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Some nontrivial two-term dilogarithm identities

JOHN M. CAMPBELL

Abstract. In 2012, Lima introduced a remarkable two-term dilogarithm identity, based on a proof for the Basel problem due to Beukers et al. Using a series transform obtained very recently via Legendre polynomial expansions, we nontrivially extend Lima's identity, and offer a new proof of this same identity.

1. Introduction

The dilogarithm function is defined as $\mathrm{Li}_2(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2}$, which converges for all complex z with $|z| \leq 1$. In this note, we derive new and nontrivial two-term dilogarithm identities, improving upon remarkable discoveries due to Lima [11].

The natural logarithm function, as defined for positive values, is, of course, very fundamental in mathematics as an elementary classical function, apart from how frequently the natural logarithm arises in science, technology, and engineering fields, outside of pure mathematics. So, this begs the question as to what may be considered as an appropriate way of extending or lifting this function, in the context of a given application, or within a given discipline in mathematics, science, etc. In this regard, the study of so-called *higher logarithm functions* forms a prominent area within the field of special functions theory, with the above defined dilogarithm as something of a prototypical instance of what is meant by a higher logarithm, in consideration as to above definition for Li_2 compared to the Maclaurin series expansion $-\ln(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$. The problem of determining a closed form for $\mathrm{Li}_2(1)$ is perhaps one of the most famous problems throughout the history mathematics: This is referred to as the *Basel problem*, as solved by Euler in 1734, with the closed form $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$. This is indicative of the importance, historically and otherwise, about the subject of symbolically evaluating expressions involving the dilogarithm mapping. This article introduces new results in this area.

There are only eight known values z such that both $\mathrm{Li}_2(z)$ and z may be expressed in closed form [13, §1]. This motivates the development of techniques for symbolically evaluating two-term linear combinations of dilogarithmic expressions (cf. [11]). For the sake of brevity, we assume familiarity with basic Li_2 identities, such as $\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2)$ and $\mathrm{Li}_2(1-x) + \mathrm{Li}_2(1-x^{-1}) = -\frac{1}{2}\ln^2 x$. The evaluation due to Lima [11] whereby

$$\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} - \frac{1}{2}\ln^2\left(1+\sqrt{2}\right) \tag{1}$$

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does not follow from previously known two-term Li_2 identities. This evaluation is proved in [11] using an argument relying on an evaluation for

$$\int_{\ln\left(1+\sqrt{2}\right)/2}^{\infty} \ln(\tanh z)\, dz,$$

which, in turn, relies on a double integral evaluation due to Beukers et al. [3]. We offer a simplified proof of (1), and extend (1) in nontrivial ways, using a series transform very recently introduced in [5]. The evaluations given in this note, as in Examples 2.2–2.5 below, are nontrivial; Mathematica and Maple, in particular, are not able to obtain these evaluations, even with the use of Mathematica commands such as `FunctionExpand` and `FullSimplify` or Maple packages such as *SumTools*; the same holds for Lima's evaluation in (1).

Apart from Lima's work in [11], there has been a considerable amount of previous research devoted to two-term dilogarithm identities, as in with the work of Bytsko in [4]. For example, two-term dilogarithm relations for Li_2 evaluated at expressions as in $\frac{1}{\lambda^2}$ for $\lambda = 2\cos\frac{\pi}{7}$ are given in [4], and an earlier two-term Li_2 evaluation due to Gordon and McIntosh [7] involving

$$\mathrm{Li}_2\left(\frac{\sqrt{3+2\sqrt{5}}-1}{2}\right)$$

is also reproduced in [4]. A main source of interest in the two-term dilogarithm identities that we prove is due to Ramanujan's two-term Li_2 evaluations, as in the following equation [2, p. 32] (cf. [10]):

$$\mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{6}\mathrm{Li}_2\left(\frac{1}{9}\right) = \frac{\pi^2}{18} - \frac{\ln^2 3}{6}.$$

Furthermore, two-term dilogarithm evaluations have been involved in applications pertaining to differential geometry, making a particular note of the remarkable identity due to Khoi [8] (cf. [12]) given as follows:

$$L\left(\frac{1}{\phi(\phi+\sqrt{\phi})}\right) - L\left(\frac{\phi}{\phi+\sqrt{\phi}}\right) = -\frac{\pi^2}{20},$$

where the Rogers dilogarithm function is such that $L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\ln(z)\ln(1-z)$, and where $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$ denotes the famous golden ratio constant.

2. Main identity and applications

For the sake of brevity, we assume basic familiarity with the orthogonal family of Legendre polynomials $P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2(x-1)^{n-k}(x+1)^k$. The key idea behind our improving upon Lima's work in [11] is given by the following identity, which was introduced in 2021 [5] using fractional calculus and Legendre polynomial expansions: If f is an analytic function over $(0,1)$, and if

$$\sum_{n\geq 0} a_n x^n = \sum_{m\geq 0} b_m P_m(2x-1)$$

holds with respect to the usual norm for functions on $(0,1)$, then

$$\sum_{n\geq 0} \frac{a_n}{(2n+1)^2\left(\frac{\left(\frac{1}{2}\right)_n}{n!}\right)^2} = \sum_{m\geq 0} \frac{(-1)^m b_m}{(2m+1)^2}, \tag{2}$$

letting the Pochhammer symbol be defined and denoted as per usual, with $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$ for a natural number n. We also recall the Euler integral $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\, du$ used to define the Γ-function, along with the Legendre duplication formula: $\Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}\left(\frac{1}{4}\right)^k \binom{2k}{k}\Gamma(k+1)$. In our applying the series transform indicated in (2), we need to make use of the famous generating function (g.f.) formula given below:

$$\frac{1}{\sqrt{1-2yz+z^2}} = \sum_{n=0}^{\infty} P_n(y) z^n. \tag{3}$$

As below, we let $\operatorname{sgn}(r)$ denote the sign function, so that, for a real value r, $\operatorname{sgn}(0)=0$, $\operatorname{sgn}(r)=1$ if r is positive, and $\operatorname{sgn}(r)=-1$ otherwise.

Theorem 2.1. *The equality whereby*

$$\frac{1}{1+z}\sum_{n=0}^{\infty}\frac{\left(\frac{16z}{(1+z)^2}\right)^n}{(2n+1)^2\binom{2n}{n}} = \operatorname{sgn}(z)\frac{i\left[\operatorname{Li}_2\left(-\sqrt{-z}\right)-\operatorname{Li}_2\left(\sqrt{-z}\right)\right]}{2\sqrt{z}} \tag{4}$$

holds if both sides converge for real z. Here i is the imaginary unit.

Proof. We rewrite the g.f. in (3) so that

$$\frac{1}{\sqrt{1+2z+z^2}}\cdot\frac{1}{\sqrt{1-x\frac{4z}{1+2z+z^2}}} = \sum_{n=0}^{\infty} P_n(2x-1)z^n$$

for suitably bounded x and z. On the other hand, rewriting the latter factor on the left-hand side, as a function of x, with its Maclaurin series, we obtain that:

$$\sum_{n=0}^{\infty}\frac{(-1)^n\binom{-\frac{1}{2}}{n}\left(\frac{4z}{1+2z+z^2}\right)^n x^n}{\sqrt{1+2z+z^2}} = \sum_{n=0}^{\infty} P_n(2x-1)z^n.$$

Through a direct application of (2) to the above equality, we obtain that

$$\frac{\pi}{4\sqrt{(z+1)^2}}\sum_{n=0}^{\infty}\frac{\left(\frac{z}{(z+1)^2}\right)^n\Gamma(2n+1)}{\Gamma^2\left(n+\frac{3}{2}\right)} = \sum_{m=0}^{\infty}\frac{(-1)^m z^m}{(2m+1)^2},$$

and we set $|z+1|>0$. Since $\sum_{m=0}^{\infty}\frac{(-1)^m y^{2m} z^m}{2m+1}$ evaluates as $\frac{\tan^{-1}\left(y\sqrt{z}\right)}{y\sqrt{z}}$, and since the antiderivative of this latter expression with respect to y is

$$\frac{i\left[\operatorname{Li}_2\left(-iy\sqrt{z}\right)-\operatorname{Li}_2\left(iy\sqrt{z}\right)\right]}{2\sqrt{z}},$$

this easily gives us the desired result. $\square$

2.1. **Applications.** We begin by applying Theorem 2.1 so as to obtain a new and simplified proof of Lima's identity in (1).

Proof of (1)*:* Setting, in Theorem 2.1, $z=-\left(\sqrt{2}-1\right)^2$, this gives us that:

$$\frac{\sum_{n=0}^{\infty}\frac{(-4)^n}{(2n+1)^2\binom{2n}{n}}}{1-\left(\sqrt{2}-1\right)^2} = \frac{\operatorname{Li}_2\left(\sqrt{2}-1\right)-\operatorname{Li}_2\left(1-\sqrt{2}\right)}{2\left(\sqrt{2}-1\right)}. \tag{5}$$

So, it remains to evaluate the above infinite series. In this direction, by using the Maclaurin series expansion

$$\sum_{n=0}^{\infty}\frac{(-4)^n t^{2n}}{(2n+1)\binom{2n}{n}} = \frac{\sinh^{-1}(t)}{t\sqrt{1+t^2}},$$

letting $\sinh^{-1}(z) = \ln(\sqrt{z^2+1}+z)$ denote the inverse hyperbolic sine, we find that we may compute the antiderivative of the right-hand side of the above equality, as below:

$$\begin{gathered}\mathrm{Li}_2\left(-e^{-\sinh^{-1}(t)}\right) - \mathrm{Li}_2\left(e^{-\sinh^{-1}(t)}\right) + \\ \sinh^{-1}(t)\left(\ln\left(1-e^{-\sinh^{-1}(t)}\right) - \ln\left(e^{-\sinh^{-1}(t)}+1\right)\right).\end{gathered}$$

This is easily seen by differentiating this symbolic form. Setting $t \to 1$ and $t \to 0$, this gives us the equality of

$$\frac{\mathrm{Li}_2\left(1-\sqrt{2}\right) - \mathrm{Li}_2\left(\sqrt{2}-1\right) + \frac{\pi^2}{4} + \left(\frac{\ln(2)}{2} - \ln\left(2+\sqrt{2}\right)\right)\sinh^{-1}(1)}{1-\left(\sqrt{2}-1\right)^2}$$

and

$$\frac{\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right)}{2\left(\sqrt{2}-1\right)}.$$

Rearranging this equality, we obtain that

$$\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} + \frac{1}{2}\sinh^{-1}(1)\ln\left(\sqrt{2}-1\right),$$

as desired. □

A relevant application of Lima's evaluation in (1) concerns a pair of classic polylogarithmic ladders due to Lewin (cf. [9, §1.6]), as below, writing α in place of $\sqrt{2}-1$:

$$4L(\alpha) - L(\alpha^2) = \frac{\pi^2}{4}, \tag{6}$$

$$4L(\alpha) + 4L(\alpha^2) - L(\alpha^4) = \frac{5\pi^2}{12}. \tag{7}$$

We see that: Thanks to Lima's identity, as in (1), we may obtain two correponding dilogarithm ladders with powers of $-\alpha$ in place of powers of α, since the powers of α other than α itself in (6) and (7) are even. The identity in (6) was used in a prominent way in [6] in a proof for a binomial-harmonic sum evaluation introduced in [6], using a Legendre polynomial-based integration technique closely related to the key identity in (2). The foregoing considerations strongly motivate further uses of Theorem 2.1 in the determination of two-term dilogarithm identities.

In order to generalize our new proof of Lima's identity shown in (1), we need to generalize how we had proved our evaluation for the infinite series on the left-hand side of (5), so as to be able to evaluate generating functions of the following form:

$$\sum_{n=0}^{\infty} \frac{x^n}{(2n+1)^2\binom{2n}{n}}. \tag{8}$$

However, it is known that this is equal to:

$$\begin{gathered}\frac{2i\mathrm{Li}_2\left(-\sqrt{1-\frac{x}{4}}-\frac{i\sqrt{x}}{2}\right)}{\sqrt{x}} - \frac{2i\mathrm{Li}_2\left(\sqrt{1-\frac{x}{4}}+\frac{i\sqrt{x}}{2}\right)}{\sqrt{x}} + \\ \frac{i\pi^2}{2\sqrt{x}} + \frac{2\ln\left(\frac{-\sqrt{1-\frac{x}{4}}-\frac{i\sqrt{x}}{2}+1}{\sqrt{1-\frac{x}{4}}+\frac{i\sqrt{x}}{2}+1}\right)\csc^{-1}\left(\frac{2}{\sqrt{x}}\right)}{\sqrt{x}}.\end{gathered}$$

This is easily verifiable, by writing

$$\sum_{n=0}^{\infty} \frac{x^n y^{2n}}{(2n+1)\binom{2n}{n}} = \frac{4\sin^{-1}\left(\frac{\sqrt{x}y}{2}\right)}{\sqrt{x}y\sqrt{4-xy^2}},$$

and by then computing the antiderivative of the right-hand side.

Example 2.2. Setting $z = \frac{1}{-9-4\sqrt{5}}$ in Theorem 2.1, we may, as explained below, obtain the following identity:

$$\mathrm{Li}_2\left(\frac{1}{\phi^3}\right) - \mathrm{Li}_2\left(-\frac{1}{\phi^3}\right) = \frac{\phi^3\left(\pi^2 - 18\ln^2(\phi)\right)}{3\left(\phi^6-1\right)}. \tag{9}$$

Inputting the above value for z into the left-hand side of the identity in Theorem 2.1, it remains to evaluate the series in (8) for $x=-1$, making use of the classically known values for $\mathrm{Li}_2\left(\frac{1}{\phi}\right)$ and $\mathrm{Li}_2\left(-\frac{1}{\phi}\right)$ [13, §1]. As indicated above, Maple and Mathematica are not able to evaluate the left-hand side of the equality in (9). For example, inputting

```
FunctionExpand[
 PolyLog[2, GoldenRatio^(-3)] - PolyLog[2, -GoldenRatio^(-3)]]
```

into Mathematica, this CAS is not able to compute any evaluation for the above input.

Example 2.3. Setting $z = 7-4\sqrt{3}$ in Theorem 2.1, the left-hand side of this Theorem involves, in this case, the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\binom{2n}{n}}$$

which we may easily evaluate according to the above identity for the generating function in (8), giving us that:

$$\mathrm{Li}_2\left(i\left(2-\sqrt{3}\right)\right) - \mathrm{Li}_2\left(-i\left(2-\sqrt{3}\right)\right) = \frac{2i\sqrt{7-4\sqrt{3}}\left(8G - \pi\ln\left(2+\sqrt{3}\right)\right)}{3\left(8-4\sqrt{3}\right)},$$

letting $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ denote Catalan's constant.

Example 2.4. Setting $z = 3-2\sqrt{2}$ in Theorem 2.1, by again making use of the known symbolic form for the power series in (8), we obtain that:

$$\mathrm{Li}_2\left(i\left(\sqrt{2}-1\right)\right) - \mathrm{Li}_2\left(-i\left(\sqrt{2}-1\right)\right)$$

evaluates as

$$\frac{1}{32}i\left(\sqrt{2}\left(\psi^{(1)}\left(\frac{1}{8}\right) + \psi^{(1)}\left(\frac{3}{8}\right)\right) + 8\pi\ln\left(\sqrt{2}-1\right) - 4\sqrt{2}\pi^2\right),$$

writing $\psi^{(1)}(z) = \frac{d^2}{dz^2}\ln\Gamma(z)$ to denote the trigamma function.

Example 2.5. Setting $z = \frac{1}{3}$ in Theorem 2.1, we obtain, again making use of the evaluation for (8), that

$$\mathrm{Li}_2\left(\frac{i}{\sqrt{3}}\right) - \mathrm{Li}_2\left(-\frac{i}{\sqrt{3}}\right)$$

equals:

$$\frac{i\left(3\psi^{(1)}\left(\frac{1}{6}\right) + 15\psi^{(1)}\left(\frac{1}{3}\right) - 6\sqrt{3}\pi\ln(3) - 16\pi^2\right)}{36\sqrt{3}}.$$

As indicated above, the left-hand side of (4) may be written as an expression involving the difference

$$\mathrm{Li}_2\left(2i\sqrt{\frac{z}{(1+z)^2}}+\sqrt{1-\frac{4z}{(1+z)^2}}\right)-\mathrm{Li}_2\left(-2i\sqrt{\frac{z}{(1+z)^2}}-\sqrt{1-\frac{4z}{(1+z)^2}}\right)$$

along with combinations of elementary functions. Let the above difference be written as:

$$\mathrm{Li}_2(\alpha(z))-\mathrm{Li}_2(-\alpha(z)). \tag{10}$$

So, according to (4), if both

$$\mathrm{Li}_2(\sqrt{-z})-\mathrm{Li}_2(-\sqrt{-z}) \tag{11}$$

and (10) are convergent, then one such expression admits a closed-form evaluation if and only if the other such expression does.

Although the focus of this article has been on two-term Li_2 identities, we may also use Theorem 2.1 to obtain identities that bear a resemblance to the dilogarithmic ladder

$$\pi^2=36\mathrm{Li}_2\left(\frac{1}{2}\right)-36\mathrm{Li}_2\left(\frac{1}{4}\right)-12\mathrm{Li}_2\left(\frac{1}{8}\right)+6\mathrm{Li}_2\left(\frac{1}{64}\right)$$

given in [1]; for example, setting $z=-\frac{1}{4}$ gives us a closed form for a rational linear combination of $\mathrm{Li}_2\left(\frac{1}{4}\right)$, $\mathrm{Li}_2\left(-\frac{1}{3}\right)$, and $\mathrm{Li}_2\left(\frac{1}{3}\right)$. Explicitly,

$$2\mathrm{Li}_2\left(-\frac{1}{3}\right)+\mathrm{Li}_2\left(\frac{1}{4}\right)-2\mathrm{Li}_2\left(\frac{1}{3}\right)=-\frac{\pi^2}{6}-2\ln^2(2)+2\ln(2)\ln(3).$$

We encourage the exploration of further uses of Theorem 2.1.

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John M. Campbell received his M.Math. in pure mathematics from the University of Waterloo, and graduated first class with distinction with a Specialized Honours B.Sc. degree in mathematics from York University. He has been awarded the prestigious Carswell Scholarship and the Irvine R. Pounder Award, and has worked as a research assistant at York University

and at the Fields Institute for Research in Mathematical Sciences. He is currently working as a researcher at York University.

Department of Mathematics and Statistics, York University
E-mail address: jmaxwellcampbell@gmail.com