# Irish Mathematical Society
# Cumann Matamaitice na hÉireann

# Bulletin

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# Irish Mathematical Society Bulletin

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

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Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

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## EDITORIAL

A number of Irish institutes have changed status and become full universities. More accurately, they have ceased to exist as independent bodies, and have been absorbed into new ones. This is reflected in the institutional affiliations listed for some of the Society's officers, where new acronyms appear. The process has been underway for a couple of years, and may continue. For the benefit of members who may not be aware of this, here are some details:
ATU is the *Atlantic Technological University,* with campuses in Donegal, Sligo, Mayo and Galway. It incorporates the former Sligo IT, Letterkenny IT and Galway-Mayo IT.
MTU is *Munster Technological University,* which has six campuses in Cork and Kerry. Its formation involved a merger of Cork IT and Tralee IT, and the end of their separate existence and acronyms.
SETU is the *South-East Technological University,* with campuses in Waterford and Carlow. It embraces the resources of the former Waterford IT and Carlow IT.
TUD is *Technological University Dublin,* with campuses in Dublin and Kildare, including all resources of the former Dublin IT.
TUS is the *Technological University of the Shannon: Midwest,* with six campuses in Westmeath, Tipperary, Limerick and Clare.

This year's Annual Scientific Meeting will be held at TUD on 1 and 2 September. Website: `https://www.tudublin.ie/mathematics/ims-2022`.

The IMS Committee shared the common distress and sympathy with Ukraine, and put a message on our webpage, and cancelled plans to support travel to the ICM. The International Mathematical union decided to cancel the St Petersburg event, and proceed with the International Congress as a virtual event, free to all. To participate in the meeting, from 6-14 July, register at `https://www.mathunion.org/`.

The document *ICM 2022 Section Descriptions* `https://www.mathunion.org/fileadmin/IMU/Publications/CircularLetters/2019-2020/IMU%20AO%20CL%2012_2020_ICM2022_structure.pdf`, prepared by an advisory committee of the IMU, gives a useful up-to-date outline summary of the current state of research right across the whole discipline.

We remind meeting organisers that the normal deadline for reports to the Bulletin is 15 December for the Winter issue.

Correction: In Bulletin 88, on page 87, the solution to problem 86.3 was given, but was mistakenly labelled as the solution to problem 85.3.

For a limited time, beginning as soon as possible after the online publication of this Bulletin, a printed and bound copy may be ordered online on a print-on-demand basis at a minimal price[1].

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[1]Go to `www.lulu.com` and search for *Irish Mathematical Society Bulletin*.

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UCC: http://www.ucc.ie/en/matsci/postgraduate/
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://sarah.mitchell@ul.ie

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor.

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# NOTICES FROM THE SOCIETY

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(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

| | |
|---|---|
| Institutional member | €200 |
| Ordinary member | €30 |
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| DMV, IMTA, NZMS or RSME reciprocity member | €15 |
| AMS reciprocity member | $20 |
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(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

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(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form, available at

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with one year's subscription to:

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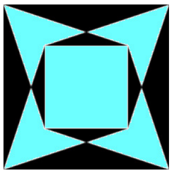

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# The life and work of David W. Lewis (1944–2021)

THOMAS UNGER

## 1. Life

David William Lewis was born in Douglas, Isle of Man, on 21st February 1944. He attended Douglas High School where he developed an interest in physics and astronomy, and ultimately mathematics. He went on to the University of Liverpool, and after completing his BSc degree in 1965 commenced doctoral studies in topology under the guidance of Terry Wall. When his PhD funding ended in 1968, he found employment in UCD where he was appointed as an assistant lecturer at the Mathematics Department. He continued working on his doctoral thesis, shifting from topology to algebra and specifically to the area of quadratic and hermitian forms.

Figure 1.1. David and Anne Lewis on the occasion of David's retirement conference in 2009. Photograph: M. Mackey.

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David was awarded the PhD degree in 1979 by the National University of Ireland. He was promoted to senior lecturer in 1987 and awarded a DSc degree in 1992. In 1997 he was promoted to associate professor and then in 2006 to full professor. From 1999 to 2002 he was head of department. Colleagues have commented on his fairness and effectiveness in this onerous role.

From the late 1980s David was involved in an Erasmus exchange with the University of Ghent. This was an offshoot of a collaboration with Jan Van Geel, and resulted later on in two of David's four PhD students. From 1997 until 2006 he was the local coordinator of two successive European research training networks for PhD students and post-docs (both called *Algebraic K-Theory, Linear Algebraic Groups and Related Structures*), expertly managed by Ulf Rehmann at the University of Bielefeld. In this context a major conference and a smaller workshop were co-organized by David in UCD in 1999 and 2004, respectively.

For more on David's early years growing up in Douglas, his university career, and his working life at UCD, see the interview [50] by Gary McGuire.

After his retirement in 2009 David remained research active as emeritus professor for a good number of years. In late 2013 he was diagnosed with Parkinson's disease. This led to a gradual decline in his health. David passed away peacefully on 20th August 2021, his wife Anne having pre-deceased him by three years. David and Anne are survived by their three sons Alan, Stephen and Gareth and their families. Their only daughter Joanne had passed away at a young age.

David will be fondly remembered for his fine qualities as a mathematician and the pleasure of collaborating with him, and for his friendship, kindness, thoughtfulness, sense of humour, humility and dedication to his family.

## 2. Work

David published more than 60 papers (including a number of surveys and expository papers), one volume of conference proceedings [2] and a book on matrix theory [41]. He also maintained a website about mathematicians from the Isle of Man and the Manx diaspora, cf. [44].

David's PhD thesis contained a number of significant results as well as the germs of ideas that where fleshed out in later papers. His early publications match up with the chapters in his thesis [28] almost one-to-one, cf. [25], [26], [27], [29], [30], [31].

David made numerous contributions to the algebraic theory of quadratic forms and related areas, such as central simple algebras with involution. Below I will describe some of those results with the aim of allowing the reader to form a reasonable impression of David's research interests and the impact of his work. My selection of topics is by no means exhaustive. I will also indicate some noteworthy extensions and generalizations by other researchers of David's work. For the benefit of the readers of the *Bulletin* I have the kept the style expository.

### 2.1. Some background material.

Consider a pair $(R,\sigma)$ where $R$ is a unital ring, not necessarily commutative, and $\sigma : R \to R$ is an involution, i.e., an anti-automorphism of order 2. With $(R,\sigma)$ we can associate the Witt group $W^\varepsilon(R,\sigma)$ of isometry classes of nonsingular $\varepsilon$-hermitian forms $\varphi : M \times M \to R$, where metabolic forms are identified with zero. In this notation $\varepsilon$ is a central element in $R$ such that $\sigma(\varepsilon)\varepsilon = 1$, $M$ is a finitely generated projective right $R$-module, $\varepsilon$-hermitian means that $\varphi$ is bi-additive and satisfies $\varphi(x\alpha, y\beta) = \sigma(\alpha)\varphi(x,y)\beta$ and $\varphi(y,x) = \varepsilon\sigma(\varphi(x,y))$ for all $x,y \in M$ and all $\alpha,\beta \in R$, nonsingular means that the $R$-linear map $M \to M^*,\ x \mapsto [y \mapsto \varphi(x,y)]$ to the dual module (considered as a right $R$-module via $f\alpha := \sigma(\alpha)f$ for all $f \in M^*$ and all $\alpha \in R$) is an isomorphism, and metabolic means that $M$ contains a direct summand that coincides with its orthogonal module with respect to $\varphi$. The group operation is induced by the orthogonal sum $\varphi_1 \perp \varphi_2$, defined on $M_1 \oplus M_2$ by

$$\varphi_1 \perp \varphi_2(x_1 + x_2, y_1 + y_2) := \varphi_1(x_1, y_1) + \varphi_2(x_2, y_2)$$

for all $x_i, y_i \in M_i$, $i = 1, 2$. Often one only considers (or needs to consider) hermitian and skew-hermitian forms, which correspond to the cases $\varepsilon = 1$ and $\varepsilon = -1$, respectively.

Here are some examples (if $\varepsilon = 1$, we write $W$ instead of $W^1$):

$$W(\mathbb{Z}, \mathrm{id}) \cong W(\mathbb{R}, \mathrm{id}) \cong W^{\pm 1}(\mathbb{C}, {}^-) \cong W(\mathbb{H}, {}^-) \cong \mathbb{Z},$$
$$W(\mathbb{C}, \mathrm{id}) \cong W^{-1}(\mathbb{H}, {}^-) \cong \mathbb{Z}/2\mathbb{Z},$$
$$W^{-1}(\mathbb{R}, \mathrm{id}) \cong W^{\pm 1}(\mathbb{R} \times \mathbb{R}, \widehat{\ }) = 0,$$
$$W(\mathbb{Q}_2, \mathrm{id}) \cong \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

where $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are the integers, real numbers and complex numbers, as usual, $\mathbb{H}$ is Hamilton's quaternion algebra, $\mathbb{Q}_2$ is the field of 2-adic numbers, ${}^-$ denotes conjugation, $\widehat{\ }$ denotes the exchange involution, and $\cong$ denotes isomorphism.

If $R$ is commutative, the tensor product of $R$-modules induces a multiplication that turns $W^\varepsilon(R, \sigma)$ into a ring. If 2 is invertible in $R$ various simplifications can be made.

If $R = F$ is a field, $\varepsilon = 1$ and $\sigma = \mathrm{id}_F$ we obtain the Witt ring $W(F) := W^1(F, \mathrm{id}_F)$ of classes of symmetric bilinear forms on finite-dimensional $F$-vector spaces. If the characteristic of $F$ is different from 2, any symmetric bilinear form $b : V \times V \to F$ on a finite-dimensional $F$-vector space $V$ can be uniquely identified with a quadratic form $q_b$ over $F$ via $q_b(x) := b(x, x)$ and vice versa via $b_q(x, y) := \frac{1}{2}(q(x + y) - q(x) - q(y))$. Let us consider a quadratic form $q : V \to F$ where $\dim_F V = n$. After choosing an $F$-basis $(e_1, \ldots, e_n)$ of $V$ we can represent $q$ by the symmetric matrix $(b_q(e_i, e_j)) \in M_n(F)$. A different choice of basis yields a congruent matrix. If $q_1$ and $q_2$ are quadratic forms over $F$ such that their associated matrices are congruent, then $q_1$ and $q_2$ are isometric, and we write $q_1 \simeq q_2$. It is a standard result that if the characteristic of $F$ is different from 2, one can find a basis of $V$ that is orthogonal with respect to $b_q$, i.e., such that the matrix of $q$ is a diagonal matrix $\mathrm{diag}(a_1, \ldots, a_n)$. We then write $q \simeq \langle a_1, \ldots, a_n\rangle$ and note that

$$\langle a_1, \ldots, a_n\rangle = \langle a_1\rangle \perp \cdots \perp \langle a_n\rangle.$$

The quadratic form $\langle a_1, \ldots, a_n\rangle$ is nonsingular if and only if $\det \mathrm{diag}(a_1, \ldots, a_n) \neq 0$ if and only if $a_1, \ldots, a_n$ are nonzero. Furthermore, $\langle a_1, \ldots, a_n\rangle$ is isotropic over $F$ if the quadratic polynomial $\sum_{i=1}^n a_i x_i^2$ has a nontrivial zero over $F$. Arbitrary permutations of the entries of $\langle a_1, \ldots, a_n\rangle$, as well as multiplication of the entries by nonzero squares, give rise to isometric forms. For example, every $a_i$ that is a nonzero square in $F$ can be replaced by 1. In particular, we can view the nonzero entries $a_i$ as elements of the square class group $F^\times / F^{\times 2}$.

Using diagonal notation, the sum and product in $W(F)$ are induced by

$$\langle a_1, \ldots, a_n\rangle \perp \langle b_1, \ldots, b_m\rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle$$

and

$$\langle a_1, \ldots, a_n\rangle \otimes \langle b_1, \ldots, b_m\rangle = \langle a_1 b_1, a_1 b_2, \ldots, a_i b_j, \ldots, a_n b_m\rangle,$$

respectively. Hyperbolic forms are finite orthogonal sums of the hyperbolic plane $\langle 1, -1\rangle$. They coincide with the metabolic forms in characteristic not 2, and so they are identified with the zero element of $W(F)$. The identity element of $W(F)$ is the class of the form $\langle 1\rangle$. If $q$ is a quadratic form over $F$, we denote its class in $W(F)$ by $[q]$. For example,

$$0 = [\langle 1, -1\rangle] = [\langle 2, -2\rangle],\ 1 = [\langle 1\rangle],\ 2 = [\langle 1, 1\rangle].$$

For later use, let's look at a more elaborate example:

**Example 2.1.** Consider the field of rational numbers $F = \mathbb{Q}$ and let $q = [\langle 2, 3\rangle] \in W(F)$. Then:

$$
\begin{aligned}
q^2 &= [\langle 2,3\rangle \otimes \langle 2,3\rangle] = [\langle 4,6,6,9\rangle] = [\langle 1,6,6,1\rangle],\\
2^2 &= [\langle 1,1\rangle \otimes \langle 1,1\rangle] = [\langle 1,1,1,1\rangle],\\
q^2 - 2^2 &= [\langle 1,6,6,1\rangle \perp \langle -1,-1,-1,-1\rangle] = [\langle 6,6,-1,-1\rangle],\\
q(q^2 - 2^2) &= [\langle 2,3\rangle \otimes \langle 6,6,-1,-1\rangle] = [\langle 12,12,-2,-2,18,18,-3,-3\rangle]\\
&= [\langle 3,3,-2,-2,2,2,-3,-3\rangle] = [\langle 2,-2\rangle \perp \langle 2,-2\rangle \perp \langle 3,-3\rangle \perp \langle 3,-3\rangle] = 0.
\end{aligned}
$$

The study of Witt rings of fields started with Witt's seminal paper from 1937 and received a major boost by Pfister in the 1960s. The resulting *algebraic theory of quadratic forms* has been a fruitful research area from its inception, with deep connections to many other areas in mathematics. The standard references are the books by Lam [23], Scharlau [56] and Knus [21]. The monograph [12] focusses on the modern geometric theory of quadratic forms. For more on the history of quadratic forms I refer the reader to [57].

Let $K$ be a field of characteristic $\neq 2$. A finite-dimensional $K$-algebra $A$ is called central simple over $K$ if $A$ has no non-trivial two-sided ideals and the centre of $A$ is $K$. Let $\sigma$ be an involution on $A$ and let $F = \{a \in K \mid \sigma(a) = a\}$ be the fixed field of $\sigma$. The pair $(A, \sigma)$ is called a *central simple $F$-algebra with involution* (in this terminology we emphasize $F$ rather than $K$ even though $Z(A) = K$). If $F = K$, then $\sigma$ is said to be of the first kind. Otherwise $\sigma$ is said to be of the second kind (or unitary), and we must have $[K : F] = 2$. In this case, it is customary to allow the possibility that $K$ is not a quadratic extension field of $F$, but a double-field isomorphic to $F \times F$. When this happens, $A$ is not simple, but a product of two simple $F$-algebras that are mapped to each other by $\sigma$. The motivation for allowing this possibility is that it can occur after scalar extension. For ease of exposition I will ignore this situation and assume that the centre of $A$ is a field. The standard reference is *The Book of Involutions* [22], which also contains many notes with historical pointers.

Real square matrices with transposition $(M_n(\mathbb{R}), t)$ and complex square matrices with conjugate transposition $(M_n(\mathbb{C}), *)$ are easy examples of central simple $\mathbb{R}$-algebras with involution of the first, and second kind, respectively. Hamilton's quaternion algebra with quaternion conjugation $(\mathbb{H}, \bar{\ })$ is a central simple $\mathbb{R}$-algebra with involution of the first kind.

If $(D, \vartheta)$ is a central simple $F$-algebra with involution where $D$ is a division algebra and if $h : V \times V \to D$ is a nonsingular $\varepsilon$-hermitian form over $(D, \vartheta)$, where $V$ is a finite-dimensional right $D$-vector space, then $(\mathrm{End}_D(V), \mathrm{ad}_h)$ is a central simple $F$-algebra with involution of the same kind as $\vartheta$. Here $\mathrm{ad}_h$, the adjoint involution of $h$, is defined by the property

$$h(x, f(y)) = h(\mathrm{ad}_h(f)(x), y),\ \forall x, y \in V,\ \forall f \in \mathrm{End}_D(V).$$

In fact, *all* central simple $F$-algebras with involution are of this form for some $D$, unique up to isomorphism, and some $h$, determined up to multiplication by a scalar in $F^\times$. For example, $(M_n(\mathbb{R}), t) \cong (\mathrm{End}_{\mathbb{R}}(\mathbb{R}^n), \mathrm{ad}_h)$, where $h = n \times \langle 1\rangle := \langle 1, \dots, 1\rangle$ ($n$ copies of 1).

Because of this correspondence between involutions and $\varepsilon$-hermitian forms, central simple algebras with involution can be thought of as generalizations of quadratic forms. However, quadratic forms also show up in other ways. Important examples are obtained via the $K$-linear reduced trace map $\mathrm{Trd}_A : A \to K$, which is defined as follows: let $\Omega$ be a splitting field of $A$, i.e., $A \otimes_K \Omega \cong M_n(\Omega)$ (such a field always exist, cf. [22, Theorem (1.1)]). Then for $a \in A$, $\mathrm{Trd}_A(a)$ is the trace of the matrix of the image of $a$

under scalar extension to $\Omega$. One can show that $\mathrm{Trd}_A(a) \in K$ and is independent of the choice of $\Omega$. The trace form of $A$ is the symmetric bilinear form

$$T_A : A \times A \to K,\ (x, y) \mapsto \mathrm{Trd}_A(xy).$$

The associated quadratic form $x \mapsto T_A(x, x) = \mathrm{Trd}_A(x^2)$ is usually also denoted by $T_A$. The involution trace form of $(A, \sigma)$,

$$T_{(A,\sigma)} : A \times A \to K,\ (x, y) \mapsto \mathrm{Trd}_A(\sigma(x)y),$$

is symmetric bilinear over $F = K$ if $\sigma$ is of the first kind and hermitian over $(K, \sigma|_K)$ if $\sigma$ is of the second kind. The forms $T_A$ and $T_{(A,\sigma)}$ are both nonsingular.

2.2. **Exact sequences of Witt groups.** In many situations, Witt groups cannot be computed explicitly but can be related to other Witt groups via exact sequences. Let $F$ be a field of characteristic not 2 and let $F(\sqrt{a})$ be a quadratic field extension of $F$. Denote the map induced by $\sqrt{a} \mapsto -\sqrt{a}$ on the field $F(\sqrt{a})$ by $^-$ as usual. Milnor and Husemoller showed that there is an exact sequence

$$0 \longrightarrow W(F(\sqrt{a}), ^-) \xrightarrow{\pi} W(F) \xrightarrow{\rho} W(F(\sqrt{a})), \tag{2.1}$$

where $\pi$ is induced by the trace $\mathrm{Tr}_{F(\sqrt{a})/F}$ and $\rho$ is induced by base change to $F(\sqrt{a})$, cf. [51, Appendix 2].

Let $b \in F$ be nonzero and let $D$ denote the generalized quaternion algebra $(a, b)_F$, i.e., the $F$-algebra generated by symbols $i$ and $j$ that satisfy $i^2 = a,\ j^2 = b$ and $ij = -ji$. We assume that $D$ is a division algebra and denote the map induced by $i \mapsto -i$, $j \mapsto -j$ by $^-$ as well since we can identify $F(\sqrt{a})$ with a subfield of $D$ in the obvious way. For example, if $F = \mathbb{R}$ and $a = b = -1$, then $D = \mathbb{H}$ and $F(\sqrt{a}) = \mathbb{C}$.

In the spirit of (2.1) David proved that the sequence

$$0 \to W(D, ^-) \to W(F(\sqrt{a}), ^-) \to W^{-1}(D, ^-) \to W(F(\sqrt{a})) \tag{2.2}$$

is exact in his 1979 paper [26]. (For ease of exposition I will not describe the maps that occur in this exact sequence and those in the remainder of this section. They are similar to the maps $\pi$ and $\rho$ in (2.1).) In [3, Appendix 2] this sequence was generalized by Parimala, Sridharan and Suresh to the exact sequence

$$W^\varepsilon(A, \sigma) \to W^\varepsilon(\widetilde{A}, \sigma_1) \to W^{-\varepsilon}(A, \sigma) \to W^\varepsilon(\widetilde{A}, \sigma_2), \tag{2.3}$$

where $(A, \sigma)$ is a central simple $F$-algebra with involution, $\widetilde{A}$ is the centralizer of a skew-symmetric unit $\lambda$ in $A$ with the property that $F(\lambda)$ is a quadratic extension of $F$, $\sigma_1 = \sigma|_A$, and $\sigma_2 = \mathrm{Int}(\mu^{-1}) \circ \sigma_1$ for a skew-symmetric unit $\mu$ in $A$ that anti-commutes with $\lambda$, where Int denotes inner automorphism. (Note that not all central simple algebras with involution contain such elements $\lambda$ and $\mu$.) The "key exact sequence" (2.3) was used by Bayer-Fluckiger and Parimala in their proof of Serre's Conjecture II for classical groups, cf. [4, 5].

In [31] David extended the sequences (2.1) and (2.2) further to the right, resulting in the exact sequences

$$0 \to W(F(\sqrt{a}), ^-) \to W(F) \to W(F(\sqrt{a})) \to W(F) \to W(F(\sqrt{a}), ^-) \to 0 \tag{2.4}$$

and

$$\begin{aligned} 0 \to W(D, ^-) \to W(F(\sqrt{a}), ^-) \to W^{-1}(D, ^-) \to W(F(\sqrt{a})) \\ \to W^{-1}(D, ^-) \to W(F(\sqrt{a}), ^-) \to W(D, ^-) \to 0. \end{aligned} \tag{2.5}$$

Anybody looking at the exact sequences (2.4) and (2.5) will surely not be able to resist folding them into exact polygons. In fact, it turns out that these sequences are special cases of an exact octagon involving Witt groups of Clifford algebras, as David showed in [34]: Let $C$ denote the Clifford algebra $C(q)$ of a nonsingular quadratic form $q$ over

$F$, and let $C' := C(q \perp \langle a \rangle)$ where $a \in F$ is nonzero. The algebra $C$ carries the two natural involutions $\sigma_1$ and $\sigma_{-1}$, where $\sigma_{\pm 1}(x) = \pm x$ for all $x \in V$, where $V$ is the finite-dimensional $F$-vector space on which $q$ is defined. The following octagon is exact:

$$\begin{array}{ccccc}
 & W(C,\sigma_{-1}) & \longrightarrow & W(C',\sigma_{-1}) & \\
\nearrow & & & & \searrow \\
W(C',\sigma_1) & & & & W^{-1}(C,\sigma_1) \\
\uparrow & & & & \downarrow \\
W(C,\sigma_1) & & & & W^{-1}(C',\sigma_1) \\
\nwarrow & & & & \swarrow \\
 & W^{-1}(C',\sigma_{-1}) & \longleftarrow & W^{-1}(C,\sigma_{-1}) &
\end{array}.$$

In [33], David obtained equivariant versions (i.e., the forms are invariant under the action of a finite group) of (2.4) and (2.5), rolled up into exact octagons. These are related to work of Ranicki on $L$-groups, cf. [55].

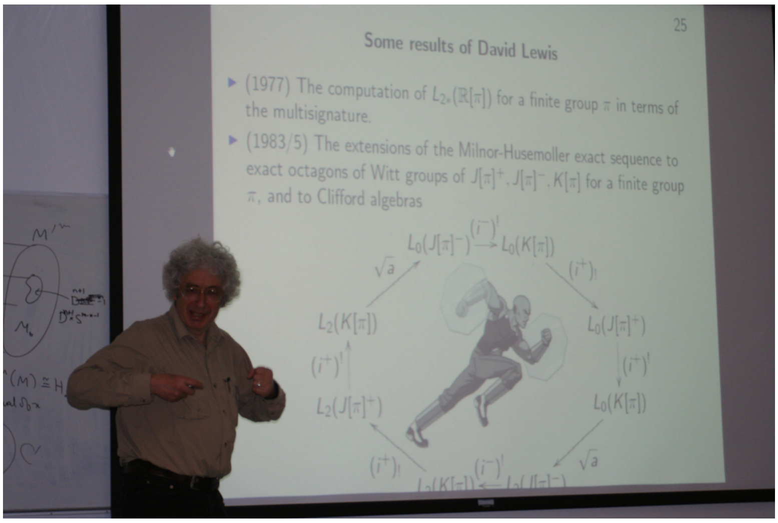


FIGURE 2.1. Andrew Ranicki (1948–2018) on the occasion of David's retirement conference in 2009. Photograph: S. McGarraghy.

Inspired by (2.4), Grenier-Boley and Mahmoudi extended (2.3) further to the right and obtained the exact octagon

$$\begin{array}{ccccc}
 & W^{\varepsilon}(A,\sigma) & \longrightarrow & W^{\varepsilon}(\widetilde{A},\sigma_1) & \\
\nearrow & & & & \searrow \\
W^{-\varepsilon}(\widetilde{A},\sigma_2) & & & & W^{-\varepsilon}(A,\sigma) \\
\uparrow & & & & \downarrow \\
W^{\varepsilon}(A,\sigma) & & & & W^{\varepsilon}(\widetilde{A},\sigma_2) \\
\nwarrow & & & & \swarrow \\
 & W^{\varepsilon}(\widetilde{A},\sigma_1) & \longleftarrow & W^{-\varepsilon}(A,\sigma) &
\end{array},$$

which remains valid in the equivariant case. As an application they proved that if $(A, \sigma)$ is a central simple algebra with involution of the first kind over $F$, then $W(F)$ is finite if and only if both $W^{\varepsilon}(A,\sigma)$ and $W^{-\varepsilon}(A,\sigma)$ are finite, generalizing a similar observation of David's for quaternion algebras, cf. [31].

The exact sequences of Lewis, Grenier-Boley and Mahmoudi, and Parimala, Sridharan and Suresh were further generalized to Witt groups of Azumaya algebras with involution by First in his impressive 2022 paper [14], which also contains a wealth of information about Azumaya algebras with involution and several important applications.

2.3. **Annihilating polynomials.** In Example 2.1 we saw that the element $q = [\langle 2,3\rangle] \in W(\mathbb{Q})$ is a root of the polynomial $p(x) = x(x^2 - 2^2)$, which is then said to be an annihilating polynomial of $q$. In his 1987 paper [38], David proved:

**Theorem 2.2.** *Let $F$ be a field of characteristic not 2. Let $\varphi$ be any quadratic form of dimension $n$ over $F$, and let $q = [\varphi] \in W(F)$. Then $p_n(q) = 0$ in $W(F)$, where $p_n(x)$ is the monic integer polynomial defined by*

$$p_n(x) := \begin{cases} x(x^2-2^2)(x^2-4^2)\cdots(x^2-n^2) & \text{if } n \text{ is even} \\ (x^2-1^2)(x^2-3^2)\cdots(x^2-n^2) & \text{if } n \text{ is odd} \end{cases}.$$

In other words, the theorem says that Witt rings of fields of characteristic not 2 are integral rings. This had been known for a long time, but David's result provided the first examples of polynomials that annihilated particular classed of quadratic forms. There are several proofs of Theorem 2.2, but the slickest is due to Leung. It goes via induction on $n$, using the recurrence relation $p_n(x) = (x+n)p_{n-1}(x-1)$, cf. [38, Comment 1].

As an application of Theorem 2.2 David obtained the standard structural properties of Witt rings, the main ones being: they have no odd torsion, no odd-dimensional zero divisors and no nontrivial idempotents; their prime ideals are determined by the orderings of the underlying field. Proofs of these facts are mostly omitted from [38], but can be found in David's 1989 paper [40]. For a description of subsequent work on annihilating polynomials by David and others up to 2000, I refer to David's survey [43]. David's doctoral students Seán McGarraghy and Stefan De Wannemacker (1971–2013) also worked on this topic, cf. [10], [11], [49], [48].

2.4. **Levels of division algebras.** A field $F$ is called real if $-1$ cannot be written as a sum of squares of elements of $F$ (real fields are sometimes called formally real fields, but this practice is slowly disappearing). Note that the characteristic of $F$ must thus be zero. By the Artin-Schreier theorem (see for example [23, VIII, Theorem 1.10]), $F$ is real if and only if $F$ has at least one ordering.

If $-1$ can be written as a sum of squares in $F$, this begs the question how many squares are needed. The level $s(F)$ of a non-real field $F$ is defined to be the smallest integer $n$ such that $-1$ is a sum of $n$ squares in $F$. The level of a real field is defined to be $\infty$. In 1932 van der Waerden posed the following problem, cf. [58]:

> *Wenn in einem Körper die Zahl* $-1$ *Summe von* 3 *Quadraten ist, so auch von* 2 *Quadraten; wenn von* 5, 6 *oder* 7*, so auch von* 4*; wenn von* 15 *oder weniger, so auch von* 8.

In other words, he asked if 1, 2, 4 and 8 are the only possible values of the level $\leq 15$. In 1934, H. Kneser proved that this is indeed the case, and that in addition all multiples of 16, except those of the form $2^{8h}g + 16h$, could occur, cf.[20, Satz 2]. The fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-7})$ have level 1, 2 and 4, respectively. Also, at the time, algebraic number fields were known to have level 1, 2, 4 or $\infty$, but there were no known fields of finite level $> 4$. It was not until 1963 that the complete solution was given by Pfister. He showed that the level of a field is either $\infty$ or a 2-power, and that every 2-power occurs as the level of some field, cf. [52].

The notion of level also makes sense for unitary rings that are not necessarily commutative, but Pfister's 2-power result no longer holds in general. For example, the ring $\mathbb{Z}/4\mathbb{Z}$ has level 3. A result of note is the topological proof of Dai, Lam and Peng of the fact that for any given positive integer $n$, the integral domain

$$\frac{\mathbb{R}[x_1,\dots,x_n]}{(1+x_1^2+\cdots+x_n^2)}$$

has level $n$, cf. [9].

For noncommutative rings several generalizations of the level have also been investigated. For example, the sublevel $\underline{s}(R)$ of a ring $R$ is the smallest positive integer $n$ such that 0 can be written as a sum of $n+1$ squares in $R$, and is $\infty$ if 0 cannot be written as a sum of squares.

In their 1985 paper [24] Leep, Shapiro and Wadsworth investigated sums of squares in central simple algebras. Recall that these are matrix algebras with entries in division algebras, finite-dimensional over their centre (assumed to be of characteristic not 2). They observed that by a result of Griffin and Krusemeyer (namely that if $R$ is a commutative ring with $2 \in R^\times$ and $n \geq 2$, then every element of $M_n(R)$ is a sum of 3 squares [16]) it suffices to consider division algebras $D$. They showed that $\underline{s}(D) < \infty$ if and only if $s(D) < \infty$ if and only if each element of $D$ is a sum of squares in $D$, cf. [24, Theorem D]. (Note that if $D$ is actually a field, then this result is an easy exercise.) In their investigations the trace form $T_D$ played a central role. David settled their Conjecture 3.6 and proved:

**Theorem 2.3** ([35])**.** *0 is a nontrivial sum of squares in $D$ if and only if the trace form $T_D$ is weakly isotropic.*

David had a particular interest in levels of quaternion division algebras. In 1989 he proved:

**Theorem 2.4** ([39, Propositions 2 and 3])**.**
(1) *There exist quaternion division algebras of level $2^k+1$ for all $k \geq 1$.*
(2) *There exist quaternion division algebras of level $2^k$ for all $k \geq 0$.*

The proof consists of constructing explicit families of quaternion division algebras whose levels are these prescribed values. For example, consider the rational function field $K = \mathbb{R}(x,y,z)$, the Laurent series field $F = K(\!(t)\!)$, and let $a = x^2+y^2+z^2$. Then $D = (a,t)_F$ is a division algebra of level 3, cf. [39, Proposition 1].

In light of Theorem 2.4, it is natural to ask a) whether values other than $2^k$ and $2^k+1$ can occur as the level of a quaternion division algebra, and b) if so, what these values are. In 2008, Hoffmann answered question a) in the affirmative. He proved that there are infinitely many quaternion division level values not of the form $2^k$ or $2^k+1$, cf. [17]. Note that this result did not yield any explicit new values. Indeed, question b) still seems to be an open problem.

In [36] David wondered if for a quaternion division algebra $D$ the level and sublevel are always related as follows: $s(D) = \underline{s}(D)$ or $s(D) = 1+\underline{s}(D)$. Hoffmann showed that this is indeed the case, cf. [18]. In fact, Hoffmann came up with the idea of the proof at the retirement conference in honour of David in 2009, where he gave a survey talk on levels and sublevels of rings.

For more information on levels, I refer to David's 1987 survey [37] in issue 19 of this *Bulletin* and the updated version [42] from 2001, as well as Hoffmann's more recent survey [19].

2.5. **Signatures of involutions.** Let $F$ be a real field and let $P$ be an ordering of $F$. We can think of $P$ as the set of nonnegative elements of $F$ with respect to some total order relation. For example, $\mathbb{Q}$ and $\mathbb{R}$ each have a unique ordering, $\mathbb{C}$ has no orderings, $\mathbb{Q}(\sqrt{2})$ has two orderings ($P_1$ where $\sqrt{2}$ is positive, and $P_2$ where $-\sqrt{2}$ is positive), $\mathbb{R}(t)$ has infinitely many orderings, and $\mathbb{R}((t))$ has two orderings (one where $t$ is positive and one where $-t$ is positive). The set of orderings of $F$ is denoted $X_F$ and called the space of orderings of $F$ (as it is a topological space). Let $q : V \to F$ be a nonsingular quadratic form over $F$. The (Sylvester) signature of $q$ at $P$ is the integer

$$\operatorname{sgn}_P q := \#\{a_i \in P\} - \#\{a_i \in -P\}.$$

For example, if $F = \mathbb{Q}(\sqrt{2})$ and $q \simeq \langle 1, \sqrt{2}\rangle$, then

$$\operatorname{sgn}_{P_1} q = 2 - 0 = 2 \quad \text{and} \quad \operatorname{sgn}_{P_2} q = 1 - 1 = 0.$$

The total signature of $q$ is the map

$$\operatorname{sgn} q : X_F \to \mathbb{Z},\ P \mapsto \operatorname{sgn}_P q.$$

The total signature yields a characterization of the torsion elements of the Witt ring: $[q] \in W_{\mathrm{tors}}(F)$ if and only if $\operatorname{sgn}_P q = 0$ for all $P \in X_F$. If $F$ is a nonreal field, then $W(F) = W_{\mathrm{tors}}(F)$, i.e., every element of $W(F)$ is torsion. Furthermore, for either type of field the torsion order is 2-primary. In other words, $[q]$ is torsion of and only if there exists a positive integer $\ell$ such that $2^\ell \times q$ is a hyperbolic form (we also say that $q$ is weakly hyperbolic). These fundamental results were established by Pfister in [53], and are referred to as *Pfister's local-global principle*. See also [23, VIII,§3].

Let $(A, \sigma)$ be a central simple $F$-algebra with involution, let $K = Z(A)$ and let $P \in X_F$ (ordering are always considered on the fixed field $F$ of $\sigma$). The involution $\sigma$ is said to be positive at $P$ if the involution trace form $T_{(A,\sigma)}$ is positive definite. This notion goes back to Weil [59]. A more fine-grained measure of positivity is given by the signature of $\sigma$ at $P$,

$$\operatorname{sgn}_P \sigma := \sqrt{\operatorname{sgn}_P T_{(A,\sigma)}},$$

introduced by David and Jean-Pierre Tignol for involutions of the first kind, cf. [46], and by Quéguiner for involutions of the second kind, cf. [54]. This definition extends the concept of signatures from quadratic forms to involutions. In the split case $(A, \sigma) = (\operatorname{End}_F(V), \operatorname{ad}_q)$ the signatures of the quadratic form $q$ and its adjoint involution $\operatorname{ad}_q$ satisfy $\operatorname{sgn}_P \operatorname{ad}_q = |\operatorname{sgn}_P q|$. Such a relationship holds more generally for signatures of hermitian forms over central simple $F$-algebras with involution and their adjoint involutions. The details are too technical to be discussed here. The interested reader may consult [1].

The involution $\sigma$ is said to be hyperbolic if there exists an element $e \in A$ such that $e^2 = e$ and $\sigma(e) = 1 - e$. Hyperbolic involutions were introduced in [6]. If $(A, \sigma) \cong (\operatorname{End}_D(V), \operatorname{ad}_h)$, then $\sigma$ is hyperbolic if and only if $h$ is hyperbolic. The involution $\sigma$ is weakly hyperbolic if there exists a positive integer $n$ such that the involution $* \otimes \sigma$ on $M_n(K) \otimes_K A \cong M_n(A)$ is hyperbolic, where $*$ denotes conjugate transposition. In 2003, David and I extended Pfister's local-global principle to central simple algebras with involution and to hermitian forms over such algebras, cf. [47]. See also [7].

2.6. **Classification of involutions.** Let $F$ be a field of characteristic $\neq 2$, let $V$ be a vector space of dimension $n$ over $F$ and let $q : V \to F$ be a nonsingular quadratic form on $V$. Assume that $q \simeq \langle a_1, \dots, a_n\rangle$. To $q$ we can associate its dimension, $\dim(q) = n$, its determinant $d(q) = a_1 \cdots a_n \cdot F^{\times 2} \in F^\times/F^{\times 2}$, and its Hasse invariant $s(q) =$ class of $\prod_{i<j}(a_i, a_j)_F$ in the Brauer group $\operatorname{Br}(F)$ (whose elements can be identified with the isomorphism classes of $F$-central division algebras, cf. [23, IV]) with the convention that $s(q) = 1$ if $n = 1$. If $F$ is in addition a real field, then $q$ has an associated total signature $\operatorname{sgn} q$, as we have seen above.

The dimension, determinant, Hasse invariant and total signature are called the "classical" invariants of quadratic form theory. They are isometry class invariants (and thus independent of the chosen diagonalization of $q$). In other words, if $q_1$ and $q_2$ are isometric, then they have the same classical invariants. The converse is false in general, but true under certain conditions on the third power of the *fundamental ideal* $I(F)$ of even-dimensional forms in the Witt ring $W(F)$. Indeed, in their seminal 1974 paper [13], Elman and Lam showed that if $I^3(F) = 0$, then quadratic forms are classified up to isometry by dim, $d$ and $s$, and if $I^3(F)$ is torsion-free, then quadratic forms are classified up to isometry by dim, $d$, $s$ and sgn.

In their 1998 paper [4] Bayer-Fluckiger and Parimala extended these classification results to isometry classes of hermitian forms over central simple algebras with involution for suitable generalizations of the classical invariants, under the assumption that $I^3(F(\sqrt{-1})) = 0$ (see [5]) and an additional assumption on $F$ when the involution is unitary.

In 1999, David and Tignol [46] obtained similar classification results for conjugacy classes of involutions on a given central simple algebra, again for suitable generalizations of the classical invariants (including signatures of involutions as described in the previous section) and under certain assumptions on $I^3(F)$ (keeping [5] in mind since they use the results of [4]) and $F$. (Two involutions $\sigma$ and $\sigma'$ on a central simple algebra $A$ are conjugate if and only if $(A, \sigma)$ and $(A, \sigma')$ are isomorphic as central simple algebras with involution.)

2.7. **Sesquilinear Morita theory.** Let $R$ be a commutative ring, and let $(A, \sigma)$ be an $R$-algebra with involution. Let $P$ be a faithful finitely generated projective right $A$-module and let $h_0 : P \times P \to A$ be a nonsingular $\varepsilon_0$-hermitian form over $(A, \sigma)$. Hermitian Morita theory asserts that the categories of (nonsingular) $\varepsilon$-hermitian forms over $(\mathrm{End}_A(P), \mathrm{ad}_{h_0})$ and of (nonsingular) $\varepsilon\varepsilon_0$-hermitian forms over $(A, \sigma)$ are equivalent. Furthermore, orthogonal sums and hyperbolic spaces are preserved under this correspondence, cf. [21, I,§9].

Restricting to central simple algebras with involution (over fields of characteristic not 2), the usefulness of hermitian Morita theory is immediately clear since questions about $\varepsilon$-hermitian forms can be reduced to forms over division algebras with involution, which is an advantage since in this situation—except in the case of skew-symmetric bilinear forms over fields—nonsingular forms are diagonalizable, cf. [32], and thus more amenable to computation.

The main result of [8] is a generalization of hermitian Morita theory on two levels: anti-automorphisms that are not assumed to be of order 2 and sesquilinear forms are considered instead of involutions and $\varepsilon$-hermitian forms, respectively. This 2013 paper, a collaboration with Anne Cortella, is David's final one.

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# Special values of Legendre's chi-function and the inverse tangent integral

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Abstract. In our recent publication in this Bulletin [88 Winter (2021), 31–37] a series transform proved via Fourier–Legendre theory and fractional operators in a 2022 article was applied to prove five two-term dilogarithm identities. One such identity gave a closed form for $\mathrm{Li}_2(\sqrt{2}-1) - \mathrm{Li}_2(1-\sqrt{2})$, and we had attributed this closed form to a 2012 article by Lima. However, as we review in our current article, there had actually been a number of previously published proofs of formulas that are equivalent to the closed-form evaluation for the equivalent expression $\chi_2(\sqrt{2}-1)$, letting $\chi_2$ denote the Legendre chi-function. We offer a brief survey of the history of special values for $\chi_2$ and the inverse tangent integral $\mathrm{Ti}_2$, in relation to the results given in our previous BIMS publication. Two of the two-term dilogarithm relations proved in this previous publication were actually introduced in 1915 by Ramanujan in an equivalent form in terms of the $\mathrm{Ti}_2$ function, which adds to the interest in the alternative proofs for these results that we had independently discovered. We also apply special values for $\chi_2$ and $\mathrm{Ti}_2$, together with a Legendre-polynomial based series transform, to obtain evaluations for rational double hypergeometric series with inevaluable single sums.

## 1. Introduction

In the 2022 article [8], the series transform reproduced as equation (2) in [7] was proved using Fourier–Legendre (FL) theory and fractional calculus, building on an FL-based integration method introduced in the 2019 research article [10]. Using this series transform from [8] together with the generating function for Legendre polynomials, we had proved in [7] five two-term dilogarithm evaluations. These five evaluations are reproduced below. We had incorrectly stated that the first out of the five equations listed below was introduced by Lima in 2012 [18], without our having been aware that an equivalent formulation of this first equation was given in terms of the Legendre chi-function in the 1958 text [15, p. 19]. Lima proved (1) in [18] and one of the main results in [18] follows from (1), but the fact that (1) was previously known, as far back as 1958 [15, p. 19], was not indicated anywhere in [18] or in the zbMATH review [2] of [18] (cf. [11]). Furthermore, while our method for proving the below results using Legendre polynomials is highly original, all of the five formulas below had been known prior to [7], without the author having been aware of this; see [21], [15, p. 19] and [12].

$$\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} - \frac{1}{2}\ln^2\left(1+\sqrt{2}\right) \tag{1}$$

$$\mathrm{Li}_2\left(\frac{1}{\phi^3}\right) - \mathrm{Li}_2\left(-\frac{1}{\phi^3}\right) = \frac{\phi^3\left(\pi^2 - 18\ln^2(\phi)\right)}{3\left(\phi^6-1\right)} \tag{2}$$

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$$\mathrm{Li}_2\left(i\left(2-\sqrt{3}\right)\right)-\mathrm{Li}_2\left(-i\left(2-\sqrt{3}\right)\right)=\frac{2i\sqrt{7-4\sqrt{3}}\left(8G-\pi\ln\left(2+\sqrt{3}\right)\right)}{3\left(8-4\sqrt{3}\right)} \tag{3}$$

$$\mathrm{Li}_2\left(i\left(\sqrt{2}-1\right)\right)-\mathrm{Li}_2\left(-i\left(\sqrt{2}-1\right)\right) \tag{4}$$
$$=\frac{1}{32}i\left(\sqrt{2}\left(\psi^{(1)}\left(\frac{1}{8}\right)+\psi^{(1)}\left(\frac{3}{8}\right)\right)+8\pi\ln\left(\sqrt{2}-1\right)-4\sqrt{2}\pi^2\right)$$

$$\mathrm{Li}_2\left(\frac{i}{\sqrt{3}}\right)-\mathrm{Li}_2\left(-\frac{i}{\sqrt{3}}\right)=\frac{i\left(3\psi^{(1)}\left(\frac{1}{6}\right)+15\psi^{(1)}\left(\frac{1}{3}\right)-6\sqrt{3}\pi\ln(3)-16\pi^2\right)}{36\sqrt{3}}. \tag{5}$$

Also, a different formulation of the main transform from our recent article [7] was included in an unpublished online note [23] from 2000, but was proved differently; also, a different formulation of this same result was given by Bradley in [3], and proved in much the same way as in [23]. The above identities for the dilogarithmic expressions in (3) and (4) had been given by Ramanujan in 1915 [1, 21] in an equivalent form in terms of the special function known as the inverse tangent integral $\mathrm{Ti}_2$. Ramanujan's approach toward evaluating (3) and (4) was very different compared to our Legendre polynomial-based proofs for equivalent evaluations [7], which further motivates the application of our methods from [7]. As indicated in Section 2.2 below, there have actually been a number of previously published proofs of identities equivalent to (1) [4, 5, 22].

The corrections to our publication [7] covered above motivate the brief survey offered in Section 2 on past literature concerning the above evaluations for the two-term dilogarithm combinations in (1), (2), (3), and (4), relative to the methods and results from [7].

**Remark 1.1.** Subsequent to the publication of [7], the five dilogarithmic identities indicated in (1)–(5) were reproduced in the Wolfram MathWorld encyclopedia entry on the dilogarithm function [25], with [7] cited as a Reference for these identities. This same MathWorld entry [25] contains links to the corresponding encyclopedia entries on the inverse tangent integral [26] and Legendre's chi-function [14], and this led the author to discover that equivalent formulas for the values in (1)–(4) had been previously recorded in mathematical literature prior to both [7] and [18]; this, in turn, had inspired the author to explore the history of special values for $\chi_2$ and $\mathrm{Ti}_2$ in relation to the material in [7] and [18], culminating in the survey offered in Section 2 below.

## 2. Survey

### 2.1. The Legendre chi-function.

The special function known as Legendre's chi-function is defined as follows [14]:

$$\chi_\nu(z)=\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)^\nu}.$$

From the above definition, it is immediate that

$$\chi_\nu(z)=\frac{1}{2}\left(\mathrm{Li}_\nu(z)-\mathrm{Li}_\nu(-z)\right).$$

So, we see that the left-hand sides of (1) and (2) may be naturally expressed with the $\chi$-function. As it turns out, the identities

$$\chi_2\left(\sqrt{2}-1\right)=\frac{1}{16}\pi^2-\frac{1}{4}\ln^2\left(\sqrt{2}+1\right) \tag{6}$$

and

$$\chi_2\left(\sqrt{5}-2\right)=\frac{1}{24}\pi^2-\frac{3}{4}\ln^2\left(\frac{\sqrt{5}+1}{2}\right), \tag{7}$$

which are easily seen to be equivalent to (1) and (2), respectively, were previously known [14] [15, p. 19], prior to the publication of [7]. New identities involving the Legendre chi-function were recently given in [24], in which the classical identity

$$\chi_2\left(\frac{1-x}{1+x}\right)+\chi_2(x)=\frac{3\zeta(2)}{4}+\frac{1}{2}\ln(x)\ln\left(\frac{1+x}{1-x}\right)$$

is reproduced from the classic text [16]. We see that (6) follows directly from the identity for $\chi_2\left(\frac{1-x}{1+x}\right)+\chi_2(x)$ given above, and this same identity may be used in a direct way to prove (7). The foregoing considerations add to the interest in the new and Legendre polynomial-based alternate proofs of (6) and (7) given in [7]. The evaluations in (6) and (7) are also reproduced in [23], again with reference to Lewin's text [16]. The formulas in (6) and (7) are well-known and were recently noted [20] in the context of applications related to the special function known as the Barnes G-function.

2.2. **Landen's identity and the Rogers $L$-function.** One of the main results in [18], as highlighted in the title of [18] and in the corresponding zbMATH review [2], is as given below:

$$\mathrm{Li}_2\left(\sqrt{2}-1\right)+\mathrm{Li}_2\left(1-\frac{1}{\sqrt{2}}\right)=\frac{\pi^2}{8}-\frac{\ln^2\left(1+\sqrt{2}\right)}{2}-\frac{1}{8}\ln^2 2. \tag{8}$$

However, this follows in a direct way from (1) together with the famous Landen identity

$$\mathrm{Li}_2(z)=-\mathrm{Li}_2\left(\frac{z}{z-1}\right)-\frac{1}{2}\ln^2(1-z),$$

but it is not indicated in [18] or its reviews [2, 11] that (1) was previously known in an equivalent way via the Legendre chi-function, as far back as Lewin's classic 1958 text [15, p. 19]. The article [18] was the main inspiration behind our publication in [7], but it is suggested in [18] that (1) was introduced in Lima's 2012 article in [18]. Part of the reason as to the confusion concerning the origins of identities as in (1) is due to a number of different special functions and notational conventions that have been used to express such identities, with reference to the $\chi_\nu$-function defined above, along with the $\mathrm{Ti}_2$-function defined below and the different definitions/notations for the Rogers dilogarithm function indicated below. Again, our published proof of (1) [7], which relied on a fractional calculus-derived transform from the 2022 article [8], is original, as is the case with our proofs in [7] of the above symbolic forms for (2), (3), (4), and (5).

The fact that the formula in (8) that was highlighted as a main result in [18] and presented as being new in Lima's paper [18] follows directly from Landen's identity together with the classically known evaluation in (1) recorded in the 1958 text [15, p. 19] has not been noted in any past literature citing [18], including [13, 17, 19]. Letting

$$L(x)=\frac{6}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}+\frac{1}{2}\ln x\ln(1-x)\right)$$

denote the normalized Rogers dilogarithm function, in the 1999 article [5], it was noted that an equivalent formulation of the above equation for $\mathrm{Li}_2\left(\sqrt{2}-1\right)+\mathrm{Li}_2\left(1-\frac{1}{\sqrt{2}}\right)$ follows in a direct way from the identity

$$L(x)+L(1-x)=1 \tag{9}$$

together with Abel's duplication formula, which follows from Abel's functional equation

$$L(x)+L(y)=L(xy)+L\left(\frac{x(1-y)}{1-xy}\right)+L\left(\frac{y(1-x)}{1-xy}\right).$$

This is also noted in [18]. So, we find that the formula in (1), which traces back to the 1958 text [15, p. 19], may also be proved using the functional relations for the Rogers dilogarithm given in (9) together with Abel's duplication formula and Landen's identity. This provides a remarkably different proof compared to our Legendre polynomial-based proof of (1) that we had introduced in [7].

Using the alternative notation/definition

$$L_R(x) = \text{Li}_2(x) + \frac{1}{2} \ln x \ln(1-x)$$

for the Rogers $L$-function indicated in [27], the formula

$$L_R\left(2-\sqrt{2}\right) - L_R\left(\frac{2-\sqrt{2}}{2}\right) = \frac{\pi^2}{24}$$

was proved in 1981 in [22] through the use of the Rogers–Ramanujan and the Andrews–Gordon identities. Using the functional relation in (9), this can be used to produce yet another proof of (1).

Bytsko [4] proved the identity

$$L_R\left(1-\frac{1}{\sqrt{2}}\right) + L_R\left(\sqrt{2}-1\right) = \frac{\pi^2}{8} \tag{10}$$

as the $k=2$ case of the formula

$$\sum_{i=1}^{k-1} L_R\left(\frac{\sin^2 \frac{\pi}{3k+2}}{\sin^2 \frac{(i+1)\pi}{3k+2}}\right) + L\left(\frac{\sin \frac{\pi}{3k+2}}{\sin \frac{(k+1)\pi}{3k+2}}\right) = \frac{\pi^2}{6}\frac{3k}{3k+2}$$

given in [4]; we see that (10) is equivalent to (8), which, as indicated above, is equivalent to (1).

2.3. **Ramanujan's inverse tangent integral.** Integrals of the form

$$\text{Ti}_2(x) = \int_0^x \frac{\arctan t}{t}\, dt$$

were of interest to Ramanujan, and remarkable results on the special function $\text{Ti}_2$ defined above were given in his 1915 article [21] (cf. [1, §17], [26]). From the series expansion

$$\text{Ti}_2(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)^2},$$

we obtain that

$$\text{Ti}_2(x) = \frac{1}{2i}\left(\text{Li}_2(ix) - \text{Li}_2(-ix)\right).$$

So, we find that the expressions in (3), (4), and (5) are naturally expressible as specific values of $\text{Ti}_2$. Ramanujan introduced the identity whereby

$$\sum_{n=0}^{\infty} \frac{\sin(4n+2)x}{(2n+1)^2} = \text{Ti}_2(\tan x) - x \ln \tan x \tag{11}$$

for $0 < x < \frac{1}{2}\pi$, and noted that this may be proved by applying term-by-term differentiation to the above series [21] (cf. [1, §17]). A corrected version [1, p. 365] of Ramanujan's formula for $\text{Ti}_2(\sqrt{2}-1)$ is such that:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}}{(2n+1)^2} = \sqrt{2}\,\text{Ti}_2\left(\sqrt{2}-1\right) + \frac{\pi}{4\sqrt{2}} \ln(1+\sqrt{2}). \tag{12}$$

Also, from Ramanujan's identity in (11), we obtain that

$$\mathrm{Ti}_2(1) = \frac{3}{2}\mathrm{Ti}_2(2-\sqrt{3}) + \frac{1}{8}\pi\ln(2+\sqrt{3}), \tag{13}$$

and we find that the above equalities due to Ramanujan in 1915 [21] (cf. [1, §17]) are equivalent to our formulas for (3) and (4), which we had proved in a completely different way in [7]. Ramanujan's formulas in (12) and (13) were recently noted in [20], again in the context of applications pertaining to the Barnes G-function. Our discovery presented in [7] given by the equality in (5) may be rewritten so that

$$\mathrm{Ti}_2\left(\frac{1}{\sqrt{3}}\right) = \frac{3\psi^{(1)}\left(\frac{1}{6}\right) + 15\psi^{(1)}\left(\frac{1}{3}\right) - 6\sqrt{3}\pi\ln(3) - 16\pi^2}{72\sqrt{3}}. \tag{14}$$

This can also be proved using Ramanujan's identity

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!}\frac{\cos^{2n+1}x + \sin^{2n+1}x}{(2n+1)^2} = \mathrm{Ti}_2(\tan x) + \frac{1}{2}\pi\ln(2\cos x)$$

for $0 < x < \frac{1}{2}\pi$ [21], but this is nontrivial in the sense that plugging $x = \frac{\pi}{6}$ into the above series produces a linear combination of the hypergeometric series

$${}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{2},\frac{3}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right] \quad \text{and} \quad {}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{2},\frac{3}{2}\end{matrix}\,\middle|\,\frac{3}{4}\right],$$

which computer algebra systems such as Maple 2020 are not able to evaluate.

2.4. **Sherman's and Bradley's formulas.** The main transform from [7], our proof of which relied on results from our 2022 article [8], is such that

$$\frac{1}{1+z}\sum_{n=0}^{\infty}\frac{\left(\frac{16z}{(1+z)^2}\right)^n}{(2n+1)^2\binom{2n}{n}} = \mathrm{sgn}(z)\frac{i\left[\mathrm{Li}_2\left(-\sqrt{-z}\right) - \mathrm{Li}_2\left(\sqrt{-z}\right)\right]}{2\sqrt{z}} \tag{15}$$

holds if both sides converge for real $z$. Our proof of this in [7] relied on the generating function for Legendre polynomials together with a fractional calculus-derived series transform from the 2022 article [8]. A different formulation of this result was given in an unpublished note by Sherman in 2000 [23]. In [23], by integrating the Maclaurin series expansion

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}}\frac{(4x)^n}{2n+1} = \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}},$$

it was shown that

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}}\frac{(4x)^n}{(2n+1)^2}$$

is expressible as a linear combination of

$$\chi_2\left(e^{i\arcsin\sqrt{x}}\right)$$

and elementary expressions, in contrast to our identity in (15) [7]. It appears that our dilogarithm transform identity indicated in [7, p. 36] had not been considered previously. With regard to our formula in (15) and its derivation in [7], the following closely related formula was proved in a different way in [3]:

$$\int_0^x \ln(\tan\theta)\,d\theta = x\ln\tan x - \frac{1}{4}\sum_{k=0}^{\infty}\frac{(2\sin 2x)^{2k+1}}{(2k+1)^2\binom{2k}{k}}. \tag{16}$$

Bradley [3] also showed that

$$L(2,\chi_6) = \frac{\pi\sqrt{3}}{18}\ln 3 + \frac{1}{2}\sum_{k=0}^{\infty}\frac{3^k}{(2k+1)^2\binom{2k}{k}},$$

which, together with (16), can be used to give an equivalent formulation of (14), where the expression $\chi_6$ denotes the non-principal Dirichlet character modulo 6. This is shown using an equivalent formulation of Ramanujan's 1915 identity in (11) together with (16), in contrast to our methods from [7].

An evaluation for $\mathrm{Ti}_2\left(\frac{\sqrt{3}}{3}\right)$ was also given in 1984 in [12], using a previously known relation [16, p. 106] involving $\mathrm{Ti}_2$ and the special function known as the Clausen integral.

## 3. Double series

We conclude by briefly considering how the special values for $\chi_2$ and $\mathrm{Ti}_2$ considered in this article may be applied using our previous work on double series [6, 9]. As a special case of a hypergeometric transform introduced in [6] using the FL-based evaluation technique from [10], it was shown that: For a suitably bounded parameter $p$,

$$\frac{\pi}{2}\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m p^n \frac{\binom{2m}{m}^2\binom{2n}{n}}{m+n+1} \tag{17}$$

equals

$$\frac{-1}{\sqrt{p}}\times\left(\mathrm{Li}_2\left(-2\sqrt{\frac{p}{\left(\sqrt{1-4p}+1\right)^2}}\right)-\mathrm{Li}_2\left(2\sqrt{\frac{p}{\left(\sqrt{1-4p}+1\right)^2}}\right)\right).$$

In [9], we had applied this identity for (17) together with the known closed form for $\mathrm{Li}_2(\sqrt{2}-1)-\mathrm{Li}_2(1-\sqrt{2})$ to obtain new bivariate hypergeometric series evaluations. Setting $p=\frac{1}{20}$ in (17) and using the closed form in (2), we obtain the remarkable formula

$$\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m\left(\frac{1}{20}\right)^n\frac{\binom{2m}{m}^2\binom{2n}{n}}{m+n+1} = \frac{\sqrt{5}\pi}{3} - \frac{6\sqrt{5}\ln^2(\phi)}{\pi}.$$

Summing over $n\in\mathbb{N}_0$, we obtain an inevaluable ${}_2F_1\left(\frac{1}{5}\right)$-series; summing over $m\in\mathbb{N}_0$, we obtain a ${}_3F_2(1)$-series with no closed form. Similarly, by setting $p=-\frac{1}{12}$ in (17) and using Ramanujan's formula in (13), we may obtain that

$$\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m\left(-\frac{1}{12}\right)^n\frac{\binom{2m}{m}^2\binom{2n}{n}}{m+n+1} = \frac{16G}{\sqrt{3}\pi} - \frac{2\ln\left(2+\sqrt{3}\right)}{\sqrt{3}}.$$

Summing over $n\in\mathbb{N}_0$, we obtain an inevaluable ${}_2F_1\left(-\frac{1}{3}\right)$-series; summing over $m\in\mathbb{N}_0$, we again obtain a ${}_3F_2(1)$-series that does not admit any closed form. We leave it to a separate project to pursue a full exploration of the application of the techniques from [6, 9] together with the special values for $\chi_2$ and $\mathrm{Ti}_2$ considered in this article.

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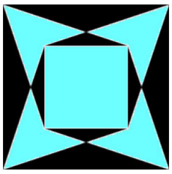

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# A proof, a consequence and an application of Boole's combinatorial identity

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Abstract. Boole's combinatorial identity is proved, and a consequence of it for analytic functions is derived that is used to evaluate a sequence of integrals in terms of Euler's secant sequence of integers.

## 1. Boole's identity

This features early on in [2], (cf. equation (6) on page 20) and states that if $n$ is a nonnegative integer, then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^n = n!. \tag{1}$$

In addition, if $n \geq 1$, and $m$ is any nonnegative integer less than $n$, then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m = 0. \tag{2}$$

Both of these statements have many proofs; consult [1], and the references cited therein.

Here's an outline of a combined proof of (1) and (2):

*Proof.* Write

$$\sigma_n(m) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m = n!\sum_{k=0}^{n}\frac{(-1)^k(n-k)^m}{k!\,(n-k)!},\ \ m,n = 0,1,2,\ldots.$$

Fix $m$, and observe that the sequence $\{\sigma_n(m)/n!,\ n = 0,1,\ldots\}$ is the convolution of the sequences $\{(-1)^n/n!,\ n = 0,1,\ldots\}$, and $\{n^m/n!,\ n = 0,1,\ldots\}$. Hence

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\sigma_n(m)}{n!}z^n &= \sum_{n=0}^{\infty}z^n\sum_{k=0}^{n}\frac{(-1)^k}{k!}\frac{(n-k)^m}{(n-k)!}\\ &= \Big(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^n\Big)\Big(\sum_{n=0}^{\infty}\frac{n^m}{n!}z^n\Big)\\ &= e^{-z}W_m(z),\end{aligned}$$

where

$$W_m(z) = \sum_{n=0}^{\infty}\frac{n^m z^n}{n!} = \Theta^m e^z,$$

$\Theta$ standing for the differential operator $z\frac{d}{dz}$, much used by Boole in his treatment of linear differential equations with variable coefficients.

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Clearly, $W_0(z) = e^z, W_1(z) = ze^z$, and the following recurrence relation holds:

$$W_{m+1}(z) = zW_m'(z) + W_m(z),\ m = 0, 1, \ldots,$$

where the prime denotes differentiation. So, $W_m(z)$ is a monic polynomial $p_m(z)$ times $e^z$, and $\deg p_m = m$, which is easy to see by induction. Hence,

$$\sum_{n=0}^{\infty} \frac{\sigma_n(m)}{n!} z^n = p_m(z),$$

from which it follows immediately that $\sigma_n(m) = 0, \forall n > m$ and $\sigma_n(n) = n!$. Thus (1) and (2) are true. □

## 2. A simple consequence

Suppose $f$ is analytic on a disc $D$ centred at 0 in the complex plane. Then, for any nonnegative integer $n$,

$$\lim_{x \to 0} \frac{1}{x^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(kx) = f^{(n)}(0). \tag{3}$$

*Proof.* Let

$$F(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(kx), \forall x \in \tfrac{1}{n} D.$$

Clearly, $F$ is analytic on a subdisc of $D$ centred at 0, on which

$$F^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m f^{(m)}(kx).$$

In particular, it follows from (2) that

$$F^{(m)}(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m f^{(m)}(0) = 0,\ m = 0, 1, \ldots, n-1, \tag{4}$$

and from (1) that

$$F^{(n)}(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^n f^{(n)}(0) = n! f^{(n)}(0). \tag{5}$$

Therefore, by integrating by parts multiple times, and applying (4) repeatedly,

$$F(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} F^{(n)}(t)\, dt = \frac{x^n}{(n-1)!} \int_0^1 (1-s)^{n-1} F^{(n)}(xs)\, ds.$$

Hence

$$F(x) - x^n \frac{F^{(n)}(0)}{n!} = \frac{x^n}{(n-1)!} \int_0^1 (1-s)^{n-1} [F^{(n)}(xs) - F^{(n)}(0)]\, ds.$$

Let $\epsilon > 0$. By hypothesis, there exists $\delta > 0$ such that $|F^{(n)}(z) - F^{(n)}(0)| < \epsilon$ whenever $|z| < \delta$, and so $|F^{(n)}(xs) - F^{(n)}(0)| < \epsilon$ whenever $|x| < \delta$ and $0 \le s \le 1$. Consequently, if $0 < |x| < \delta$,

$$\begin{aligned}
\left| \frac{F(x)}{x^n} - \frac{F^{(n)}(0)}{n!} \right| &\le \frac{1}{(n-1)!} \int_0^1 (1-s)^{n-1} |F^{(n)}(xs) - F^{(n)}(0)|\, ds \\
&\le \frac{\epsilon}{(n-1)!} \int_0^1 (1-s)^{n-1}\, ds \\
&= \frac{\epsilon}{n!}.
\end{aligned}$$

In other words,

$$\lim_{x\to 0}\frac{F(x)}{x^n}=f^{(n)}(0),$$

by (5), as claimed. □

In particular, if $f$ has a power series expansion about 0 so that, for some $r>0$,

$$f(x)=\sum_{m=0}^{\infty}a_m x^m,\ \forall |x|<r,$$

then

$$\lim_{x\to 0}\frac{1}{x^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(kx)=n!a_n$$

by (3).

## 3. An application

Consider the sequence of integrals

$$I_n=\int_0^\infty \frac{(\ln(x))^n}{1+x^2}\,dx,\ n=0,1,2\ldots$$

It's familiar that $I_0=\pi/2$, and clear that

$$\begin{aligned} I_n &= \int_0^1 \frac{(\ln(x))^n}{1+x^2}\,dx+\int_1^\infty \frac{(\ln(x))^n}{1+x^2}\,dx \\ &= \int_0^1 \frac{(\ln(x))^n}{1+x^2}\,dx+\int_0^1 \frac{(\ln(\frac{1}{x}))^n}{1+x^2}\,dx \\ &= (1+(-1)^n)\int_0^1 \frac{(\ln(x))^n}{1+x^2}\,dx. \end{aligned}$$

Hence, $I_{2n+1}=0,\ n=0,1,2,\ldots$. It's an exercise on page 134 in [3] (Titchmarsh's Theory of Functions) that $I_2=\pi^3/8$, while the computer package MAPLE spews out values of $I_{2n}$ for $n=2,3,4,5,6$, according to which

$$I_4=\frac{5\pi^5}{2^5}, I_6=\frac{61\pi^7}{2^7}, I_8=\frac{1385\pi^9}{2^9},\ I_{10}=\frac{50521\pi^{11}}{2^{11}},\ I_{12}=\frac{13936098\pi^{13}}{2^{13}}.$$

The numbers 1, 5, 61, 1385, 50521, 139360981 are the first six terms of the integer sequence named Euler's secant sequence, and numbered A000364 in [4] (Sloane's online encyclopedia of integer sequences). If $a(n)$ denotes the $n$th term of this sequence, it's tempting to conjecture that

$$I_{2n}=\frac{a(n)\pi^{2n+1}}{2^{2n+1}}, n=0,1,2,\ldots.$$

One way to confirm this is as follows.

*Proof.* Recall that, for $x>0$, $\ln x$ is the limit of the decreasing sequence, $m(\sqrt[m]{x}-1),\ m=1,2,\ldots$. Hence

$$\begin{aligned} I_n &= \lim_{m\to\infty} m^n\int_0^\infty \frac{(x^{1/m}-1)^n}{1+x^2}\,dx \\ &= \lim_{m\to\infty} m^n\int_0^\infty \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{x^{k/m}}{1+x^2}\,dx \\ &= \lim_{m\to\infty} m^n\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}J(k/m), \end{aligned}$$

where, for $|\Re\alpha| < 1$,

$$J(\alpha) = \int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \frac{\pi}{2}\sec\left(\frac{\pi\alpha}{2}\right).$$

Since sec admits of a power series expansion about 0 of the form

$$\sec x = \sum_{n=0}^{\infty} \frac{a(n)}{(2n)!} x^{2n},$$

that is valid for all $|x| < \pi/2$, it follows that

$$\begin{aligned} I_n &= \frac{\pi}{2} \lim_{m\to\infty} m^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sec\left(\frac{k\pi m}{2n}\right) \\ &= \frac{\pi^{2n+1}}{2^{2n+1}} \sec^{(n)}(0), \end{aligned}$$

by (3), and so, in particular, $I_{2n+1} = 0$, $n = 0, 1, \ldots$, as we noted above, and

$$I_{2n} = \frac{a(n)\pi^{2n+1}}{2^{2n+1}},$$

as desired. □

**Remark 3.1.** The connection between the values of the sequence $I_n$ of integrals, and terms of the sequence A000364, doesn't appear to have been noticed before.

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# A Survey of Research on the Impact of the COVID-19 Closures on the Teaching and Learning of Mathematics at University Level in Ireland

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Abstract. In March 2020, Irish higher education institutions were forced to close their campuses because of the COVID-19 pandemic and all teaching activities moved online. We survey the research carried out on the effects of the COVID-19 university closures on the teaching and learning of mathematics in Ireland.

## 1. Introduction

On Thursday 12 March 2020 the Taoiseach announced that due to the COVID-19 pandemic all school and higher education institutions in Ireland would close the following day. At this time educators in Ireland, and indeed around the world, were faced with unprecedented challenges and were forced to completely change their teaching methods overnight. It is to their credit that with just a few days to prepare, most institutions moved their courses online by the start of the following week. To begin with, the closures were expected to last for a few weeks, but in the case of universities most classes did not return to campus until the start of the 2021/22 academic year. In this article I will outline how institutions responded, as well as surveying some research on the impact of the closures on the teaching and learning of mathematics at university level in Ireland.

In 2020 and 2021, research was carried out around the world into the impact of the COVID-19 closures on teaching and learning. Much of this work was at school level, for example Riemers [31] has information about the consequences of the pandemic for primary and secondary education systems in 11 countries. Organisations such as the OECD have issued wide-ranging reports on this topic [32]. In Ireland, the ESRI has published detailed reports on the implications of the pandemic for children [4] as well as highlighting the effects of school closures on widening inequality [5]. The importance of a numerate society in order to deal with issues affecting the health of a nation (such as a major pandemic) was discussed by O'Sullivan, O'Meara, Goos and Conway [29]. Amongst the studies conducted at school level in Ireland are those by Dempsey and Burke on the impact of educational closures on Irish teachers ([6]) and principals ([7]) at primary and secondary level. Of course one of the major consequences of the pandemic on second level education in Ireland was the cancellation of the Leaving Certificate in 2020; Doyle, Lysaght and O'Leary [8] report on how teachers navigated the calculated grades system.

The effects of such replacements of end-of-school state examinations on entry standards to university have been studied in the UK by Hodds ([11]). There have been many other studies of the effects of the closures on mathematics education at university

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level internationally; for example on lecturers and students in Norwegain universities [30], on graduate student programmes in the US ([15]), and on adapting courses for pre-service mathematics teachers in Australia [18]. Researchers have also written about the issue of assessment ([2], [14] ). The situation in Ireland was studied extensively by various researchers in 2020. In this article, I will attempt to give an overview of their findings. I have grouped the relevant papers thematically into four categories: the lecturer perspective; the provision of mathematics support and tutorials; the student perspective; and assessment. I will give a brief overview of the research in each of these categories in the following sections.

## 2. The Lecturers' Perspective

Before we begin, let us acknowledge that third level institutions' response to the COVID-19 closures is very different from planned online delivery of courses. In fact educators coined the phrase *Emergency Remote Teaching* (ERT) to describe the rapid move online. Hodges, Moore, Lockee, Trust and Bond [12] define ERT as *a temporary shift of instructional delivery to an alternate delivery mode due to crisis circumstances. It involves the use of fully remote teaching solutions for instruction or education that would otherwise be delivered face-to-face or as blended or hybrid courses and that will return to that format once the crisis or emergency has abated. The primary objective in these circumstances is not to re-create a robust educational ecosystem but rather to provide temporary access to instruction and instructional supports in a manner that is quick to set up and is reliably available during an emergency or crisis.* Thus they make a distinction between ERT and learning experiences that are carefully planned and designed to be online in advance.

Lishchynska and Palmer [16] describe the experience of mathematics lecturers across the country (and indeed the globe) of waking up one day in March 2020 to find that their job had changed completely overnight. They recall that after the initial shock the community realised that they had to move their teaching online, that they had very little time to do it, and that it probably would not be perfect. In the summer of 2020, Ní Fhloinn and Fitzmaurice conducted an online survey of mathematics lecturers to gather information on how they coped with this rapid change. They used various mailing lists to invite lecturers to take part in their study and received 257 responses from academics in 29 different countries. More than 30% of the responses came from mathematicians working in Ireland. The results of the analysis of the survey data have been published in [24], [25], [26] and [27].

One of the first decisions facing lecturers in March 2020 was how to replace their face-to-face lectures in the online environment. Some chose to livestream their lectures using Teams or Zoom. Others made short videos or pre-recorded entire lectures. Three quarters of the respondents to Ní Fhloinn and Fitzmaurice's survey included some form of live online session in their teaching, more than 60% made recordings and over 40% had both [26]. Lecturers who chose to have live sessions said that they did so in order to facilitate students' questions and to try to keep the format as close to that of their regular classes as possible. They also emphasised the importance of giving structure to students' days by sticking to the lecture timetable. One reason given for not having live sessions was the worry that some students did not have access to fast broadband. Reasons for using recorded videos included the flexibility it offered to lecturers and students, and the fact that students could replay them as often as they wanted. Quality control was also cited as an advantage of making videos since the recording could be edited or redone if mistakes were made, however this was also seen as a disadvantage since this process could be very time-consuming. There was an increase in the use of Virtual Learning Environments (VLEs) such as Moodle or Blackboard, with most lecturers uploading notes and examples.

Ní Fhloinn and Fitzmaurice [24] found that 90% of respondents had no previous experience of teaching online. It is no surprise then that the vast majority of them felt high levels of stress associated to the initial move online, although many of the lecturers reported that their stress levels decreased towards the end of the semester. Almost all of the participants found that teaching online was very time-consuming, and about two thirds of them said that they worked more hours than usual.

Apart from the stress and extra workload, lecturers encountered other challenges while conducting emergency remote teaching. Some of these were technical in nature; the main difficulty seemed to be the problem of replacing the ability to write mathematics on chalkboards or whiteboards during classes [24]. Ní Fhloinn and Fitzmaurice [26] found that the participants in their study were very resourceful in this regard, making use of tablets and stylus pens, visualisers, and even pen and paper recorded using their smartphones. Lecturers found it more difficult to translate other aspects of their teaching to the online setting however, with many reporting that conducting discussions or groupwork was problematic. Gauging student understanding during online classes was seen as a major challenge as lecturers missed being able to see their students' faces and reactions. They felt that communication with students was more difficult online not least because of the problems students faced when trying to type mathematical expressions when asking a question in an email or in a discussion forum [24]. More than a third of respondents to the Ní Fhloinn and Fitzmaurice survey were concerned about a lack of interaction in their classes along with problems with student engagement. Lishchynska and Palmer [16] noted that students were reluctant to take an active part in online discussions but that the majority of students did view these discussions. In contrast, some lecturers felt that the anonymity of tools such as online polling helped to increase student involvement over what might be expected in in-person lectures [24].

Lecturers also reported some advantages of online teaching [24]. Some liked the extra flexibility in their timetables and the fact that they did not have to spend time commuting to campus. Others felt that the resources created for online learning, such as short videos, allowed students to work at their own pace and they saw the fact that students had increased responsibility for their own learning as a benefit. In addition, some lecturers felt that the resources developed for emergency remote teaching could be incorporated in modules in future years. Ní Fhloinn and Fitzmaurice [25] summarise the practical advice of the lecturers in their study on issues of concern such as: technology options; specific online teaching approaches, ways of supporting students, and ways of reducing stress for teaching staff.

Lishchynska and Palmer [16] indicated that they saw a shift in emphasis for students from the familiar structure of on-campus classes and supports to the need to be a self-directed learner. They expressed the worry that students were expected to make this transition very quickly at the beginning of the COVID-19 closures. Many of their colleagues around the country had the same concerns and much work was done to provide supports to students. In the next section we will review some of the findings on these initiatives.

## 3. Provision of Online Support

Tutorials have traditionally been one of the main supports offered to mathematics students at university as they offer students the opportunity to learn in a small group setting. Lishchynska, Palmer and Cregan [17] outline the benefits of this teaching method, for example students can ask for help and get instant feedback on their work, in addition to interacting meaningfully with their peers. Lecturers and tutors also benefit since they get good information on the difficulties that students are experiencing and can identify problematic topics and misconceptions in real-time. Lishchynska et al. [17] report on how a range of alternatives to in-person tutorials were put in place in MTU during the COVID-19 closures, and on the views of students and academics on these alternatives. Staff at MTU replaced the traditional in-person problem-solving tutorial sessions in six modules with combinations of: live tutorials delivered through Zoom, group-work conducted in Zoom breakout rooms, discussion fora, automated formative feedback on online quizzes, and written individual feedback on students' submitted homework. For the latter, lecturers encouraged students to submit their solutions to a set of exercises and also allowed them to indicate if they had problems with questions so that the lecturer could offer advice. The authors were the lecturers of the modules in question [17]. They reflected on the positive and negative aspects of the various alternatives from their own point of view. They noted that the live tutorials (on Zoom) were similar in some respects to in-person tutorials in that interaction with students was possible, however they found it difficult to see students' written work and to interpret silences. Lishchynska et al. note that the silences could mean that students have no questions or that they do not feel comfortable asking questions. They had similar issues with the groupwork tutorials via Zoom breakout rooms; although students could help each other and share their screens, good interaction was not guaranteed and progress could be slow. Lishchynska et al. [17] note that the discussion fora were popular with students however very few of them were willing to ask questions and most students only accessed the forum to see replies to others' inquiries. The authors valued the online quizzes and associated formative feedback but found that both the creation of good quiz questions and the creation of constructive feedback was very time-consuming. Similarly, giving written feedback on students' assignments was a heavy burden, however this process gave the lecturers insight into student thinking, allowed them to give targeted assistance, and enabled them to foster a connection with students. Lishchynska et al. conclude that no one tutorial alternative was found to match the learning experience of in-person tutorials but they suggest that a combination of such approaches may be beneficial. In particular they saw that the formative feedback initiatives (either in written or automatically-generated form) helped to engage students and inform lecturers.

Lishchynska et al. [17] surveyed the 264 students who experienced the range of tutorial alternatives to gather their views on the supports. Of these, 139 students responded. The students were very positive about the live tutorials saying that they liked having access to the lecturer, and having their questions answered. They made similar comments about the groupwork tutorials. Some students enjoyed working with their peers but others found this difficult and felt that they would benefit from more time with the tutor. The students who had the opportunity to get written feedback on assignments said that this initiative helped them build understanding as well as confidence in their work. In addition, they liked having a regular schedule of exercises to work on. Similar comments were made about the online quizzes and students appreciated the opportunity to practice and to receive instant feedback. Students were also asked whether they used other supports; nearly half of respondents said that they did not seek further help while 39% sought help from their peers and 13% used MLS or private tuition. The majority of students who interacted with other students did so through messaging apps with a minority using video conferencing facilities. When asked to rank potential future supports the majority chose live tutorials and homework with feedback, however nearly 40% felt that in-person tutorials were more beneficial than any form of online support.

Prior to March 2020, mathematics learning support (MLS) was common in most higher education institutions in Ireland [3] but the provision of online supports was limited [19]. Thus the COVID-19 closures necessitated a drastic change in the provision of mathematics support. Many of the Mathematics Learning Support centres aimed to replicate their in-person drop-in services using web conferencing platforms such as Zoom, Teams, or their VLE [1], [20], [23]. This was the case in UCD where Mullen and Cronin ([22], [23]) conducted a study with colleagues Pettigrew, Rylands, and Shearman from Western Sydney University (WSU). In this project they investigated student and tutor views on online MLS in Ireland and Australia. Six tutors and six students from UCD were interviewed, along with seven students and four tutors from WSU. Mullen et al. categorised the comments of the participants into five themes: usage of mathematics and statistics support, pedagogical changes, social interaction, 'Maths is different', and the future of online mathematics and statistics support. In both institutions, the tutors described changes to their usual pedagogy because of the move online; in particular they tended to spend more time giving detailed answers instead of using their usual techniques such as guided questioning. They found it more difficult to interact with students and especially to diagnose difficulties. This was in part because of the lack of non-verbal cues (exacerbated when students did not turn on their cameras), and not being able to see students' work. Students also commented on the lack of interaction and the subsequent loss of rapport with their tutors, as well as the difficulty of showing their work. The participants expressed the view that this was a particular problem in Mathematics. Tutors usually had access to tablets and stylus pens and so were able to write mathematics in real-time and share their screens with students, but most students did not have access to this technology. Students also found it difficult to type questions involving mathematical notation in chat facilities. However online support did offer certain advantages, and both groups mentioned positive aspects of online MLS; for example some tutors reported that students seemed to be better prepared for the online sessions than they might have been in the past, while some students said that they felt more confident asking questions in an online environment than they would in person. Students and tutors appreciated the increased flexibility and accessibility of online MLS.

It was notable that in both universities involved in the Mullen et al. study that the numbers of students availing of MLS decreased significantly during the COVID-19 closures ([22], [23]). A similar drop in attendance was seen in the Mathematics Support Centre in Maynooth University [20]. Mac an Bhaird, McGlinchey, Mulligan, O'Malley, and O'Neill reported on the introduction of online study groups at the beginning of the 2020/21 academic year as a means of encouraging students to engage with online MLS [20]. More than 700 students registered to take part in the initiative. They were assigned to groups of four or five students who were studying the same material. These groups met once per week on Teams and had access to a tutor during their meeting time. About 60% of the registered students eventually participated in the study groups with 220 students attending at least half of the sessions. In December 2020, Mac an Bhaird et al. [20] surveyed students who were registered for mathematics modules at Maynooth University. The survey had 114 responses of which 88 were from students who had availed of online MLS. Seventy one of the respondents had been involved in the study group initiative. The majority of these students felt that the study groups helped them to increase their understanding of and engagement with their mathematics modules. They appreciated the help from tutors, the opportunity to work with their peers in a small group setting, and that the process was student-led. Some expressed disappointment that attendance in their group was often low and that the group did not work well as a result. Other students said that it was sometimes difficult to interact with their groups online. The students suggested that the group size should be increased, and that efforts should be made to help group-members get to know each other at the outset. Some students also wanted more tutor involvement in the sessions. The students who had not taken part in the study group initiative gave a variety of reasons for not engaging with it. Some did not know about the scheme, others said that they had no time, did not need the extra help, or preferred to work alone. Mullen et al. [22] reported that tutors in WSU encouraged group work in their online sessions but sometimes found it difficult to get students to engage. Some students said that it was easier to avoid contributing to discussions online than it would be in-person. However students in WSU valued the groupwork sessions because they offered a chance to interact with their peers, and some noted that they did not realise how important these kinds of interactions were for their learning until they were gone.

Mac an Bhaird et al. [20] also asked students about their experience of online drop-in mathematics support. About one-third of the respondents to their survey had availed of this; they commented positively about the flexibility of the service and in particular about the help received from tutors. The reasons given for not attending drop-in sessions were similar to those cited above. In addition, some students felt that they had enough support within their module and did not need the drop-in service, while others reported that timetable clashes meant that they could not attend. Nearly one-third of respondents said that they did not have access to good broadband, which had implications for their engagement.

Students in the Mac an Bhaird et al. [20] study were divided on whether in-person MLS was preferable to online MLS. The students who prefered in-person support said that they found it easier to ask questions in-person and that they missed working in the atmosphere of the Mathematics Support Centre. Mullen et al. [23] noted that the future of MLS is likely to include in-person and online elements. The tutors in their study were keen to return to campus but felt that the online resources developed during the pandemic should be re-used. Students missed face-to-face interactions, however some wanted to keep elements of online MLS as it is useful for when they cannot make it to campus [23].

As well as synchronous support most institutions around the country also offered asynchronous support in the form of notes, videos, practice questions etc. O'Sullivan, Casey and Crowley [28] describe a project undertaken at MTU which aimed to use learning analytics to study students' engagement with online asynchronous support. The authors focused on a set of resources called *Maths Online* which was offered through their institution's VLE. The resources were organised by topic and by degree programme. They consisted of notes, auto-corrected quiz questions, software (MAPLE, SPSS and Minitab), links to other websites, a discussion area, and a facility to book online MLS consultations. Solutions to previous examination papers relating to one module were available through *Maths Online*. O'Sullivan et al. [28] used student interaction data gathered by the VLE to study how students engaged with these resources. They found that engagement was high, with nearly three-quarters of students enrolled in mathematics and statistics modules accessing the *Maths Online* course. However less than a third of these students accessed content on three or more days, and more than four-fifths used *Maths Online* for a total of 30 minutes or less. The most popular features were the software downloads and the examination solutions. The discussion forum was also viewed by a high percentage of students even though they seemed reluctant to actively participate in discussions. The quizzes were used by a minority of students. O'Sullivan, Casey and Crowley [28] comment that students' first impression of an online resource is crucial to their continued engagement with it and thus the design and presentation of online learning objects are vitally important. They advise that a home page for a resource such as *Maths Online* needs to catch the attention of students as well as being clear and informative.

## 4. The Student Perspective

Although we have seen some of the views of students on the provision of online support during the COVID-19 closures, we have concentrated up to now on the views of lecturers and tutors. In this section we will summarise the findings of three studies that surveyed mathematics undergraduate students during the summer of 2020.

The first is a study carried out by Meehan and Howard which investigated the perceptions of mathematics students in UCD of online teaching and learning during the initial lockdown period [21]. They emailed a survey to 900 students in May 2020 and received 156 responses. One of the aims of this project was to elicit students' views on the aspects of online lecture and tutorial formats that were beneficial for their learning. Meehan and Howard [21] gathered students' comments into three categories relating to: the online environment in general; the online environment for learning; the online delivery of lectures and tutorials. Each of these categories contained both positive and negative experiences. For example students liked the fact that the move online meant that they did not have to spend long hours commuting to university, but some students found working at home difficult either because of a lack of a quiet place to work or because of poor internet connections. This created problems for students when downloading large video files, when trying to participate in a live lecture, and most particularly when taking an online examination. Students liked having the ability to watch and re-watch recordings of lectures and shorter videos. They mentioned that they used these resources to review material and liked the flexibility involved as well as being able to work at their own pace. However the move online meant that they lost the structure of their usual timetable and some of them had difficulty scheduling their work. Some felt that it was easy to fall behind in this learning environment and mentioned that having a regular schedule of short quizzes helped to keep them on-track and to allow them to gauge their own understanding. Many students missed interaction with their peers, lecturers and tutors and the consequent loss of learning opportunities. Some said that it was more difficult to carry out group work and ask questions online, although for some it was easier to do this. The students liked discussion boards and especially the possibility of seeing others' questions. Some asked for more anonymity when asking questions in this format, and also that questions be answered promptly. In regards to the delivery of lectures online, some felt that the live lectures provided a structure for their days and allowed students to ask questions. Some students wanted more opportunities for interaction in lectures while others felt that this was not useful and was distracting. When lectures were recorded and delivered asynchronously, students preferred having a sequence of short videos rather than one long one. This was partly due to the problems of downloading a large video but also because students felt that shorter videos aided concentration and motivation, and were easier to navigate to find material. In some modules, the lecturers did not provide recordings or live lectures and the students in these modules were adamant that providing notes alone was not enough. Students said that they liked live interactive tutorials although they had difficulties sharing their work and writing mathematics online. They also liked when solutions to assignments were provided.

Meehan and Howard [21] asked students about their ideal blended learning experience. Many students responded by saying that they hoped that they would be fully back on-campus in the future. Others stressed the need for more interaction in the online environment. Some students described something similar to a flipped-classroom model where students would be provided with pre-recorded videos, notes and exercises in advance of small-group problem-solving sessions with lecturers and tutors. Based on their analysis, Meehan and Howard [21] make some recommendations at the end of their report. They advocate for maintaining some elements of the flexibility afforded by the online environment, especially in an effort to avoid long commutes for students. They also highlight the need to plan for better interactions between peers and between students and teaching staff. They advise that each module should have clear weekly schedules and that carefully organised and labelled recordings be made available to students.

In a similar project, Hyland and O'Shea [13] carried out a national survey of mathematics undergraduates in the summer of 2020. The survey consisted of questions in three broad areas: teaching and learning, assessment, and personal experience. The aim was to get information on the impact of the COVID-19 closures on students' learning experiences as well as students' view on the future provision of teaching and assessment. In all, there were 263 responses from students in six universities. To begin with, students were asked about their access to equipment and infrastructure; almost all participants said that they had a laptop or PC, three quarters of them had a quiet place to work, but more than one third of them had poor internet access [13]. The survey asked students how their lectures were delivered during the university closures. The answers to this question highlighted the range of resources that lecturers around the country put in place for their classes. About 90% of respondents said that they were provided with recorded lectures or short videos, about half of them had live lectures online, while four fifths of them had access to lecture notes. In addition the majority of students said that their lecturers created practice quizzes for them and gave them solutions to assignments or past examination papers. Even with these resources at hand, nearly 60% of students said that the COVID-19 closures had a negative impact on their capacity to learn mathematics. Many of the students said that they missed in-person classes especially tutorials. More than three quarters of the students had some form of tutorial support during the initial closures. The majority of these students said that they had live tutorials facilitated through Teams or Zoom, while some of their modules had discussion boards manned by tutors who were able to answer mathematical questions. The students seem to prefer online live tutorials to the discussion board format as they said that they found it difficult to ask questions and were sometimes embarrassed because everyone could see their query. This may be one of the reasons for the findings we saw in the O'Sullivan et al. [28], Lishchynska et al. [17], and Meehan and Howard [21] studies that showed that students often viewed discussion boards but were reluctant to participate themselves. The students in the Hyland and O'Shea [13] study also missed the usual interactions with tutors and students in in-person tutorials and the resulting learning opportunities that these interactions afford.

It was notable that very few students complained about the quality of teaching during this period with most of them citing the loss of interaction and communication with their lecturers, tutors and especially their peers as reasons for their difficulties [13]. This lack of interaction may be the reason why more than half of the respondents in the Hyland and O'Shea study said that they felt more isolated than usual. The university closures seemed to have a large impact on students' well-being and mental health as about two thirds of students said that they felt more anxious and found it more difficult to motivate themselves during that time. In addition, the participants echoed the views of their peers in the Meehan and Howard [21] study that it was difficult to pace their learning without the help of a set timetable and structure.

Students also found positives in their experience of learning during the initial COVID-19 closures [13]. Some of them liked the flexibility of being able to study at home (without a commute) and whenever was convenient. Some students liked working at their own pace and were proud of their new study skills. Others mentioned that the extra resources that were put in place for them were very helpful. Students expressed mixed views when asked what kind of learning experience they would like in the future.

Some wanted all teaching to be back on-campus but asked that resources be made available to students who could not attend. Others recognised that large-group lectures were unlikely to take place on campus in Autumn 2020 but asked that small-group tutorials return to in-person delivery.

Lishchynska and Palmer [16] used engagement data from their VLE and end of module feedback forms to gather information on students' preferences for learning resources in three modules in MTU. These were a second-year module, a fourth-year module and an MSc module. They found that overall 63% of students used both notes and videos when studying however a much higher proportion of MSc students did this in comparison with the undergraduates. The postgraduate students also found remote learning more difficult than the other students in this study. The vast majority of students surveyed said that they preferred asynchronous to synchronous learning resources, When asked about their preferences for future course delivery nearly 90% did not want a fully online experience but over half of them said that they would like a mix of in-person and remote learning. The students were asked to describe the advantages and disadvantages of learning online during the university closures and their opinions were strikingly similar to those that we have seen in the two studies above. They liked having the flexibility to study at their own pace and in their own time but some missed the structure of their usual timetable. They also found it harder to motivate themselves to work and some had no access to a quiet place to study. They liked the recordings that were available to them and having the ability to pause and re-watch segments. However they missed having the opportunity to ask questions in classes and to interact with their peers. Some of the students in this study felt that although they had learned how to use methods during the COVID-19 closures, they worried that they did not fully understand the reasoning behind the methods. How to assess student understanding in an online environment was a difficult problem for lecturers; we will consider this issue in the next section.

## 5. Assessment

One of the most significant implications of the COVID-19 closures in 2020 was that traditional on-campus examinations were impossible forcing universities to react swiftly to modify their assessment methods. Ní Fhloinn and Fitzmaurice [27] report on how the mathematics lecturers in their study tackled this issue. They found that four fifths of them gave some form of online assessment while the remainder did not. Some of the people who did not use online assessment said that their examinations had not yet happened, others replaced examinations by coursework, and for some the examinations were canceled completely. Of the lecturers who did use online assessment, one fifth gave formative assessment only (such as written assignments or online quizzes which did not contribute to grades), two fifths gave summative assessments only (such as open-book exams or multiple choice quizzes which did contribute to grades), and the remainder used a combination of both. The participants were asked whether they saw a difference in grade profiles compared to previous years. About one quarter saw no difference and the remainder observed some differences ranging from small to large, however less than 10% reported very large differences. The lecturers reported grade increases in some modules and decreases in others. When asked for possible reasons for these differences, some said that the stress of doing examinations online could have led to decreases in overall grades, while others thought that increases may have been due to having open book assessments, having more time allotted to each examination, or changes in marking guidelines. A small number attributed increases to cheating on the part of the students with some lecturers worried that it was difficult to vouch for the legitimacy of grades in an unproctored setting. Others highlighted the (sometimes physical) challenge of grading written examinations online [24]. Despite these issues, most of the lecturers were satisfied with their assessment regime.

Some of the studies that we saw earlier give information on the students' views of changes to traditional assessment methods. Meehan and Howard [21] found that students in UCD liked having open-book examinations and that having examinations at home was less stressful for most students. However when students did not have a quiet place to work or had poor internet connections, having the examination on campus was preferable. Hyland and O'Shea [13] also found that students seemed to like the open-book examinations and thought that they were fair. However the initial uncertainty in March 2020 about the format of assessment was disconcerting for students and some mentioned that they had technical problems during exams or when submitting their work which caused them much stress.

## 6. Conclusion

In this article, we have seen many themes recurring. In particular, we have seen that lecturers, tutors, and students value the connections that are fostered in in-person classes. These interactions give teaching staff valuable insights into student thinking, and give students opportunities to ask questions and receive feedback. This is particularly true in the tutorial and MLS settings, but also holds in the case of lectures. This may come as a surprise to those who view traditional mathematics lectures as very static. In any case, when designing any online teaching experiences, care should be taken to incorporate design features which enable meaningful communication between teaching staff and students, and between peers.

We have seen that students' difficulty in writing mathematics and sharing their work in an online environment was one of the reasons for the lack of interaction. It seems that access to specific types of technology can really help here. Heraty et al. [10] outline various different methods that tutors in Maynooth University used to communicate mathematics effectively to their students. It is vital that students also have access to appropriate technology. Many of the studies above found that although students usually had laptops or PCs, they may not have tablets and stylus pens and a large proportion of them do not have access to reliable broadband. These facts must be taken into consideration when designing future provisions.

There seems to be little appetite from lecturers or students for fully online courses, however both groups saw benefits from aspects of the teaching and learning experience over the last two years. In particular, many resources have now been created and can be profitably re-used. Staff and students both liked the element of flexibility that the move online facilitated, however the lack of a timetable was problematic for some students. A major concern is the heavy workload that lecturers had to bear during ERT. It is clear that creating good resources is very time-consuming and this must be included in any planning.

Apart from an increased workload, staff often found ERT stressful. This was the case for students too, many of whom felt more isolated and more anxious than usual. This highlights the importance of paying attention to the mental health of staff and students.

The studies that we have reviewed above have shown the effects of the COVID-19 closures on teaching staff and on students. They share many common threads, but perhaps the main message conveyed is that, despite the best efforts of all concerned, it remains difficult to recreate the atmosphere of in-person mathematical learning opportunities in an online setting. Engelbrecht, Linares and Borba ([9]) expressed the view that the international COVID-19 university closures have hastened the advent of online and blended learning becoming more prevalent. If they are correct, it would be prudent to use the experience that we have gained from ERT over the last two years when designing any future online courses or resources.

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# Corrigendum: Modular Metric Spaces

HANA ABOBAKER AND RAYMOND A. RYAN

Abstract. We indicate a corrected version of the paper [1], which had some errors. We are grateful to V. V. Chistyakov for bringing these to our attention.

## 1. Introduction

The paper [1] had some errors. We are grateful to V. V. Chistyakov for bringing these to our attention.

A corrected version of the paper has been posted on arXiv [2].

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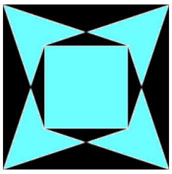

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# Some simple proofs of Lima's two-term dilogarithm identity

SEÁN M. STEWART

Abstract. Recently, Lima found a remarkable two-term dilogarithm identity whose proof was based on a hyperbolic form of a proof for the Basel problem given by Beukers, Kolk, and Calabi. A number of simple proofs for this identity that make use of known functional relations for the dilogarithm function are given and an application of Lima's identity to another two-term dilogarithm evaluation is presented.

## 1. Introduction

The dilogarithm function defined by $\mathrm{Li}_2(x) := \sum_{n=1}^\infty x^n/n^2$ and valid for $|x| \leqslant 1$ is a classical function of mathematical physics. Introduced by Leibniz in 1696 [8, p. 351] and thoroughly discussed by Euler some seventy years later [5, pp. 124–126], it has subsequently been well studied in the literature (for further historical details concerning the function see, for example, [12]). The canonical integral representation for the dilogarithm is

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt, \quad x \leqslant 1, \tag{1}$$

an integral that cannot be expressed in terms of elementary functions. Only at a handful of values is the dilogarithm known to reduce to simpler constants. These occur for the eight arguments: $0, \frac{1}{2}, \pm 1, -\varphi, \pm\frac{1}{\varphi}$, and $\frac{1}{\varphi^2}$ [9, pp. 4, 6–7]. Here $\varphi := (1+\sqrt{5})/2$ denotes the golden ratio.

Despite the paucity of special values found for the dilogarithm function it satisfies a multitude of functional relations. Some of these functional relations which we will have a need for are [9, p. 6, Eq. (1.15); p. 5, Eq. (1.11); p. 5, Eq. (1.12); p. 4, Eq. (1.7)]:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2), \quad -1 \leqslant x \leqslant 1 \quad \text{(duplication formula)} \tag{2}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log(x)\log(1-x), \quad 0 < x < 1 \quad \text{(Euler's reflexion formula)} \tag{3}$$

$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2\left(1-\frac{1}{x}\right) = -\frac{1}{2}\log^2(x), \quad x > 0 \quad \text{(Landen's identity)} \tag{4}$$

$$\mathrm{Li}_2(-x) + \mathrm{Li}_2\left(-\frac{1}{x}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(x), \quad 0 < x \leqslant 1 \quad \text{(inversion formula)} \tag{5}$$

Euler's reflexion formula, Landen's identity, and the inversion formula are examples of two-term dilogarithm identities. Replacing $x$ with $1-x$ in Landen's identity results in

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the following alternative form

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{x-1}\right) = -\frac{1}{2}\log^2(1-x), \quad x < 1. \tag{6}$$

while substituting $x = \frac{1}{2}$ into Euler's reflexion formula leads to the special value

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2). \tag{7}$$

Many other functional relations for the dilogarithm can be found. Here the reader is encouraged to consult the works of Kirillov [7] and Gordon and McIntosh [6].

The dilogarithm function today can be found in a wide variety of applications ranging from algebraic $K$-theory [13], Euler sums [16, 14], to conformal field theory [3]. For those unacquainted with the function it is best summed up in the words of Don Zagier who writes [17, p. 6]:

> ...the dilogarithm is one of the simplest non-elementary functions one can imagine. It is also one of the strangest. It occurs not quite often enough, and in not quite an important enough way, to be included in the Valhalla of the great transcendental functions ... [A]nd yet it occurs too often, and in far too varied contexts, to be dismissed as a mere curiosity. ... Almost all of its appearances in mathematics, and almost all the formulas relating to it, have something of the fantastical in them, as if this function alone among all others possessed a sense of humor.

New results found for the function therefore remain important. One such result was recently given by Lima who gave the remarkable two-term dilogarithm identity [10, Eq. (11)]

$$\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} - \frac{1}{2}\log^2\left(\sqrt{2}+1\right). \tag{8}$$

It was obtained by evaluating an integral that stemmed from a double integral used in a proof for the Basel problem given by Beukers, Kolk, and Calabi [2] where a non-trivial trigonometric change of variables is used, except with the trigonometric change of variables changed to its analogous hyperbolic form. What makes Lima's identity so interesting is that it is thought to not follow trivially from any previously known two-term dilogarithm identities [4].

Recently Campbell gave a new proof for Lima's identity using a series transformation obtained via Legendre polynomial expansions [4]. In this note we give three separate simple proofs for this same result. The first follows from the three functional relations (2) to (4), the second from a four-term dilogarithm functional relation, while the third from the evaluation of a definite integral in two different ways. As one application of Lima's identity, we will use it to show that

$$\mathrm{Li}_2\left(-\sqrt{2}\right) + \mathrm{Li}_2\left(-1-\sqrt{2}\right) = -\frac{5\pi^2}{24} - \frac{1}{2}\log\left(1+\sqrt{2}\right)\log\left(2+2\sqrt{2}\right). \tag{9}$$

Other non-trivial two-term dilogarithm identities due to Ramanujan can be found listed in [1, pp. 324–325] and still others are given by Loxton in [11]. Here by non-trivial we mean those two-term dilogarithm identities that do not directly follow on substituting for some value of $x$ into one of the two-term functional relations for the dilogarithm function.

## 2. Simple proofs of Lima's identity using functional relations

The two proofs we give here for Lima's identity make use of various functional relations for the dilogarithm function.

2.1. **Using Landen's identity, Euler's reflexion formula, and the duplication formula.** For the first of the proofs we give for Lima's identity we proceed by employing Landen's identity. For the first dilogarithm term appearing in (8) we have

$$\begin{aligned}\mathrm{Li}_2\left(\sqrt{2}-1\right) &= \mathrm{Li}_2\left(1-(2-\sqrt{2})\right) = -\mathrm{Li}_2\left(1-\frac{1}{2-\sqrt{2}}\right) - \frac{1}{2}\log^2\left(2-\sqrt{2}\right)\\ &= -\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\log^2\left(\frac{2}{2+\sqrt{2}}\right)\\ &= -\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left(\log(2)-\log(2+\sqrt{2})\right)^2.\end{aligned}$$

Noting that $\log(2+\sqrt{2}) = \frac{1}{2}\log(2) + \log(1+\sqrt{2})$, then

$$\begin{aligned}\mathrm{Li}_2\left(\sqrt{2}-1\right) &= -\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left(\frac{1}{2}\log(2)-\log(1+\sqrt{2})\right)^2\\ &= -\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{8}\log^2(2) + \frac{1}{2}\log(2)\log\left(1+\sqrt{2}\right) - \frac{1}{2}\log^2\left(1+\sqrt{2}\right).\end{aligned} \tag{10}$$

And for the second dilogarithm term appearing in (8), applying Landen's identity followed by Euler's reflexion formula one obtains

$$\begin{aligned}\mathrm{Li}_2\left(1-\sqrt{2}\right) &= -\mathrm{Li}_2\left(1-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\log^2\left(\sqrt{2}\right)\\ &= -\left[\frac{\pi^2}{6} - \log\left(\frac{1}{\sqrt{2}}\right)\log\left(1-\frac{1}{\sqrt{2}}\right) - \mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right)\right] - \frac{1}{8}\log^2(2)\\ &= \mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi^2}{6} + \frac{1}{2}\log(2)\log(2+\sqrt{2}) - \frac{1}{8}\log^2(2)\\ &= \mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi^2}{6} + \frac{1}{2}\log(2)\log(1+\sqrt{2}) + \frac{1}{8}\log^2(2),\end{aligned} \tag{11}$$

where again the result $\log(2+\sqrt{2}) = \frac{1}{2}\log(2) + \log(1+\sqrt{2})$ has been used. Taking the difference between (10) and (11) we see that

$$\begin{aligned}\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{6} - \frac{1}{2}\log^2(1+\sqrt{2}) - \frac{1}{4}\log^2(2)\\ - \left[\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) + \mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right)\right].\end{aligned} \tag{12}$$

A value for the dilogarithm term appearing within the square brackets on the right of the equality in (12) can be found from the duplication formula. Setting $x = \frac{1}{\sqrt{2}}$ in (2) we see that

$$\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) + \mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{24} - \frac{1}{4}\log^2(2),$$

where the result given in (7) has been used. Thus (12) reduces to (8) and completes the first of our proofs for Lima's identity.

2.2. **Using a four-term dilogarithm functional relation.** We first give a four-term functional relation involving dilogarithms.

**Theorem 2.1.** *For $-1 \leqslant x \leqslant 1$ the following four-term functional relation involving dilogarithms holds:*

$$\mathrm{Li}_2\left(\frac{1-x}{1+x}\right) - \mathrm{Li}_2\left(-\frac{1-x}{1+x}\right) = \frac{\pi^2}{4} + \mathrm{Li}_2(-x) - \mathrm{Li}_2(x) + \log(x)\log\left(\frac{1+x}{1-x}\right). \tag{13}$$

*Proof.* In view of (1) it is immediate that $\frac{d}{dx}\operatorname{Li}_2(x) = -\log(1-x)/x$. Consider

$$\frac{d}{dx}\left[\operatorname{Li}_2\left(\frac{1-x}{1+x}\right) - \operatorname{Li}_2\left(-\frac{1-x}{1+x}\right)\right] = \frac{2}{1-x^2}\log\left(\frac{2x}{1+x}\right) - \frac{2}{1-x^2}\log\left(\frac{2}{1+x}\right)$$
$$= \frac{2}{1-x^2}\log(x).$$

Integrating the above expression with respect to $x$ gives

$$\operatorname{Li}_2\left(\frac{1-x}{1+x}\right) - \operatorname{Li}_2\left(-\frac{1-x}{1+x}\right) = 2\int \frac{\log(x)}{1-x^2}\,dx = \int \frac{\log(x)}{1-x}\,dx + \int \frac{\log(x)}{1+x}\,dx + C, \quad (14)$$

after a partial fraction decomposition has been employed. Here $C$ is an arbitrary constant of integration. Making the change of variable of $t \mapsto 1-t$ in (1) we see that the first integral appearing in (14) is

$$\int \frac{\log(t)}{1-t}\,dt = \operatorname{Li}_2(1-t), \quad (15)$$

where, for convenience, we have dropped the arbitrary constant of integration. For the second integral appearing in (14), integrating by parts followed by a change of variable of $t \mapsto -t$ leads to

$$\int \frac{\log(t)}{1+t}\,dt = \log(t)\log(1+t) + \operatorname{Li}_2(-t), \quad (16)$$

where once more for convenience the arbitrary constant of integration has been dropped. Thus (14) becomes

$$\operatorname{Li}_2\left(\frac{1-x}{1+x}\right) - \operatorname{Li}_2\left(-\frac{1-x}{1+x}\right) = \operatorname{Li}_2(1-x) + \operatorname{Li}_2(-x) + \log(x)\log(1+x) + C. \quad (17)$$

To find the constant $C$, we set $x = 0$. Doing so we find

$$C = -\operatorname{Li}_2(-1) = \frac{\pi^2}{12}.$$

Here the value for $\operatorname{Li}_2(-1)$ is found on setting $x = 1$ in the inversion formula of (5). Substituting the value found for $C$ into (17), after applying Euler's reflexion formula to the term $\operatorname{Li}_2(1-x)$ the desired result then follows. □

**Remark 2.2.** The identity given by (13) is not new. It is listed, for example, online at The Wolfram Functions Site [15].

If one sets $x = \sqrt{2} - 1$ in (13), as

$$\frac{1-x}{1+x} = x = \sqrt{2} - 1,$$

one finds

$$\operatorname{Li}_2\left(\sqrt{2}-1\right) - \operatorname{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{4} + \operatorname{Li}_2\left(1-\sqrt{2}\right) - \operatorname{Li}_2\left(\sqrt{2}-1\right) - \log^2\left(\sqrt{2}-1\right),$$

or

$$\operatorname{Li}_2\left(\sqrt{2}-1\right) - \operatorname{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} - \frac{1}{2}\log^2\left(\sqrt{2}-1\right) = \frac{\pi^2}{8} - \frac{1}{2}\log^2\left(1+\sqrt{2}\right),$$

since $\log(\sqrt{2}-1) = -\log(1+\sqrt{2})$ and completes the second of our proofs for Lima's identity.

## 3. An application

As an application we now give a two-term dilogarithm identity that makes use of Lima's identity. This is identity (9). To prove this result, setting $x = -\sqrt{2}$ in identity (6) yields

$$\operatorname{Li}_2(-\sqrt{2}) = -\operatorname{Li}_2(2-\sqrt{2}) - \frac{1}{2}\log^2(1+\sqrt{2}). \tag{18}$$

Applying Euler's reflexion formula to the $\operatorname{Li}_2(2-\sqrt{2})$ term produces

$$\begin{aligned}\operatorname{Li}_2(2-\sqrt{2}) &= \operatorname{Li}_2\left(1-(\sqrt{2}-1)\right) = \frac{\pi^2}{6} - \operatorname{Li}_2(\sqrt{2}-1) - \log(\sqrt{2}-1)\log(2-\sqrt{2})\\ &= \frac{\pi^2}{6} - \operatorname{Li}_2(\sqrt{2}-1) + \frac{1}{2}\log(2)\log(1+\sqrt{2}) - \log^2(1+\sqrt{2}),\end{aligned}$$

since $\log(2-\sqrt{2}) = \frac{1}{2}\log(2) + \log(\sqrt{2}-1)$ and $\log(\sqrt{2}-1) = -\log(1+\sqrt{2})$. Thus (18) becomes

$$\operatorname{Li}_2(-\sqrt{2}) = -\frac{\pi^2}{6} - \frac{1}{2}\log(2)\log(1+\sqrt{2}) + \frac{1}{2}\log^2(1+\sqrt{2}) + \operatorname{Li}_2(\sqrt{2}-1). \tag{19}$$

Next, setting $x = 1+\sqrt{2}$ in the inversion formula yields

$$\operatorname{Li}_2(-1-\sqrt{2}) = -\operatorname{Li}_2(1-\sqrt{2}) - \frac{\pi^2}{6} - \frac{1}{2}\log^2(1+\sqrt{2}). \tag{20}$$

Adding (19) and (20) gives

$$\operatorname{Li}_2(-\sqrt{2}) + \operatorname{Li}_2(-1-\sqrt{2}) = \operatorname{Li}_2(\sqrt{2}-1) - \operatorname{Li}_2(1-\sqrt{2}) - \frac{\pi^2}{3} - \frac{1}{2}\log(2)\log(1+\sqrt{2}). \tag{21}$$

On substituting Lima's identity into (21) the two-term dilogarithm identity given in (9) immediately follows.

While the result given in (21) is interesting in its own right, it is important for another reason. If a method that is independent of Lima's identity can be found which gives the value for the dilogarithm sum appearing to the left of the equality in (21), it will give a third proof for Lima's identity. This will now be shown using a definite integral that is evaluated in two different ways.

The definite integral we consider is

$$J = \int_0^1 \frac{\operatorname{arcsinh}(x)}{x\sqrt{1+x^2}}\,dx.$$

Substituting $x = \sinh(t)$ followed by substituting $t = \log(u)$ we find

$$J = \int_0^{\log(1+\sqrt{2})} \frac{t}{\sinh(t)}\,dt = 2\int_1^{1+\sqrt{2}} \frac{\log(u)}{u^2-1}\,du,$$

or

$$J = -\int_1^{1+\sqrt{2}} \frac{\log(u)}{1-u}\,du - \int_1^{1+\sqrt{2}} \frac{\log(u)}{1+u}\,du,$$

after a partial fraction decomposition has been made. The first of the integrals to the right of the equality is (15), the second is (16). Thus

$$\begin{aligned}J &= -\operatorname{Li}_2(1-u)\Big|_1^{1+\sqrt{2}} - \Big[\log(u)\log(1+u) + \operatorname{Li}_2(-u)\Big]_1^{1+\sqrt{2}}\\ &= -\frac{\pi^2}{12} - \frac{1}{2}\log(2)\log(1+\sqrt{2}) - \log^2(1+\sqrt{2}) - \operatorname{Li}_2(-\sqrt{2}) - \operatorname{Li}_2(-1-\sqrt{2}).\end{aligned}$$

Evaluating the definite integral for $J$ a second time but in a different way, noting that for $x > 0$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\operatorname{arccoth}(\sqrt{1+x^2}) = -\operatorname{arctanh}\left(\frac{1}{\sqrt{1+x^2}}\right) = -\operatorname{arcsinh}\left(\frac{1}{x}\right),$$

where, for convenience, the various arbitrary constants of integration have been dropped, integrating by parts we have

$$J = -\operatorname{arcsinh}(x)\operatorname{arcsinh}\left(\frac{1}{x}\right)\bigg|_0^1 + \int_0^1 \frac{\operatorname{arcsinh}\left(\frac{1}{x}\right)}{\sqrt{1+x^2}}\,dx = -\log^2(1+\sqrt{2}) + \int_0^1 \frac{\operatorname{arcsinh}\left(\frac{1}{x}\right)}{\sqrt{1+x^2}}\,dx.$$

Here the result $\operatorname{arcsinh}(1) = \log(1+\sqrt{2})$ has been used. Enforcing a substitution of $x \mapsto \frac{1}{x}$ produces

$$J = -\log^2(1+\sqrt{2}) + \int_1^\infty \frac{\operatorname{arcsinh}(x)}{x\sqrt{1+x^2}}\,dx = -\log^2(1+\sqrt{2}) + \int_0^\infty \frac{\operatorname{arcsinh}(x)}{x\sqrt{1+x^2}}\,dx - J,$$

or

$$J = -\frac{1}{2}\log^2(1+\sqrt{2}) + \frac{1}{2}\int_0^\infty \frac{\operatorname{arcsinh}(x)}{x\sqrt{1+x^2}}\,dx = -\frac{1}{2}\log^2(1+\sqrt{2}) + I.$$

A value for the remaining integral $I$ can be readily found. Substituting $x = \sinh(u)$ gives

$$I = \frac{1}{2}\int_0^\infty \frac{u}{\sinh(u)}\,du = \int_0^\infty \frac{ue^{-u}}{1-e^{-2u}}\,du,$$

where the definition for the hyperbolic sine function in terms of exponentials has been used. Expanding the denominator as an infinite geometric series one has

$$I = \sum_{n=0}^\infty \int_0^\infty ue^{-(2n+1)u}\,du.$$

The interchange that has been made here between the integration sign and the summation is permissible due to the posititivity of all terms involved. Integrating by parts we find

$$I = \sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \sum_{n=1}^\infty \frac{1}{n^2} - \sum_{n=1}^\infty \frac{1}{(2n)^2} = \left(1-\frac{1}{4}\right)\sum_{n=1}^\infty \frac{1}{n^2} = \frac{3}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

Here the well-known result for the Basel problem of $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ has been used. The absolute convergence of the series allows for the rearrangement of its terms.

Returning to the integral $J$, we find

$$J = \frac{\pi^2}{8} - \frac{1}{2}\log^2(1+\sqrt{2}).$$

Equating the two values found for $J$ leads to the result given in (9), which when substituted into (21) leads to Lima's identity, thereby providing our third proof for this remarkable result.

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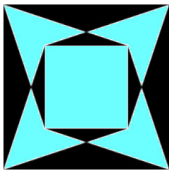

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# How two hundred years ago William Rowan Hamilton turned into a mathematician

ANNE VAN WEERDEN

Abstract. In 1822, the year he turned seventeen, William Rowan Hamilton became aware of his enormous mathematical talents, and wrote his first original mathematical papers. As a celebration of the two hundredth anniversary of the transition from the orientalist, theologian or statesman he was expected to become into the famous mathematician he would become, this is an overview of that remarkable year.

## 1. Introduction

On 26 January 1823 a central eclipse of the moon occurred; such eclipses have long durations and the moon is very dark. Two days later William Rowan Hamilton wrote from Trim, where he lived with his uncle James Hamilton, to his sister Eliza[1] that in the summer of 1822 he had "made calculations of all the circumstances" of the eclipse. When the time of emersion approached he could not find the moon, but shortly thereafter he saw Jupiter's moon Io through his telescope and knew that also our moon had started to emerge. "For it is a remarkable coincidence that Jupiter's moon emerged from a total eclipse only three minutes and a-half before ours did. At the same time Saturn was on the meridian" [1, 126].

On 23 February 1823 William wrote to 'Cousin Arthur',[2] who lived in Dublin, "[What] struck me was the near coincidence in point of time between the eclipse of our moon and that of the first Satellite of Jupiter. By an investigation founded on the successive propagation of light, I ascertained that there were places (not in this earth) at which the emersion of Jupiter's moon and the middle of the eclipse of ours would have appeared to synchronise, and also that these places are all contained in a hyperboloid of revolution, Jupiter being in one focus, the earth in the other, and the axis equal to the space that light traverses in the difference of the times of the phenomena: about ninety millions of miles. The result is remarkable" [1, 128-129].[3]

This observation obviously not being something most amateur astronomers would make, it is one of the indications that William had become a mathematician, a transition which had happened in 1822.

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[1]Both their parents died young. In 1817 Sarah Hamilton née Hutton died at thirty-seven; thereafter all five Hamilton siblings lived with relatives. Their father Archibald died in 1819, he was forty-one.

[2]Arthur Hamilton was a cousin of James and Archibald Hamilton.

[3]To verify William's results, data from the free planetarium program *Stellarium*, http://stellarium.org, and the website *Eclipsewise*, http://eclipsewise.com/lunar/LEcatalog/LEcatalog.html, were combined. Mentioning ninety million miles, William apparently had calculated that the light of Io's emersion started eight minutes after the time of greatest eclipse of our moon and that, travelling for about 38 minutes, it arrived on Earth 3.5 minutes before the emersion of our moon. From this result, in combination with his remark about Saturn, it appears that his calculations for the emersions of our moon and of Io were indeed very accurate.

## 2. From the Classics to Mathematics

In 1808 young William had been brought to Trim to be educated, and uncle James, linguist, curate and schoolmaster, had immersed him in the Classics from the moment he arrived. In 1810, when William was four, his mother Sarah wrote in an astonished and very proud letter to her sister Mary Hutton that he read "Latin, Greek and Hebrew!!" [1, 36-37]. Thereafter European languages were added, and "with a view to India," as was his father's wish [1, 57], also Oriental ones; Graves notes that "when thirteen years old [he] was in different degrees acquainted with thirteen languages."[4]

Early in 1821 the Classics still had been William's most important subject; in May 1821 he had written to Cousin Arthur, "You are not to imagine that because astronomical calculations take up the greater part of my letters to you, they therefore occupy the principal portion of my time; it is employed in the study of the classics as my serious business, and only occasionally in the sciences by way of recreation, in which light I consider them, however closely I may pursue them for a time" [1, 90]. But in August 1821[5] Uncle James gave him Lloyd's 1819 *Analytic Geometry*,[6] to which William reacted strongly, in September 1822 writing to Cousin Arthur, "Ill-omened gift! it was the commencement of my present course of mathematical reading, which has in so great a degree withdrawn my attention, I may say my affection, from the Classics" [1, 112]. Still, late in 1821 or early in 1822 he had written a long poem, according to Graves of "the Prize-poem order," with as its subject "the Literature of Rome" [1, 105-108].

In March 1822 William made "a great many calculations about the next eclipse of the moon: part of it will fall on August 3, my birthday. I have also made a view of the progress for Dublin," as he later wrote to Eliza. And on 31 March he wrote an Essay, 'On the value of 0/0, with preliminary remarks on Division.' Graves adds, "[which] by a subsequent annotation of his own is discredited as 'unnecessary'," yet he thought it worthwhile to give the calculations, to show William's "early interest in the elementary notions of science"[1, 101].

In April William contracted whooping-cough, which for adults is usually not acutely severe but it can last for months, and the coughing fits can be exhaustive. But William also suffered from chronic bronchitis which may have aggravated his symptoms; early in May he was allowed to go to Dublin to stay with Cousin Arthur for a 'required change of air', because he "had been for some time forbidden to read, coughed much, and had to struggle with great difficulty of breathing" [1, 99, 100, 303].

On 31 May 1822, while still in Dublin and reading the first volume of Laplace's *Mécanique Céleste*,[7] William found a "flaw in the reasoning by which Laplace demonstrates the parallelogram of forces" and gave a more general proof [1, 661-662]. According to Graves the document was found by Henry Hennessy "inserted at the pages it refers to in the copy of the *Mécanique Céleste* which belonged to Dr. Brinkley, and which subsequently came into the possession of Mr. Hennessy" [1, 103]. Although it was not the direct cause, it did lead to the chain of events which would bring William, towards the end of the year, in contact with Brinkley, who then was Royal Astronomer of Ireland and therefore lived at Dunsink Observatory.

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[4]R. P. Graves: *Our Portrait Gallery*, Dublin University Magazine (19) (1842), 94–110. https:/www.maths.tcd.ie/pub/HistMath/People/Hamilton/Gallery/Gallery.html. See for the discussion about this claim https://annevanweerden.nl/docs/Sir_William_Rowan_Hamilton_-_hyperpolyglot.pdf.

[5]William wrote that he had received Lloyd's book "in August, while the King was in Dublin." George IV visited Ireland from 12 August until 3 September 1821, https://georgianpapers.com/2021/10/18/erins-king-the-politics-and-pageantry-of-george-ivs-visit-to-ireland-in-1821, he therefore must have received it in the second half of August.

[6]B. Lloyd: *Analytic Geometry*, s.n., Dublin, 1819. https://doi.org/10.48495/qj72pf99m.

[7]P. S. Laplace: *Traité de Mécanique Céleste*, Vol. 1, J. B. M. Duprat, Paris, 1798. https://archive.org/details/traitdemcaniquec01lapl. The 'flaw' is on p. 6.

In June 1822 William was back in Trim, and in July again in Dublin. On 11 July he solved a mathematical problem in Analytic Geometry, posed as the Prize Question for 1822 in the *Gentleman's Mathematical Companion* of November 1821.[8] It had been shown to him by his later college tutor, the mathematician Charles Boyton, who was a son of a family friend and had become a Fellow on 13 July 1821; he solved it before Boyton did [1, 81, 90, 108]. William then still expected to enter College in October, or perhaps November.[9]

On 4 September 1822 William gave an overview of 1821, in a letter to Cousin Arthur. "I was amused this morning, looking back on the eagerness with which I began different branches of the Mathematics, and how I always thought my present pursuit the most interesting. I believe it was seeing Zerah Colburn[10] that first gave me an interest in those things. For a long time afterwards I liked to perform long operations in Arithmetic in my mind; extracting the square and cube root, and everything that related to the properties of numbers. It is now a good while since I began Euclid. Do you remember when I used to go to breakfast with you, and we read two or three propositions together every morning? I was then so fond of it, that when my uncle wished me to learn Algebra, he said he was afraid I would not like its uphill work after the smooth and easy path of Geometry. However, I became equally fond of Algebra" [1, 111].

This is also the letter in which William mentioned the "Ill-omened gift", the book by Lloyd he had received in 1821, and that he would become a mathematician was now inevitable; on 26 August 1822 he wrote a letter to his aunt Mary for which she apparently reprimanded him.[11] Graves comments on this "remarkable letter," "After having entered upon the study of Newton, Laplace, and Lagrange, he began to feel that he possessed powers akin to theirs; perhaps, too, he had floating notions of some of the discoveries which lay before him, for to this year he himself assigns the composition of an Essay which contains the germ of his investigations respecting Systems of Rays,[12] which were begun in the following year" [1, 110].

What William wrote to aunt Mary was, "I have been continuing my Classics, as usual, with my uncle. But I fear I shall never be so fond of them as of the Mathematics that I am now reading. I know that an intimate acquaintance with Classical literature is of the greatest importance both in College and in society: that nothing contributes more to form and refine one's taste; but still, in human literature, I think there is nothing that so exalts the mind, or so raises one man above his fellow-creatures, as the researches of Science. Who would not rather have the fame of Archimedes than that of his conqueror Marcellus, or than any of those learned commentators on the Classics, whose highest ambition was to be familiar with the thoughts of other men? If indeed I could hope to become myself a Classic, or even to approach in any degree to those great masters of ancient poetry, I would ask no more; but since I have not the presumption

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[8] J. Hampshire: *The Gentleman's mathematical companion*, vol. 5, Davis and Dickson, London, 1821-1826, xxv, 160 (question), xxvi, 447–452 (question and answers). Due to incoherent page numbering, page numbers refer to pages of the scanned volume. https://babel.hathitrust.org/cgi/pt?id=mdp.39015065321062. The Prize Question posed in 1822 as no 36 was won by P. P. and EPSILON (p. 319), one of William's answers is given on [1, 109].

[9] On 1 July one of the main entrance exams had been held, the next ones were on 14 October and 4 November. Trinity College Dublin Admissions Records, 1769-1825, https://doi.org/10.48495/6q182n74x.

[10] In 1817 William and Zerah had "engaged in trials of arithmetical skill," in which William's "antagonist was generally the victor." They met again in 1819, and Zerah "seems to have very freely imparted to Hamilton the methods used by him in calculation" [1, 77].

[11] Having been praised from very early childhood, all his life Hamilton had to work hard not to become vain. The support and criticisms of his family seems to have laid the foundation for his perseverance.

[12] See also page 54. For the 'Theory of Systems of Rays', leading to Hamiltonian mechanics and to his knighthood, see his 'Mathematical Papers', https://www.maths.tcd.ie/pub/HistMath/People/Hamilton/Papers.html.

to think so,[13] I must enter on that field which is open for me. Mighty minds in all ages have combined to rear upon a lofty eminence the vast and beautiful temple of Science, and inscribed their names upon it in imperishable characters; but the edifice is not completed: it is not yet too late to add another pillar or another ornament. I have yet scarcely arrived at its foot, but I may aspire one day to reach its summit" [1, 110-111].

## 3. A Mathematician

On 23 September 1822 William wrote to Eliza who had just entered the school of the Misses Hincks in Dublin,[14] "I have some curious discoveries - at least they are so to me - to show Charles Boyton when next we meet: he will be my Tutor soon. No lady reads a novel with more anxious interest than a mathematician investigates a problem, particularly if in any new or untried field of research. All the energies of his mind are called forth, all his faculties are on the stretch for the discovery. Sometimes an unexpected difficulty starts up, and he almost despairs of success. Often, if he be as inexperienced as I am, he will detect mistakes of his own, which throw him back. But when all have been rectified, when the happy clue has been found and followed up, when the difficulties, perhaps unusually great, have been completely overcome, what is his rapture! Such in kind, though not in degree, as Newton's, when he found the one simple and pervading principle which governs the motions of the universe, from the fall of an apple to the orbits of the stars" [1, 114-115].

About the 'curious discoveries' Graves writes, "There exists a Paper of twenty-one folio pages entitled "Essay on Equations representing Systems of Right lines in a given Plane. Part I.: On the manner in which they arise from problems determining a right line, which admit of more than one solution. By William Hamilton." To this title is appended a note which I transcribe. ("This curious old Paper, found by me to-day in settling my study, must have been written at least as early as 1822. It contains the germ of my investigations respecting Systems of Rays, begun in 1823. W.R.H., February 27, 1834.") [1, 115].

Apparently early in October William's entrance into College was postponed until the next year; William mentioned it to Eliza in a letter written on 9 October [1, 116]. Graves writes, "This decision was arrived at after much discussion between his uncle and his Cousin Arthur, the determining motive being the state of his health, which during the spring and the summer had caused much uneasiness" [1, 115]. It did not keep William away from his mathematical researches however.

On 31 October William wrote to Cousin Arthur, "When was Mr. Kiernan's letter left at Cumberland-street?[15] He tells me that "I forgot your 'queries about Laplace' for a long time" [...]; "but at last I laid them before Dr. Brinkley, who said he thought them ingenious, and he was so good as to say that he would write an explanation for you. He also desired me to bring you to him, and that he would be happy to know you, and to show you the Observatory. This of course, you know, is a great honour"" [1, 119]. Graves remarks that he could not "supply any information" about the 'queries', which William seems to have written when reading Laplace.

[13]While at college Hamilton twice won the Chancellor's Prize; for 'The Ionian Islands' and 'Eustace de St. Pierre' [1, 154], like his 1821 potentially prize winning poem both history poems. It made no difference.

[14]Bithia and Frances Hincks were aunts of the Reverend Edward Hincks, the famous Egyptologist and Assyriologist, https://www.dib.ie/biography/hincks-edward-a4021. They were related by marriage to the Huttons.

[15]Cousin Arthur lived at South Cumberland-street. George Shirley Kiernan was a family friend, State Apothecary, and a member of the Royal Irish Academy, https://archive.org/details/transactionsofro13iris/page/n133.

Graves then writes the concluding remarks, “I find among the early mathematical manuscripts of Hamilton one entitled ‘Example of an Osculating Circle determined without any consideration repugnant to the utmost rigour of Analysis,’ and dated November 14, 1822; a second, [of the same date],[16] entitled ‘Osculating Parabola to Curves of Double Curvature’; and a third, dated December, 1822, of which the title is, ‘On Contacts between Algebraic Curves and Surfaces.’ These papers mark the year 1822, when he attained the seventeenth year of his age, as that in which Hamilton entered upon the path of original mathematical discovery. With the second and third of them in his hand, availing himself of the kind permission of Dr. Brinkley, he paid his first visit to him at the Observatory.[17] Dr. Brinkley was impressed by their value, and desired to see some of the investigations in a more developed form; with this request Hamilton complied, by forwarding to him in the following month a paper entitled ‘Developments’” [1, 124]. Unfortunately, this paper is most likely lost; Graves remarks, “It was returned by him to Hamilton, and was in possession of the latter in the year 1841, but I have not discovered it among the manuscripts entrusted to me, nor I believe is it to be found in the Hamilton collection deposited in the manuscript-room of the Library of Trinity College” [1, 124].

On 23 February 1823 William wrote to Cousin Arthur, in the same letter in which he had written about the hyperboloid formed by the places of synchronicity of the emersion of Io and the greatest eclipse of our moon with which this narrative started, “Perhaps you heard that Dr. Brinkley expressed his full approbation of my “Developments”” [1, 128]. He finally entered College on 7 July 1823.[18]

## References

[1] R. P. Graves: *Life of Sir William Rowan Hamilton*, Vol. I, Hodges, Figgis, & Co., Dublin, 1882. https://archive.org/details/lifeofsirwilliam01gravuoft

**Anne van Weerden** As an information specialist working in Utrecht University Library and just having started her master’s program in theoretical physics, while enrolled in a seminar on the History of Vector Analysis she came across the distorted descriptions of Hamilton’s private life as an unhappily married alcoholic. She decided to stop her studies and find out what had happened.

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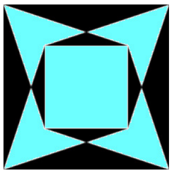
[16] Graves did not give a date; it was taken from the overview of Trinity College Dublin Library’s manuscript collection IE TCD MSS 7773-6, which contains the latter two papers, but not the first.

[17] William had visited Dunsink Observatory three years before, on 8 July 1819, but Brinkley had not been at home [1, 62].

[18] TCD Admissions Records, 1769-1825, pp. 343 and 344, https://doi.org/10.48495/6q182n74x. The second page contains some errors; William was born in Dublin, and his father Archibald had died before he entered College. See also the *Atlas blog* of November 2019, http://www.mathsireland.ie/blog/2019_11_cm.



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# Abel's limit theorem, its converse, and multiplication formulae for $\Gamma(x)$

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Abstract. Abel's well-known limit theorem for power series, and its corrected converse due to J. E. Littlewood, form the basis for a general identity that is prresented here, which is shown to be equivalent to Gauss's multiplication theorem for the Gamma function.

## 1. Introduction

An incomplete solution of a problem of mine, numbered Problem 86.3 in [4], (that was presented in [5]) prompted this note about Abel's limit theorem on power series, and two of its partial converses due, respectively, to Tauber and Littlewood. These are landmark results in the development of Real and Complex Analysis. For instance, Abel's theorem initiated the study of the boundary behaviour of analytic functions on the unit disc, and, in conjunction with Cesáro's consistency theorem on the convergence of arithmetic means of a convergent sequence, paved the way for summing series by different methods dealt with in [2], while the theorems of Tauber and Littlewood gave rise to the beautiful sub-topic of Wiener's Tauberian Analysis, also exposed in [2].

Students of Analysis who are desirous of learning "the tricks of the trade" would do well to study proofs of Abel's theorem and Tauber's, and at least acquaint themselves with the more profound result of Littlewood. All three theorems are simply expounded in [6].

In this note, we'll state and provide standard proofs of the theorems of Abel and Tauber, and state, but not prove, Littlewood's deeper result; instead, we'll illustrate its utility by means of a simple example. These theorems will be discussed in Sections 2, 3 and 4, respectively. As an illustration of the underlying ideas we'll derive a general theorem in Section 5, which is motivated by the aforementioned journal problem, and show that a special case of it is equivalent to Gauss's multiplication formula for the Gamma function (see the example in Section 9.56 of [1]).

## 2. Abel's limit theorem

Throughout the note, $f$ stands for a generic power series $\sum_{n=0}^{\infty} a_n x^n$ whose radius of convergence is 1, though the coefficients will differ from time to time.

According to Abel: if the series $\sum_{n=0}^{\infty} a_n$ is convergent, then

$$\lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} a_n.$$

We sketch the standard proof of this.

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*Proof.* Let

$$s_n = \sum_{k=0}^{n} a_k,\ \ n = 0, 1, 2 \ldots,$$

and $x \in [0, 1)$. As a first step we express $f(x)$ as a convex combination of the sequence $s_0, s_1, s_2, \ldots$. This is easy to do since by pointwise multiplication of two absolutely convergent power series

$$\begin{aligned}(1 + x + x^2 + \cdots)(a_0 + a_1 x + a_2 x^2 + \cdots) &= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots \\ &= s_0 + s_1 x + s_2 x^2 + \cdots\end{aligned}$$

so that

$$f(x) = (1 - x)\Big(\sum_{n=0}^{\infty} x^n\Big)\Big(\sum_{n=0}^{\infty} a_n x^n\Big) = (1 - x)\sum_{n=0}^{\infty} s_n x^n.$$

Accordingly, if $s = \lim_{n\to\infty} s_n$, and $0 \le x < 1$,

$$f(x) - s = (1 - x)\sum_{n=0}^{\infty} s_n x^n - s(1 - x)\sum_{n=0}^{\infty} x^n = (1 - x)\sum_{n=0}^{\infty}(s_n - s)x^n,$$

from which it follows that

$$|f(x) - s| \le (1 - x)\sum_{n=0}^{\infty} |s_n - s| x^n \le \sup\{|s_n - s| : n = 0, 1, 2, \ldots\}.$$

Hence

$$\sup\{|f(x) - s| : 0 \le x < 1\} \le \sup\{|s_n - s| : n = 0, 1, 2, \ldots\},$$

a step in the right direction, but not the final one! To obtain the desired result, we refine the argument just given by splitting the sum $(1-x)\sum_{n=0}^{\infty} |s_n - s| x^n$ in two appropriately. To achieve this, let $\epsilon > 0$, and choose an integer $n_0$ so that $|s_n - s| < \epsilon$, $\forall n > n_0$, whence for any $x \in (0, 1)$,

$$(1 - x)\sum_{n=n_0+1}^{\infty} |s_n - s| x^n \le \epsilon(1 - x)\sum_{n=n_0+1}^{\infty} x^n \le \epsilon.$$

Consequently,

$$|f(x) - s| \le (1 - x)\sum_{n=0}^{n_0} |s_n - s| x^n + \epsilon,$$

and so, on letting $x$ tend to 1 from the left,

$$\limsup_{x\to 1^-} |f(x) - s| \le \epsilon.$$

Since $\epsilon$ is an arbitrary positive number, this means that $\lim_{x\to 1^-} f(x) = s = \sum_{n=0}^{\infty} a_n$, as claimed. □

## 3. Tauber's converse

As the example

$$\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n,\ |x| < 1,$$

shows, the direct converse of Abel's theorem is false.

Tauber proved a conditional converse according to which, if $\lim_{x\to 1^-} f(x) = s$, and $\lim_{n\to\infty} n a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ is convergent and its sum is $s$.

*Proof.* To see this, note that

$$f(x) - s_n = \sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{n} a_k = \sum_{k=1}^{n} a_k(x^k - 1) + \sum_{k=n+1}^{\infty} a_k x^k,$$

for any $x \in (0,1)$, and any positive integer $n$. Now

$$\begin{aligned} \Big|\sum_{k=n+1}^{\infty} a_k x^k\Big| &= \Big|\sum_{k=n+1}^{\infty} (ka_k)\frac{1}{k}x^k\Big| \\ &\leq \frac{1}{n+1}\sum_{k=n+1}^{\infty} k|a_k|x^k \\ &\leq \frac{1}{(n+1)(1-x)}\max\{k|a_k| : k \geq n+1\}, \end{aligned}$$

and

$$\Big|\sum_{k=1}^{n} a_k(x^k - 1)\Big| \leq \sum_{k=1}^{n} |a_k|(1 - x^k) \leq (1-x)\sum_{k=1}^{n} k|a_k|.$$

Combining these estimates we have that

$$|f(x) - s_n| \leq (1-x)\sum_{k=1}^{n} k|a_k| + \frac{1}{(n+1)(1-x)}\max\{k|a_k| : k \geq n\}.$$

Bearing in mind that $x$ and $n$ are at our disposal, to be chosen as we see fit, it's now convenient to set $x \equiv x_n = 1 - \frac{1}{n+1}$. With this choice we have

$$|f(x_n) - s_n| \leq \frac{1}{n+1}\sum_{k=1}^{n} k|a_k| + \max\{k|a_k| : k \geq n\},$$

an expression that tends to zero as $n \to \infty$, its first term by Cesáro's theorem, and its second by hypothesis. Therefore

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} f(x_n) = \lim_{x\to 1^-} f(x) = s,$$

as we wanted to show. □

## 4. Littlewood's converse

Tauber's result was considerably strengthened by Littlewood [3] who proved that if $\lim_{x\to 1^-} f(x) = s$, and the sequence $na_n$ is merely bounded, then the series $\sum_{n=0}^{\infty} a_n$ is convergent and its sum is $s$. We won't give the proof of this, but instead provide a simple example to illustrate its utility.

**Example 4.1.** Suppose $0 < \theta < 2\pi$. Then

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\ln\Big(2\sin\frac{\theta}{2}\Big).$$

*Proof.* Fix $\theta \in (0, 2\pi)$, and consider the power series expansion about the origin of $f(x) = \ln(1 - 2\cos\theta x + x^2)$, namely, if $|x| < 1$, then

$$\begin{aligned} f(x) &= \ln\left[(1-e^{i\theta}x)(1-e^{-i\theta}x)\right] \\ &= -\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} x^n - \sum_{n=1}^{\infty} \frac{e^{-in\theta}}{n} x^n \\ &= -2\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} x^n. \end{aligned}$$

Clearly, $f(x) \to \ln(2 - 2\cos\theta)$ as $x \to 1^-$, and the coefficients of the last displayed power series satisfy Littlewood's condition. Hence, when $x = 1$ the displayed series is convergent and its sum is $\ln\left(4\sin^2\frac{\theta}{2}\right) = 2\ln\left(2\sin\frac{\theta}{2}\right)$, which yields the result. □

## 5. Gauss's multiplication formula for $\Gamma(x)$

As a precursor to this, we first establish the next result which relies on the theorems just described of both Abel and Littlewood.

**Theorem 5.1.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Suppose the sequence $a_n$ satisfies Littlewood's condition, and $m$ is a positive integer. Then $f(x) - f(x^m)$ converges to $s$ as $x \to 1^-$ iff the series*

$$\sum_{n=0}^{\infty}\left[\left(\sum_{r=0}^{m-1} a_{nm+r}\right) - a_n\right]$$

*is convergent and its sum is $s$.*

*Proof.* For $|x| < 1$,

$$\begin{aligned} f(x) - f(x^m) &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{nm} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{m-1} a_{nm+r} x^{nm+r} - \sum_{n=0}^{\infty} a_n x^{nm} \\ &= \sum_{n=0}^{\infty} x^{mn}\Big(\sum_{r=0}^{m-1} a_{mn+r} x^r - a_n\Big) \\ &= \sum_{n=0}^{\infty} x^{mn}\Big((a_{mn} - a_n) + \sum_{r=1}^{m-1} a_{mn+r} x^r\Big) \\ &= \sum_{n=0}^{\infty} c_n x^n, \end{aligned}$$

where, for $n = 0, 1, \ldots,$

$$c_{nm+r} = \begin{cases} a_{mn} - a_n, & \text{if } r = 0, \\ a_{nm+r}, & \text{if } r = 1, \ldots, m-1. \end{cases}$$

Suppose now that $f(x) - f(x^m)$ converges to $s$ as $x \to 1^-$. Then the series $\sum_{n=0}^{\infty} c_n x^n$ satisfies the hypotheses of Littlewood's theorem, and so $s = \sum_{n=0}^{\infty} c_n$. In other words, if $C_n$ denotes the $n$th partial sum of this series, $C_n \to s$, whence, in particular, $s = \lim_{n\to\infty} C_{mn}$, i.e.

$$s = \lim_{n\to\infty}\sum_{k=0}^{n}\sum_{r=0}^{m-1} c_{km+r} = \sum_{n=0}^{\infty}\Big(\sum_{r=0}^{m-1} a_{nm+r} - a_n\Big),$$

as desired. Conversely, suppose the last displayed series is convergent. Let

$$b_n = \sum_{r=0}^{m-1} a_{nm+r} - a_n,\ n = 0, 1, \ldots,$$

and

$$F(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then from above,

$$\begin{aligned} f(x) - f(x^m) &= \sum_{n=0}^{\infty} x^{mn}\Big(\sum_{r=0}^{m-1} a_{mn+r}x^r - a_n\Big) \\ &= \sum_{n=0}^{\infty} b_n x^{mn} + \sum_{n=0}^{\infty} x^{mn} \sum_{r=1}^{m-1} a_{mn+r}(x^r - 1) \\ &= F(x^m) + \sum_{r=1}^{m-1}(x^r - 1)\sum_{n=0}^{\infty} a_{mn+r}x^{mn} \\ &= F(x^m) + \sum_{r=1}^{m-1}(x^r - 1)h_r(x), \end{aligned}$$

where, for $r = 1, .., m - 1$,

$$h_r(x) = \sum_{n=0}^{\infty} a_{mn+r}x^{mn} = O(1)\log\frac{1}{1-x},\ (x \to 1^-).$$

As a result,

$$\sum_{r=1}^{m-1}(x^r - 1)h_r(x) = O(1)(1-x)\log\frac{1}{1-x} = o(1),\ (x \to 1^-).$$

Thus

$$f(x) - f(x^m) = F(x^m) + o(1)\ (x \to 1^-).$$

By Abel, $\lim_{x\to 1^-} F(x^m) = s$, and so $f(x) - f(x^m)$ converges to $s$ as $x \to 1^-$. This completes the proof. □

The folowing example is a direct consequence of this theorem.

**Example 5.2.** Let $m$ be any positive integer. Then, for all $a > 0$,

$$\sum_{n=0}^{\infty}\Big[\Big(\sum_{r=0}^{m-1}\frac{1}{nm+r+a}\Big) - \frac{1}{n+a}\Big] = \ln m. \tag{1}$$

*Proof.* Let $a_n = 1/(n + a)$, $n = 0, 1, 2, \ldots$, and $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Since the series in (1) is plainly convergent, by the theorem its sum is equal to the limit of $f(x) - f(x^m)$ as $x \to 1^-$. To calculate this, notice first that if $|x| < 1$, then

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n}x^n + \frac{1}{a} + \sum_{n=1}^{\infty}\Big(\frac{1}{n+a} - \frac{1}{n}\Big)x^n = \ln\frac{1}{1-x} + g(x),$$

say, and then that $f(x) - f(x^m) = \ln(1 + x + \cdots + x^{m-1}) + g(x) - g(x^m)$ which converges to $\ln m$ as $x \to 1^-$, since, by Abel, $\lim_{x\to 1^-} g(x)$ exists. □

The special case of this example, with $m = 3$ and $a = 1$, leads to the conclusion that

$$\sum_{n=0}^{\infty} \frac{9n+5}{9n^3 + 18n^2 + 11n + 2} = 3 \ln 3,$$

a proof of which was sought in [4].

What's noteworthy about (1), and surprising perhaps, is that, for each fixed integer $m > 1$, the series is convergent and its sum function is independent of $a$! What's the explanation for that? The reason is because—as we shall proceed to demonstrate—it's equivalent to Gauss's multiplication theorem for the Gamma function, $\Gamma(x)$, according to which if $m$ is a positive integer, then

$$m^{mx-\frac{1}{2}} \prod_{r=0}^{m-1} \Gamma(x + \frac{r}{m}) = (2\pi)^{\frac{m-1}{2}} \Gamma(mx), \, \forall x > 0. \tag{2}$$

This is an extension of the more familiar duplication formula due to Legendre:

$$2^{2x-1}\Gamma(x)\Gamma(x + \frac{1}{2}) = \sqrt{\pi}\Gamma(2x).$$

To explain the connection between (1) and (2), recall that the reciprocal of $\Gamma(z)$ is an entire function of the complex variable $z$, with simple zeros at the integers $0, -1, -2, \ldots$, that admits of the canonical factorization

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^{-z/n},$$

where $\gamma$ is Euler's constant $\lim_{n\to\infty} \Big( \sum_{k=1}^{n} \frac{1}{k} - \ln n \Big)$. Hence, denoting by $\psi$ the derivative of $\ln \Gamma$,

$$-\psi(x) = -\frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x} + \gamma + \sum_{n=1}^{\infty} \Big( \frac{1}{n+x} - \frac{1}{n} \Big).$$

Therefore, if $m$ is a positive integer, and $x > 0$, then

$$\begin{aligned}
& -m\frac{\Gamma'(mx)}{\Gamma(mx)} + \sum_{r=0}^{m-1} \frac{\Gamma'(x+\frac{r}{m})}{\Gamma(x+\frac{r}{m})} \\
= \;& \frac{1}{x} + m\gamma + m\sum_{n=1}^{\infty}\Big(\frac{1}{n+mx} - \frac{1}{n}\Big) - \sum_{r=0}^{m-1}\Big[\frac{1}{x+\frac{r}{m}} + \gamma + \sum_{n=1}^{\infty}\Big(\frac{1}{n+x+\frac{r}{m}} - \frac{1}{n}\Big)\Big] \\
= \;& -\sum_{r=1}^{m-1}\frac{1}{x+\frac{r}{m}} + m\sum_{n=1}^{\infty}\Big(\frac{1}{n+mx} - \frac{1}{n}\Big) - \sum_{r=0}^{m-1}\sum_{n=1}^{\infty}\Big(\frac{1}{n+x+\frac{r}{m}} - \frac{1}{n}\Big) \\
= \;& -\sum_{r=1}^{m-1}\frac{1}{x+\frac{r}{m}} - \sum_{n=1}^{\infty}\Big[\sum_{r=0}^{m-1}\frac{1}{n+x+\frac{r}{m}} - \frac{m}{n+mx}\Big] \\
= \;& -m\Big(\sum_{r=1}^{m-1}\frac{1}{mx+r} + \sum_{n=1}^{\infty}\Big[\Big(\sum_{r=0}^{m-1}\frac{1}{mn+r+mx}\Big) - \frac{1}{n+mx}\Big]\Big) \\
= \;& -m\sum_{n=0}^{\infty}\Big[\Big(\sum_{r=0}^{m-1}\frac{1}{mn+r+mx}\Big) - \frac{1}{n+mx}\Big] \\
= \;& -m\ln m,
\end{aligned}$$

by (1), with $a = mx$. In other words, for $x > 0$, assuming (1) holds,

$$\frac{d}{dx}\Big(\sum_{r=0}^{m-1} \ln\Gamma(x+\frac{r}{m}) - \ln\Gamma(mx)\Big) = -m\ln m.$$

Thus, for some constant $C(m)$,

$$\frac{\prod_{r=0}^{m-1}\Gamma(x+\frac{r}{m})}{\Gamma(mx)} = m^{-mx}C(m),\ \forall x > 0.$$

But, from the product formula for $1/\Gamma(x)$, it's clear that

$$\lim_{x\to 0^+}\frac{1}{x\Gamma(x)} = 1, \text{ whence } \lim_{x\to 0^+}\frac{\Gamma(x)}{\Gamma(mx)} = m.$$

Hence

$$C(m) = m\prod_{r=1}^{m-1}\Gamma(\frac{r}{m}).$$

It remains to compute the product $p(m) = \prod_{k=1}^{m-1}\Gamma(\frac{k}{m})$. To do this, we adapt Gauss's ploy (which legend says he used in kindergarten one day to add the first 100 natural numbers) and determine the geometric mean of $p(m)$ and the product of its factors in reverse order, namely, $\prod_{k=1}^{m-1}\Gamma(\frac{m-k}{m})$, also $p(m)$, of course. So, we consider

$$\begin{aligned} p(m)^2 &= \prod_{k=1}^{m-1}\Gamma(\frac{k}{m})\Gamma(1-\frac{k}{m}) \\ &= \prod_{k=1}^{m-1}\Big(\frac{\pi}{\sin\frac{k\pi}{m}}\Big) \\ &= \pi^{m-1}\prod_{k=1}^{m-1}\frac{1}{\sin\frac{k\pi}{m}}, \end{aligned}$$

by the reflection property of the Gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}.$$

To compute the product of the numbers $\sin\frac{k\pi}{n}, k = 1, 2, \ldots, n-1$, note that

$$\begin{aligned} 4^{m-1}\Big(\prod_{k=1}^{m-1}\sin\frac{k\pi}{m}\Big)^2 &= \prod_{k=1}^{m-1}\Big(4\sin^2\frac{k\pi}{m}\Big) \\ &= \prod_{k=1}^{m-1}|1-e^{\frac{2ik\pi}{m}}|^2. \end{aligned}$$

But the $m$ numbers $e^{\frac{2ik\pi}{m}}, k = 0, 1, \ldots, m-1$, are precisely the $m$th roots of unity, and so

$$z^m - 1 = (z-1)\prod_{k=1}^{m-1}(z - e^{\frac{2ik\pi}{m}}).$$

Hence,

$$m = \prod_{k=1}^{m-1}(1-e^{\frac{2ik\pi}{m}}),\ m^2 = \prod_{k=1}^{m-1}|1-e^{\frac{2ik\pi}{m}}|^2.$$

Consequently,

$$2^{2(m-1)}\Big(\prod_{k=1}^{m-1}\sin\frac{k\pi}{m}\Big)^2 = m^2,$$

from which it follows that

$$\prod_{k=1}^{m-1}\sin\frac{k\pi}{m} = \frac{m}{2^{m-1}},$$

since $\sin\frac{k\pi}{m} > 0$ for $k = 1, 2, \dots, m-1$. Hence

$$p(m)^2 = \frac{\pi^{m-1}2^{m-1}}{m},\ p(m) = \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}},$$

and so $C(m) = \sqrt{m}(2\pi)^{\frac{m-1}{2}}$, whence we obtain Gauss's formula:

$$\prod_{r=0}^{m-1}\Gamma(x+\frac{r}{m}) = (2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mx}\Gamma(mx).$$

Thus, with $a = mx$, the identity (1) implies (2). Since we can easily reverse the steps just made from (1) to (2), it should be clear that (1) is a consequence of (2).

To sum up: if $m$ is any positive integer, Gauss's multiplication statement for the Gamma function that

$$m^{mx}\prod_{r=0}^{m-1}\Gamma(x+\frac{r}{m}) = \prod_{r=1}^{m-1}\Gamma(\frac{r}{m})\Gamma(mx) = \sqrt{m}(2\pi)^{\frac{m-1}{2}}\Gamma(mx),\ \forall x > 0,$$

is equivalent to the statement that

$$\sum_{n=0}^{\infty}\Big(\sum_{r=0}^{m-1}\frac{1}{mn+r+x} - \frac{1}{n+x}\Big) = \ln m,\ \forall x > 0.$$

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# A Characterization of Cyclic Groups via Indices of Maximal Subgroups

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Abstract. We show that cyclic groups are the only finitely generated groups with the property that distinct maximal subgroups have distinct indices.

## 1. Introduction

It's well known that distinct subgroups of a cyclic group $G$ have distinct indices in $G$. It's also well known that if a finite group of order $n$ has at most one subgroup of order $m$—and so of index $n/m$—for each divisor $m$ of $n$, then the group is cyclic (see, for example, [1, p. 192]). The property that distinct subgroups have distinct indices thus characterizes the class of finite cyclic groups. It's less well known that the same property characterizes the infinite cyclic group: if distinct subgroups of an infinite group have distinct indices (as cardinal numbers) then the group is cyclic—see [7] which also covers the case of finite groups. The proof relies crucially on a result of Schur: if the center of a group $G$ has finite index then the commutator subgroup of $G$ is finite. Schur's result also underpins a similar characterization of the infinite cyclic group as the only infinite group in which each nontrivial subgroup has finite index (see [2] or [8, p. 446], or the more elementary treatment in [4]).

Recall that a *maximal subgroup* of a group is a proper subgroup that is not strictly contained in another proper subgroup. We prove an analogous characterization of cyclic groups to that in [7] using maximal subgroups in place of arbitrary subgroups. To be precise, we establish the following.

**Theorem.** *A finitely generated group is cyclic if and only if distinct maximal subgroups have distinct indices.*

To be more precise, we take the well-known "only if" direction as given and prove the "if" direction. The result seems to be new. At least, we've not been able to find it in the literature. Whether the gap we (appear to) fill was much needed, you, dear reader, can decide (see [3, p. 332]). Our proof hinges on a property of the Frattini subgroup. We discuss this and other background material in Section 2. In the case of finite groups, we give a second proof of the theorem using the inclusion-exclusion principle. The result fails without the hypothesis that $G$ is finitely generated—see Section 3.

There are several characterizations of families of groups in terms of properties of maximal subgroups. For instance, a finite group is a product of its Sylow subgroups (equivalently, is nilpotent) if and only if each maximal subgroup is normal [5, Cor. 10.3.4]. Another example: a finite group is supersolvable[1] if and only if each maximal subgroup

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[1]A group $G$ is *supersolvable* if it admits a chain of normal subgroups $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_r = \{1\}$ (for some positive integer $r$) such that each quotient $G_{i-1}/G_i$ is cyclic.

has prime index [5, Cor. 10.5.1 and Thm. 10.5.8]. Our characterization of cyclic groups is a modest companion to these classical observations.

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## 2. Preliminaries.

For ease of reference, we collect several results which feature in the proofs of the theorem. It may be best to just skim this section, referring back to look more closely as needed.

**Finitely generated groups and maximal subgroups.** A finite group has only finitely many subgroups, and thus each proper subgroup is contained in a maximal subgroup. By the magic of Zorn's lemma, the same conclusion holds for a finitely generated group.

**Lemma 1.** *Each proper subgroup of a finitely generated group is contained in a maximal subgroup.*

*Proof.* Let $G$ be a finitely generated group, say with generating set $\{g_1, \ldots, g_r\}$. For $H$ a proper subgroup of $G$, write $\mathcal{C}$ for the collection of proper subgroups of $G$ containing $H$, ordered by inclusion. If we show that every chain in $\mathcal{C}$ has an upper bound, then by Zorn's lemma $\mathcal{C}$ has a maximal element, that is, there is a maximal subgroup of $G$ containing $H$.

For $\{H_\lambda\}_{\lambda\in\Lambda}$ a chain in $\mathcal{C}$ (so that $H_\lambda \subseteq H_{\lambda'}$ or $H_{\lambda'} \subseteq H_\lambda$ for $\lambda, \lambda' \in \Lambda$), we set $\widetilde{H} = \bigcup_{\lambda\in\Lambda} H_\lambda$. Then $\widetilde{H}$ is a subgroup of $G$ containing $H$. If $\widetilde{H}$ is a proper subgroup, then it's an upper bound in $\mathcal{C}$ for $\{H_\lambda\}_{\lambda\in\Lambda}$ and the proof is complete. Now if $\widetilde{H} = G$, then $g_1 \in H_{\lambda_1}, \ldots, g_r \in H_{\lambda_r}$ for suitable $\lambda_1, \ldots, \lambda_r \in \Lambda$. As the subgroups $H_\lambda$ $(\lambda \in \Lambda)$ form a chain, it follows that there is a *single* $\lambda_k$ such that each $H_{\lambda_i}$ is contained in $H_{\lambda_k}$. Hence each $g_i$ belongs to $H_{\lambda_k}$ and $H_{\lambda_k} = G$, a contradiction. Thus $\widetilde{H}$ *is* a proper subgroup, and we've proved the lemma. □

**The Frattini subgroup.** Next we introduce a core notion.

**Definition.** The *Frattini subgroup* $\Phi = \Phi(G)$ of a group $G$ is the intersection of the maximal subgroups of $G$. If $G$ has no maximal subgroups then $\Phi(G) = G$.

Since each automorphism of a group permutes the maximal subgroups in the group, $\Phi$ is characteristic in $G$ (that is, stable under each automorphism of $G$). In particular, $\Phi$ is always a normal subgroup. We set $\overline{G} = G/\Phi$. Further, for $H$ a subgroup of $G$, we write $\overline{H}$ for the image of $H$ under the canonical quotient map $g \mapsto g\Phi : G \to \overline{G}$. That is, if $H$ is a subgroup of $G$, then $\overline{H} = H\Phi/\Phi$.

We can now record the crucial property of $\Phi$ that we exploit.

**Proposition 1.** *Let $G$ be a finitely generated group and let $H$ be a subgroup of $G$. Then:*

*(a) $G = H$ if and only if $\overline{G} = \overline{H}$;*
*(b) $G$ is cyclic if and only if $\overline{G}$ is cyclic.*

*Proof.* It's obvious that $G = H$ implies $\overline{G} = \overline{H}$. For the other direction in part (a), suppose $\overline{G} = \overline{H}$, so that $G = H\Phi$. Suppose also that a maximal subgroup $M$ of $G$ contains $H$. Since $\Phi \subseteq M$, we then have $G = H\Phi \subseteq M$ which is absurd. Hence $H$ is not contained in a maximal subgroup of $G$. Using Lemma 1, we see that $G = H$, as required.

Part (b) follows from part (a). In detail, $G$ is cyclic if and only if $G = \langle g\rangle$ for some $g \in G$. By part (a), this is equivalent to $\overline{G} = \langle g\Phi\rangle$ for some $g\Phi \in \overline{G}$. □

**Maximal subgroups and distinct indices.** We note a quick consequence of the hypothesis that distinct maximal subgroups of a group have distinct indices.

**Lemma 2.** *Suppose that distinct maximal subgroups of a group $G$ have distinct indices in $G$ (as cardinal numbers). Then each maximal subgroup is normal in $G$ of prime index.*

*Proof.* Let $M$ be a maximal subgroup of $G$. For $g \in G$, the conjugate $gMg^{-1}$ is again maximal and has the same index in $G$ as $M$. Thus $gMg^{-1} = M$ for all $g \in G$, that is, $M$ is normal in $G$.

We can therefore consider the quotient group $G/M$. Its subgroups have the form $H/M$ as $H$ varies through the subgroups of $G$ containing $M$. Hence $G/M$ has no nontrivial proper subgroups, and so is cyclic of prime order. Indeed, for $g \notin M$, the group $\langle gM \rangle$ is a nontrivial subgroup of $G/M$, and thus $\langle gM \rangle = G/M$; moreover, $gM$ must have prime order as otherwise $G/M$ would admit a nontrivial proper subgroup. □

**Relatively prime indices.** In the case of finite groups, our second proof of the theorem uses the index formula of the next lemma. The formula applies equally to infinite groups and is no harder to prove in this generality.

**Lemma 3.** *Let $G$ be a group and let $H_1, \ldots, H_r$ be finite index subgroups of $G$ whose indices are pairwise relatively prime, that is,* $\gcd([G : H_i], [G : H_j]) = 1$, *for $i \neq j$. Then $H_1 \cap \cdots \cap H_r$ has finite index in $G$ and*

$$[G : H_1 \cap \cdots \cap H_r] = [G : H_1] \cdots [G : H_r]. \tag{1}$$

*Proof.* To simplify the notation, we set $K = H_1 \cap \cdots \cap H_r$. Consider the map of left coset spaces

$$gK \mapsto (gH_1, \ldots, gH_r) : G/K \to G/H_1 \times \cdots \times G/H_r.$$

Observe that this map is injective. Indeed, if $gH_i = g'H_i$ for each $i$, then $g'^{-1}g \in H_i$ for each $i$, and so $g'^{-1}g \in K$ and $gK = g'K$. Thus $K$ has finite index in $G$ and

$$[G : K] \le [G : H_1] \cdots [G : H_r]. \tag{2}$$

On the other hand,

$$[G : K] = [G : H_i][H_i : K] \quad (\text{for } i = 1, \ldots, r).$$

As the indices $[G : H_i]$ are pairwise relatively prime, the product $[G : H_1] \cdots [G : H_r]$ divides $[G : K]$. In particular,

$$[G : H_1] \cdots [G : H_r] \le [G : K]. \tag{3}$$

Comparing (2) and (3), we've proved (1). □

**The Inclusion-Exclusion Principle.** We recall the statement and give a short proof.

**Proposition 2.** *For finite sets $S_1, \ldots, S_r$,*

$$\left| \bigcup_{i=1}^{r} S_i \right| = \sum_i |S_i| - \sum_{j<k} |S_j \cap S_k| + \cdots + (-1)^{r-1} |S_1 \cap \cdots \cap S_r|. \tag{4}$$

*Proof.* Let $S = \bigcup_{i=1}^r S_i$. Given $T \subseteq S$, we write $\mathbb{1}_T$ for the characteristic function of $T$. In this notation, we'll establish the equality of functions

$$\mathbb{1}_{\bigcup_{i=1}^r S_i} = \sum_i \mathbb{1}_{S_i} - \sum_{j<k} \mathbb{1}_{S_j \cap S_k} + \cdots + (-1)^{r-1} \mathbb{1}_{S_1 \cap \cdots \cap S_r}. \tag{5}$$

The identity (4) then follows by taking the integral of each side with respect to the counting measure on $S$ (the one that gives each element of $S$ measure 1).

We set $\mathbb{1} = \mathbb{1}_S$. For $T \subseteq S$, we also write $T^c$ for the complement of $T$ in $S$, so that

$$\mathbb{1}_{T^c} = \mathbb{1} - \mathbb{1}_T. \tag{6}$$

Further, for $T_i \subseteq S$ (for $i = 1, 2$), we have

$$\mathbb{1}_{T_1 \cap T_2} = \mathbb{1}_{T_1} \mathbb{1}_{T_2}. \tag{7}$$

Now, taking $T = (\bigcup_{i=1}^r S_i)^c$ in (6) gives

$$\begin{aligned} \mathbb{1}_{\bigcup_{i=1}^r S_i} &= \mathbb{1} - \mathbb{1}_{(\bigcup_{i=1}^r S_i)^c} \\ &= \mathbb{1} - \mathbb{1}_{\bigcap_{i=1}^r S_i^c} \\ &= \mathbb{1} - \mathbb{1}_{S_1^c} \cdots \mathbb{1}_{S_r^c} && \text{(by (7))} \\ &= \mathbb{1} - (\mathbb{1} - \mathbb{1}_{S_1}) \cdots (\mathbb{1} - \mathbb{1}_{S_r}) && \text{(by (6))}. \end{aligned}$$

Expanding the product on the last line, we see that

$$\begin{aligned} \mathbb{1}_{\bigcup_{i=1}^r S_i} &= \sum_i \mathbb{1}_{S_i} - \sum_{j<k} \mathbb{1}_{S_j} \mathbb{1}_{S_k} + \cdots + (-1)^{r-1} \mathbb{1}_{S_1} \cdots \mathbb{1}_{S_r} \\ &= \sum_i \mathbb{1}_{S_i} - \sum_{j<k} \mathbb{1}_{S_j \cap S_k} + \cdots + (-1)^{r-1} \mathbb{1}_{S_1 \cap \cdots \cap S_r}. \end{aligned}$$

We've shown that (5) holds and hence also (4). □

## 3. Two Examples.

We prove the theorem in Sections 4 (finite groups) and 5 (infinite groups). In this section, we give two examples of groups that are not finitely generated, and so certainly not cyclic, but have the property that distinct maximal subgroups have distinct indices. Thus the theorem fails if we drop the hypothesis that our groups are finitely generated.

First, let $M$ be a maximal subgroup of an abelian group $A$, so that the quotient $A/M$ has no proper nontrivial subgroups. As noted in the proof of Lemma 2, it follows that $A/M \cong \mathbb{Z}/p\mathbb{Z}$ for some prime $p$.

**Example 1.** Suppose an abelian group $A$ is such that $nA = A$ for all nonzero integers $n$ (using additive notation). Abelian groups with this property are called *divisible*. For example, the additive group $\mathbb{Q}$ is divisible. Further, a quotient of a divisible group is divisible. In particular, $A$ can never have $\mathbb{Z}/p\mathbb{Z}$ as a quotient (for $p$ a prime), and so $A$ has no maximal subgroups. Thus the maximal subgroups of $A$ have distinct indices—vacuously. By Lemma 1, if $A$ is nontrivial then it is not finitely generated.

**Example 2.** Consider the additive group $A = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ where the sum is over all primes. By construction, $A$ is not finitely generated.

Its maximal subgroups are the subgroups $lA$ as $l$ varies through the primes. Indeed, for $l$ prime, multiplication by $l$ is an isomorphism on $\mathbb{Z}/p\mathbb{Z}$ for $p \neq l$ and is the zero map on $\mathbb{Z}/l\mathbb{Z}$. Thus $lA = \bigoplus_{p \neq l} \mathbb{Z}/p\mathbb{Z}$. The projection map from $A$ to $\mathbb{Z}/l\mathbb{Z}$ therefore induces an isomorphism

$$A/lA \cong \mathbb{Z}/l\mathbb{Z}, \tag{8}$$

whence $lA$ is a maximal subgroup of $A$. On the other hand, say $M$ is a maximal subgroup of $A$. Then $A/M \cong \mathbb{Z}/l\mathbb{Z}$ for some prime $l$, so that $lA \subseteq M$. Maximality of $lA$ now implies that $lA = M$.

By (8), distinct maximal subgroups of $A$ have distinct indices in $A$.

## 4. Finite Groups.

We now prove the theorem for finite groups—in two ways.

Let $G$ be a nontrivial finite group and write $M_1, \ldots, M_r$ for the maximal subgroups of $G$. By hypothesis, the indices $[G : M_1], \ldots, [G : M_r]$ are distinct. By Lemma 2, each $M_i$ is normal in $G$ and there exist primes $p_1, \ldots, p_r$ such that $G/M_i \cong \mathbb{Z}/p_i\mathbb{Z}$ (for $i = 1, \ldots, r$).

*First proof.* Consider the homomorphism

$$g \mapsto (gM_i)_{i=1,\ldots r} : G \to \prod_{i=1}^{r} G/M_i.$$

Its kernel is $M_1 \cap \cdots \cap M_r = \Phi$, and thus $\overline{G} = G/\Phi$ embeds in $\prod_{i=1}^{r} G/M_i$. Now

$$\prod_{i=1}^{r} G/M_i \cong \prod_{i=1}^{r} \mathbb{Z}/p_i\mathbb{Z} \cong \mathbb{Z}/p_1 \cdots p_r\mathbb{Z}.$$

Hence $\overline{G}$ embeds in a cyclic group, and so is cyclic. Using Proposition 1 (b), we conclude that $G$ is cyclic. □

We need a preparatory observation for the second proof. Let $i_1, \ldots, i_k$ be distinct indices between 1 and $r$. Since the various (group) indices $[G : M_{i_j}]$ are (pairwise) relatively prime (for $j = 1, \ldots, k$), it follows from Lemma 3 that

$$[G : M_{i_1} \cap \cdots \cap M_{i_k}] \ = \ [G : M_{i_1}] \cdots [G : M_{i_k}] \ = p_{i_1} \cdots p_{i_k}. \tag{9}$$

We set $n = |G|$ and rewrite (9) as

$$|M_{i_1} \cap \cdots \cap M_{i_k}| = \frac{n}{p_{i_1} \cdots p_{i_k}}. \tag{10}$$

*Second proof.* The strategy of the proof is to show that there are elements of $G$ that lie outside each maximal subgroup. For any such $g \in G$, the cyclic subgroup $\langle g \rangle$ cannot be proper, so that $G$ is cyclic with generator $g$. To implement the strategy, we count the number of elements in $\bigcup_{i=1}^{r} M_i$. A trivial estimate then shows that this number is less than $n = |G|$, whence $G$ is cyclic.

Using the inclusion-exclusion principle and (10), we have

$$\begin{aligned}
\left| \bigcup_{i=1}^{r} M_i \right| &= \sum_i |M_i| - \sum_{j<k} |M_j \cap M_k| + \cdots + (-1)^{r-1} |M_1 \cap \cdots \cap M_r| \\
&= \sum_i \frac{n}{p_i} - \sum_{j<k} \frac{n}{p_j p_k} + \cdots + (-1)^{r-1} \frac{n}{p_1 \cdots p_r} \\
&= n \left( \sum_i \frac{1}{p_i} - \sum_{j<k} \frac{1}{p_j p_k} + \cdots + (-1)^{r-1} \frac{1}{p_1 \cdots p_r} \right) \\
&= n \left[ 1 - \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right) \right] \\
&< \ n.
\end{aligned}$$

We've proved (again) that $G$ is cyclic. □

**Remark 1.** For $n$ a positive integer, let $p_1, \ldots, p_r$ be the distinct prime divisors of $n$. Recall that $\phi(n)$ is the number of integers between 1 and $n$ that are relatively prime to

$n$. Applying the above proof to a cyclic group of order $n$ gives the well-known formula

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_r}\right). \tag{11}$$

The inclusion-exclusion principle also yields the formula directly—without an appeal to group theory. To spell this out, write $\mathbf{n}$ for the set of integers between 1 and $n$ and $\mathbf{n}/\mathbf{p}_i$ for the set of elements of $\mathbf{n}$ that are divisible by $p_i$, so that $|\mathbf{n}/\mathbf{p}_i| = n/p_i$ (for $i = 1, \ldots, r$). Then the union $\bigcup_{i=1}^r \mathbf{n}/\mathbf{p}_i$ consists of the integers between 1 and $n$ that are divisible by *some* prime divisor of $n$—that is, the integers between 1 and $n$ that are not relatively prime to $n$. Thus

$$\left|\bigcup_{i=1}^{r} \mathbf{n}/\mathbf{p}_i\right| = n - \phi(n).$$

Counting $|\bigcup_{i=1}^r \mathbf{n}/\mathbf{p}_i|$ via the inclusion-exclusion principle (exactly as above), we obtain (11) once more.

## 5. Infinite Groups.

Next we prove the theorem for infinite groups. We treat abelian groups first (finite and infinite) and then reduce to this case.

To start, note that if distinct maximal subgroups of a group $G$ have distinct indices then each quotient of $G$ inherits the property. Indeed, for $N$ a normal subgroup of $G$, a maximal subgroup of $G/N$ has the form $M/N$ for a unique maximal subgroup $M$ of $G$ containing $N$ and $[G/N : M/N] = [G : M]$.

**Lemma 4.** *Let $A$ be a finitely generated abelian group such that distinct maximal subgroups of $A$ have distinct indices in $A$. Then $A$ is cyclic.*

*Proof.* By (a part of) the fundamental theorem of finitely generated abelian groups, there is a nonnegative integer $r$ and a finite abelian group $T$ such that $A \cong \mathbb{Z}^r \times T$. As quotients of $A$, the groups $\mathbb{Z}^r$ and $T$ have the property that distinct maximal subgroups have distinct indices.

If $r = 0$, then $A$ is finite and hence cyclic (since we've proved the theorem for finite groups).

Suppose $r > 0$. We need to show that $T$ is trivial and $r = 1$. If $T$ is nontrivial, then it admits a maximal subgroup, say $T_{\max}$, and $[T : T_{\max}] = p$ for some prime $p$. In this case, the subgroups $\mathbb{Z}^r \times T_{\max}$ and $p\mathbb{Z} \times \mathbb{Z}^{r-1} \times T$ would each have index $p$ in $\mathbb{Z}^r \times T$. We conclude that $T$ is trivial, so that $A \cong \mathbb{Z}^r$. In the same way, we have $r = 1$: for $r > 1$ and $p$ a prime, the subgroups $p\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^{r-2}$ and $\mathbb{Z} \times p\mathbb{Z} \times \mathbb{Z}^{r-2}$ each have index $p$ in $\mathbb{Z}^r$. □

Combining Lemma 4 and earlier arguments, the theorem for infinite groups follows readily.

*Proof.* Let $G$ be a finitely generated infinite group such that distinct maximal subgroups of $G$ have distinct indices in $G$. We want to show that $G$ is cyclic.

Write $\{M_i\}_{i\in I}$ for the family of maximal subgroups of $G$. By Lemma 2, each $M_i$ is normal in $G$ and each quotient $G/M_i$ is cyclic, hence abelian. As in the first proof for finite groups, we consider the homomorphism

$$g \mapsto (gM_i)_{i\in I} : G \to \prod_{i\in I} G/M_i.$$

Since the kernel is $\Phi$, we see that $\overline{G} = G/\Phi$ embeds in the abelian group $\prod_{i\in I} G/M_i$, and so $\overline{G}$ is abelian. Moreover, $\overline{G}$ is finitely generated and distinct maximal subgroups of $\overline{G}$

have distinct indices. Hence, by Lemma 4, $\overline{G}$ is cyclic. Appealing to Proposition 1 (b), we conclude that $G$ is cyclic. □

## 6. A Comment on Cyclic $p$-groups.

The maximal subgroups of a finite cyclic group $G$ are the subgroups of prime index, one for each prime divisor of $|G|$. In particular, a nontrivial finite cyclic $p$-group ($p$ a prime) has a unique maximal subgroup. Conversely, if a finite group $G$ has a unique maximal subgroup $M$, then any element of $G$ that is not in $M$ is a generator, so $G$ is cyclic, and the order of $G$ is a $p$-power for $p = [G : M]$. That is, the nontrivial cyclic groups of prime-power order are precisely the finite groups that have a single maximal subgroup. This characterization of cyclic groups of prime-power order can be slightly augmented as follows.

($\alpha$) *Suppose the maximal subgroups of a finite group $G$ form a single conjugacy class. Then $G$ is cyclic of prime-power order.*

*Proof.* Let $M$ be a maximal subgroup of $G$. By hypothesis, the union of the maximal subgroups of $G$ is $\bigcup_{g \in G} gMg^{-1}$. Now, an elementary counting argument shows that a finite group is never a union of conjugates of a proper subgroup (see, for example, [9, Lemma 6.1]). Thus there is an element of $G$ that is not contained in a maximal subgroup, whence $G$ is cyclic. Moreover, $M$ must be the unique maximal subgroup of $G$, and so $G$ has prime-power order. □

**Remark 2.** What happens if you replace "finite" in ($\alpha$) by "finitely generated"? The statement is then false—in spectacular fashion. In fact, for each sufficiently large prime $p$, there is an infinite group $G$ such that

(a) each nontrivial proper subgroup has order $p$ and so is maximal,

(b) the maximal subgroups (that is, the nontrivial proper subgroups) form a single conjugacy class.

Note that any such $G$ is generated by two elements: for $h \neq 1$ and $g \notin \langle h \rangle$, the subgroup $\langle h, g \rangle$ has more than $p$ elements, and hence $G = \langle h, g \rangle$.

An infinite group that satisfies (a) is called a *Tarski monster*. These ghoulish groups were shown to exist by A. Y. Olshanskii and independently by E. Rips (for details, see Chap. 9 of Olshanskii's bestiary [6]). Among them are ones that also satisfy (b).

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# Daniel Rosenthal, David Rosenthal and Peter Rosenthal: A Readable Introduction to Real Mathematics (2nd Edition), Springer, 2018. ISBN:978-3-030-80731-3, GBP 44.99, 218+xviii pp.

REVIEWED BY ROBIN HARTE

In this second edition of their delightful "readable introduction" the Rosenthal family have preserved the structure, Chapters 1-12, of the first edition, and meticulously gone through them picking out typos and minor infelicities, and then (the Operator Theory beginning to show through!) added two new chapters: on infinite series and then higher dimensional spaces, including norms and inner products.

Back in the early chapters, Customs Officials have searched the alleged prime number 100,000,559 and uncovered its cargo of prime factors 53, 223 and 8,461. As in the first edition, Chapters 1 to 3 introduce the natural numbers, the principle of mathematical induction, and then modular arithmetic. Chapters 4 and 5 are devoted to the fundamental theorem of arithmetic, prime factorisation, and then the theorems of Fermat and Wilson.

Chapter 6 is about the RSA method of "public key cryptography". Here the "recipient" publicly announces a number $N = pq$ which is the product of two very large and distinct prime numbers $p$ and $q$, which are not revealed. Now a "message" is just a number $M < N$. To receive such messages the recipient announces another number $E$, the "encryptor", and asks the sender to compute, and send, the remainder $R$ which $M^E$ leaves on division by $N$. The recipient, wishing to determine $M$, must now find a decryptor $D$, such that for every $0 \leq L \leq N$, $L^{ED} \equiv L \bmod N$. With a little help from Fermat's theorem, it follows $R^D \equiv M \bmod N$.

Chapters 7 to 9 discuss the Euclidean algorithm, rational and irrational numbers, and then complex numbers. Chapter 10 is about cardinal numbers and infinite sets, and finally Chapters 11 to 12, "Euclidean plane geometry" and ruler-and-compass "constructibility", leading to (quadratic) "surds".

Towards the end, in the new Chapter 13, they offer the decomposition of a natural number as the product of a square free and a perfect square, which they deploy to prove that the reciprocals of the prime numbers combine to form a divergent series. Finally, in Chapter 14, they offer their introduction to norms and inner products.

This picky reviewer would have appreciated an appendix listing, in palatable form, axioms for the real numbers, and indeed "Euclidean plane geometry"; but the Rosenthals have between them produced a very fine, and very readable, introduction to "real" mathematics. Local readers should, as a supplement, also read the very beautiful notes ("Prime numbers", 2006 Course 4281 TCD tinyurl.com/4puvvr8p), from our own very much lamented T.G. Murphy.

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# Philippe Zaouati: Perelman's Refusal: A Novel, American Mathematical Society, 2021. ISBN: 978-1-4704-6304-5, $29.00 ($23.20 to members), 133 pp.

REVIEWED BY PETER LYNCH

The background to this novel will be known to many *Bulletin* readers. For a century, a conjecture made by Henri Poincaré in 1904 eluded all attempts at proof. In 1982, William Thurston, a Princeton mathematician, proposed a taxonomy for classifying three-dimensional manifolds. His theory, known as the geometrization conjecture, describes all such manifolds. Over a period beginning in November 2002, Grigori (Grisha) Perelman, who had been completely out of contact with the mathematical community for seven years, posted three papers on `arXiv.org` with a proof of Thurston's geometrization conjecture. Perelman's papers did not mention Poincaré but, in fact, the Poincaré conjecture is a special case of Thurston's conjecture.

The Poincaré conjecture is that all closed, simply-connected three-dimensional manifolds are topological 3-spheres. It is a key result in topology and also has important implications for cosmology: the universe is perhaps the largest three-dimensional manifold, so the conjecture is relevant to the "shape of the universe".

In 2006 the International Mathematical Union (IMU) nominated Perelman for a Fields Medal. The award was to be made at the quadrennial International Congress of Mathematicians (ICM) in Madrid in August 2006. The IMU Newsletter predicted that the congress would be the occasion when Poincaré's conjecture would become a theorem. However, Perelman indicated his intention to decline the award and IMU feared that this would cast a shadow over the congress. The IMU President of the time, Professor Sir John Ball, travelled to St. Petersburg to meet Perelman, in the hope of persuading him to accept the prize.

The above sketch sets the scene for *Perelman's Refusal*. The action of the book takes place over a few days in June 2006. The author, Philippe Zaouati, met with Professor Ball in 2014 to discuss the entire affair. While Ball was positive about the plan for a book and provided valuable input, he did not comment on Perelman's personal circumstances or on the content of their conversation, which he said was strictly confidential. The extensive conversations in the book are products of the author's imagination, but they have a great semblance of authenticity and credibility.

John Ball and Grigori Perelman met on 11th June 2006. They spent the morning in a conference centre by the Neva River and the afternoon walking around the magnificent city of St. Petersburg. The two characters interacted with empathy, each man fully aware of the sincerity and honesty of the other. Ball tried, using a number of clever and persuasive arguments, to convince Perelman that, in everyone's interest, he should come to Madrid and accept the Fields Medal. Failing that, he should permit it to be awarded *in absentia*. However, it seemed evident from the outset that Perelman's decision had already been made. This was not the first time he had declined a prize: in 1996, he had refused a prestigious award from the European Mathematical Society, and he would later reject the $1 million Millennium Prize of the Clay Mathematics Institute.

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Late in the evening of June 11th, John Ball, relaxing in an easy chair in his hotel room, falls into a reverie, imagining the thoughts of Perelman. Clearly, he has a deep respect for, understanding of and sympathy with the complex Russian. But why would Perelman turn down the honour of a Fields Medal? There seemed to be several reasons. Fame meant nothing to Perelman. He had resigned from the Steklov Institute and no longer considered himself a mathematician. He felt that he could not accept a prize intended to encourage mid-career mathematicians. He wanted nothing to do with the ICM, which he regarded as a circus, or to accept an award from the King of Spain. A more domestic reason bubbled up in Ball's reverie, forcing him to conclude that Perelman was determined: "Mamma won't go to Madrid; I won't ask Mamma to go to Madrid. No, I won't go."

The description of Ball's reverie is a worthy, and successful, attempt to provide a window on the mental workings of a mathematical genius. But is the genius Perelman or an archetype conjured up by the author? In either case, the italicized passages in the chapter make for fascinating reading. The reverie strives to plumb the mind of Perelman, to understand what enthuses him, what irks him, what infuriates him.

Mathematics was the spiritual force that impelled Perelman. As a Jew, he faced major obstacles to his mathematical development: in the Russian university system, there was systematic discrimination against Jews. However, competing in the International Mathematical Olympiad in 1982, Perelman achieved a perfect score, winning a gold medal. This gained him access, at the age of sixteen, to the School of Mathematics and Mechanics at the Leningrad State University, without the requirement to take the discriminatory admission examinations.

The Fields Medal held no value for Perelman. Money was of little interest to him; indeed, he feared it. The 1990s was a time of great economic upheaval in Russia, and he witnessed some of the unsavoury consequences: "In Russia, money always leads to violence". This alone was reason enough for him to decline the $1 million Millennium Prize.

The following morning, the two men met once more and walked together again through the streets of St. Petersburg. Anyone planning to attend the ICM in July should enjoy the narrative detail provided by the author in his descriptions of that splendid city. Although the prospects seemed remote, Ball wondered whether there was any circumstance in which the Russian would come to Madrid? Before they parted, he put one last question to Perelman; he proposed an imaginative, if highly improbable, scenario. He batted off Perelman's objection that it was hypothetical, asking him to treat the proposal in a *reductio ad absurdum* way, at which point Perelman finally said "Yes". However, the condition — which I shall not reveal — was never satisfied.

This book contains little about the mechanics of the Poincaré conjecture. It discusses Ricci flow only in a general way and readers seeking details must look elsewhere. However, an excellent popular account, with many endnotes pointing to further sources, is available [1]. The mathematical development of Grigori Perelman, his career in America, his return to Russia and his withdrawal from the mathematical community are touched upon but again a more detailed source is available [2]. Finally, an extensive article in *The New Yorker* [3] includes a detailed account of the sorry story of a paper published by mathematicians Cao and Zhu in the *Asian Journal of Mathematics*. They claimed credit for the proof of Poincaré, but their claim did not survive scrutiny: passages of their paper were plagiarised and it brought no honour to its authors.

The stimulating re-imagination of the encounter between Grigori Perelman and John Ball makes this book well worth reading. I enjoyed it greatly and can recommend it to *Bulletin* readers and, indeed, to anyone interested in the world of mathematics.

## References

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[2] Gessen, Masha, 2011: *Perfect Rigour*, Icon Books, 242pp. ISBN:978-1-8483-1301-9.
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[4] Podcast, 2006: In Our Time: the Poincaré Conjecture. Melvyn Bragg and guests discuss a puzzle that may explain the shape of the universe. `https://www.bbc.co.uk/sounds/play/p0038x8l`

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# Paul J. Nahin: In Pursuit of Zeta-3, Princeton University Press, 2021. ISBN:9780691206073, USD 26.95, 344 pp.

REVIEWED BY JOHN E. McCARTHY

When I was a young mathematician, the harmonic analyst Henry Helson gave me some excellent advice: *"Every mathematician should study the Zeta function. It is a mirror in which you see yourself."* He was right; no matter what field of mathematics you study, there is some connection to the Zeta function.

The main subject of the book under review is the Zeta function, defined for complex numbers $s$ with Re(s) > 1 by the Dirichlet series

$$\zeta(s) \ = \ \sum_{n=1}^{\infty} n^{-s},$$

and extending by analytic continuation to $\mathbb{C} \setminus \{1\}$.

Euler calculated $\zeta(s)$ exactly when $s$ is a positive even integer, and the author gives us derivations of some of the formulas, including

$$\begin{aligned} \zeta(2) &= \textstyle\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \\ \zeta(4) &= \textstyle\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}. \end{aligned}$$

The title of the book refers to the question: is there a closed form formula, involving the standard mathematical constants, for

$$\zeta(3) \ = \ \sum_{n=1}^{\infty} \frac{1}{n^3}?$$

Discussing this question, the author gives us a tour of some classical topics in mathematics such as the Gamma function, Euler's constant $\gamma$, and Fourier series. He demonstrates various ingenious identities, such as the following (due to Euler):

$$\zeta(3) \ = \ \frac{2\pi^2}{7} \ln 2 + \frac{16}{7} \int_0^{\pi/2} x \big[ \ln(\sin x) \big] dx,$$

and gives a lovely proof that

$$\gamma \ = \ - \int_0^{\infty} e^{-x} \ln x \, dx.$$

The author has an engaging writing style, and comes across as a jovial uncle entertaining his nieces and nephews with stories and magic tricks. There are lots of challenging problems, with worked solutions at the back of the book, which will appeal to some readers. There are interesting digressions, both mathematical and historical. I enjoyed reading the book, and yet it bothered me in several ways.

First is the question of who the audience is. Nahin writes in the introduction that he hopes the audience will be enthusiastic readers of mathematics at the level of high school AP calculus—that is about the same as Honours Leaving Certificate Maths. I

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think back to when I had taken the Leaving Cert, and imagine what my reactions would have been to this book. I believe they would have been delight, confusion, and intimidation, in that order.

The delight would come from the many clever tricks to evaluate integrals, and the glimpses of the mathematical world beyond the garden wall. The confusion would come from the author's eschewment of rigour. He regularly differentiates under integral signs, and interchanges orders of integration, with a remark that this can often be justified but not always. Why not just state some theorems that give sufficient conditions, so that the reader at least knows what needs to be checked?

Worse is the author's cavalier treatment of divergent series. He never defines what he means by convergence of such a series (implicitly, he means by some form of analytic continuation, but this is not defined, nor are the problems associated with it discussed). Closely related to the Zeta function is the Eta function, defined by a Dirichlet series and extended analytically to the entire plane:

$$\eta(s) \;=\; \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s} \;=\; (1-2^{1-s})\zeta(s), \qquad \text{for Re(s)} > 0. \tag{1}$$

His proof that $\eta(0) = -\zeta(0) = \frac{1}{2}$ is that if you put $s = 0$ in (1) you get

$$\eta(0) \;=\; 1-1+1-1+1\ldots, \tag{2}$$

and if you put $x = 1$ in the series

$$\frac{1}{1+x} \;=\; 1-x+x^2-x^3+\ldots$$

you get

$$\frac{1}{2} \;=\; 1-1+1-1+1\ldots. \tag{3}$$

Comparing (2) and (3), he deduces that $\eta(0) = \frac{1}{2}$. But in the late $18^{\text{th}}$ century, Callet pointed out that if you let $x = 1$ in the series

$$\frac{1}{1+x+x^2} \;=\; 1-x+x^3-x^4+x^6-x^7+\ldots$$

you would get

$$\frac{1}{3} \;=\; 1-1+1-1+1\ldots.$$

Nahin does not discuss why divergent series can be used to get valid formulas, or why $\frac{1}{2}$ really is the correct value for $\eta(0)$.

Intimidation would follow, because I would have felt stupid for not really understanding the arguments laid out in front of me. Today I am a professional mathematician (something that, to the relief of my mother and my eternal wonderment, turns out to be an actual job). I know how to justify differentiating under integral signs, and I understand that divergent series make sense as long as you are clear about definitions. So I, and other readers of the this Bulletin, can enjoy this book. But with a bit more care about rigour, and more effort in justifying why the steps in an argument follow a logical strategy, instead of just attributing them to the genius of Euler, this could have been a book that did indeed hit the target audience. Ultimately, we don't want just to marvel at magic tricks, we want to know how they are done.

**John E. McCarthy** grew up in Limerick, and graduated from Trinity College Dublin in 1983. He got a Ph.D. from U.C. Berkeley, and is currently a professor at Washington University

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# PROBLEMS 89

IAN SHORT

## Problems

The first problem in this issue was posed by Des MacHale of University College Cork.

**Problem 89.1.** It is well known that it is possible to dissect a square into a finite number of different squares, but that it is not possible to dissect an equilateral triangle into a finite number of different equilateral triangles. Determine whether it is possible to dissect an isosceles right-angled triangle into a finite number of different isosceles right-angled triangles.

In this problem, 'dissect' means 'partition into two or more pieces', and 'different' means that no two of the shapes considered are congruent.

The second problem was suggested by Toyesh Prakash Sharma, Agra College, India.

**Problem 89.2.** Prove that

$$\int_{-\pi/2}^{\pi/2} \cos^2(\tan x)\, dx = \frac{\pi}{2}(1+e^{-2}).$$

The third problem comes from Finbarr Holland of University College Cork.

**Problem 89.3.** Let $a_k$ and $b_k$ be real numbers with $a_k < b_k$, for $k = 1, 2, \dots, n$, and let

$$r_n(z) = \prod_{k=1}^{n} \frac{b_k + z}{a_k + z}.$$

Prove that

$$\int_{-\infty}^{\infty} \log|r_n(ix)|\, dx = \pi \sum_{k=1}^{n} (b_k - a_k).$$

## Solutions

Here are solutions to the problems from *Bulletin* Number 87.

I learned the first problem from a paper by Boris Springborn (*Enseign. Math.* 63, 2017, 333–373). It was solved by Riccardo Della Martera and the North Kildare Mathematics Problem Club, who offered two solutions. I present one of the solutions from the Problem Club.

*Problem 87.1***.** Determine the maximum distance between a straight line intersecting a triangle and the vertices of that triangle.

*Solution 87.1.* Let our triangle be $ABC$. Imagine drawing circles of radius $r$ about each of $A$, $B$ and $C$. Our question is, how big can $r$ be so that a line $L$ can still be drawn between (and possibly tangential to) these circles.

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If the line does not touch any of the circles, then we have some wiggle room, and $r$ could be enlarged. Similarly, if the line touched only one of the circles, then the line could be moved slightly away from the corresponding vertex and all the circles could be enlarged. Thus, the best line must touch at least two circles. Without loss of generality, suppose they are centred at $A$ and $B$.

*Case 1:* If $L$ does not cross the side $AB$, then as $L$ touches circles of the same radius centered at $A$ and $B$, the line will be parallel to $AB$ on the same side as $C$. It is obvious that the radius will be half the perpendicular height of $C$ from $AB$, in which case $L$ touches all three circles. The largest such possible radius would arise when $L$ is parallel to the shortest side of the triangle.

*Case 2:* Alternatively, if the line crosses the side $AB$, then it must also cross another side, say $AC$. If the angle $CAB$ is obtuse, then we can expand the radius until it is half of $|AB|$. Likewise if $CBA$ is obtuse. If both $CAB$ and $CBA$ are acute, it is possible that the circle at $C$ can get in the way of expanding $r$ to half of $|AB|$. In this case, as we have assumed that $L$ touches the circles centered $A$ and $B$, it will also touch the circle centered at $C$, and so we are back in Case 1.

So, the answer is half of the tallest perpendicular height, if the triangle's angles are all acute. If any angle is obtuse, then half the length of a longest adjacent side is the answer. □

The second problem was solved by the proposer, Des MacHale, and by the North Kildare Mathematics Problem Club. We present the solution of the Problem Club.

Des provided a second challenge, to prove that if each pair of elements $x$ and $y$ of a ring satisfies $(x^4 - x)y = y(x^4 - x)$ then the ring is commutative. There is a prize of Des's recent book *The Poetry of George Boole* for the first correct, elementary solution, which has yet to be claimed.

*Problem 86.2***.** Prove that if each element $x$ of a ring satisfies $x^4 + x = 2x^3$ then the ring is commutative.

*Solution 87.2.* First, replace $x$ by $-x$ to get a second identity. If we add and subtract the second identity from the original, then we obtain the pair of identities

$$2x^4 = 0 \quad \text{and} \quad 2x = 4x^3.$$

Hence

$$2x^3 = 4x^5 = 2x \times 2x^4 = 0,$$

so the original identity (with $-x$ in place of $x$) becomes $x^4 = x$.

Observe that if $a^2 = 0$, then $a = a^4 = (a^2)^2 = 0$.

Next, let $e$ be an idempotent element of the ring (that is, $e^2 = e$). Observe that

$$(ex - exe)^2 = 0 \quad \text{and} \quad (xe - exe)^2 = 0.$$

By the observation just mentioned, we have $ex = exe = xe$. Hence idempotents belong to the centre $C$ of the ring.

Next, notice that $x + x^2$ is an idempotent element, because

$$(x + x^2)^2 = x^2 + 2x^3 + x^4 = x^2 + 0 + x.$$

Hence $x + x^2 \in C$.

Choose any elements $x$ and $y$ of the ring. Expand $x+y+(x+y)^2$ and subtract $x+x^2$ and $y + y^2$ to see that $xy + yx \in C$. In particular, this element commutes with $x$, so we deduce that $x^2 \in C$. Since $x + x^2 \in C$, we deduce that $x \in C$, as required. □

The third problem was posed by Finbarr Holland of University College Cork. It was solved by Henry Ricardo of the Westchester Area Math Circle, NY, USA, Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia, Riccardo Della Martera, Eugene Gath of the University of Limerick, the North Kildare Mathematics Problem Club, and the proposer. We are spoilt for choice with solutions, all excellent. We opt for that of Henry Ricardo, which was similar to others.

*Problem 87.3*. Determine the sums of the series

$$\sum_{m,n=1}^{\infty} \frac{1}{mn(m+n+1)} \quad \text{and} \quad \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{mn(m+n+1)}.$$

*Solution 87.3.* Let $S$ be the sum of the first series. Then

$$\begin{aligned} S &= \sum_{m,n=1}^{\infty} \frac{1}{mn} \int_0^1 x^{m+n}\,dx \\ &= \int_0^1 \left(\sum_{m=1}^{\infty} \frac{x^m}{m}\right)\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right) dx \\ &= \int_0^1 (\ln(1-x))^2\,dx, \end{aligned}$$

where we have applied Fubini's theorem to interchange sums and integrals. The integral can be evaluated by substituting $x = 1 - e^y$ and then integrating by parts twice. We obtain $S = 2$.

Let $T$ be the sum of the second series. Then

$$\begin{aligned} T &= \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{mn} \int_0^1 x^{m+n}\,dx \\ &= \int_0^1 \left(\sum_{m=1}^{\infty} \frac{(-x)^m}{m}\right)\left(\sum_{n=1}^{\infty} \frac{(-x)^n}{n}\right) dx \\ &= \int_0^1 (\ln(1+x))^2\,dx. \end{aligned}$$

Applying the substitution $x = e^y - 1$ and integrating by parts twice gives $T = 2(\ln 2)^2 - 4\ln 2 + 2$. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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