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Special values of Legendre's chi-function and the inverse tangent integral

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Abstract. In our recent publication in this Bulletin [88 Winter (2021), 31–37] a series transform proved via Fourier–Legendre theory and fractional operators in a 2022 article was applied to prove five two-term dilogarithm identities. One such identity gave a closed form for $\mathrm{Li}_2(\sqrt{2}-1) - \mathrm{Li}_2(1-\sqrt{2})$, and we had attributed this closed form to a 2012 article by Lima. However, as we review in our current article, there had actually been a number of previously published proofs of formulas that are equivalent to the closed-form evaluation for the equivalent expression $\chi_2(\sqrt{2}-1)$, letting χ_2 denote the Legendre chi-function. We offer a brief survey of the history of special values for χ_2 and the inverse tangent integral Ti_2, in relation to the results given in our previous BIMS publication. Two of the two-term dilogarithm relations proved in this previous publication were actually introduced in 1915 by Ramanujan in an equivalent form in terms of the Ti_2 function, which adds to the interest in the alternative proofs for these results that we had independently discovered. We also apply special values for χ_2 and Ti_2, together with a Legendre-polynomial based series transform, to obtain evaluations for rational double hypergeometric series with inevaluable single sums.

1. Introduction

In the 2022 article [8], the series transform reproduced as equation (2) in [7] was proved using Fourier–Legendre (FL) theory and fractional calculus, building on an FL-based integration method introduced in the 2019 research article [10]. Using this series transform from [8] together with the generating function for Legendre polynomials, we had proved in [7] five two-term dilogarithm evaluations. These five evaluations are reproduced below. We had incorrectly stated that the first out of the five equations listed below was introduced by Lima in 2012 [18], without our having been aware that an equivalent formulation of this first equation was given in terms of the Legendre chi-function in the 1958 text [15, p. 19]. Lima proved (1) in [18] and one of the main results in [18] follows from (1), but the fact that (1) was previously known, as far back as 1958 [15, p. 19], was not indicated anywhere in [18] or in the zbMATH review [2] of [18] (cf. [11]). Furthermore, while our method for proving the below results using Legendre polynomials is highly original, all of the five formulas below had been known prior to [7], without the author having been aware of this; see [21], [15, p. 19] and [12].

$$\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} - \frac{1}{2}\ln^2\left(1+\sqrt{2}\right) \tag{1}$$

$$\mathrm{Li}_2\left(\frac{1}{\phi^3}\right) - \mathrm{Li}_2\left(-\frac{1}{\phi^3}\right) = \frac{\phi^3\left(\pi^2 - 18\ln^2(\phi)\right)}{3\left(\phi^6-1\right)} \tag{2}$$

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$$\mathrm{Li}_2\left(i\left(2-\sqrt{3}\right)\right)-\mathrm{Li}_2\left(-i\left(2-\sqrt{3}\right)\right)=\frac{2i\sqrt{7-4\sqrt{3}}\left(8G-\pi\ln\left(2+\sqrt{3}\right)\right)}{3\left(8-4\sqrt{3}\right)} \tag{3}$$

$$\mathrm{Li}_2\left(i\left(\sqrt{2}-1\right)\right)-\mathrm{Li}_2\left(-i\left(\sqrt{2}-1\right)\right) \tag{4}$$

$$=\frac{1}{32}i\left(\sqrt{2}\left(\psi^{(1)}\left(\frac{1}{8}\right)+\psi^{(1)}\left(\frac{3}{8}\right)\right)+8\pi\ln\left(\sqrt{2}-1\right)-4\sqrt{2}\pi^2\right)$$

$$\mathrm{Li}_2\left(\frac{i}{\sqrt{3}}\right)-\mathrm{Li}_2\left(-\frac{i}{\sqrt{3}}\right)=\frac{i\left(3\psi^{(1)}\left(\frac{1}{6}\right)+15\psi^{(1)}\left(\frac{1}{3}\right)-6\sqrt{3}\pi\ln(3)-16\pi^2\right)}{36\sqrt{3}}. \tag{5}$$

Also, a different formulation of the main transform from our recent article [7] was included in an unpublished online note [23] from 2000, but was proved differently; also, a different formulation of this same result was given by Bradley in [3], and proved in much the same way as in [23]. The above identities for the dilogarithmic expressions in (3) and (4) had been given by Ramanujan in 1915 [1, 21] in an equivalent form in terms of the special function known as the inverse tangent integral Ti_2. Ramanujan's approach toward evaluating (3) and (4) was very different compared to our Legendre polynomial-based proofs for equivalent evaluations [7], which further motivates the application of our methods from [7]. As indicated in Section 2.2 below, there have actually been a number of previously published proofs of identities equivalent to (1) [4, 5, 22].

The corrections to our publication [7] covered above motivate the brief survey offered in Section 2 on past literature concerning the above evaluations for the two-term dilogarithm combinations in (1), (2), (3), and (4), relative to the methods and results from [7].

Remark 1.1. Subsequent to the publication of [7], the five dilogarithmic identities indicated in (1)–(5) were reproduced in the Wolfram MathWorld encyclopedia entry on the dilogarithm function [25], with [7] cited as a Reference for these identities. This same MathWorld entry [25] contains links to the corresponding encyclopedia entries on the inverse tangent integral [26] and Legendre's chi-function [14], and this led the author to discover that equivalent formulas for the values in (1)–(4) had been previously recorded in mathematical literature prior to both [7] and [18]; this, in turn, had inspired the author to explore the history of special values for χ_2 and Ti_2 in relation to the material in [7] and [18], culminating in the survey offered in Section 2 below.

2. Survey

2.1. The Legendre chi-function.

The special function known as Legendre's chi-function is defined as follows [14]:

$$\chi_\nu(z)=\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)^\nu}.$$

From the above definition, it is immediate that

$$\chi_\nu(z)=\frac{1}{2}\left(\mathrm{Li}_\nu(z)-\mathrm{Li}_\nu(-z)\right).$$

So, we see that the left-hand sides of (1) and (2) may be naturally expressed with the χ-function. As it turns out, the identities

$$\chi_2\left(\sqrt{2}-1\right)=\frac{1}{16}\pi^2-\frac{1}{4}\ln^2\left(\sqrt{2}+1\right) \tag{6}$$

and

$$\chi_2\left(\sqrt{5}-2\right)=\frac{1}{24}\pi^2-\frac{3}{4}\ln^2\left(\frac{\sqrt{5}+1}{2}\right), \tag{7}$$

which are easily seen to be equivalent to (1) and (2), respectively, were previously known [14] [15, p. 19], prior to the publication of [7]. New identities involving the Legendre chi-function were recently given in [24], in which the classical identity

$$\chi_2\left(\frac{1-x}{1+x}\right)+\chi_2(x)=\frac{3\zeta(2)}{4}+\frac{1}{2}\ln(x)\ln\left(\frac{1+x}{1-x}\right)$$

is reproduced from the classic text [16]. We see that (6) follows directly from the identity for $\chi_2\left(\frac{1-x}{1+x}\right)+\chi_2(x)$ given above, and this same identity may be used in a direct way to prove (7). The foregoing considerations add to the interest in the new and Legendre polynomial-based alternate proofs of (6) and (7) given in [7]. The evaluations in (6) and (7) are also reproduced in [23], again with reference to Lewin's text [16]. The formulas in (6) and (7) are well-known and were recently noted [20] in the context of applications related to the special function known as the Barnes G-function.

2.2. **Landen's identity and the Rogers L-function.** One of the main results in [18], as highlighted in the title of [18] and in the corresponding zbMATH review [2], is as given below:

$$\mathrm{Li}_2\left(\sqrt{2}-1\right)+\mathrm{Li}_2\left(1-\frac{1}{\sqrt{2}}\right)=\frac{\pi^2}{8}-\frac{\ln^2\left(1+\sqrt{2}\right)}{2}-\frac{1}{8}\ln^2 2. \tag{8}$$

However, this follows in a direct way from (1) together with the famous Landen identity

$$\mathrm{Li}_2(z)=-\mathrm{Li}_2\left(\frac{z}{z-1}\right)-\frac{1}{2}\ln^2(1-z),$$

but it is not indicated in [18] or its reviews [2, 11] that (1) was previously known in an equivalent way via the Legendre chi-function, as far back as Lewin's classic 1958 text [15, p. 19]. The article [18] was the main inspiration behind our publication in [7], but it is suggested in [18] that (1) was introduced in Lima's 2012 article in [18]. Part of the reason as to the confusion concerning the origins of identities as in (1) is due to a number of different special functions and notational conventions that have been used to express such identities, with reference to the χ_ν-function defined above, along with the Ti_2-function defined below and the different definitions/notations for the Rogers dilogarithm function indicated below. Again, our published proof of (1) [7], which relied on a fractional calculus-derived transform from the 2022 article [8], is original, as is the case with our proofs in [7] of the above symbolic forms for (2), (3), (4), and (5).

The fact that the formula in (8) that was highlighted as a main result in [18] and presented as being new in Lima's paper [18] follows directly from Landen's identity together with the classically known evaluation in (1) recorded in the 1958 text [15, p. 19] has not been noted in any past literature citing [18], including [13, 17, 19]. Letting

$$L(x)=\frac{6}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}+\frac{1}{2}\ln x\ln(1-x)\right)$$

denote the normalized Rogers dilogarithm function, in the 1999 article [5], it was noted that an equivalent formulation of the above equation for $\mathrm{Li}_2\left(\sqrt{2}-1\right)+\mathrm{Li}_2\left(1-\frac{1}{\sqrt{2}}\right)$ follows in a direct way from the identity

$$L(x)+L(1-x)=1 \tag{9}$$

together with Abel's duplication formula, which follows from Abel's functional equation

$$L(x)+L(y)=L(xy)+L\left(\frac{x(1-y)}{1-xy}\right)+L\left(\frac{y(1-x)}{1-xy}\right).$$

This is also noted in [18]. So, we find that the formula in (1), which traces back to the 1958 text [15, p. 19], may also be proved using the functional relations for the Rogers dilogarithm given in (9) together with Abel's duplication formula and Landen's identity. This provides a remarkably different proof compared to our Legendre polynomial-based proof of (1) that we had introduced in [7].

Using the alternative notation/definition

$$L_R(x) = \mathrm{Li}_2(x) + \frac{1}{2}\ln x \ln(1-x)$$

for the Rogers L-function indicated in [27], the formula

$$L_R\left(2-\sqrt{2}\right) - L_R\left(\frac{2-\sqrt{2}}{2}\right) = \frac{\pi^2}{24}$$

was proved in 1981 in [22] through the use of the Rogers–Ramanujan and the Andrews–Gordon identities. Using the functional relation in (9), this can be used to produce yet another proof of (1).

Bytsko [4] proved the identity

$$L_R\left(1-\frac{1}{\sqrt{2}}\right) + L_R\left(\sqrt{2}-1\right) = \frac{\pi^2}{8} \tag{10}$$

as the $k=2$ case of the formula

$$\sum_{i=1}^{k-1} L_R\left(\frac{\sin^2\frac{\pi}{3k+2}}{\sin^2\frac{(i+1)\pi}{3k+2}}\right) + L\left(\frac{\sin\frac{\pi}{3k+2}}{\sin\frac{(k+1)\pi}{3k+2}}\right) = \frac{\pi^2}{6}\frac{3k}{3k+2}$$

given in [4]; we see that (10) is equivalent to (8), which, as indicated above, is equivalent to (1).

2.3. **Ramanujan's inverse tangent integral.** Integrals of the form

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\arctan t}{t}\, dt$$

were of interest to Ramanujan, and remarkable results on the special function Ti_2 defined above were given in his 1915 article [21] (cf. [1, §17], [26]). From the series expansion

$$\mathrm{Ti}_2(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k-1}}{(2k-1)^2},$$

we obtain that

$$\mathrm{Ti}_2(x) = \frac{1}{2i}\left(\mathrm{Li}_2(ix) - \mathrm{Li}_2(-ix)\right).$$

So, we find that the expressions in (3), (4), and (5) are naturally expressible as specific values of Ti_2. Ramanujan introduced the identity whereby

$$\sum_{n=0}^{\infty}\frac{\sin(4n+2)x}{(2n+1)^2} = \mathrm{Ti}_2(\tan x) - x\ln\tan x \tag{11}$$

for $0 < x < \frac{1}{2}\pi$, and noted that this may be proved by applying term-by-term differentiation to the above series [21] (cf. [1, §17]). A corrected version [1, p. 365] of Ramanujan's formula for $\mathrm{Ti}_2(\sqrt{2}-1)$ is such that:

$$\sum_{n=0}^{\infty}\frac{(-1)^{n(n-1)/2}}{(2n+1)^2} = \sqrt{2}\mathrm{Ti}_2\left(\sqrt{2}-1\right) + \frac{\pi}{4\sqrt{2}}\ln(1+\sqrt{2}). \tag{12}$$

Also, from Ramanujan's identity in (11), we obtain that

$$\mathrm{Ti}_2(1) = \frac{3}{2}\mathrm{Ti}_2(2-\sqrt{3}) + \frac{1}{8}\pi \ln(2+\sqrt{3}), \tag{13}$$

and we find that the above equalities due to Ramanujan in 1915 [21] (cf. [1, §17]) are equivalent to our formulas for (3) and (4), which we had proved in a completely different way in [7]. Ramanujan's formulas in (12) and (13) were recently noted in [20], again in the context of applications pertaining to the Barnes G-function. Our discovery presented in [7] given by the equality in (5) may be rewritten so that

$$\mathrm{Ti}_2\left(\frac{1}{\sqrt{3}}\right) = \frac{3\psi^{(1)}\left(\frac{1}{6}\right) + 15\psi^{(1)}\left(\frac{1}{3}\right) - 6\sqrt{3}\pi\ln(3) - 16\pi^2}{72\sqrt{3}}. \tag{14}$$

This can also be proved using Ramanujan's identity

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} \frac{\cos^{2n+1}x + \sin^{2n+1}x}{(2n+1)^2} = \mathrm{Ti}_2(\tan x) + \frac{1}{2}\pi\ln(2\cos x)$$

for $0 < x < \frac{1}{2}\pi$ [21], but this is nontrivial in the sense that plugging $x = \frac{\pi}{6}$ into the above series produces a linear combination of the hypergeometric series

$${}_3F_2\!\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2}\end{matrix} \,\middle|\, \frac{1}{4}\right] \quad \text{and} \quad {}_3F_2\!\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2}\end{matrix} \,\middle|\, \frac{3}{4}\right],$$

which computer algebra systems such as Maple 2020 are not able to evaluate.

2.4. **Sherman's and Bradley's formulas.** The main transform from [7], our proof of which relied on results from our 2022 article [8], is such that

$$\frac{1}{1+z}\sum_{n=0}^{\infty} \frac{\left(\frac{16z}{(1+z)^2}\right)^n}{(2n+1)^2\binom{2n}{n}} = \mathrm{sgn}(z)\frac{i\left[\mathrm{Li}_2\left(-\sqrt{-z}\right) - \mathrm{Li}_2\left(\sqrt{-z}\right)\right]}{2\sqrt{z}} \tag{15}$$

holds if both sides converge for real z. Our proof of this in [7] relied on the generating function for Legendre polynomials together with a fractional calculus-derived series transform from the 2022 article [8]. A different formulation of this result was given in an unpublished note by Sherman in 2000 [23]. In [23], by integrating the Maclaurin series expansion

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}} \frac{(4x)^n}{2n+1} = \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}},$$

it was shown that

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}} \frac{(4x)^n}{(2n+1)^2}$$

is expressible as a linear combination of

$$\chi_2\left(e^{i\arcsin\sqrt{x}}\right)$$

and elementary expressions, in contrast to our identity in (15) [7]. It appears that our dilogarithm transform identity indicated in [7, p. 36] had not been considered previously. With regard to our formula in (15) and its derivation in [7], the following closely related formula was proved in a different way in [3]:

$$\int_0^x \ln(\tan\theta)\,d\theta = x\ln\tan x - \frac{1}{4}\sum_{k=0}^{\infty} \frac{(2\sin 2x)^{2k+1}}{(2k+1)^2\binom{2k}{k}}. \tag{16}$$

Bradley [3] also showed that

$$L(2,\chi_6) = \frac{\pi\sqrt{3}}{18}\ln 3 + \frac{1}{2}\sum_{k=0}^{\infty}\frac{3^k}{(2k+1)^2\binom{2k}{k}},$$

which, together with (16), can be used to give an equivalent formulation of (14), where the expression χ_6 denotes the non-principal Dirichlet character modulo 6. This is shown using an equivalent formulation of Ramanujan's 1915 identity in (11) together with (16), in contrast to our methods from [7].

An evaluation for $\mathrm{Ti}_2\left(\frac{\sqrt{3}}{3}\right)$ was also given in 1984 in [12], using a previously known relation [16, p. 106] involving Ti_2 and the special function known as the Clausen integral.

3. Double series

We conclude by briefly considering how the special values for χ_2 and Ti_2 considered in this article may be applied using our previous work on double series [6, 9]. As a special case of a hypergeometric transform introduced in [6] using the FL-based evaluation technique from [10], it was shown that: For a suitably bounded parameter p,

$$\frac{\pi}{2}\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m p^n \frac{\binom{2m}{m}^2\binom{2n}{n}}{m+n+1} \tag{17}$$

equals

$$\frac{-1}{\sqrt{p}} \times \left(\mathrm{Li}_2\left(-2\sqrt{\frac{p}{\left(\sqrt{1-4p}+1\right)^2}}\right) - \mathrm{Li}_2\left(2\sqrt{\frac{p}{\left(\sqrt{1-4p}+1\right)^2}}\right)\right).$$

In [9], we had applied this identity for (17) together with the known closed form for $\mathrm{Li}_2(\sqrt{2}-1) - \mathrm{Li}_2(1-\sqrt{2})$ to obtain new bivariate hypergeometric series evaluations. Setting $p = \frac{1}{20}$ in (17) and using the closed form in (2), we obtain the remarkable formula

$$\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m\left(\frac{1}{20}\right)^n\frac{\binom{2m}{m}^2\binom{2n}{n}}{m+n+1} = \frac{\sqrt{5}\pi}{3} - \frac{6\sqrt{5}\ln^2(\phi)}{\pi}.$$

Summing over $n \in \mathbb{N}_0$, we obtain an inevaluable ${}_2F_1\left(\frac{1}{5}\right)$-series; summing over $m \in \mathbb{N}_0$, we obtain a ${}_3F_2(1)$-series with no closed form. Similarly, by setting $p = -\frac{1}{12}$ in (17) and using Ramanujan's formula in (13), we may obtain that

$$\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m\left(-\frac{1}{12}\right)^n\frac{\binom{2m}{m}^2\binom{2n}{n}}{m+n+1} = \frac{16G}{\sqrt{3}\pi} - \frac{2\ln\left(2+\sqrt{3}\right)}{\sqrt{3}}.$$

Summing over $n \in \mathbb{N}_0$, we obtain an inevaluable ${}_2F_1\left(-\frac{1}{3}\right)$-series; summing over $m \in \mathbb{N}_0$, we again obtain a ${}_3F_2(1)$-series that does not admit any closed form. We leave it to a separate project to pursue a full exploration of the application of the techniques from [6, 9] together with the special values for χ_2 and Ti_2 considered in this article.

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