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Paul J. Nahin: In Pursuit of Zeta-3, Princeton University Press, 2021. ISBN:9780691206073, USD 26.95, 344 pp.

REVIEWED BY JOHN E. MCCARTHY

When I was a young mathematician, the harmonic analyst Henry Helson gave me some excellent advice: *"Every mathematician should study the Zeta function. It is a mirror in which you see yourself."* He was right; no matter what field of mathematics you study, there is some connection to the Zeta function.

The main subject of the book under review is the Zeta function, defined for complex numbers s with Re(s) > 1 by the Dirichlet series

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} n^{-s},$$

and extending by analytic continuation to $\mathbb{C} \setminus \{1\}$.

Euler calculated $\zeta(s)$ exactly when s is a positive even integer, and the author gives us derivations of some of the formulas, including

$$\begin{aligned} \zeta(2) &= \textstyle\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \\ \zeta(4) &= \textstyle\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}. \end{aligned}$$

The title of the book refers to the question: is there a closed form formula, involving the standard mathematical constants, for

$$\zeta(3) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^3}?$$

Discussing this question, the author gives us a tour of some classical topics in mathematics such as the Gamma function, Euler's constant γ, and Fourier series. He demonstrates various ingenious identities, such as the following (due to Euler):

$$\zeta(3) \;=\; \frac{2\pi^2}{7}\ln 2 + \frac{16}{7}\int_0^{\pi/2} x\big[\ln(\sin x)\big]\,dx,$$

and gives a lovely proof that

$$\gamma \;=\; -\int_0^{\infty} e^{-x}\ln x\,dx.$$

The author has an engaging writing style, and comes across as a jovial uncle entertaining his nieces and nephews with stories and magic tricks. There are lots of challenging problems, with worked solutions at the back of the book, which will appeal to some readers. There are interesting digressions, both mathematical and historical. I enjoyed reading the book, and yet it bothered me in several ways.

First is the question of who the audience is. Nahin writes in the introduction that he hopes the audience will be enthusiastic readers of mathematics at the level of high school AP calculus—that is about the same as Honours Leaving Certificate Maths. I

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think back to when I had taken the Leaving Cert, and imagine what my reactions would have been to this book. I believe they would have been delight, confusion, and intimidation, in that order.

The delight would come from the many clever tricks to evaluate integrals, and the glimpses of the mathematical world beyond the garden wall. The confusion would come from the author's eschewment of rigour. He regularly differentiates under integral signs, and interchanges orders of integration, with a remark that this can often be justified but not always. Why not just state some theorems that give sufficient conditions, so that the reader at least knows what needs to be checked?

Worse is the author's cavalier treatment of divergent series. He never defines what he means by convergence of such a series (implicitly, he means by some form of analytic continuation, but this is not defined, nor are the problems associated with it discussed). Closely related to the Zeta function is the Eta function, defined by a Dirichlet series and extended analytically to the entire plane:

$$\eta(s) \;=\; \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s} \;=\; (1-2^{1-s})\zeta(s), \qquad \text{for Re(s)} > 0. \tag{1}$$

His proof that $\eta(0) = -\zeta(0) = \frac{1}{2}$ is that if you put $s = 0$ in (1) you get

$$\eta(0) \;=\; 1-1+1-1+1\ldots, \tag{2}$$

and if you put $x = 1$ in the series

$$\frac{1}{1+x} \;=\; 1-x+x^2-x^3+\ldots$$

you get

$$\frac{1}{2} \;=\; 1-1+1-1+1\ldots. \tag{3}$$

Comparing (2) and (3), he deduces that $\eta(0) = \frac{1}{2}$. But in the late 18^{th} century, Callet pointed out that if you let $x = 1$ in the series

$$\frac{1}{1+x+x^2} \;=\; 1-x+x^3-x^4+x^6-x^7+\ldots$$

you would get

$$\frac{1}{3} \;=\; 1-1+1-1+1\ldots.$$

Nahin does not discuss why divergent series can be used to get valid formulas, or why $\frac{1}{2}$ really is the correct value for $\eta(0)$.

Intimidation would follow, because I would have felt stupid for not really understanding the arguments laid out in front of me. Today I am a professional mathematician (something that, to the relief of my mother and my eternal wonderment, turns out to be an actual job). I know how to justify differentiating under integral signs, and I understand that divergent series make sense as long as you are clear about definitions. So I, and other readers of the this Bulletin, can enjoy this book. But with a bit more care about rigour, and more effort in justifying why the steps in an argument follow a logical strategy, instead of just attributing them to the genius of Euler, this could have been a book that did indeed hit the target audience. Ultimately, we don't want just to marvel at magic tricks, we want to know how they are done.

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