatrix} e & f \\ h & i \end{pmatrix}.$$

Find an expression for $\det M$ in terms of $\det A$, $\det B$, $\det C$, $\det D$, and e.

The second problem is from Finbarr Holland of University College Cork.

Problem 90.2. Prove that

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{a_k a_{n-k}}{(2k+1)(2(n-k)+1)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{x^2}{\sin^2 x}\, dx = \log 4,$$

where

$$a_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)}, \quad n = 0, 1, 2, \ldots .$$

The third problem was proposed by Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia.

Problem 90.3. Evaluate

$$\sum_{m,n=0}^{\infty} \binom{2m}{m}^2 \binom{2n}{n} \frac{1}{2^{4m+2n}(m+n+1)}.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 88.

The first problem was solved by Kee-Wai Lau of Hong Kong, China, Seán Stewart, the North Kildare Mathematics Problem Club, and the proposer Anthony O'Farrell, editor of this Bulletin. We present the solution of Kee-Wai Lau.

Problem 88.1. Consider the sequence $x_0, x_1, \ldots$ defined by $x_0 = \sqrt{5}$ and $x_n = \sqrt{2 + x_{n-1}}$, for $n = 1, 2, \ldots$. Prove that

$$\prod_{n=1}^{\infty} \frac{2}{x_n} = 2 \log\left(\frac{1+\sqrt{5}}{2}\right).$$

Received 19-1-2021
DOI:10.33232/BIMS.0090.42.45.

©2022 Irish Mathematical Society

Solution 88.1. Let $c = \log((1+\sqrt{5})/2)$. Using the identity $1+\cosh(2x) = 2\cosh^2 x$, it can be proven by induction that

$$x_n = 2\cosh\left(\frac{c}{2^n}\right), \quad \text{for } n = 0, 1, \dots.$$

Then, using the identity $\sinh(2x) = 2\sinh x \cosh x$, it can be proven by induction that

$$\prod_{k=1}^{n} \frac{2}{x_k} = 2^{n+1} \sinh\left(\frac{c}{2^n}\right), \quad \text{for } n = 1, 2, \dots.$$

Hence

$$\prod_{k=1}^{\infty} \frac{2}{x_k} = 2c \times \lim_{n\to\infty} \frac{\sinh(c/2^n)}{c/2^n} = 2c,$$

as required. □

The second problem was solved by the North Kildare Mathematics Problem Club and the proposer, J.P. McCarthy of Munster Technological University. Here is the solution of the Problem Club.

*Problem 88.2***.** Let P be a 3-by-3 matrix each entry of which is an n-by-n complex Hermitian matrix; that is, each entry P_{ij} is an n-by-n complex matrix equal to its own conjugate transpose P_{ij}^*. Suppose that the sum along any row or column of P is the n-by-n identity matrix I_n:

$$\sum_{k=1}^{3} P_{kj} = \sum_{k=1}^{3} P_{ik} = I_n.$$

Suppose also that the entries of P along rows and columns satisfy

$$P_{ik}P_{il} = \delta_{kl}P_{ik} \quad \text{and} \quad P_{kj}P_{lj} = \delta_{kl}P_{kj},$$

where δ_{kl} is 1 if k and l are equal and otherwise it is 0 (and no summation convention should be applied). Prove that the entries of P commute with one another.

Solution 88.2. The given data imply that, regarded as linear operators on $\mathbb{C}^n$, the matrices P_{ij} are idempotents, orthogonal projections, and each row or column gives an orthogonal decomposition of $\mathbb{C}^n$ as a sum of three corresponding subspaces $V_{ij} = \operatorname{im}(P_{ij})$. We are also given that P_{ij} commutes with every P_{ik} and P_{kj}, that is, P_{ij} commutes with every entry in its row and every entry in its column. So it remains to show that each entry commutes with the entries outside its row and column.

The given conditions remain valid under permutation of rows or columns, so it suffices to prove that P_{11} commutes with P_{22}.

If $v \in V_{22}$, then $P_{12}v = 0$ so $P_{11}v + P_{13}v = v$, and also $P_{23}v = 0$ so $P_{13}v + P_{33}v = v$. Hence $P_{11}v = P_{33}v$. Thus for any $v \in \mathbb{C}^n$, $P_{11}P_{22}v = P_{33}P_{22}v$. Thus $P_{11}P_{22} = P_{33}P_{22}$. Taking the conjugate transpose, we also have $P_{22}P_{11} = P_{22}P_{33}$. Permuting indices, we deduce that

$$P_{ii}P_{jj} = P_{ii}P_{kk} \quad \text{and} \quad P_{ii}P_{jj} = P_{kk}P_{jj}$$

whenever i, j and k are mutually-distinct indices. Thus

$$P_{11}P_{22} = P_{33}P_{22} = P_{33}P_{11} = P_{22}P_{11}.$$

So P_{11} commutes with P_{22}, as required. □

The third problem was posed by Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia. It was solved by Ankush Kumar Parcha, a student from Indira Gandhi National Open University, New Delhi, India, Finbarr Holland, the North Kildare Mathematics Problem Club, and the proposer. We present the solution of the problem club.

Problem 88.3. Prove that

$$\sum_{n=2}^{\infty} \frac{(-1)^n H_{\lfloor n/2 \rfloor}}{n} = (\log 2)^2,$$

where H_n denotes the nth harmonic number

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

and $\lfloor \cdot \rfloor$ denotes the floor function.

Solution 88.3. The sum of the series is

$$\begin{aligned}
&\int_0^1 \left(H_1 y - H_1 y^2 + H_2 y^3 - H_2 y^4 + H_3 y^5 - H_3 y^6 + \cdots\right) dy \\
&= \int_0^1 \left(y + \frac{y^3}{2} + \frac{y^5}{3} + \cdots\right) \left(1 - y + y^2 - y^3 + \cdots\right) dy \\
&= -\int_0^1 \frac{\log(1-y^2)}{y(1+y)}\, dy \\
&= -\int_0^1 (\log(1-y) + \log(1+y)) \left(\frac{1}{y} - \frac{1}{y+1}\right) dy.
\end{aligned}$$

This gives us four integrals, and we proceed to evaluate them in terms of the dilogarithm

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt, \quad x \leqslant 1$$

(cf. Seán Stewart's article in BIMS89). We have

$$\begin{aligned}
&\int_0^1 \frac{\log(1-y)}{y}\, dy = -\mathrm{Li}_2(1), \\
&\int_0^1 \frac{\log(1+y)}{y}\, dy = \int_0^{-1} \frac{\log(1-t)}{t}\, dt = -\mathrm{Li}_2(-1), \\
&\int_0^1 \frac{\log(1+y)}{1+y}\, dy = \int_0^1 \log(1+y)\, d\log(1+y) = \tfrac{1}{2}(\log 2)^2,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \frac{\log(1-y)}{1+y}\, dy &= \int_1^2 \frac{\log(2-x)}{x}\, dx \\
&= \int_1^2 \frac{\log 2 + \log(1-\frac{x}{2})}{x}\, dx \\
&= (\log 2)^2 + \int_{\frac{1}{2}}^1 \frac{\log(1-t)}{t}\, dt \\
&= (\log 2)^2 - \mathrm{Li}_2(1) + \mathrm{Li}_2(\tfrac{1}{2}).
\end{aligned}$$

Combining these values, we see that the sum of the series is

$$\mathrm{Li}_2(1) + \mathrm{Li}_2(-1) + \tfrac{1}{2}(\log 2)^2 + (\log 2)^2 - \mathrm{Li}_2(1) + \mathrm{Li}_2(\tfrac{1}{2}).$$

Putting in the known values

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6}, \quad \mathrm{Li}_2(-1) = -\frac{\pi^2}{12}, \quad \mathrm{Li}_2(\tfrac{1}{2}) = \frac{\pi^2}{12} - \tfrac{1}{2}(\log 2)^2,$$

we obtain the solution $(\log 2)^2$. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

School of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, United Kingdom