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Abel's limit theorem, its converse, and multiplication formulae for $\Gamma(x)$

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Abstract. Abel's well-known limit theorem for power series, and its corrected converse due to J. E. Littlewood, form the basis for a general identity that is prresented here, which is shown to be equivalent to Gauss's multiplication theorem for the Gamma function.

1. Introduction

An incomplete solution of a problem of mine, numbered Problem 86.3 in [4], (that was presented in [5]) prompted this note about Abel's limit theorem on power series, and two of its partial converses due, respectively, to Tauber and Littlewood. These are landmark results in the development of Real and Complex Analysis. For instance, Abel's theorem initiated the study of the boundary behaviour of analytic functions on the unit disc, and, in conjunction with Cesáro's consistency theorem on the convergence of arithmetic means of a convergent sequence, paved the way for summing series by different methods dealt with in [2], while the theorems of Tauber and Littlewood gave rise to the beautiful sub-topic of Wiener's Tauberian Analysis, also exposed in [2].

Students of Analysis who are desirous of learning "the tricks of the trade" would do well to study proofs of Abel's theorem and Tauber's, and at least acquaint themselves with the more profound result of Littlewood. All three theorems are simply expounded in [6].

In this note, we'll state and provide standard proofs of the theorems of Abel and Tauber, and state, but not prove, Littlewood's deeper result; instead, we'll illustrate its utility by means of a simple example. These theorems will be discussed in Sections 2, 3 and 4, respectively. As an illustration of the underlying ideas we'll derive a general theorem in Section 5, which is motivated by the aforementioned journal problem, and show that a special case of it is equivalent to Gauss's multiplication formula for the Gamma function (see the example in Section 9.56 of [1]).

2. Abel's limit theorem

Throughout the note, f stands for a generic power series $\sum_{n=0}^{\infty} a_n x^n$ whose radius of convergence is 1, though the coefficients will differ from time to time.

According to Abel: if the series $\sum_{n=0}^{\infty} a_n$ is convergent, then

$$\lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} a_n.$$

We sketch the standard proof of this.

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Proof. Let

$$s_n = \sum_{k=0}^{n} a_k, \ n = 0, 1, 2 \ldots,$$

and $x \in [0,1)$. As a first step we express $f(x)$ as a convex combination of the sequence $s_0, s_1, s_2, \ldots$. This is easy to do since by pointwise multiplication of two absolutely convergent power series

$$\begin{aligned}(1 + x + x^2 + \cdots)(a_0 + a_1 x + a_2 x^2 + \cdots) &= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots \\ &= s_0 + s_1 x + s_2 x^2 + \cdots\end{aligned}$$

so that

$$f(x) = (1-x)\Big(\sum_{n=0}^{\infty} x^n\Big)\Big(\sum_{n=0}^{\infty} a_n x^n\Big) = (1-x)\sum_{n=0}^{\infty} s_n x^n.$$

Accordingly, if $s = \lim_{n\to\infty} s_n$, and $0 \leq x < 1$,

$$f(x) - s = (1-x)\sum_{n=0}^{\infty} s_n x^n - s(1-x)\sum_{n=0}^{\infty} x^n = (1-x)\sum_{n=0}^{\infty}(s_n - s)x^n,$$

from which it follows that

$$|f(x) - s| \leq (1-x)\sum_{n=0}^{\infty} |s_n - s| x^n \leq \sup\{|s_n - s| : n = 0, 1, 2, \ldots\}.$$

Hence

$$\sup\{|f(x) - s| : 0 \leq x < 1\} \leq \sup\{|s_n - s| : n = 0, 1, 2, \ldots\},$$

a step in the right direction, but not the final one! To obtain the desired result, we refine the argument just given by splitting the sum $(1-x)\sum_{n=0}^{\infty} |s_n - s| x^n$ in two appropriately. To achieve this, let $\epsilon > 0$, and choose an integer n_0 so that $|s_n - s| < \epsilon$, $\forall n > n_0$, whence for any $x \in (0,1)$,

$$(1-x)\sum_{n=n_0+1}^{\infty} |s_n - s| x^n \leq \epsilon(1-x)\sum_{n=n_0+1}^{\infty} x^n \leq \epsilon.$$

Consequently,

$$|f(x) - s| \leq (1-x)\sum_{n=0}^{n_0} |s_n - s| x^n + \epsilon,$$

and so, on letting x tend to 1 from the left,

$$\limsup_{x\to 1^-} |f(x) - s| \leq \epsilon.$$

Since ϵ is an arbitrary positive number, this means that $\lim_{x\to 1^-} f(x) = s = \sum_{n=0}^{\infty} a_n$, as claimed. ☐

3. Tauber's converse

As the example

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \ |x| < 1,$$

shows, the direct converse of Abel's theorem is false.

Tauber proved a conditional converse according to which, if $\lim_{x\to 1^-} f(x) = s$, and $\lim_{n\to\infty} n a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ is convergent and its sum is s.

Proof. To see this, note that

$$f(x) - s_n = \sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{n} a_k = \sum_{k=1}^{n} a_k(x^k - 1) + \sum_{k=n+1}^{\infty} a_k x^k,$$

for any $x \in (0, 1)$, and any positive integer n. Now

$$\begin{aligned} \Big| \sum_{k=n+1}^{\infty} a_k x^k \Big| &= \Big| \sum_{k=n+1}^{\infty} (k a_k) \frac{1}{k} x^k \Big| \\ &\leq \frac{1}{n+1} \sum_{k=n+1}^{\infty} k|a_k| x^k \\ &\leq \frac{1}{(n+1)(1-x)} \max\{k|a_k| : k \geq n+1\}, \end{aligned}$$

and

$$\Big| \sum_{k=1}^{n} a_k(x^k - 1) \Big| \leq \sum_{k=1}^{n} |a_k|(1 - x^k) \leq (1 - x) \sum_{k=1}^{n} k|a_k|.$$

Combining these estimates we have that

$$|f(x) - s_n| \leq (1 - x) \sum_{k=1}^{n} k|a_k| + \frac{1}{(n+1)(1-x)} \max\{k|a_k| : k \geq n\}.$$

Bearing in mind that x and n are at our disposal, to be chosen as we see fit, it's now convenient to set $x \equiv x_n = 1 - \frac{1}{n+1}$. With this choice we have

$$|f(x_n) - s_n| \leq \frac{1}{n+1} \sum_{k=1}^{n} k|a_k| + \max\{k|a_k| : k \geq n\},$$

an expression that tends to zero as $n \to \infty$, its first term by Cesáro's theorem, and its second by hypothesis. Therefore

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} f(x_n) = \lim_{x\to 1^-} f(x) = s,$$

as we wanted to show. □

4. Littlewood's converse

Tauber's result was considerably strengthened by Littlewood [3] who proved that if $\lim_{x\to 1^-} f(x) = s$, and the sequence na_n is merely bounded, then the series $\sum_{n=0}^{\infty} a_n$ is convergent and its sum is s. We won't give the proof of this, but instead provide a simple example to illustrate its utility.

Example 4.1. Suppose $0 < \theta < 2\pi$. Then

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\ln\Big(2 \sin \frac{\theta}{2}\Big).$$

Proof. Fix $\theta \in (0, 2\pi)$, and consider the power series expansion about the origin of $f(x) = \ln(1 - 2\cos\theta x + x^2)$, namely, if $|x| < 1$, then

$$\begin{aligned} f(x) &= \ln\left[(1 - e^{i\theta}x)(1 - e^{-i\theta}x)\right] \\ &= -\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} x^n - \sum_{n=1}^{\infty} \frac{e^{-in\theta}}{n} x^n \\ &= -2\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} x^n. \end{aligned}$$

Clearly, $f(x) \to \ln(2 - 2\cos\theta)$ as $x \to 1^-$, and the coefficients of the last displayed power series satisfy Littlewood's condition. Hence, when $x = 1$ the displayed series is convergent and its sum is $\ln\left(4\sin^2\frac{\theta}{2}\right) = 2\ln\left(2\sin\frac{\theta}{2}\right)$, which yields the result. □

5. Gauss's multiplication formula for $\Gamma(x)$

As a precursor to this, we first establish the next result which relies on the theorems just described of both Abel and Littlewood.

Theorem 5.1. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Suppose the sequence a_n satisfies Littlewood's condition, and m is a positive integer. Then $f(x) - f(x^m)$ converges to s as $x \to 1^-$ iff the series*

$$\sum_{n=0}^{\infty}\left[\left(\sum_{r=0}^{m-1} a_{nm+r}\right) - a_n\right]$$

is convergent and its sum is s.

Proof. For $|x| < 1$,

$$\begin{aligned} f(x) - f(x^m) &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{nm} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{m-1} a_{nm+r}x^{nm+r} - \sum_{n=0}^{\infty} a_n x^{nm} \\ &= \sum_{n=0}^{\infty} x^{mn}\Big(\sum_{r=0}^{m-1} a_{mn+r}x^r - a_n\Big) \\ &= \sum_{n=0}^{\infty} x^{mn}\Big((a_{mn} - a_n) + \sum_{r=1}^{m-1} a_{mn+r}x^r\Big) \\ &= \sum_{n=0}^{\infty} c_n x^n, \end{aligned}$$

where, for $n = 0, 1, \ldots$,

$$c_{nm+r} = \begin{cases} a_{mn} - a_n, & \text{if } r = 0, \\ a_{nm+r}, & \text{if } r = 1, \ldots, m-1. \end{cases}$$

Suppose now that $f(x) - f(x^m)$ converges to s as $x \to 1^-$. Then the series $\sum_{n=0}^{\infty} c_n x^n$ satisfies the hypotheses of Littlewood's theorem, and so $s = \sum_{n=0}^{\infty} c_n$. In other words, if C_n denotes the nth partial sum of this series, $C_n \to s$, whence, in particular, $s = \lim_{n\to\infty} C_{mn}$, i.e.

$$s = \lim_{n\to\infty}\sum_{k=0}^{n}\sum_{r=0}^{m-1} c_{km+r} = \sum_{n=0}^{\infty}\Big(\sum_{r=0}^{m-1} a_{nm+r} - a_n\Big),$$

as desired. Conversely, suppose the last displayed series is convergent. Let

$$b_n = \sum_{r=0}^{m-1} a_{nm+r} - a_n, \; n = 0, 1, \ldots,$$

and

$$F(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then from above,

$$\begin{aligned} f(x) - f(x^m) &= \sum_{n=0}^{\infty} x^{mn} \Big(\sum_{r=0}^{m-1} a_{mn+r} x^r - a_n \Big) \\ &= \sum_{n=0}^{\infty} b_n x^{mn} + \sum_{n=0}^{\infty} x^{mn} \sum_{r=1}^{m-1} a_{mn+r}(x^r - 1) \\ &= F(x^m) + \sum_{r=1}^{m-1} (x^r - 1) \sum_{n=0}^{\infty} a_{mn+r} x^{mn} \\ &= F(x^m) + \sum_{r=1}^{m-1} (x^r - 1) h_r(x), \end{aligned}$$

where, for $r = 1, .., m - 1$,

$$h_r(x) = \sum_{n=0}^{\infty} a_{mn+r} x^{mn} = O(1) \log \frac{1}{1-x}, \; (x \to 1^-).$$

As a result,

$$\sum_{r=1}^{m-1} (x^r - 1) h_r(x) = O(1)(1-x) \log \frac{1}{1-x} = o(1), \; (x \to 1^-).$$

Thus

$$f(x) - f(x^m) = F(x^m) + o(1) \; (x \to 1^-).$$

By Abel, $\lim_{x \to 1^-} F(x^m) = s$, and so $f(x) - f(x^m)$ converges to s as $x \to 1^-$. This completes the proof. □

The folowing example is a direct consequence of this theorem.

Example 5.2. Let m be any positive integer. Then, for all $a > 0$,

$$\sum_{n=0}^{\infty} \Big[\Big(\sum_{r=0}^{m-1} \frac{1}{nm + r + a} \Big) - \frac{1}{n+a} \Big] = \ln m. \tag{1}$$

Proof. Let $a_n = 1/(n + a)$, $n = 0, 1, 2, \ldots$, and $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Since the series in (1) is plainly convergent, by the theorem its sum is equal to the limit of $f(x) - f(x^m)$ as $x \to 1^-$. To calculate this, notice first that if $|x| < 1$, then

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n + \frac{1}{a} + \sum_{n=1}^{\infty} \Big(\frac{1}{n+a} - \frac{1}{n} \Big) x^n = \ln \frac{1}{1-x} + g(x),$$

say, and then that $f(x) - f(x^m) = \ln(1 + x + \cdots + x^{m-1}) + g(x) - g(x^m)$ which converges to $\ln m$ as $x \to 1^-$, since, by Abel, $\lim_{x \to 1^-} g(x)$ exists. □

The special case of this example, with $m = 3$ and $a = 1$, leads to the conclusion that

$$\sum_{n=0}^{\infty} \frac{9n+5}{9n^3+18n^2+11n+2} = 3\ln 3,$$

a proof of which was sought in [4].

What's noteworthy about (1), and surprising perhaps, is that, for each fixed integer $m > 1$, the series is convergent and its sum function is independent of a! What's the explanation for that? The reason is because—as we shall proceed to demonstrate—it's equivalent to Gauss's multiplication theorem for the Gamma function, $\Gamma(x)$, according to which if m is a positive integer, then

$$m^{mx-\frac{1}{2}} \prod_{r=0}^{m-1} \Gamma(x + \frac{r}{m}) = (2\pi)^{\frac{m-1}{2}} \Gamma(mx), \ \forall x > 0. \tag{2}$$

This is an extension of the more familiar duplication formula due to Legendre:

$$2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2}) = \sqrt{\pi}\Gamma(2x).$$

To explain the connection between (1) and (2), recall that the reciprocal of $\Gamma(z)$ is an entire function of the complex variable z, with simple zeros at the integers $0, -1, -2, \ldots$, that admits of the canonical factorization

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^{-z/n},$$

where γ is Euler's constant $\lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \ln n \Big)$. Hence, denoting by ψ the derivative of $\ln \Gamma$,

$$-\psi(x) = -\frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x} + \gamma + \sum_{n=1}^{\infty} \Big(\frac{1}{n+x} - \frac{1}{n} \Big).$$

Therefore, if m is a positive integer, and $x > 0$, then

$$\begin{aligned}
& -m\frac{\Gamma'(mx)}{\Gamma(mx)} + \sum_{r=0}^{m-1} \frac{\Gamma'(x+\frac{r}{m})}{\Gamma(x+\frac{r}{m})} \\
= \ & \frac{1}{x} + m\gamma + m\sum_{n=1}^{\infty} \Big(\frac{1}{n+mx} - \frac{1}{n} \Big) - \sum_{r=0}^{m-1} \Big[\frac{1}{x+\frac{r}{m}} + \gamma + \sum_{n=1}^{\infty} \Big(\frac{1}{n+x+\frac{r}{m}} - \frac{1}{n} \Big) \Big] \\
= \ & -\sum_{r=1}^{m-1} \frac{1}{x+\frac{r}{m}} + m\sum_{n=1}^{\infty} \Big(\frac{1}{n+mx} - \frac{1}{n} \Big) - \sum_{r=0}^{m-1}\sum_{n=1}^{\infty} \Big(\frac{1}{n+x+\frac{r}{m}} - \frac{1}{n} \Big) \\
= \ & -\sum_{r=1}^{m-1} \frac{1}{x+\frac{r}{m}} - \sum_{n=1}^{\infty} \Big[\sum_{r=0}^{m-1} \frac{1}{n+x+\frac{r}{m}} - \frac{m}{n+mx} \Big] \\
= \ & -m\Big(\sum_{r=1}^{m-1} \frac{1}{mx+r} + \sum_{n=1}^{\infty} \Big[\Big(\sum_{r=0}^{m-1} \frac{1}{mn+r+mx} \Big) - \frac{1}{n+mx} \Big] \Big) \\
= \ & -m\sum_{n=0}^{\infty} \Big[\Big(\sum_{r=0}^{m-1} \frac{1}{mn+r+mx} \Big) - \frac{1}{n+mx} \Big] \\
= \ & -m\ln m,
\end{aligned}$$

by (1), with $a = mx$. In other words, for $x > 0$, assuming (1) holds,

$$\frac{d}{dx}\Big(\sum_{r=0}^{m-1} \ln\Gamma(x+\frac{r}{m}) - \ln\Gamma(mx)\Big) = -m\ln m.$$

Thus, for some constant $C(m)$,

$$\frac{\prod_{r=0}^{m-1}\Gamma(x+\frac{r}{m})}{\Gamma(mx)} = m^{-mx}C(m),\ \forall x > 0.$$

But, from the product formula for $1/\Gamma(x)$, it's clear that

$$\lim_{x\to 0^+}\frac{1}{x\Gamma(x)} = 1, \text{ whence } \lim_{x\to 0^+}\frac{\Gamma(x)}{\Gamma(mx)} = m.$$

Hence

$$C(m) = m\prod_{r=1}^{m-1}\Gamma(\frac{r}{m}).$$

It remains to compute the product $p(m) = \prod_{k=1}^{m-1}\Gamma(\frac{k}{m})$. To do this, we adapt Gauss's ploy (which legend says he used in kindergarten one day to add the first 100 natural numbers) and determine the geometric mean of $p(m)$ and the product of its factors in reverse order, namely, $\prod_{k=1}^{m-1}\Gamma(\frac{m-k}{m})$, also $p(m)$, of course. So, we consider

$$\begin{aligned} p(m)^2 &= \prod_{k=1}^{m-1}\Gamma(\frac{k}{m})\Gamma(1-\frac{k}{m}) \\ &= \prod_{k=1}^{m-1}\Big(\frac{\pi}{\sin\frac{k\pi}{m}}\Big) \\ &= \pi^{m-1}\prod_{k=1}^{m-1}\frac{1}{\sin\frac{k\pi}{m}}, \end{aligned}$$

by the reflection property of the Gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}.$$

To compute the product of the numbers $\sin\frac{k\pi}{n}, k = 1, 2, \ldots, n-1$, note that

$$\begin{aligned} 4^{m-1}\Big(\prod_{k=1}^{m-1}\sin\frac{k\pi}{m}\Big)^2 &= \prod_{k=1}^{m-1}\Big(4\sin^2\frac{k\pi}{m}\Big) \\ &= \prod_{k=1}^{m-1}|1-e^{\frac{2ik\pi}{m}}|^2. \end{aligned}$$

But the m numbers $e^{\frac{2ik\pi}{m}}, k = 0, 1, \ldots, m-1$, are precisely the mth roots of unity, and so

$$z^m - 1 = (z-1)\prod_{k=1}^{m-1}(z - e^{\frac{2ik\pi}{m}}).$$

Hence,

$$m = \prod_{k=1}^{m-1}(1-e^{\frac{2ik\pi}{m}}),\ m^2 = \prod_{k=1}^{m-1}|1-e^{\frac{2ik\pi}{m}}|^2.$$

Consequently,

$$2^{2(m-1)}\Big(\prod_{k=1}^{m-1}\sin\frac{k\pi}{m}\Big)^2 = m^2,$$

from which it follows that

$$\prod_{k=1}^{m-1}\sin\frac{k\pi}{m} = \frac{m}{2^{m-1}},$$

since $\sin\frac{k\pi}{m} > 0$ for $k = 1, 2, \ldots, m-1$. Hence

$$p(m)^2 = \frac{\pi^{m-1}2^{m-1}}{m},\ p(m) = \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}},$$

and so $C(m) = \sqrt{m}(2\pi)^{\frac{m-1}{2}}$, whence we obtain Gauss's formula:

$$\prod_{r=0}^{m-1}\Gamma(x+\frac{r}{m}) = (2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mx}\Gamma(mx).$$

Thus, with $a = mx$, the identity (1) implies (2). Since we can easily reverse the steps just made from (1) to (2), it should be clear that (1) is a consequence of (2).

To sum up: if m is any positive integer, Gauss's multiplication statement for the Gamma function that

$$m^{mx}\prod_{r=0}^{m-1}\Gamma(x+\frac{r}{m}) = \prod_{r=1}^{m-1}\Gamma(\frac{r}{m})\Gamma(mx) = \sqrt{m}(2\pi)^{\frac{m-1}{2}}\Gamma(mx),\ \forall x > 0,$$

is equivalent to the statement that

$$\sum_{n=0}^{\infty}\Big(\sum_{r=0}^{m-1}\frac{1}{mn+r+x} - \frac{1}{n+x}\Big) = \ln m,\ \forall x > 0.$$

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