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The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

Correspondence concerning books for review in the *Bulletin* should be directed to

`mailto://reviews.ims@gmail.com`

All other correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

`mailto://ims.bulletin@gmail.com`

and only if not possible in electronic form to the address

The Editor
Irish Mathematical Society Bulletin
Department of Mathematics and Statistics
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Co. Kildare W23 HW31

Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

Irish Mathematical Society Bulletin 88, Winter 2021

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EDITORIAL

David Malone comes to the end of his term as IMS Secretary as the New Year 2022 begins. I would like to acknowledge some other contributions he makes to the Society. He arranged our registration with the Charities Regulator, and takes care of our continuing compliance with the legal requirements. Our Charity Number is 20020279, and our submissions are publicly available on the regulator's website. David also mediated the allocation of official Digital Object Identifiers (DOIs) to the Bulletin articles. This was also facilitated by Maynooth Universty's Library staff. From the present issue, DOI's will be printed at the bottom of the first page of each article. David has also retrospectively generated and uploaded DOI's for all previous articles, and this will enable the location of articles in internet searches by DOI.

Noteworthy online material from 2021 includes Colm Mulcahy's blog on the first 50 Irish women in mathematics[1] and the panel discussion held to mark the eightieth anniversay of the foundation of the Dublin Institute for Advanced Studies[2]

The 2022 Annual Scientific Meeting of the Society (aka the "September Meeting") will be hosted by Technical University, Dublin.

We remind organisers and other contributors that the normal deadline for submissions to the Bulletin is 15 December for the Winter issue and 15 May for the Summer issue.

The task of editing the Bulletin is greatly helped by the support of the Editorial Board and the officers of the Society. Special thanks are due to Ian Short, who manages the problem page. This year Eleanor Lingham agreed to join the Board, and has taken over the management of the book reviews.

Note that the website serves up pdf files of the individual articles, as well as the pdf file of the whole Bulletin. I want to acknowledge the efficient and unfailing support of our website manager, Michael Mackey.

For a limited time, beginning as soon as possible after the online publication of this Bulletin, a printed and bound copy may be ordered online on a print-on-demand basis at a minimal price[3].

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E-mail address: ims.bulletin@gmail.com

[1] http://www.mathsireland.ie/blog/2021_05_cm
[2] https://www.dias.ie/2021/07/30/panel-discussion-dias-and-devalera/
[3] Go to www.lulu.com and search for *Irish Mathematical Society Bulletin*.

LINKS FOR POSTGRADUATE STUDY

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: mailto://maths@dcu.ie
DIT: mailto://chris.hills@dit.ie
IT Sligo: mailto://creedon.leo@itsligo.ie
IT Tralee:
http://www.ittralee.ie/en/CareersOffice/StudentsandGraduates/PostgraduateStudy/
NUIG: mailto://james.cruickshank@nuigalway.ie
MU: mailto://mathsstatspg@mu.ie
QUB:
http://web.am.qub.ac.uk/wp/msrc/msrc-home-page/postgrad_opportunities/
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: http://www.ucc.ie/en/matsci/postgraduate/
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://sarah.mitchell@ul.ie

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor.

E-mail address: ims.bulletin@gmail.com

NOTICES FROM THE SOCIETY

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Officers and Committee Members 2022

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Dr R. Flatley, Dr R. Gaburro, Dr D. Mackey, Prof. M. Mathieu, Prof. A. O'Shea, Dr R. Ryan, Assoc. Prof. H. Šmigoc, Dr N. Snigireva.

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Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€200
Ordinary member	€30
Student member	€15
DMV, IMTA, NZMS or RSME reciprocity member	€15
AMS reciprocity member	$20
LMS reciprocity member (paying in Euro)	€15
LMS reciprocity member (paying in Sterling)	£15

The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 40.
If paid in sterling then the subscription is £30.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 40.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form, available at
`http://www.irishmathsoc.org/links/apply.pdf`
with one year's subscription to:

Dr Cónall Kelly
School of Mathematical Sciences
Western Gateway Building, Western Road
University College Cork
Cork, T12 XF62
Ireland

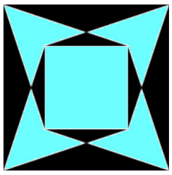
Deceased Members

It is with regret that we report the deaths of members:

David W. Lewis, of UCD, died on 20 August 2021.

Niall Ó Murchadha, of UCC, died on 7 October 2021.

E-mail address: subscriptions.ims@gmail.com



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Lawrence J. Crane (1931–2021)

JOHN J. H. MILLER (COMPILER)

1. Brief Overview

Lawrence John Crane, who died in March 2021, aged 89, was one of the leading applied mathematicians of his day in Ireland. His lifelong enthusiasm for the subject was appreciated by his students and colleagues in Trinity College, where he lectured for over forty years.

Born on 11 July 1931 in Neilston, near Glasgow, the youngest of four, he graduated from Glasgow University in 1954 with a BSc in Mathematics and Natural Philosophy. As a Mathematics student, his skills found applications in the Torpedo Experimental Establishment in Greenock, near Glasgow.

He went to Dublin for the first time in 1954-55 as a scholar at the school of Cosmic Physics in the Dublin Institute of Advanced Studies. In 1955 he was awarded the James Caird scholarship in Aeronautics at the University of Strathclyde. From 1957-61 he lectured in Applied Mathematics in Strathclyde and completed his Ph.D. on 'Properties of Jets and Wakes' under the supervision of Prof Donald Pack. In 1959 he spent a year in Germany as a scholar in Freiburg supervised by Prof. Henry Görtler. He was awarded a D.Sc in 1976 and the William Jack Prize in 1977 for his research, both from Glasgow University .

He moved to Dublin in 1961 and joined the TCD School of Mathematics, where he worked until his retirement in 2001. Students – especially in engineering – appreciated his ability to make complex material tractable. He applied his mathematics skills to solve real world problems in engineering, physics and pharmacy. In his early years in Trinity he conducted USAF-sponsored research with Percy McCormack on the break-up of liquid jets.

He started a successful Erasmus programme in Trinity in the 1980s in conjunction with TU Darmstadt, Germany where he was a visiting professor in 1985. Laterally, Lawrence continued to work on boundary layer and slip flow problems. He continued to contribute in a meaningful way to advances in the field of fluid dynamics. This commitment continued into his 80s with ongoing collaborations on wave energy research and the supervision of students. He published his last paper in 2017 and made notable advances in research up to a few weeks before his death.

He was a master at transforming a complex problem to the most simple, elegant form, while preserving its essential features. His favourite maxim "the best is the enemy of the good" was shown by his ability to find a close approximation to almost any problem, in his head. Notably, he discovered one of the few exact solutions of the Navier-Stokes equations. His professional approach to applied mathematics and his published works are highly regarded by his peers.

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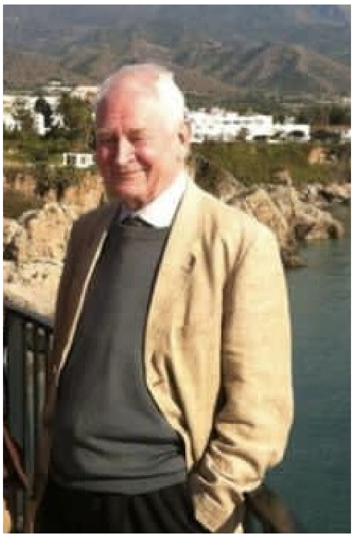

On holiday in Spain

With INCA student Nopparat Pochai

Since his twenties Lawrence enjoyed travelling. Germany was a favourite and frequently-visited destination with many business trips and numerous holidays there. In 2010 he completed the final section of the Camino de Santiago on his own. He enjoyed attending and contributing to international mathematics conferences. He was an avid reader on a wide range of topics including military history, scripture, history of mathematics and science. All his life he was a regular churchgoer with deeply held beliefs. He was very interested in theology and scripture and his faith gave him solace in his final months.

He was generous with his time and always willing to help other people, tutoring disadvantaged students in mathematics and statistics for the Ballymun JUST Programme and delivering 'meals on wheels' for many years in Clontarf. He connected easily with people from all walks of life. He had a friendly and engaging personality and frequently struck up conversations with people anywhere – often in different languages! His great enthusiasm for life made him an eternal optimist. He was a very modest man who wore his achievements lightly and placed high value on family and friendship.

He was predeceased by his wife Mary and is survived by his children Stephen, Martin and Alison and his four grandchildren.

May he rest in peace.

2. Some brief reminiscences by colleagues

Lawrence is fondly remembered as an inspiring, influential Adviser whose warm and generous nature made him a pleasure to work with. Forever meticulous in style and methods, he always radiated enthusiasm for his work and had a natural ability to provide deep insights into approaches to many technical problems. Clearly, Lawrence loved mathematics ("it's great fun!") and constantly stirred thought and enquiry – always striving to extract solutions in their most elegant form. Following his final doctoral supervision duties with INCA, Lawrence continued to work on boundary layer and slip flow problems. Throughout the past decade, the quality and quantity of his research never flagged and, up to 2016, he continued to collaborate and contribute to advance the field of fluid dynamics. Up to recent years, Lawrence continued to enjoy international travel – especially to mathematics conferences. One of his most memorable occasions was accepting an invitation to attend a mathematics ceremony in the Royal Society, London. Typically in such events, Lawrence actively engaged with attendees and was always eager to participate in discussions on historic and technical matters. In all locations visited, Lawrence would acquaint himself with the historic backdrops and seek out buildings and architecture of interest. Of course, no trip would be complete without a visit to an old bookstore or two and occasionally, to some unassuming local restaurant! He will be deeply missed by all students and researchers who had the honour of knowing and working with him. *Tony McVeigh.*

I will let others talk about his work in academia, and his expertise in Fluid Mechanics. What I noticed a number of times was his genuine affection for some of his former PhD students. Some of them he kept in touch with, and some even came to visit him in Ireland.

In his later years, he became very interested in family history. We worked on this together. He was quite happy for me to work the PC, while he did the thinking. In this area, he showed amazing creativity, flexibility, and a genuine delight when we established another good connection with his past. By far his most endearing trait in this area was his dogged determination to follow all the leads until he was satisfied that we had tied down a cast iron conclusion.

He was a born optimist and if you happened to meet him on a bad weather day, you had better be ready with a positive response, or you were not at the races! *Diarmuid Herlihy.*

When I think back over the time for which I have known Lawrence – almost sixty years, dating from my arrival in Trinity as an undergraduate, which coincided with his as a lecturer – I find that the last ten years or so have provided some of the most distinctive memories. We came together naturally at events such as the School of Mathematics Christmas lunch: recalling past times and former students, or simply chatting as friends. We also met with past classmates of mine when they were back in Trinity for the Scholars' Dinner in 2014, and enjoyed a very happy afternoon tea and chat in the Davenport Hotel. Thus, the picture of Lawrence that first comes to mind now is of a friend among a circle of friends, enjoying good food and equally good conversation.

Of course there are more academic memories also. Lawrence lectured me in mechanics and fluid dynamics. I think of him often when I am flying (or when I was flying, in the days when air travel was normal), with reference to the fluid dynamics; the flow round aerofoils, in this case aeroplane wings, is keeping the plane in the air. He contributed also to the world of school mathematics, for instance through his role in cooperating on the selection of a teacher of Applied Mathematics to receive the Victor Graham trophy, presented annually. While Lawrences family and his research colleagues will miss him most, all his friends and colleagues regret the loss of his friendly and calm presence. *Elizabeth Oldham.*

Lawrence and I worked together in INCA for more than 20 years, after we had both retired from TCD, in 2001 and 2000 respectively. It was a happy period for us both; we were delighted to be in a stable, secure environment that we controlled. The only pressure on us was the self-imposed drive to design and construct a technically and commercially feasible device to harvest the power of ocean waves for the generation of electricity. The device we chose was a Masuda buoy, a floating axi-symmetric buoy containing an oscillating water column. This apparently simple task had not been completed in 2001, and this is still the case 20 years later. Personally, our pensions gave us financial security and, academically, we had done our best in teaching and research during our years in TCD. We were now free and we let the technical demands of the daily tasks determine our actions. We were able to spend the time necessary to deal with each technical question as it arose and we had no need to spend time seeking extra funding. Pencils, paper, a pocket calculator, a desktop computer and knowhow were all that we needed. From time to time colleagues in the same field made interesting observations about our work: one young colleague said that all we were doing was schoolboy mathematics , although later the boss said that he wished he had been aware of our work before he built his own device. On another occasion Lawrence was working quietly in the visitors' room at a wave tank facility. A senior technician was making a phonecall to his boss from an adjoining room separated by a partition that did not quite reach the ceiling. Thinking that he was talking privately he was heard to say that "the two chaps from Dublin were getting on with their testing and although they were rather old, they seemed to know what they were doing". We took this as one of the best compliments we had ever received! It is a pleasure to have worked with Lawrence over such a long period and we had great fun. *John Miller.*

In his lectures, Lawrence had a particular skill in teaching an understanding of differential equations, their solution and their application to real-life problems, whether in an introductory 'methods course or a more advanced course on fluid dynamics. He found TCD an extremely friendly environment. In conversation, a few weeks before he died, he told me that he considered the university had high standards in mathematics, and higher expectations of students than was the norm elsewhere. At that time, women were excluded from senior academic positions at the university (including Fellowship); moreover, they were obliged to have left the campus by 6 p.m.! He admired how the mathematics students were devoted to running their own activities in the Dublin University Mathematical Society (DUMS), and took great care in preparing their meetings. He had a strong interest in the history of mathematics, and especially in the contributions of Scottish and Irish mathematicians. He was shocked to learn that, fairly recently, students in the DUMS had decided to dump their library (including the Fry and Rowe collections) in a skip. *Maurice O'Reilly.*

Lawrence's research interest throughout his career was Fluid Mechanics, an important subject in which he had several research students. By all accounts Lawrence was an excellent and caring lecturer. Apart from the lectures on his favourite Fluid Mechanics, he lectured on Mathematical Methods and shared a lecture course with me on Dynamics for the engineering department. Engineering graduates of the 1970s and 1980s, who themselves reached enviable heights in their subsequent research, still speak most highly of the Dynamics lectures and how inspiring and essential they are in their work. Lawrence played his part in the life of the mathematics department and of the College, in general. Mention must be made of the highly successful Erasmus Program between Trinity College and the University of Darmstadt; it was all due to Lawrences hard work. He was a wonderful colleague and friend in every way. He will long be remembered with great fondness and admiration. *Petros Florides.*

I have known Lawrence for over 50 years; he was already in the Department when I came in 1966. He was a welcoming and helpful presence to me then, and through the years continued to be a much valued and supportive colleague. I have always held him in high regard. Whenever I needed help, for example, as sometimes happened, to fill in on a course, I knew that I could turn to him; he would always, willingly, and unhesitatingly, step in even at short notice. Lawrence was a very good teacher; he was responsive to the needs of his students, and they benefited from and appreciated his courses. He was always accessible to any colleague, not only from Mathematics but also from Engineering and from other departments, who came to him with a mathematical or technical problem; he would either help them to solve their problem directly, or if it were more involved would engage with it and sometimes collaborate in resolving it. He was a seriously professional mathematician with a deep commitment to his subject, which was sensed by his students, and which added to the credibility of his lectures. This commitment and engagement continued after his retirement, in particular through an ongoing collaboration in INCA which was a serious interest and source of satisfaction and fulfilment for him. *David Spearman.*

Lawrence gave magnificent courses in mechanics and the partial differential equations of mathematical physics which have been a source of inspiration through my professional life. I always found him to be very kind in all my dealings with him likewise with other students. My only regret is not being able to attend his course on Fluid Mechanics which was not part of the Engineering degree at the time. He had the very valuable knack of rendering the most difficult of subjects rather simple. I consider it a great privilege to have known him as a person and as an applied mathematican who engendered an abiding interest in his subject. *William Coffey.*

3. Academic Career

Studies, Qualifications and Awards
1950-54 Student in Glasgow University.
1954 B.Sc. (First Class Honours) in Mathematics and Natural Philosophy.
1954-55 Scholar, Dublin Institute of Advanced Studies, School of Cosmic Physics.
1955-57 Sir James Caird Scholar in Aeronautics at the University of Strathclyde, Glasgow.
1957-58 Assistant Lecturer in Mathematics, University of Strathclyde, Glasgow.
1958-61 Lecturer in Mathematics, University of Strathclyde, Glasgow.
1959 Scholar, Deutsche Versuchsanstalt für Luftfahrt, Freiburg, West Germany.
1960 Ph.D, Glasgow University.
1961 Lecturer in Applied Mathematics, Trinity College, Dublin.
1976 D.Sc, Glasgow University.
1977 Awarded William Jack Prize in Mathematics by Glasgow University.

Research Experience
1957-58 Research (leading to the degree of Ph.D) in viscous jets and wakes under the supervision of Professor D.C. Pack, at the University of Strathclyde.
1959 Research in secondary flows under Professor H. Görtler, D.V.L. Institute, Freiburg.
1962-65 Research in collaboration with P.D. McCormack on the break-up of liquid jets; this work was sponsored by the U.S.A.F.
1966-69 Research Consultant with I.C.I. Fibres Ltd., Harrowgate, Yorkshire. Research was carried out on fluid flow and heat transfer on spinning fibres.
1973-77 Research contract with U.K. Atomic Energy Authority on Entry and Exit flows in Nuclear Reactors.
1981 DAAD supported research project on Rotating flows.
1989-93 Research contract with AKZO-NOBEL, Klingenberg, Bavaria on air flow in fibre bundles.
1998-99 Research project for Siemens A.G. Mülheim, Germany on deposition of particles on turbine blades.
1993-97 Brite Euram SISCO project on detection of cracks in thin plates by Rayleigh-Lamb waves.
1998 onwards. Research Consultant with Freudenberg, KG, Weinheim, Germany, topic: the self suction fibre spinning process.
2000 onwards. Participant in EU Fourth framework PSUDO project on parallel simulation of drug release, then continuing in collaboration with the School of Computer Applications D.C.U. and with the Department of Pharmacy T.C.D.
2003 Sustainable Energy Ireland supported project on air flow past slender towers, in association with Airtricity Ireland.

Consultant over a period of years to Performance Fluid Dynamics Ltd., Dublin on various fluid dynamical projects.

Academic and other positions
1965-70 Tutor, TCD
1969 Elected Fellow, TCD
1995 Elected Senior Fellow and Member of Board, TCD
1980 Elected Member of G.A.M.M., (Gesellschaft für Angewandte Mathematik und Mechanik, Germany).
1980 Member of INCA
2000- Treasurer of INCA

Ph.D theses supervised
B. Siddappa, (1971) Newtonian flows in viscoelasticity.
L.C. Tan, (1977) Pressure Losses in Ducts, Bifurcations and Manifolds.
P. Carragher, (1978) Heat Transfer on continuous solid surfaces.
P. Lynch, (1982) Planetary Scale Hydrodynamic Instability.
B. Redmond, (1994) Drag and Heat Transfer in Melt Spinning.
John Harding, (2007) Incompressible and Compressible Boundary Layers on a Fibre in the Melt Spinning Process, jointly with Dr. Brendan Redmond.
Sandra Spillane, (2008) A Study of Boundary Layer Flow with no-slip and slip Boundary Conditions, jointly with Dr. Brendan Redmond.
A. G. McVeigh (2011) Interactions between Fluid Flow and Cylinders.
David McDonnell, (2012) An Analysis of Drug Dissolution in vivo, jointly with Dr. Brendan Redmond.

M.Sc theses supervised
S. Siddappa, (1972) Boundary Layer on cylinders.
N.Bach, (1994) Air Flow in fibre bundles.
E. Kenny, (1994) A moving boundary finite element solution of a pharmaceutical drug delivery problem.
C. Schafer, (1995) Finite Difference Approach to Boundary Layer Phenomena.
M. Brennan, (1996) Numerical Study of riblets on drag.
K. Mayes, (1997) Paper motion in printing machines.
P. Cooke, (1997) Steady Laminar Flow in narrow annuli.

4. Mathematical career following retirement from TCD

During the more than twenty years that followed his retirement from Trinity College, Lawrence remained an enthusiastic and productive research mathematician. During this time he based himself in INCA, the Institute for Numerical Computation and Analysis in Dublin. Founded in 1980, this is an independent non-profit research body, which has official charitable status; it has no paid employees. In the early years of this period Lawrence was one of the two INCA researchers appointed by the Royal College of Surgeons in Ireland as Research Fellows. These were two-year positions without pay.
He was also involved on behalf of INCA in the research training of postgraduate students from Ireland and abroad, notably as the PhD thesis advisor of a mature student who was awarded a doctorate by the University of Buckingham.
Around 2010 INCA became seriously interested in renewable energy. It was decided that an appropriate approach would be to support the creation of a commercial company, Waveforce Energy Limited, to develop new devices to harvest electric power from ocean waves. It quickly became evident that this was an area in which mathematical and computational modelling could be of great help. As a result, a period of intense

activity followed, which involved the modelling of the motion of floating oscillating water column buoys in a wide range of ocean conditions from benign wave tanks to the extreme conditions off the west coast of Ireland. In his final years he began work on models of turbines, which he continued to work on until a few weeks of the end. He particularly enjoyed his visits to the wave tanks in Cork, Galway and Glasgow, to JFC in Tuam where the first Waveforce buoys were manufactured and to the ocean test site off Belmullet.

List of Publications

Below is a list of publications compiled mainly by Lawrence himself in about 2000 around the time of his retirement from Trinity College. Later papers have been added. ZAMM stands for Zeitschrift für Angewandte Mathematik und Mechanik.
ZAMP stands for Zeitschrift für Angewandte Mathematik und Physik.

(A) Jets and wakes
A1 L.J. Crane, D.C. Pack. The laminar and turbulent mixing of jets of compressible fluid. Part 1. Flow far from the orifice. Journal of Fluid Mechanics 2, p.449, 1957.
A2 L.J. Crane, The laminar and turbulent mixing of jets of compressible fluid. Part 11. The mixing of two semi-infinite streams. Journal of Fluid Mechanics 3, p. 81, 1957.
A3 L.J. Crane, D.C. Pack. The mixing of a jet gas with an atmosphere of a different gas at large distances from the orifice. Part 1, The Plane Jet. Quarterly Journal of Mechanics and Applied Mechanics and Applied Mathematics, 14, p.385, 1961.
A4 L.J. Crane, The mixing of a jet gas with an atmosphere of a different gas at large distances from the orifice. Part 11. The round jet. Quarterly Journal of Mechanics and Applied Mathematics, 14, p. 393, 1961.
A5 L.J. Crane, A note on Stewartsons paper "On asymptotic expansions in the theory of boundary layers". Journal of Mathematics and Physics 38, p. 172, 1959.
A6 P.D. McCormack, D. Cochran, L. Crane, Periodic vorticity and its effect on jet mixing. Physics of Fluids. 9, p. 1055, 1966.
A7 L.J. Crane, Error estimate for the Schlichting round jet. Z.A.M.M. 54, p. 591, 1974.

(B) The breakup of liquid jets
B1 L.J. Crane, S. Birch, P.D. McCormack, The effect of mechanical vibration on the break-up of a cylindrical water jet in air, British J. Applied Physics, 15, p. 743, 1964.
B2 P.D. McCormack, L.J. Crane, S. Birch, An experimental and theoretical analysis of cylindrical liquid jets subject to vibration. British J. Applied Physics, 16, p. 395, 1965.
B3 P.D. McCormack, L.J. Crane, S. Birch, Derivation of jet velocity modulation caused by injector vibration. British J. Applied Physics, 16, p. 1911, 1965.

(C) Properties of continuous boundary layers
C1 L.J. Crane, Flow past a stretching plate. Z.A.M.P., 21, p.645, 1970.
C2 L.J. Crane, Boundary layer flow on a circular cylinder moving in a fluid at rest. Z.A.M.P., 23, p. 201, 1972.
C3 L.J. Crane, Heat transfer on continuous solid surfaces, Ingenieur - Archive., 43 pp, 203-214, 1974.
C4 L.J. Crane, Boundary layer flow due to a stretching cylinder. Z.A.M.P., 26, p. 619, 1975.
C5 P. Carragher, L.J. Crane, Heat transfer on a continuous stretching sheet. ZAMM 62, p. 564, 1982.

(D) Plumes

D1 L.J. Crane, Thermal convection from a horizontal wire, Z.A.M.P.,10, p. 453, 1959.
D2 L.J. Crane, Axially symmetric plumes at large Prandtl numbers. Z.A.M.P., 25, p. 365, 1974.
D3 L.J. Crane, Axially symmetric plumes at very small Prandtl numbers. Z.A.M.P., 26, p. 427, 1975.
D4 L.J. Crane, The round plume due to a falling point source of heat. Z.A.M.P., 28, p. 599, 1977.
D5 L.J. Crane, The plane turbulent plume in a magnetic field. Z.A.M.P., 40, 69-77, 1989.

(E) Natural convection on cylinders

E1 L.J. Crane, Natural convection from a vertical cylinder at very large Prandtl numbers. Journal of Engineering Mathematics, 10, p.115, 1976.
E2 L.J. Crane, Natural convection from a vertical cylinder at very small Prandtl numbers. Z.A.M.P., 27, p. 61, 1976.

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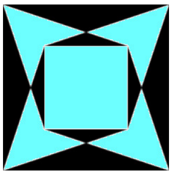

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David John Simms (1933–2018)

SIDDHARTHA SEN AND DAVID MALONE

David Simms was on the faculty in Trinity College Dublin from 1964 to 2003, serving as lecturer, professor and head of department. He was a dedicated lecturer who continued to teach after his retirement, and taught nearly every student who passed through the department for over fifty years. He passed away on June 24, 2018 following a long illness.

1. Early Life

David was born in 1933 in Sankeshwar, Mysore, India and proudly described himself as Indian-born. His father, J Gerald Simms, worked in the Indian civil service. His mother, Eileen (née Goold-Verschoyle), and father were both from Donegal. The family moved back-and-forth between Ireland and India several times during David's early years. David remembered speaking with children in Hindi at a young age. Later, he boarded at the English-speaking Bishop Cotton School in the Himalayas at age 8. At one stage, he contracted whooping cough, but seemed to consider his stay in the school sanatorium as something of a highlight.

In 1942, his mother began a trip back to Ireland, with her young family. As the Suez Canal was closed at the time, they traveled around the coast of Africa on the SS City of Cairo[1]. This ship was targetted by a German U-boat, and sunk. We will quote the 10-year old David's account of what happened.

> *When the torpedo hit us I was lying in the bottom bunk reading. I remember hearing a deafening boom, so I jumped out of bed but I had only gone a short way before I remembered my lifebelt. I dashed back grabbed up my lifebelt which was behind the door, took up my coat which was hooked on the cupboard handle, and tore off a blanket from the end of the bed. When I got up to the next deck I found myself squashed between a crowd of grown-ups swaying with the motion of the ship. At last I got to the top deck and went towards our lifeboat which was No.8. In the middle I stopped to put on my coat and lifebelt and then hurried to the lifeboat. I had only been there a few minutes when some sailors started shouting "Mrs. Simms!" After a time my mother came and we got into the boat with her up a ladder. . . . When we got away from the ship we had to talk in whispers and the men could not smoke cigarettes because we saw the submarine, which was clear against the sky. In the distance I heard shrieks and shouts, as I was looking I saw an explosion where the*

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[1]The story of the SS City of Cairo is fascinating and something that David had a great deal of interest in. In 1984, when a book [2] was published on the topic, David met Karl-Friedrich Merten, the captain of the U-Boat.

> *boat was. I was told at that moment it was another torpedo, it looked as if the ship was blown to pieces but it had not although it sank soon after. ...*
>
> *On the thirteenth day someone sighted a ship on the horizon. Everyone was very excited and went for the water in the barrels. The man in charge of the boat put some round tins, which floated, and sent up smoke, on the water, to attract the ship's attention. Soon the ship came and let down rope ladders like nets. Some officers came and helped a few who had grown weak.*
>
> *After we got on the boat we were given coffee which, although it had no sugar, tasted very sweet. In the cabin I drank so much water that I did not like it for a few days after. In about seven days we arrived in Cape Town and here we are now.*

The family survived this ordeal, and arrived in Cape Town. Initially, they stayed with one of David's uncles. David attended Somerset West Primary School, Hottentots-Holland High School and then a Marist Brothers school in Capetown. He even played a little junior soccer for De Beers AFC, a team he recalled being weak, but the goalie did go on to play for the South African National Team.

In 1947, his father moved from the Indian Civil Service to the Home Civil Service and took a job in London. Here David attended Berkhamsted School, where Graham Green's father was his house master. The family moved to Dublin in 1949, but David stayed on as a boarder to finish his schooling.

In 1951, David entered Trinity College Dublin to study mathematics. He won an entrance scholarship, and then took the undergraduate scholarship examinations in his first year, at the encouragement of his tutor. Trinity, at this time, was much like it had been in the 1890s: lecture rooms were heated by coal fires, water came from taps in the squares and there was just one lavatory! David got involved with various activities, including the Cumann Gaelach, despite having no Irish. This seems to have made him something of a social hit, earning several mentions in the *Four & Six* column of Trinity News. He graduated in 1955 at the top of his class with a gold medal.

David wanted to go to Princeton, but he was persuaded by one of his lecturers, Arthur Allen, to do his Ph.D. in Cambridge with William Hodge. He went to Peterhouse, where Allen had been a Bye-Fellow, and was the first Robert Gardiner scholar. Hodge's previous student had been Michael Atiyah, who was visiting the Institute for Advanced Study when David first arrived in Cambridge, but he recalls Atiyah's enthusiasm bringing the place to life when he returned. David and Atiyah remained good friends for the rest of their lives. David sometimes joked that with Hodge being the greatest British mathematician of the first half of the 20^{th} century and Atiyah being the greatest British mathematician of the second half of the 20^{th} century, he was surrounded by people that he couldn't hope to live up to.

In his first year at Cambridge, Hodge told David to read a book by a young German mathematician on New Topological Methods in Algebraic Geometry. The book was by Fritz Hirzebruch, and David felt it introduced him to a new era of mathematics[2]. David and Hirzebruch later met at the British Mathematical Colloquium meeting in Liverpool. They also became good friends. David later spent two spells as a visiting professor in Bonn, delivering a course on Lie Groups and quantum mechanics.

David completed his thesis on *The Spectral Sequence of a Pair of Coverings, and Related Cohomology Groups*, and then, in 1958, he moved to the Department of Mathematics at the University of Glasgow, initially as an assistant, but was encouraged to apply for a lecturing post after 2 years. He got the lecturing post, though he contested

[2] David later reviewed an English translation [16].

that he was never *really* interviewed for it. While in Glasgow, he met Anngret Erichson, later to become his wife. He also fulfilled his aim to go to Princeton, spending 1962–1963 as an instructor there while on leave of absence from Glasgow.

2. Teaching and Service

In 1964, David returned to Trinity to the post of Lecturer in the Department of Pure Mathematics. This was a relatively unstable time for the department, as after the long tenure of T.S. Broderick, two professors, H. Halberstam and G.A. Dirac, had passed through quickly. David had expressed an interest in returning to Dublin to Halberstam and it was between the tenure of these two professors that he returned (again, he claimed, without much of an interview). Trinity had evolved a lot in the intervening years, and there was now a physical department with staff offices.

While professor, G.A. Dirac seems to have decided that the mathematics books in the TCD library would be better housed in the mathematics department. It is unclear what the library thought of this, but regardless, Dirac organised for the books to be moved (or possibly abducted!). David took over the care of these books in 1967, and grew the library for over fifty years. In 2012, he reckoned it had over 20,000 books, which he procured, made available in the coffee room as they arrived, filed in the basement and carefully indexed (eventually using the department computer system [41]).

David's main undergraduate teaching was second year courses. In particular, for many years he taught the core senior freshman modules on algebra and analysis. Consequently, almost every mathematics student who passed through TCD was taught by David. These covered coordinate-free linear algebra, tensors, vector fields, classification of operators, calculus on manifolds, Lebesgue Integration and some complex analysis[3]. He carefully revised the courses every year, experimenting with different types of delivery. Students remember these modules as being an important basis for the mathematics that they did later, influencing those who went on to lecture these subjects. One student recalled them, simply, as the best lectures that they ever had.

In 1967, David delivered public lectures as part of a scheme for adults with a Leaving Cert to refresh their knowledge of mathematics. The Trinity News noted that over 30 nuns were among the 120 attending these lectures, something of a novelty in TCD at the time. The lectures were regarded as a success. Many of the attendees were teachers, possibly updating their knowledge given the significant syllabus changes introduced at the Leaving Certificate around that time.

One interesting side effect of David teaching all the students was that he knew everyone. Some students of other subjects regularly frequented the School of Maths, for access to the Maths Society or computers. These students could be stopped on the stairs so that David could quiz them and find out who they were. One tradition that he established, once settled in Dublin, was inviting all the students from his classes to dinner at his house towards the end of term. These dinners[4] had something of a legendary nature among the students, who often weren't sure how to respond to such an invite. However, they seem to have left fond memories of conversations, party pieces, food and even antics on the way home. These dinners continued, with an occasional hiatus, until David officially retired. David's interest in his students extended beyond their graduation, and he was always pleased to get updates on where and what students had moved on to.

A computer committee was set up with David Simms as chairperson in 1985. While the technical details were addressed by others (including Tim Murphy as director of the

[3]There are course notes for these modules, as transcribed by students, available online. I hope to provide a tidied version on the arXiv. DM

[4]With catering organised by Anngret.

Figure 1. David Simms, with some of his class, after delivering his last lecture in 2015.

computer system [41] and Richard Timoney [3]), David oversaw funding and expenditure for the system, providing reports to the college on it and justifying its costs. David made great use of the computer system for filing away useful information. He rarely deleted any e-mails[5] and had his e-mail indexed and searchable long before this became a fashionable way to access e-mail.

Over this period, David became a Fellow of the College in 1972 and an Associate Professor in 1973. He was elected as a member of the Royal Irish Academy in 1987 and was a member of the editorial board for the Mathematical Proceedings of the Royal Irish Academy.

In 1991, David became head of department. While the Department of Pure Mathematics and Department of Applied Mathematics had operated in tandem as a School of Mathematics for many years, the departments notionally had Brian Murdoch and David Spearman as independent professors. In 1988, they had merged and now David Simms was head of the School of Mathematics, covering both departments.

The college had proposed that the department was to move from House 39, at the centre of college. This move was not universally popular within the department, as there could have been reduced library space and reduced access for students to computers. There was a protest, involving a sit-in and case being brought to the Visitors. The department was eventually moved to Westland Row, near Pearse Street Station, in the summer of 1991, with allowances for more space for the department's facilities. For the first year after the move, the Hamilton Building was not yet open, and so lectures would largely take place at the far end of college in the Arts Block or Museum Building. Several additional concessions were secured from the college, including a computer room near the centre of college for 24-hour student use. The Hamilton Building opened for lectures in 1992, with two new lecture theaters available to deliver many of the department's courses.

As head of department, David introduced Theoretical Physics as a denominated entry route for the CAO in 1992. Theoretical Physics had been introduced by David

[5]His large mailbox was used as a template for a benchmark during the design of the filesystem index `dirhash` [4], which is now a part of several Unix systems.

Spearman in the 1960s and 70s, but students who wanted to take this option applied to the moderatorship in science. This new entry route proved popular for applicants, and similar specialist entry routes were introduced by other universities. David Simms was horrified about the situation in which a student could get an A1 in Maths, Applied Maths, Physics and Chemistry and still not have enough points to secure entry to these courses. He managed to get the quota for the courses raised to prevent the points becoming unreasonably high[6].

David continued as head of school until 1997. He became a Senior Fellow in 2002 and, although he officially retired in 2003, he continued to teach as an Emeritus Fellow until 2015.

3. Research

Let us now give some highlights from David's research. David very quickly established himself as a leading researcher in the field of geometric quantization. Here his mastery of the abstract ideas of bundles and connections was used to show how the general ideas of Kostant and Sourau could be used to carry out the quantization of specific physical systems in a geometric way. The idea of geometric quantization starts from Hamilton's symplectic formulation of mechanics in which the even dimensional symplectic manifold X with a closed two form ω that belongs to an integer cohomology class underpins the dynamics. On such a space X, a Hermitian line bundle L with a $U(1)$ connection ∇ and curvature given by $i\frac{\omega}{2\pi}$ can be constructed, where L is square integrable and defines a "prequantized Hilbert space H_0". On this space, classical observables, f, that are maps $f : X \to R$, become operators $O(f)$ given by $O(f) = i\hbar\nabla_{v(f)} + M_f$, where $v(f)$ is the Hamiltonian vector field associated with f. Prequantization takes the Poisson brackets of mechanics to commutators on the prequantized Hilbert space. It is a faithful map and just tells us how to reformulate classical mechanics in a novel geometric way.

To introduce quantum mechanics in this geometric setting, an additional mathematical idea is necessary. In quantum theory the full range of classical variables are no longer observables. Thus, for example, for a particle moving in one dimension, $\omega = dx \wedge dp$ where (x, p) are the coordinate and momentum of the particle in X and are the classical observable variables, but in quantum theory such a description is not allowed. One is only allowed to describe a state using either x or p. This requires introducing the new idea of "polarization" to the symplectic space X. David carried out this procedure in a very elegant way to determine the Kepler hydrogen atom energy states and later showed that this procedure was equivalent to the way Pauli had solved the hydrogen atom problem using purely algebraic methods. The interesting point of David's analysis was that he showed that to properly construct the polarized space, half forms and thus a metaplectic structure had to be introduced to X.

David wrote a book with Woodhouse [40] and published several lecture notes on different aspects of Geometric Quantization, Lie Groups and Quantum Mechanics (e.g. [30, 15, 35]). These works are standard references in the field.

One can say that David had a taste for using a mix of abstract geometric ideas with group theory to solve concrete problems. His work with John Miller on the Harish Chandra c-function provides another example of this trait, and that he enjoyed explaining mathematical ideas clearly and directly to those who asked him[7].

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[6]David always had an interest in encouraging able students. Thus he helped Gerard Murphy study for A-levels [5] and met DM to discuss CAO options.

[7]I have been a beneficiary of this. I was also told by Peter Horvathy how David spent a great deal of time explaining geometric quantization to him. SS

4. Family

It would be remiss of us not to briefly mention more about David's family, as he was dedicated to them and took pride in their achievements. After returning to Dublin, David's father decided to pursue a Ph.D. in History, became assistant to T.W. Moody and later a TCD Fellow. His wife, Anngret, who he encouraged to become an academic, became a professor in Geography in UCD. He has three sons, Brendan, Daniel and Ciaran, who all attended Trinity. The first of whom is now at Peterhouse Cambridge, the second a barrister who held an adjunct professor position in TCD, the last is now a TCD Fellow, making David the middle generation of three generations of TCD Fellows.

It is hard to cover all of David's concerns. For example, we have not touched on his interest in Finnegans Wake, his coining of *Broomsday* for 16$^{\text{th}}$ October [10], meeting part of the Goold-Verschoyle family as part of an RTÉ documentary or his involvement with the Irish Mathematics Society. The passing of David marks the end of an era of Trinity mathematics.

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We would like to thank David's family for providing the photograph and supplying details of Davis'd life. We would also like to thank David's former students who provided their reminiscences.

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PRESIDENT'S REPORT 2021

Committee changes: David Malone (MU) will have served a maximum period of 6 years as secretary at the end of 2021. Many thanks to David for his sterling service over three terms. The mantle of secretary has passed to Derek Kitson (MIC/UL) to whom we are very grateful for taking on this role. We wish him well. Since Derek joined the committee as a result of his election as secretary and since all committee members who were eligible to serve a further term were re-elected, these were the only changes to the committee this year.

IMS Bulletin: I renew the Society's appreciation of Tony O'Farrell's work as Editor of the Bulletin and to the Editorial team. The Bulletin is freely available online from the Society's homepage while attractive printed copies are available from lulu.com. In future, institutional members will receive a complimentary printed copy of each issue.

On the initiative of J.P. McCarthy (MTU), the process of arranging for the Bulletin to be indexed on SCOPUS is in train and proceeding well. In that regard, a 'publication ethics and publication malpractice statement' approved by the IMS Committee has been added to the Bulletin's webpage.

Thanks to David Malone, all articles in the Bulletin now have a DOI (Digital Object Identifier). Both the Scopus and the DOI initiatives raise the standing of the Bulletin and make publishing therein more attractive.

Eleanor Lingham (Sheffield-Hallam) is a most welcome addition to the Editorial Board as Book Reviews Editor.

IMS website and twitter feed: Michael Mackey (UCD), in addition to maintaining an up-to-date IMS website, has refreshed the site and introduced a more streamlined and modern design. Cliff Gilmore (Manchester) has taken over the Society's twitter feed, which had fallen somewhat into abeyance. In this era of social media, it is important that the Society has a robust online presence. I thank Michael and Cliff for their work in this regard.

IMS meetings: The Society's annual 'September Meeting', that was held over from 2020 due to the coronavirus pandemic, was hosted online by Brien Nolan and Niamh O'Sullivan (DCU) from 13 to 15 January 2021. The meeting, at which several of the talks focused on epidemiological and environmental modelling, was an unequivocal success.

The 2021 annual meeting was capably and smoothly organised by Cónall Kelly (UCC) and J.P. McCarthy (MTU) and jointly hosted by these two Munster universities, this in spite of the many additional complications consequent on ever changing public health guidelines. The highly successful meeting took place from 2 to 3 September in a hybrid face-to-face/online format that brought some most welcome normality into our pandemic-dominated lives.

The annual meeting in 2022 is scheduled to be held at TU Dublin, organised by Dana Mackey, Milena Venkova and Mercedes Jordan-Santana.

Thanks to all organisers and their institutions, past, present and future, for hosting the Society's annual meeting with such admirable aplomb.

IMS conference support: Due to the coronavirus pandemic, there were no applications for conference support in 2021. The Treasurer, Cónall Kelly, has issued a call for applications to support conferences taking place between January and June 2022.

EMS Meeting of Presidents: I attended the annual European Mathematical Society (EMS) Meeting of Presidents that was held online on Saturday 29 May and on which I subsequently reported to the membership by email. Of particular note are the revamped EMS Magazine, formerly the EMS Newsletter, including its occasional Young Mathematicians Column; the 'events' and 'jobs' listings on the EMS website; the EMS publishing house, EMS Press, which has a number of journals that operate under a *Subscribe to Open (S2O)* model. Note also that the Zentralblatt reviewing service is now free online as *zbMATH Open*.

Kovalevskaya Grants and IMS Travel Grants: The Intenational Congress of Mathematicians (ICM22) will be held, under the auspices of the International Mathematical Union, in Saint Petersburg from 6 to 14 July 2022. The Local Organising Committee of ICM22, in the person of Professor Dmitry Belyaev (Oxford), contacted the IMS to ask if the Society would nominate early career researchers for 'Kovalevskaya Grants' to attend ICM22 and then support these researchers through grants covering travel to Saint Petersburg. Kovalevskaya Grants are generous, covering the registration fee and all local expenses including accommodation and meals. Grant holders will receive assistance with logistics in Saint Petersburg and visa-free entry.

The IMS sought applications for Kovalevskaya Grants, as well as accompanying IMS Travel Grants, with a closing date of 30 September. Following evaluation by the IMS Committee, 5 grants were awarded. A further 5 candidates are on a reserve list in case further Kovalevskaya Grants become available during December.

International Mathematical Union: I wrote in March, on behalf of the Society, to Minister Simon Harris T.D., Department of Further and Higher Education, Research, Innovation and Science, requesting Government endorsement of the Irish Mathematical Society to continue to act as adhering body for the IMU and, consequent on such decision, that the Government would fund Ireland's membership of the IMU in 2021 and in subsequent years. The Department of FHERIS replied to say that the correspondence had been forwarded to the Department of Education as the matter was deemed more appropriate to that department. I wrote a follow-up letter to the Department of Eduction in June which was acknowledged but otherwise no response has been received.

Ireland's membership of the International Astronomical Union (IAU) shares a similar predicament to that of the IMU. It was agreed at the IMS Committee meeting in September that the IMS and the Astronomical Society of Ireland, whose president is Dr Deirdre Coffey (UCD) and is now the Adhering Organisation for the IAU, should aim to make common cause. It was further agreed at the Committee Meeting and at the AGM that the Society would pay Ireland's IMU membership fee for 2021.

At the start of November, I wrote to Dr Jennifer Brennan, Director of Research, Development and Innovation at the Technological Higher Education Association (THEA) and Professor Jim Livesey (VP for Research and Innovation at NUIG and Chair of IUA VP/Deans of Research Standing Group) seeking the support of the research community for our campaign. Carbon copies were sent to Dr Deirdre Coffey, President of the Astronomical Society of Ireland; Professor Mark W. J. Ferguson, Director General of SFI and Chief Scientific Advisor to the Irish Government; Dr Lisa Keating, Director of Research & Innovation, IUA; and Professor Philip Nolan, Chair of IEMAG and incoming Director General of SFI. At time of writing, no response to this correspondence has been received.

It is a priority for the Society that this matter be progressed at Government level and that the long term sustainability of Ireland's membership of the IMU be put on a firm footing.

Tom Carroll, December 2021.
E-mail address: t.carroll@ucc.ie

Draft Minutes of the Irish Mathematical Society Annual General Meeting held on September 3, 2021 in MTU and on Zoom (due to Covid)

Present: C. Boyd, S. Buckley, M. Bustamante, T. Carroll, L. Creedon, N. Dobbs, R. Flatley, R. Gaburro, B. Guilfoyle, J. Granell, C. Kelly, M. Kerin, B. Kreußler, G. Lessells, E. Lingham, E. Li (guest), D. Mackey, M. Mackey, JP. McCarthy, N. Madden, D. Malone, M. Mathieu, A. O'Farrell, C. Ó Luasa, M. OReilly, A. O'Shea, N. Snigireva, C. Stack, H. Šmigoc, E. Xie (guest), T. Xie (guest).

1 Minutes
The minutes of the AGM held on December 11, 2020 and published in Bulletin 86 were accepted, with a discussion around the education subcommittee that was taken under matters arising.

2 Matters Arising
C. Stack asked a question about the disbandment of the educational subcommittee, as she had a concern that it was not in line with the subcommittee's constitution. The committee noted that it had received C. Stack's correspondence on this issue, and would be happy to respond. In summary, the proposal for the subcommittee had been approved at the 2014 AGM, and the brief for the subcommittee was described in the minutes of that meeting. C. Stack suggested that there may have been a more extensive set of rules. The committee agreed to investigate this and respond to C. Stack in detail.

3 Correspondence

- The committee had received a query about the disbandment of the educational subcommittee, which had been discussed under matters arising.
- The committee had also received a suggestion from C. Stack that it would be appropriate to remember David Lewis of UCD, who had recently passed away. The meeting observed a moment of silence in his memory.

4 Membership Applications
The new and renewed members thus far for 2021 were A. Ní Shúilleabháin, M. O'Connell, M. Leitner, M. Golitsyna, M. Mc Gettrick, N. McElduff, R. Smith, T. Lynch, E. Bach, M. Manolaki. J. Hayes had also been approved as a member, though the treasurer was investigating if he was also eligible for institutional student membership.

5 President's Report
The president gave a summary of his report (available in full in the Bulletin), covering the May 2021 EMS Presidents Meeting, IMU Membership, Kovalevskaya travel grants for the ICM, P. Lynch's article on the annual IMS scientific meeting. M. Mackey thanked the president for keeping the matter of funding of IMU membership alive. A discussion followed on if the lack of funding for IMU membership might be regarded as a reduction in status of mathematics. It was noted that there may be multiple issues at play, and physics, chemistry and astronomy were in a similar position.

6 Treasurer's Report
A report on finances was presented. Since last year, €4,000 has been placed in savings certs. There had been little expenditure on conferences and meetings due to Covid. Last year's IMU dues had been paid with the support of mathematical research institutes and departments. This year, savings due to Covid had made it possible for the society to pay directly. M. Mackey asked if it might be possible to support more meetings in the following year. This was generally agreed to be a good idea if possible.

7 International Mathematical Union
The issues around IMU membership had been dealt with as part of the President's Report and the Treasurer's Report.

8 Bulletin
A. O'Farrell extended his thanks to M. Mackey for maintaining the Bulletin's website and to the editorial board (T. Carroll, J. Cruickshank, D. Mackey, P. Mellon, A. O'Shea, I. Short and T. Unger) for their support in the Bulletin's operation. He noted that E. Lingham would be joining as a new editorial board member.

It was noted that at the committee meeting there had been a proposal to send a printed copy of the Bulletin to institutional members.

9 Educational Subcommittee
A. O'Shea reported that the new educational subcommittee had begun its work in January by electing M. Kerin as secretary. Four main issues has been discussed over the year.

- **Textbook quality assurance:** Following on from the work of the previous subcommittee, the subcommittee will endeavour to make sure that second level mathematics textbooks are subject to quality assurance.
- **Online teaching at tertiary level:** There is a plan to organise a workshop later this year on the role of hybrid/online learning in future teaching practices.
- **Supporting exceptional talent:** A subgroup has been formed which is considering measures to challenge and stimulate high-performing students newly arrived in third level. This subgroup will also investigate methods to raise the profile of mathematics in the media in order to encourage more students to study mathematics.
- **Developing links between mathematicians and researchers in mathematics education:** It is hoped to hold a series of online talks on aspects of mathematics education research relevant to those teaching at third level.

J. Granell's excellent work with the Professional Development Service for Teachers and the State Examinations Commission in relation to the new Applied Mathematics Leaving Certificate course was also noted.

10 Subcommittee on Equal Opportunities
R. Gaburro was seeking feedback on the possibility of establishing a subcommittee on equal opportunities. At recent meetings of the European Women in Mathematics (EWM) and the IMU's Committe for Women in Mathematics, it has become apparent that many national societies are forming subcommittees with interests around equal opportunities, diversity and women in mathematics. The possibility of a subcommittee in this area had been raised with the committee and it was supportive. The subcommittee could have function in supporting meetings such as the Women in Mathematics Day and providing a contact point for outreach and international activities.

T. Carroll welcomed this development. The meeting was supportive of the suggestion, and a more detailed proposal will be formulated for the committee's Christmas meeting.

11 Elections
C. Kelly, R. Flatley, M. Mathieu, R. Ryan, H. Smigoc and N. Snigireva have reached the end of two or four year terms, but were willing to stand for re-election. D. Malone has reached the end of a six year term, and is consequently not eligible for re-election.

C. Kelly was proposed as treasurer by T. Carroll and seconded by D. Malone. D. Kitson was proposed for secretary by M. Mathieu and seconded by T. Carroll. R. Flatley,

M. Mathieu, R. Ryan, H. Smigoc and N. Snigireva were proposed for general committee membership by T. Carroll and seconded by D. Malone. Election to these positions was approved by the meeting.

D. Malone was thanked for his service as secretary for the last six years.

12 AOB

- J.P. McCarthy asked about arrangements for next year's scientific meeting. This year's meeting was to have been in TU Dublin, but because of complications due to Covid and relocation of the TU Dublin Campuses, UCC and CIT (now MTU) had stepped in. D. Mackey was hoping to be able welcome the 2022 meeting to the TU Dublin campus at Grangegorman.
- The Fergus Gaines Cup was virtually presented to the 2020 and 2021 winners of the Irish Mathematical Olympiad (IrMO), as Covid had prevented the presentation last year.

  Fergus Gaines, who served as the leader of the Irish team at the IMO several times, passed away in 2001. The cup has been presented as the Fergus Gaines Cup since 2005 and has been on many adventures, including being recovered by Tim Murphy after it went missing. The cup is currently safe with G. Lessells in Limerick.

  The 2020 winner was Tianyiwa Xie who had represented Ireland three times. The 2021 winner was Ellen Lee, who had just received her Leaving Certificate results. The meeting wished both of the well-deserved winners all the best in their future studies.

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IMS Annual Scientific Meeting 2021
University College Cork and Munster Technological University

September 2nd-3rd

The 34rd annual scientific meeting of the society - the 2021 September meeting - was held on September 2-3, 2021, jointly organised by the School of Mathematical Sciences, UCC, and the Department of Mathematics, MTU. The organising committee members were Cónall Kelly (UCC) and J.P. McCarthy (MTU).

Recognising the continuing disruption caused by the COVID-19 pandemic, but seeking to take a step back in the direction of normal academic activity, the meeting had a hybrid format, with speakers on-site at the Berkeley Library, Munster Technological University, and other delegates able to attend talks virtually if they wished. This hybrid format extended to the IMS Committee meeting and AGM held as part of the event on September 2nd and 3rd respectively.

Opening welcome remarks were made by Cónall Kelly and by Prof. Maggie Cusack, the President of MTU.

The meeting consisted of invited 40 minute talks on a wide variety of mathematical topics with speakers from across Ireland, which were organised into six sessions of two talks each. Students and early career researchers attending in person were invited to display a poster on their research, with a prize awarded for best student poster. A full list of talks and posters is given below.

Judging of student posters took place on Thursday September 2 and Michael Rosbotham, of Queen's University Belfast, was presented with the award during the closing session on Friday September 3 by the President of the Irish Mathematical Society, Dr. Tom Carroll.

The web page for the meeting, which includes a link to the Programme and Book of Abstracts, as well as a gallery of pictures taken during the event, is available at

`https://www.ucc.ie/en/matsci/events/ims2021/`

The organisers are grateful to all who participated in the meeting, and we are especially grateful to our speakers and poster presenters:

Thursday 2nd September

Pauline Mellon (UCD)
- *Interplay of holomorphy and algebra in Jordan structures.*

Mark Howard (NUIG)
- *A resource theory for quantum computation.*

John Appleby (DCU)
- *Mean Square Characterisation of Discrete and Continuous Linear Stochastic Equations with Memory.*

Anna Zhigun (QUB)
- *Cell migration in fibrous environments: a multiscale approach.*

Spyridon Dendrinos (UCC)
- *Affine invariant measures in harmonic analysis*

Violeta Morari (MTU)
- *SPIRIT Maths: From student perceptions to targeted digital resources*

Friday 3rd September

Alberto Caimo (TUD)
- *Statistical modelling advances for valued networks.*

Linda Daly (UCC)
- *Seasonal Mortality at Older Ages in Ireland 1986 to 2017 and the implications for Longevity Risk.*

Natalia Kopteva (UL)
- *Numerical solution of time-fractional parabolic equations.*

Brendan Guilfoyle (MTU)
- *From CAT scans to 4-manifold topology.*

Adamaria Perotta (UCD)
- *Why computation matters? An active student-led and practice-based approach to design a computational finance module in higher education.*

Tony Lyons (WIT)
- *Geophysical flows in the presence of underlying currents.*

Posters

Arinjoy Bhanja and Christopher Noonan (UCC)
- *Stochastic Volatility Modelling and Sensitivity Analysis with ORE.*

Guillem Cobos (MTU)
- *Admissible Line Complexes.*

Fearghus Downes (IT Sligo)
- *Applying a Predator-Prey model to the production of Progeterone and Estradiol from Pregnenolone.*

Michael Rosbotham (QUB)
- *A Global Dimension Theorem for C^*-algebras.*

Yen Thuan Trinh (UCC)
- *Option Pricing and CVA Calculations using the Monte Carlo-Tree (MC-Tree) Method with the Distribution Correction Factor.*

Report by C. Kelly (email: conall.kelly@ucc.ie)

Reports of Sponsored Meetings

Just one sponsored meeting was held in 2021.

GROUPS IN GALWAY 2021
DECEMBER 2–3, 2021, NUI GALWAY

Groups in Galway 2021 was organised by Angela Carnevale and Tobias Rossmann (both from NUI Galway). The conference took place online. There were two sessions, spread over as many days. Over 100 people registered to virtually attend the meeting, and the total number of active participants at any given time exceeded 40.

The conference featured a total of seven invited talks covering a wide range of topics in contemporary group theory and related fields:

(1) Cristina Acciarri (University of Brasilia):
A stronger version of Neumann's BFC-theorem
(2) Jesús Hernández Hernández (Universidad Nacional Autónoma de México):
Conjugacy classes of big mapping class groups
(3) Caroline Lassueur (RWTH Aachen University/TU Kaiserslautern):
On the trivial source character tables of finite groups
(4) Paula Lins (KU Leuven):
Twisted conjugacy for soluble matrix groups
(5) Alastair Litterick (University of Essex):
Geometric invariant theory and Generating Sets in Finite Simple Groups
(6) Rachel Quinlan (NUI Galway):
Pleats, twists and wallpaper: a backlit story
(7) Bernardo Rodrigues (University of Pretoria):
2-*modular representations of the Conway simple groups as binary codes*

The conference website `https://torossmann.github.io/gig21/` contains abstracts of the talks and further information.

Report by Tobias Rossmann, NUI Galway
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Some nontrivial two-term dilogarithm identities

JOHN M. CAMPBELL

ABSTRACT. In 2012, Lima introduced a remarkable two-term dilogarithm identity, based on a proof for the Basel problem due to Beukers et al. Using a series transform obtained very recently via Legendre polynomial expansions, we nontrivially extend Lima's identity, and offer a new proof of this same identity.

1. INTRODUCTION

The dilogarithm function is defined as $\mathrm{Li}_2(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2}$, which converges for all complex z with $|z| \leq 1$. In this note, we derive new and nontrivial two-term dilogarithm identities, improving upon remarkable discoveries due to Lima [11].

The natural logarithm function, as defined for positive values, is, of course, very fundamental in mathematics as an elementary classical function, apart from how frequently the natural logarithm arises in science, technology, and engineering fields, outside of pure mathematics. So, this begs the question as to what may be considered as an appropriate way of extending or lifting this function, in the context of a given application, or within a given discipline in mathematics, science, etc. In this regard, the study of so-called *higher logarithm functions* forms a prominent area within the field of special functions theory, with the above defined dilogarithm as something of a prototypical instance of what is meant by a higher logarithm, in consideration as to above definition for Li_2 compared to the Maclaurin series expansion $-\ln(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$. The problem of determining a closed form for $\mathrm{Li}_2(1)$ is perhaps one of the most famous problems throughout the history mathematics: This is referred to as the *Basel problem*, as solved by Euler in 1734, with the closed form $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$. This is indicative of the importance, historically and otherwise, about the subject of symbolically evaluating expressions involving the dilogarithm mapping. This article introduces new results in this area.

There are only eight known values z such that both $\mathrm{Li}_2(z)$ and z may be expressed in closed form [13, §1]. This motivates the development of techniques for symbolically evaluating two-term linear combinations of dilogarithmic expressions (cf. [11]). For the sake of brevity, we assume familiarity with basic Li_2 identities, such as $\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2)$ and $\mathrm{Li}_2(1-x) + \mathrm{Li}_2(1-x^{-1}) = -\frac{1}{2}\ln^2 x$. The evaluation due to Lima [11] whereby

$$\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} - \frac{1}{2}\ln^2\left(1+\sqrt{2}\right) \tag{1}$$

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does not follow from previously known two-term Li_2 identities. This evaluation is proved in [11] using an argument relying on an evaluation for

$$\int_{\ln\left(1+\sqrt{2}\right)/2}^{\infty} \ln(\tanh z)\, dz,$$

which, in turn, relies on a double integral evaluation due to Beukers et al. [3]. We offer a simplified proof of (1), and extend (1) in nontrivial ways, using a series transform very recently introduced in [5]. The evaluations given in this note, as in Examples 2.2–2.5 below, are nontrivial; Mathematica and Maple, in particular, are not able to obtain these evaluations, even with the use of Mathematica commands such as `FunctionExpand` and `FullSimplify` or Maple packages such as *SumTools*; the same holds for Lima's evaluation in (1).

Apart from Lima's work in [11], there has been a considerable amount of previous research devoted to two-term dilogarithm identities, as in with the work of Bytsko in [4]. For example, two-term dilogarithm relations for Li_2 evaluated at expressions as in $\frac{1}{\lambda^2}$ for $\lambda = 2\cos\frac{\pi}{7}$ are given in [4], and an earlier two-term Li_2 evaluation due to Gordon and McIntosh [7] involving

$$\mathrm{Li}_2\left(\frac{\sqrt{3+2\sqrt{5}}-1}{2}\right)$$

is also reproduced in [4]. A main source of interest in the two-term dilogarithm identities that we prove is due to Ramanujan's two-term Li_2 evaluations, as in the following equation [2, p. 32] (cf. [10]):

$$\mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{6}\mathrm{Li}_2\left(\frac{1}{9}\right) = \frac{\pi^2}{18} - \frac{\ln^2 3}{6}.$$

Furthermore, two-term dilogarithm evaluations have been involved in applications pertaining to differential geometry, making a particular note of the remarkable identity due to Khoi [8] (cf. [12]) given as follows:

$$L\left(\frac{1}{\phi(\phi+\sqrt{\phi})}\right) - L\left(\frac{\phi}{\phi+\sqrt{\phi}}\right) = -\frac{\pi^2}{20},$$

where the Rogers dilogarithm function is such that $L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\ln(z)\ln(1-z)$, and where $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$ denotes the famous golden ratio constant.

2. Main identity and applications

For the sake of brevity, we assume basic familiarity with the orthogonal family of Legendre polynomials $P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2(x-1)^{n-k}(x+1)^k$. The key idea behind our improving upon Lima's work in [11] is given by the following identity, which was introduced in 2021 [5] using fractional calculus and Legendre polynomial expansions: If f is an analytic function over $(0,1)$, and if

$$\sum_{n\geq 0} a_n x^n = \sum_{m\geq 0} b_m P_m(2x-1)$$

holds with respect to the usual norm for functions on $(0,1)$, then

$$\sum_{n\geq 0} \frac{a_n}{(2n+1)^2\left(\frac{\left(\frac{1}{2}\right)_n}{n!}\right)^2} = \sum_{m\geq 0} \frac{(-1)^m b_m}{(2m+1)^2}, \tag{2}$$

letting the Pochhammer symbol be defined and denoted as per usual, with $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$ for a natural number n. We also recall the Euler integral $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\, du$ used to define the Γ-function, along with the Legendre duplication formula: $\Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}\left(\frac{1}{4}\right)^k \binom{2k}{k}\Gamma(k+1)$. In our applying the series transform indicated in (2), we need to make use of the famous generating function (g.f.) formula given below:

$$\frac{1}{\sqrt{1-2yz+z^2}} = \sum_{n=0}^{\infty} P_n(y) z^n. \tag{3}$$

As below, we let $\operatorname{sgn}(r)$ denote the sign function, so that, for a real value r, $\operatorname{sgn}(0)=0$, $\operatorname{sgn}(r)=1$ if r is positive, and $\operatorname{sgn}(r)=-1$ otherwise.

Theorem 2.1. *The equality whereby*

$$\frac{1}{1+z}\sum_{n=0}^{\infty}\frac{\left(\frac{16z}{(1+z)^2}\right)^n}{(2n+1)^2\binom{2n}{n}} = \operatorname{sgn}(z)\frac{i\left[\operatorname{Li}_2\left(-\sqrt{-z}\right)-\operatorname{Li}_2\left(\sqrt{-z}\right)\right]}{2\sqrt{z}} \tag{4}$$

holds if both sides converge for real z. Here i is the imaginary unit.

Proof. We rewrite the g.f. in (3) so that

$$\frac{1}{\sqrt{1+2z+z^2}}\cdot\frac{1}{\sqrt{1-x\frac{4z}{1+2z+z^2}}} = \sum_{n=0}^{\infty} P_n(2x-1)z^n$$

for suitably bounded x and z. On the other hand, rewriting the latter factor on the left-hand side, as a function of x, with its Maclaurin series, we obtain that:

$$\sum_{n=0}^{\infty}\frac{(-1)^n\binom{-\frac{1}{2}}{n}\left(\frac{4z}{1+2z+z^2}\right)^n x^n}{\sqrt{1+2z+z^2}} = \sum_{n=0}^{\infty} P_n(2x-1)z^n.$$

Through a direct application of (2) to the above equality, we obtain that

$$\frac{\pi}{4\sqrt{(z+1)^2}}\sum_{n=0}^{\infty}\frac{\left(\frac{z}{(z+1)^2}\right)^n\Gamma(2n+1)}{\Gamma^2\left(n+\frac{3}{2}\right)} = \sum_{m=0}^{\infty}\frac{(-1)^m z^m}{(2m+1)^2},$$

and we set $|z+1|>0$. Since $\sum_{m=0}^{\infty}\frac{(-1)^m y^{2m} z^m}{2m+1}$ evaluates as $\frac{\tan^{-1}(y\sqrt{z})}{y\sqrt{z}}$, and since the antiderivative of this latter expression with respect to y is

$$\frac{i\left[\operatorname{Li}_2\left(-iy\sqrt{z}\right)-\operatorname{Li}_2\left(iy\sqrt{z}\right)\right]}{2\sqrt{z}},$$

this easily gives us the desired result. □

2.1. **Applications.** We begin by applying Theorem 2.1 so as to obtain a new and simplified proof of Lima's identity in (1).

Proof of (1)*:* Setting, in Theorem 2.1, $z=-\left(\sqrt{2}-1\right)^2$, this gives us that:

$$\frac{\sum_{n=0}^{\infty}\frac{(-4)^n}{(2n+1)^2\binom{2n}{n}}}{1-\left(\sqrt{2}-1\right)^2} = \frac{\operatorname{Li}_2\left(\sqrt{2}-1\right)-\operatorname{Li}_2\left(1-\sqrt{2}\right)}{2\left(\sqrt{2}-1\right)}. \tag{5}$$

So, it remains to evaluate the above infinite series. In this direction, by using the Maclaurin series expansion

$$\sum_{n=0}^{\infty}\frac{(-4)^n t^{2n}}{(2n+1)\binom{2n}{n}} = \frac{\sinh^{-1}(t)}{t\sqrt{1+t^2}},$$

letting $\sinh^{-1}(z) = \ln(\sqrt{z^2+1}+z)$ denote the inverse hyperbolic sine, we find that we may compute the antiderivative of the right-hand side of the above equality, as below:

$$\text{Li}_2\left(-e^{-\sinh^{-1}(t)}\right) - \text{Li}_2\left(e^{-\sinh^{-1}(t)}\right) +$$
$$\sinh^{-1}(t)\left(\ln\left(1-e^{-\sinh^{-1}(t)}\right) - \ln\left(e^{-\sinh^{-1}(t)}+1\right)\right).$$

This is easily seen by differentiating this symbolic form. Setting $t \to 1$ and $t \to 0$, this gives us the equality of

$$\frac{\text{Li}_2\left(1-\sqrt{2}\right) - \text{Li}_2\left(\sqrt{2}-1\right) + \frac{\pi^2}{4} + \left(\frac{\ln(2)}{2} - \ln\left(2+\sqrt{2}\right)\right)\sinh^{-1}(1)}{1-\left(\sqrt{2}-1\right)^2}$$

and

$$\frac{\text{Li}_2\left(\sqrt{2}-1\right) - \text{Li}_2\left(1-\sqrt{2}\right)}{2\left(\sqrt{2}-1\right)}.$$

Rearranging this equality, we obtain that

$$\text{Li}_2\left(\sqrt{2}-1\right) - \text{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} + \frac{1}{2}\sinh^{-1}(1)\ln\left(\sqrt{2}-1\right),$$

as desired. □

A relevant application of Lima's evaluation in (1) concerns a pair of classic polylogarithmic ladders due to Lewin (cf. [9, §1.6]), as below, writing α in place of $\sqrt{2}-1$:

$$4L(\alpha) - L(\alpha^2) = \frac{\pi^2}{4}, \tag{6}$$

$$4L(\alpha) + 4L(\alpha^2) - L(\alpha^4) = \frac{5\pi^2}{12}. \tag{7}$$

We see that: Thanks to Lima's identity, as in (1), we may obtain two correponding dilogarithm ladders with powers of $-\alpha$ in place of powers of α, since the powers of α other than α itself in (6) and (7) are even. The identity in (6) was used in a prominent way in [6] in a proof for a binomial-harmonic sum evaluation introduced in [6], using a Legendre polynomial-based integration technique closely related to the key identity in (2). The foregoing considerations strongly motivate further uses of Theorem 2.1 in the determination of two-term dilogarithm identities.

In order to generalize our new proof of Lima's identity shown in (1), we need to generalize how we had proved our evaluation for the infinite series on the left-hand side of (5), so as to be able to evaluate generating functions of the following form:

$$\sum_{n=0}^{\infty} \frac{x^n}{(2n+1)^2\binom{2n}{n}}. \tag{8}$$

However, it is known that this is equal to:

$$\frac{2i\text{Li}_2\left(-\sqrt{1-\frac{x}{4}} - \frac{i\sqrt{x}}{2}\right)}{\sqrt{x}} - \frac{2i\text{Li}_2\left(\sqrt{1-\frac{x}{4}} + \frac{i\sqrt{x}}{2}\right)}{\sqrt{x}} +$$
$$\frac{i\pi^2}{2\sqrt{x}} + \frac{2\ln\left(\frac{-\sqrt{1-\frac{x}{4}}-\frac{i\sqrt{x}}{2}+1}{\sqrt{1-\frac{x}{4}}+\frac{i\sqrt{x}}{2}+1}\right)\csc^{-1}\left(\frac{2}{\sqrt{x}}\right)}{\sqrt{x}}.$$

This is easily verifiable, by writing

$$\sum_{n=0}^{\infty} \frac{x^n y^{2n}}{(2n+1)\binom{2n}{n}} = \frac{4\sin^{-1}\left(\frac{\sqrt{x}y}{2}\right)}{\sqrt{x}y\sqrt{4-xy^2}},$$

and by then computing the antiderivative of the right-hand side.

Example 2.2. Setting $z = \frac{1}{-9-4\sqrt{5}}$ in Theorem 2.1, we may, as explained below, obtain the following identity:

$$\mathrm{Li}_2\left(\frac{1}{\phi^3}\right) - \mathrm{Li}_2\left(-\frac{1}{\phi^3}\right) = \frac{\phi^3\left(\pi^2 - 18\ln^2(\phi)\right)}{3\left(\phi^6-1\right)}. \tag{9}$$

Inputting the above value for z into the left-hand side of the identity in Theorem 2.1, it remains to evaluate the series in (8) for $x=-1$, making use of the classically known values for $\mathrm{Li}_2\left(\frac{1}{\phi}\right)$ and $\mathrm{Li}_2\left(-\frac{1}{\phi}\right)$ [13, §1]. As indicated above, Maple and Mathematica are not able to evaluate the left-hand side of the equality in (9). For example, inputting

```
FunctionExpand[
 PolyLog[2, GoldenRatio^(-3)] - PolyLog[2, -GoldenRatio^(-3)]]
```

into Mathematica, this CAS is not able to compute any evaluation for the above input.

Example 2.3. Setting $z = 7-4\sqrt{3}$ in Theorem 2.1, the left-hand side of this Theorem involves, in this case, the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\binom{2n}{n}}$$

which we may easily evaluate according to the above identity for the generating function in (8), giving us that:

$$\mathrm{Li}_2\left(i\left(2-\sqrt{3}\right)\right) - \mathrm{Li}_2\left(-i\left(2-\sqrt{3}\right)\right) = \frac{2i\sqrt{7-4\sqrt{3}}\left(8G - \pi\ln\left(2+\sqrt{3}\right)\right)}{3\left(8-4\sqrt{3}\right)},$$

letting $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ denote Catalan's constant.

Example 2.4. Setting $z = 3-2\sqrt{2}$ in Theorem 2.1, by again making use of the known symbolic form for the power series in (8), we obtain that:

$$\mathrm{Li}_2\left(i\left(\sqrt{2}-1\right)\right) - \mathrm{Li}_2\left(-i\left(\sqrt{2}-1\right)\right)$$

evaluates as

$$\frac{1}{32}i\left(\sqrt{2}\left(\psi^{(1)}\left(\frac{1}{8}\right) + \psi^{(1)}\left(\frac{3}{8}\right)\right) + 8\pi\ln\left(\sqrt{2}-1\right) - 4\sqrt{2}\pi^2\right),$$

writing $\psi^{(1)}(z) = \frac{d^2}{dz^2}\ln\Gamma(z)$ to denote the trigamma function.

Example 2.5. Setting $z = \frac{1}{3}$ in Theorem 2.1, we obtain, again making use of the evaluation for (8), that

$$\mathrm{Li}_2\left(\frac{i}{\sqrt{3}}\right) - \mathrm{Li}_2\left(-\frac{i}{\sqrt{3}}\right)$$

equals:

$$\frac{i\left(3\psi^{(1)}\left(\frac{1}{6}\right) + 15\psi^{(1)}\left(\frac{1}{3}\right) - 6\sqrt{3}\pi\ln(3) - 16\pi^2\right)}{36\sqrt{3}}.$$

As indicated above, the left-hand side of (4) may be written as an expression involving the difference

$$\mathrm{Li}_2\left(2i\sqrt{\frac{z}{(1+z)^2}}+\sqrt{1-\frac{4z}{(1+z)^2}}\right)-\mathrm{Li}_2\left(-2i\sqrt{\frac{z}{(1+z)^2}}-\sqrt{1-\frac{4z}{(1+z)^2}}\right)$$

along with combinations of elementary functions. Let the above difference be written as:

$$\mathrm{Li}_2(\alpha(z))-\mathrm{Li}_2(-\alpha(z)). \tag{10}$$

So, according to (4), if both

$$\mathrm{Li}_2(\sqrt{-z})-\mathrm{Li}_2(-\sqrt{-z}) \tag{11}$$

and (10) are convergent, then one such expression admits a closed-form evaluation if and only if the other such expression does.

Although the focus of this article has been on two-term Li_2 identities, we may also use Theorem 2.1 to obtain identities that bear a resemblance to the dilogarithmic ladder

$$\pi^2=36\mathrm{Li}_2\left(\frac{1}{2}\right)-36\mathrm{Li}_2\left(\frac{1}{4}\right)-12\mathrm{Li}_2\left(\frac{1}{8}\right)+6\mathrm{Li}_2\left(\frac{1}{64}\right)$$

given in [1]; for example, setting $z=-\frac{1}{4}$ gives us a closed form for a rational linear combination of $\mathrm{Li}_2\left(\frac{1}{4}\right)$, $\mathrm{Li}_2\left(-\frac{1}{3}\right)$, and $\mathrm{Li}_2\left(\frac{1}{3}\right)$. Explicitly,

$$2\mathrm{Li}_2\left(-\frac{1}{3}\right)+\mathrm{Li}_2\left(\frac{1}{4}\right)-2\mathrm{Li}_2\left(\frac{1}{3}\right)=-\frac{\pi^2}{6}-2\ln^2(2)+2\ln(2)\ln(3).$$

We encourage the exploration of further uses of Theorem 2.1.

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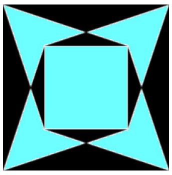

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Ireland's Participation in the 62nd International Mathematical Olympiad

MARK FLANAGAN

Abstract. An account of the Irish team's participation in IMO 2021 in St Petersburg, including team selection and preparation, the problems set at the Olympiad, the performance of the contestants, and acknowledgement of sponsors.

The 62$^{\text{nd}}$ International Mathematical Olympiad (IMO) took place in St. Petersburg, Russia, from 14–24 July 2021. A total of 619 students (64 of whom were girls) participated from 107 countries. As in 2020, due to the Covid pandemic, contestants were unable to travel to Russia in person, instead sitting invigilated examinations in their home country.

The Irish delegation consisted of six students (see Table 1) accompanied by Andrew Smith (Deputy Leader, UCD), Mark Flanagan (Team Leader, UCD), Anca Mustata (Observer, UCC) and Eugene Gath (Observer, UL).

Name	School	Year
Ellen Li	The Institute of Education, Dublin	6$^{\text{th}}$
Evan Grealish	Calasanctius College, Oranmore, Co. Galway	6$^{\text{th}}$
Adam Crowe	St Benildus College, Kilmacud, Dublin 14	6$^{\text{th}}$
James Chen	Castletroy College, Newtown, Limerick	4$^{\text{th}}$
Fionn Kimber O'Shea	Coliste an Phiarsaigh, Glanmire, Co. Cork	3$^{\text{rd}}$
Taiga Murray	St Benildus College, Kilmacud, Dublin 14	5$^{\text{th}}$

Table 1. The Irish contestants at the 62$^{\text{nd}}$ IMO

1. Team selection and preparation

The team detailed in Table 1 consisted of those six students (in order) who scored highest in the Irish Mathematical Olympiad (IrMO), which was held for the 34$^{\text{th}}$ time on Saturday, 8$^{\text{th}}$ May, 2021. The IrMO contest consists of two 3-hour papers on one day with five problems on each paper. In 2021, due to Covid restrictions, students sat the exam at home, under supervision of a trusted adult, usually a parent.

During the spring, students who participate in the IrMO usually attend extracurricular Mathematics Enrichment classes, which are offered at six Mathematics Enrichment Centres (UCC, UCD, NUIG, UL, MU and IT Sligo). These classes run each year from January until April and are offered by volunteer academic mathematicians from these universities or nearby third-level institutions. More information on the organisation of these classes, as well as links to the individual maths enrichment centres, can be found at the Irish Maths Enrichment / IrMO website, http://www.irmo.ie/.

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However, in 2021 these in-person maths enrichment activities were not possible due to the Covid pandemic. Instead, in 2021 a collaborative maths enrichment / IMO training programme (involving all six centres mentioned above) was made available online; trainers developed video-based lecture materials and associated problem sets, and this material was posted online on a password-accessible webpage on the website of the Irish Mathematical Trust (IMT). This training included an entire course on olympiad problem-solving in geometry, organised and delivered by Anca Mustata (UCC).

The selection and training for IMO 2021 followed procedures which are by now well-established. First, an Irish Maths Olympiad "Squad" was identified, consisting of the top performers in IrMO 2020 who were eligible to qualify for the Irish IMO team in 2021. For these students, various training activities were organised. First, the members of the Irish IMO Squad were directed to engage with the online training materials mentioned in the previous paragraph, and for each online training session the students could return their solutions or attempts to the homework problems by email to the proposer of the problems before a given deadline. The problem proposer then provided feedback on their work, as well as either full solutions or else hints on how to make further progress. This type of training is important for the successful engagement of "returning" students, and helps to develop the students' independence in mathematical problem solving.

Other training activities usually take the form of training camps, where students would travel to a central location in Ireland to engage in problem-solving activities. Such training activities are important, as during these mathematically intense 3–5 day events, students have the opportunity to socialise with their peers, exchange their mathematical ideas, and increase their motivation for their work throughout the year. However, due to Covid-related travel restrictions as well as restrictions on indoor activities, this year the training activities were exclusively in an online and remote format.

At the beginning of June, each of the team members was assigned a personal mentor whose role was to help the team member to structure their own personal training programme during the weeks prior to the IMO. This allowed each individual team member to tailor their efforts on each problem-solving area according to their relative strength or weakness in that particular area.

A final online team training camp took place from 12–16 July 2021, consisting of two four-hour sessions on each day. Each session provided a mix of presentation of new problem-solving materials and techniques, as well as independent problem-solving activities for the students. Typically each session would end with the students presenting and discussing their own solutions to the assigned problems.

2. Jury, Problem Selection and Examinations

In normal years, a "jury" consisting of national team leaders meets to select the IMO contest problems from a shortlist developed by the host country. Team leaders are then sequestered in a location sufficiently distant from that of the team and forbidden from communicating with the team or Deputy Leader until the end of the second day of examinations.

In 2021, as in 2020, such in-person jury meetings were not possible, and the problems were selected entirely by the problem selection committee, which was appointed by the host country. While this deprived national team leaders of the opportunity to contribute to problem selection, it also allowed us to relax the communication embargo, so that team leaders could continue to be involved in the logistics of organising local contests.

Unlike in 2020, in 2021 Covid-related travel restrictions were sufficiently relaxed in Ireland to allow all team members to travel to a central location to take the IMO exams. The IMO exams were held in the Clayton Hotel in Ballsbridge, Merrion Road, Dublin. For the two contest days, the IMO organisers required the appointment of an independent commissioner who was not involved in the team training or selection processes, a resident but not a citizen of the competing country. The responsibilities of the commissioner included receiving the examination question papers, invigilating the examination and scanning scripts for transmission to Russia at the end of each examination. In Ireland, the IMO Commissioner was Jon-Ivar Skullerud, who also served as IMO Commissioner in 2020.

The final marks for each contestant are agreed in a process known as coordination. This important part of the IMO is well-established and ensures that the scripts of the students from so many different nations are marked fairly and consistently. The decisions in this process are based on detailed and strict marking schemes prepared by the coordination teams.

The marking of the scripts of each participating country is undertaken by two independent groups. One group consists of the Team Leader, the Deputy Leader and Official Observers. The second group consists of the coordinators, who are appointed by the local organisers. In contrast to the usual IMO procedures, in 2021 the coordination process was undertaken entirely online, but the online process represented the usual face-to-face coordination meetings very faithfully.

3. The problems

The two exams took place on the 19^{th} and 20^{th} of July, with each exam having to start within a prescribed time window so that no student anywhere in the world would start an IMO exam after another student somewhere else had finished it. In Ireland, the starting time for each exam was 10am. On each day, $4\frac{1}{2}$ hours were available to solve three problems. The problems are published at http://www.imo-official.org/problems.aspx after each examination.

First Day

Problem 1. Let $n \geqslant 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

(Australia)

Problem 2. Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots, x_n$.

(Canada)

Problem 3. Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

(Ukraine)

Second Day

Problem 4. Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

(Poland)

Problem 5. Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

(Spain)

Problem 6. Let $m \geqslant 2$ be an integer, A be a finite set of (not necessarily positive) integers, and $B_1, B_2, B_3, \dots, B_m$ be subsets of A. Assume that for each $k = 1, 2, \dots, m$ the sum of the elements of B_k is m^k. Prove that A contains at least $m/2$ elements.

(Austria)

4. The results

The contest problems are chosen such that Problems 1 and 4 are the most accessible, while Problems 2 and 5 are more challenging. Problems 3 and 6 are usually the most difficult problems, whose existence on the paper is justified in posing a sizeable challenge even to the top students in the IMO competition. Table 2, which shows the scores achieved by all contestants on the 6 problems, illustrates that this gradient of difficulty was generally maintained this year also. However, Problem 2 was significantly more difficult than usual, its difficulty level being more aligned with Problems 3 and 6. From the table it can also be seen that for many problems, the marks were either close to 0 or close to 7, with partial marks difficult to obtain. This is a reflection of the marking schemes for the problems, which were somewhat unforgiving this year. These are indicators that this was a particularly difficult IMO contest.

In fact, from the available statistical information it can be concluded that IMO 2021 was the most difficult IMO in which an Irish team has ever participated. This may be seen by considering the *average percentage score* in the IMO contest, calculated as the total number of points scored altogether by the 619 participants (i.e., 7175) divided by the maximum possible total score (i.e., 42×619). This year, this was 27.6%. This lies in stark contrast to the average percentage scores for the past 5 years (2016–2020), which were 35.2%, 34.7%, 36.8%, 37.8% and 37.0% respectively. In fact, this is the lowest average percentage score in any IMO since Ireland's first participation in 1988.

Table 3 shows the results of the Irish contestants. One student (Taiga Murray) was awarded an Honourable Mention for his perfect solution to Problem 5 – at such a difficult IMO, this is a fantastic achievement. However, the scores are generally lower than in recent years – this is partly due to the difficulty of this year's contest (and in particular the marking schemes for the problems), but may also be due to the effect of the Covid

	P1	P2	P3	P4	P5	P6
0	131	522	488	218	404	562
1	36	61	110	33	12	12
2	41	12	4	39	13	2
3	10	2	1	2	4	3
4	41	3	1	12	2	1
5	39	1	0	1	5	2
6	35	2	0	5	4	0
7	286	16	15	309	175	37
average	4.393	0.375	0.372	3.817	2.152	0.481

Table 2. The number of contestants achieving each possible number of points on Problems 1–6

Name	P1	P2	P3	P4	P5	P6	total	ranking
Taiga Murray	0	0	0	1	7	0	8	348
James Chen	2	0	0	0	0	0	2	470
Ellen Li	1	0	0	0	0	0	1	499
Evan Grealish	1	0	0	0	0	0	1	499
Adam Crowe	0	0	0	0	0	0	0	533
Fionn Kimber O'Shea	0	0	0	0	0	0	0	533

Table 3. The results of the Irish contestants

restrictions on our national Mathematics Enrichment programme over the last two years, which has no doubt impacted on the students' ability to train for this challenging contest. Having said this, the Irish contestants had some good ideas on some of the problems which were unfortunately not rewarded by the marking schemes. For example, for Problem 4, all of the points in the official marking scheme hinged on spotting and proving an initial symmetry ($IX = IY$, $IT = IZ$, $XT = YZ$, $XZ = YT$) that was difficult to spot. However, while not necessarily rewarded by the marking scheme, the Irish contestants had good ideas on this problem, such as using the tangency points with the incircle of the quadrilateral $ABCD$ to simplify the required relation, finding similar triangles and ratios involving the segments in the required relation, and recognising the relevance of the incentre-excentre lemma to this problem. Also, for Problem 1 James Chen managed to solve the case $n = 100$ by showing the existence of a 5-cycle in the graph representing the problem, the difficulty of which task was appreciated by the Russian coordinators.

The figures in Table 4 have the following meaning. The first figure after the topic indicates the percentage of all points scored out of the maximum possible. The second number is the same for the Irish team and the final column indicates the Irish average score as a percentage of the overall average.

5. Outlook

The next countries to host the IMO will be

2022 Norway
2023 Japan
2024 Ukraine
2025 Australia

Problem	topic	all countries	Ireland	relative
1	number theory	62.8	9.5	15.2
2	algebra	5.4	0.0	0.0
3	geometry	5.3	0.0	0.0
4	geometry	54.5	2.4	4.4
5	combinatorics	30.7	16.7	54.2
6	algebra	6.9	0.0	0.0
all		27.6	4.8	17.3

Table 4. Relative results of the Irish team for each problem

6. Conclusions

The difficulty of the problems at IMO 2021, as well as the corresponding marking schemes, made it difficult this year for students to solve the contest problems as well as to obtain partial credit for their ideas. This level of difficulty is indicative that there is a need for the IMO Jury to be reinstated in the problem selection process. Despite this, one Irish student obtained an Honorable Mention, which is a great achievement considering the difficulty of the contest as well as a lack of face-to-face training in problem-solving. It is hoped that regular face-to-face maths enrichment and IMO training will again become possible in Ireland in the not-too-distant future. This training, as well as Ireland's continued participation in these contests, require sustained funding.

7. Acknowledgements

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The principal foundation for the success of the contestants is the work done with the students in the Mathematics Enrichment Programmes at six third level institutions. This work is carried out for free by volunteers in their spare time. Many thanks to all those involved in the Mathematics Enrichment activities and training camps, which were all held virtually in 2021. The remote Mathematics Enrichment activities were delivered in 2021 by Kazim Buyukboduk, Mark Flanagan, Mark Fortune, Eugene Gath, Mayya Golitsyna, Jonathan Grant Peters, David Malone, Myrto Manolaki, John Murray, Seoirse Murray, Anca Mustata, Andrei Mustata, Andrew Smith, Rob Sparkes, Maria Tieder and Jessica Weitbrecht. The organisation of these sessions and the collation of all video materials, problem sets, etc., as well as their presentation on the IMT website, was managed by Anca Mustata. At the IMO team training camp, which was organised by Mark Flanagan, the trainers were Mark Flanagan, Eugene Gath, Bernd Kreussler, Gordon Lessells, Myrto Manolaki, Anca Mustata and Andrew Smith. The trainers who served as personal mentors to the IMO team members during the weeks prior to the IMO contest were Mark Flanagan, Mayya Golitsyna, Bernd Kreussler, Gordon Lessells, Seoirse Murray and Anca Mustata. Thanks also to UCC, UCD, NUIG, UL, MU and IT Sligo for permitting the use of their facilities in the delivery of the national Maths Enrichment Programme.

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during the contest days at IMO 2021. Finally, thanks to the hosts for organising this year's IMO virtually from Russia, and thanks to the Russian guides Leyla Kanbarova and Anastasiya Shurpitskaya, who helped to make the Irish team members' experience at IMO 2021 a truly international one.

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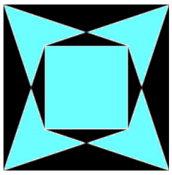

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On p-adic modules with isomorphic endomorphism algebras

BRENDAN GOLDSMITH AND NOEL WHITE

Abstract. We investigate pairs of modules over the ring of p-adic integers having isomorphic endomorphism algebras. In many cases this forces the modules to be isomorphic but there are two exceptional situations where isomorphism does not follow.

1. Introduction

Throughout this work we shall focus on modules over the ring R of p-adic integers, where p is a fixed but arbitrary prime. Recall that elements of R may be considered as infinite sums $\sum_{i=0}^{\infty} r_i p^i$, where each r_i is an integer with $0 \leq r_i < p$. Addition and multiplication are defined in the usual way and it is easy to check that R then becomes an integral domain. It is possible to topologise R with a linear topology having the submodules $p^j R$ $(j = 0, 1, \dots)$ as a base of neighbourhoods of 0; the resulting 'p-adic topology' is metrisable and R is complete with respect to this topology. R is an example of a *complete discrete valuation ring* and such rings may be regarded as a not too complicated generalisation of a field. Associated with R we have its field of quotients Q and for each integer n, the cyclic module R/p^nR which is just the usual additively written cyclic group of integers modulo p^n. In keeping with the usual notation for Abelian groups, we often write $R(p^n)$ instead of R/p^nR. There is one further significant module associated with R *viz.* the quotient module Q/R. This module is *divisible* in the sense that given any element x, the equation $x = p^n y$ has a solution y. It is not difficult to show that Q/R is isomorphic to the usual Prüfer quasi-cyclic group $\mathbb{Z}(p^\infty)$; recall that $\mathbb{Z}(p^\infty)$ is the additively written version of the infinite multiplicative group of all p^nth complex roots of unity and is generated by elements $c_1, c_2, \dots, c_n, \dots$ such that $pc_1 = 0, pc_2 = c_1, \dots, pc_{n+1} = c_n, \dots$. Hence each c_n has order p^n and every element of $\mathbb{Z}(p^\infty)$ is a multiple of some c_n. Consequently, all proper submodules of Q/R are finite and they form a chain under inclusion. It is well known that every R-module G can be expressed as a direct sum $G = D \oplus X$, where D is a divisible module and X is *reduced* in the sense that X contains no divisible submodules. Furthermore, a divisible R-module is of the form $\bigoplus_\lambda Q \oplus \bigoplus_\mu Q/R$ for cardinals $\lambda, \mu \geq 0$. We shall reserve the notation R for the ring of p-adic integers and all modules (unless specified to the contrary) will be left modules over R.

It is a long-standing question in algebra as to what extent the algebra of endomorphisms of an Abelian group or module (over an arbitrary ring) determines the module itself. In the simplest case of vector spaces over a field, it is an easy exercise in linear algebra to show that if V, W are finite-dimensional spaces over a field F then $V \cong W$ if and only if the corresponding rings of linear transformations (i.e. the endomorphism rings) are isomorphic as rings. It is less trivial, but nonetheless true, that the corresponding

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result remains true if one removes the restriction of finite dimensionality: however, a simplistic appeal to a matrix argument will result in a need for some additional set theory such as the Generalised Continuum Hypothesis (GCH).

The question that we wish to consider in the rest of this paper is simply: If G, H are R-modules and the endomorphism algebras $E_R(G), E_R(H)$ are isomorphic (as R-algebras), what can we say about the relationship (if any) between G and H? Note that in one direction the argument is straightforward (even for modules over an arbitrary ring S): if A, B are isomorphic S-modules via the isomorphism $\phi : A \to B$, then the mapping $\Phi : E_S(A) \to E_S(B)$ defined for an arbitrary endomorphism α of A by $\Phi(\alpha) = \phi\alpha\phi^{-1}$ is easily seen to be an S-algebra isomorphism.

It will be clear from our arguments that many of our results can be generalised to include modules over rings other than the rings of p-adic integers. However, it is well known that a question such as that above has no hope of being answered in, for example, the category of all $\mathbb{Z}$-modules (Abelian groups). In particular, the situation where torsion-free modules are involved is, in many cases, totally intractable: for example, it is known that there exist arbitrary large torsion-free Abelian groups all having endomorphism ring isomorphic to the ring of integers $\mathbb{Z}$. A specific advantage of working with p-adic modules is that the only indecomposable modules are then $Q, R, Q/R$ and $R(p^n)$ for each positive integer n. Moreover, the corresponding endomorphism algebras are easily calculated to be precisely $Q, R, R, R(p^n)$; the fact that $E_R(R) \cong E_R(Q/R)$ will, of course, be a difficulty for us in addressing the relationship between R-modules and their endomorphism algebras.

The first significant result in this area is probably that of Baer [1] and this was significantly simplified and generalised by Kaplansky [11, Theorem 28] who showed, *inter alia*, that torsion R-modules have the property that any isomorphism between their endomorphism algebras is induced by an endomorphism of the modules themselves. (Kaplansky's argument worked in the more general context of complete discrete valuation rings.) Consequently, results of this form are often referred to in the literature as 'Baer-Kaplansky Theorems'- see, for example, [3, 12] or [14] for a small selection of such results. Subsequently Hauptfleisch [10] and Wolfson [16] extended the result to certain classes of torsion-free Abelian groups and modules; in particular Wolfson showed that if G, H are reduced torsion-free R-modules then $E_R(G) \cong E_R(H)$ if and only if $G \cong H$. The situation for mixed modules and Abelian groups is difficult but some interesting results were obtained in [15] and [4]. Recently the authors, working on an unpublished idea of Corner, have considered the corresponding problem where isomorphism of the endomorphism algebras is replaced by anti-isomorphism, [9].

Our notation is largely standard and this, and relevant ideas from Abelian group and module theory, may be found in the texts [5, 6, 7] or [11]. In particular, for an R-module G, tG will denote the torsion submodule of G: $tG = \{g \in G | p^n g = 0, \text{ for some } n \geq 0\}$.

Mappings are written on the left and, when there is no risk of confusion, we shall denote the R-algebra of R-endomorphisms of an R-module G by simply $E(G)$; in the sequel when we write $E(G) \cong E(H)$ for R-modules G, H, we shall always mean that the isomorphism is an R-algebra isomorphism. In certain special cases we shall also make use of our knowledge of the structure of the module of R-homomorphism between modules G, H, which we denote simply by $\mathrm{Hom}(G, H)$ and the main results we need are directly analogous to those that may be found for Abelian groups in [5, Chapter VIII].

In the next section, Preliminaries, we consider the basic ideas centring on the use of idempotents and introduce an *ad hoc* notion of an *endo-root* which allows us to present a uniform treatment of some of the simpler cases that we wish to consider; an interesting by-product of this is an easy proof that vector spaces of arbitrary dimension over a fixed field are isomorphic if and only if their rings of linear transformations are isomorphic as rings.

In the third section we consider R-modules which are not reduced, i.e., modules having a direct summand isomorphic to either Q or Q/R. In particular, in this section we are able to show that if $E(H) \cong E(G)$, where G is a divisible R-module, then either $H \cong G$ or else $G = \bigoplus_{\mu} Q/R$ and $H = \widehat{\bigoplus_{\mu} R}$, the p-adic completion of the free R-module of rank μ, an arbitrary cardinal.

In the final section we show that the situation is similar to that for divisible modules but that there is one additional possibility: the modules G, H have isomorphic torsion basic submodules but are not themselves necessarily isomorphic, We indicate by means of an example that the situation is extremely complex which strongly suggests the unlikelihood of obtaining a complete classification of all outcomes.

The material in this work is part of the first chapter of a research thesis being prepared by the second author for submission to TU Dublin.

2. Preliminaries.

A key tool in approaching the relationship between a module and its endomorphism algebra is to utilise idempotents in the latter. Our first two results hold for modules over arbitrary rings and the straightforward proofs may be found in [6, Section 106 (c) (d)].

Proposition 2.1. *If M is a module over an arbitrary ring S and e is an idempotent in $E(M)$, then the rings $eE(M)e$ and $E(eM)$ are isomorphic.*

Proposition 2.2. *Suppose A, A' are modules over an arbitrary ring S and there is an algebra isomorphism $\Phi : E(A) \to E(A')$. If A has a decomposition $A = B \oplus C$, then A' has a decomposition $A' = B' \oplus C'$ and Φ induces isomorphisms $E(B) \cong E(B')$, $E(C) \cong E(C')$.*

We will use these results repeatedly in our proofs and in the latter stages of the paper we will often refer to such usage as 'by the standard argument '.

The next result also holds for modules over an arbitrary ring S and is based on an idea of Corner which we have previously exploited in [9].

Lemma 2.3. *Suppose that Φ is an algebra isomorphism between the endomorphism algebras $E(G), E(H)$. Let α_1, α_2 be projections of G onto the summands $A_i = \alpha_i(G)$, $(i = 1, 2)$ and let $\beta_i = \Phi(\alpha_i)$ be the corresponding projections of H onto summands $B_i = \beta_i(H)$, $(i = 1, 2)$. Then if $\mathrm{Hom}(A_1, A_2) = 0$ so too is $\mathrm{Hom}(B_1, B_2)$.*

Proof. Let $\beta : B_1 \to B_2$ be an arbitrary homomorphism and extend it to an endomorphism of H by setting $\beta(1 - \beta)1)(H) = 0$. Observe that if x is an arbitrary element of H then $\beta(x) = \beta(\beta_1(x) + (1 - \beta_1)(x) = \beta\beta_1(x)$ so that $\beta = \beta\beta_1$. Moreover, as $\beta(x) \in B_2$, $\beta_2\beta(x) = \beta(x)$ from which it follows that $\beta = \beta_2\beta\beta_1$. Taking images under the algebra-isomorphism Φ^{-1} we get that $\Phi^{-1}(\beta) = \alpha = \alpha_2\alpha\alpha_1$ and as this is a homomorphism : $A_1 \to A_2$, we have that $\alpha = 0$. But then $\beta = \Phi(\alpha) = 0$, as required. □

In our work in [9], we found it convenient and useful to introduce relationships that we termed the 'Fundamental Relations'; these relationships enabled us to transfer information from endomorphism algebras of decomposable modules into information relating to homomorphism modules. Suppose that G, H are R-modules with $E(G) \cong E(H)$ via an isomorphism Φ and π is an idempotent of $E = E(G)$, so that from Proposition 2.2 we get an idempotent $\pi' \in E(H) = E'$ and $E\pi \cong E'\pi'$; similarly we also will have $\pi E \cong \pi" E'$. However, the relationships $\mathrm{Hom}(\pi(G), G) \cong E\pi$ and

$\mathrm{Hom}(G, \pi(G)) \cong \pi E$ always hold so we get the following relations for an idempotent $\pi \in E(G)$ and the corresponding image of π under an algebra-isomorphism, $\pi' \in E(H)$:

$$\mathrm{Hom}(\pi(G), G) \cong \mathrm{Hom}(\pi'(H), H) \quad \text{and} \quad \mathrm{Hom}(G, \pi(G)) \cong \mathrm{Hom}(H, \pi'(H).$$

We remark that in the situation where $E(G)$ and $E(H)$ are anti-isomorphic as in [9], these relationships are 'twisted' and result in quite strong restrictions on G, H. Surprisingly, in the current situation we shall only need to appeal to this result in the proof of Proposition 3.1 ; otherwise Lemma 2.3 will suffice for our purposes.

As a first application of the ideas above we wish to consider the relationship between modules and their endomorphism algebras in some simple cases, including the case of a vector space and its ring of linear transformations. To derive a uniform approach which we believe illustrates the fundamental idea, we introduce an *ad hoc* concept: a cyclic direct summand, generated by x say, of an R-module G is said to be an *endo-root* if, given any $y \in G$, there is an endomorphism of G, ϕ_y say, with $\phi_y(x) = y$.

Note that in a vector space over a field, every non-zero element is an endo-root and for a reduced R-module G, any summand isomorphic to R has a generator which is an endo-root. It is also reasonably straightforward to show that if G is is a bounded R-module then any element of exponent equal to that of G is also an endo-root: such an element generates a direct summand - see, for example [11, Lemma 4] - and the maximality of order of such an element permits one to define an endomorphism of G mapping this element onto any other element of G. A simple example of an R-module without an endo-root is the direct sum of the cyclic modules $R(p^n)$, one copy for each positive integer n.

Theorem 2.4. *Let G, H be reduced R-modules and suppose that G has an endo-root x with $Rx = \pi(G)$ for some idempotent $\pi \in E(G)$. If $\Phi : E(G) \to E(H)$ is an algebra isomorphism with the property that $\Phi(\pi)(H)$ is generated by an endo-root of H, then $G \cong H$.*

Proof. If G is indecomposable then either (i) $G = R$ or (ii) $G = R(p^n)$ for some positive integer n. In either case we have that$E(G) \cong G$ and $E(G)$ is an integral domain. Thus in either case as $E(H) \cong E(G)$, we must have that H is indecomposable as its endomorphism ring is an integral domain. In case (i) $E(H) \cong R$ and, as observed in the introduction, H is either R or Q/R, with the latter being excluded since, by hypothesis, H is reduced so that $H \cong G$. In case (ii) the only possibility for H is $R(p^n)$ and again $H \cong G$.

Now if G is decomposable we have $G = A \oplus B$ for $A = \langle x \rangle$ and some $B \neq 0$. Then it follows from Proposition 2.2 that there is a decomposition $H = A' \oplus B'$ with $E(A) \cong E(A'), E(B) \cong E(B')$; furthermore, we have that $A' = \Phi(\pi)(H)$ so by hypothesis $A' = \langle z \rangle$ for some endo-root z of H.

Given any $g \in G$, we can find an endomorphism α of G with $\alpha(x) = g$. Now define a map $\phi : G \to H$ by setting

$$\phi(g) = (\Phi(\alpha))(z).$$

The first thing we must check is that ϕ is well defined, so suppose that there is another endomorphism β with $g = \beta(x)$. Then $(\alpha - \beta)(x) = 0$, so that $(\alpha - \beta)\pi$ is the zero map in $E(G)$ and hence $\Phi((\alpha - \beta)\pi) = \Phi(\alpha - \beta)\Phi(\pi)$ is the zero map in $E(H)$. Thus $\Phi(\alpha - \beta)(z) = 0$ so that $\Phi(\alpha)(z) = \Phi(\beta)(z)$ and the mapping ϕ is well defined.

Since Φ is additive, so too is ϕ; it remains to show that ϕ is bijective.

Suppose then that $\phi(g) = 0$ for some $g \in G$, so that $\Phi(\alpha)(z) = 0$. Since $\Phi(\pi)(x) = z$ this means that $\Phi(\alpha)\Phi(\pi) = 0$ and, as Φ is an algebra isomorphism, $\Phi(\alpha\pi) = 0$, from which it follows that $\alpha\pi$ is the zero map in G. Hence $g = \alpha(x) = \alpha\pi(x) = 0$ and ϕ is monic.

Finally let $h \in H$ be arbitrary. Since z is, by hypothesis, an endo-root of H, there is an endomorphism, η' say, with $\eta'(z) = h$. Let $\eta = \Phi^{-1}(\eta') \in E(G)$ and set $g = \eta(x)$. Observe that $\phi(g) = \Phi(\eta)(z) = \eta'(z) = h$, so that ϕ is surjective . □

We remark that the restriction to R-modules is not necessary and a key idea for a more general result is that we want our modules to have the property that two indecomposable modules are isomorphic if and only if their endomorphism algebras are isomorphic.

Corollary 2.5. *If V, W are vector spaces over a field and the rings of linear transformations are isomorphic as rings, then $V \cong W$.*

Proof. If V is a vector space then the argument in the first paragraph of the proof of Theorem 2.4 simplifies to the single possibility that G is isomorphic to the field itself. Furthermore, as observed above, every non-zero element of a vector space is an endo-root, so the requirement in Theorem 2.4 that $\Phi(\pi)$ gives an endo-root in W is trivially satisfied. The result then follows directly from Theorem 2.4. □

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Corollary 2.6. *Let G, H be reduced R-modules such that G has a direct summand isomorphic to R. If $E(G) \cong E(H)$, then $G \cong H$. In particular, if G, H are reduced torsion-free R-modules with isomorphic endomorphism algebras, then G, H are isomorphic.*

Proof. This follows immediately from Theorem 2.4 and our earlier observation that a generator of a summand isomorphic to R is always an endo-root. □

Corollary 2.7. *If G, H are bounded R-modules with $E(G) \cong E(H)$, then $G \cong H$.*

Proof. Observe firstly that if G is p^n-bounded and $\Phi : E(G) \to E(H)$ is an algebra isomorphism, then H is also p^n-bounded: if $p^nG = 0$ then $p^nE(G) = 0$ and the isomorphism Φ then forces $p^nE(H) = 0$, giving $p^n 1_H = 0$ and hence $p^nH = 0$. Now choose an element $x \in G$ of maximal order and note, as observed above, that such an element x is an endo-root of G. If $\pi : G \to \langle x \rangle$ is a projection, then $\Phi(\pi)$ gives rise to a summand $\langle z \rangle$ of H with $E(\langle x \rangle) \cong E(\langle z \rangle)$, so that $\langle z \rangle$ is also an element of order p^n, and hence a summand of H. The maximality of its order then means that $\langle z \rangle$ is an endo-root of H and the result now follows directly from Theorem 2.4. □

3. Modules which are not reduced

In this section we consider R-modules which are not reduced and show that we can restrict our considerations to reduced modules. We begin by investigating the endomorphism algebras of divisible R-modules.

Proposition 3.1. *Let G be a divisible R-module and H an R-module with $E(G) \cong E(H)$. Then*

(i) If $G = \bigoplus_{\lambda} Q$ for some non-zero cardinal λ, then $H \cong G$;

(ii) If $G = \bigoplus_{\mu} Q/R$ for some non-zero cardinal μ, then either $H \cong G$ or $H = \widehat{\bigoplus_{\mu} R}$;

(iii) If $G = \bigoplus_{\lambda} Q \oplus \bigoplus_{\mu} Q/R$ with λ, μ non-zero cardinals, then $G \cong H$.

Proof. (i) If G is torsion-free divisible then it follows from standard properties of homomorphism groups that $E(G)$ is torsion-free and divisible as an R-module. But then H cannot have a summand isomorphic to any of $R, Q/R$ or $R(p^n)$ since an application of Proposition 2.2 would result in the contradiction that $E(G)$ has such a summand.

(ii) Let π be a projection of G onto a single summand isomorphic to Q/R and denote by π' the corresponding projection in $E(H)$. Then since $E(\pi(G)) \cong R \cong E(\pi'(H))$, we have that $\pi'(H)$ is either R or Q/R. By the first of our fundamental relations we get that $\mathrm{Hom}(Q/R, \bigoplus_\mu Q/R)$ is isomorphic to either $\mathrm{Hom}(R, H)$ or $\mathrm{Hom}(Q/R, H)$. We know from [5, Proposition 44.3] that $\mathrm{Hom}(Q/R, \bigoplus_\mu Q/R) \cong \widehat{\bigoplus_\mu R}$ so that $H \cong \mathrm{Hom}(R, H) \cong \widehat{\bigoplus_\mu R}$.

Suppose then $\pi'(H) \cong Q/R$. Note firstly that H cannot have a summand isomorphic to Q since this would result in the contradiction that G has such a summand. So if $H = D \oplus X$ with D divisible and X reduced,, then $D \cong \bigoplus_\alpha Q/R$ for some cardinal α. Furthermore, X is reduced and so $\mathrm{Hom}(Q/R, H) = \mathrm{Hom}(Q/R, D) \cong \widehat{\bigoplus_\alpha R} \cong \widehat{\bigoplus_\mu R}$. Reducing modulo p and taking vector space dimensions we get that $\alpha = \mu$. It remains to show that X is necessarily zero. If not, then we have a decomposition of $H = D \oplus X$ and $\mathrm{Hom}(D, X) = 0$ and this would result in a decomposition of G of the form $G = A \oplus B$ with A, B both non-zero and, by Lemma 2.3, $\mathrm{Hom}(A, B)$ would have to be zero - this is impossible since any summand of G is necessarily a sum of copies of Q/R.

(iii) Let π be a projection from G onto $\bigoplus_\mu Q/R$ along $Y = \bigoplus_\lambda Q$. Then from Proposition 2.2 we get a projection π' of H and a decomposition $H = \pi'(H) \oplus X$, and $E(X) \cong E(Y)$, $E(\pi(G)) \cong E(\pi'(H))$; note that it follows from part(i) that $X \cong Y$. It follows from part (ii) that $\pi'(H)$ is isomorphic to either $\bigoplus_\mu Q/R$ or to $\widehat{\bigoplus_\mu R}$. However, $\mathrm{Hom}(\pi(G), Y) = 0$ while if the second possibility holds, $\mathrm{Hom}(\pi'(H), X)$ contains a copy of $\mathrm{Hom}(R, X)$ and is non-zero, contrary to Lemma 2.3. So the only possibility is that $\pi'(H) \cong \bigoplus_\mu Q/R$ and $H \cong G$. □

With the information from Proposition 3.1 at hand we can now take the necessary steps to reduce our problem to one concerning reduced modules only. First we need a simple fact regarding the four types of indecomposable R-module.

Lemma 3.2. *If X is an indecomposable R-module then* $\mathrm{Hom}(X, Q/R) \neq 0$.

Proof. Since Q/R contains a copy of $R(p^n)$ for every positive integer n, the result is immediate if $X = R(p^n)$; if $X = R$ then $\mathrm{Hom}(X, Q/R)$ is isomorphic to Q/R itself, while if $X = Q/R$ then the homomorphism module is R. The final case to consider is then $X = Q$. Apply the functor $\mathrm{Hom}(Q, -)$ to the short exact sequence $0 \to R \to Q \to Q/R \to 0$ and note that the completeness of R forces the term $\mathrm{Ext}^1(Q, R)$ to be zero, so that $\mathrm{Hom}(Q, Q) \cong \mathrm{Hom}(Q, Q/R) \neq 0$. □

So suppose now that G is an R-module of the form $G = D \oplus G_1$ where D is divisible, G_1 is reduced and both are non-zero. If H is an R-module with $E(G) \cong E(H)$ then:

(a) If $D = \bigoplus_\lambda Q$ and π is the projection of G onto D along G_1, then it follows from Proposition 2.2 that H decomposes as $H = D' \oplus H_1$ with $E(D) \cong D'$ and $E(G_1) \cong E(H_1)$; it now follows from Proposition 3.1 that $D \cong D'$. Furthermore, if H_1 has a summand isomorphic to Q, then a further application of Proposition 2.2 to H_1 would give that G_1 has a summand isomorphic to Q, contrary to G_1 being reduced. On the other hand , if H_1 has a summand isomorphic to Q/R then $\mathrm{Hom}(\bigoplus_\lambda Q, H_1) \geq \mathrm{Hom}(\bigoplus_\lambda Q, Q/R) \neq 0$, the last inequality coming from Lemma 3.2. Now $G = \bigoplus_\lambda Q \oplus G_1$

and $\mathrm{Hom}(\bigoplus_{\lambda} Q, G_1) = 0$ since G_1 is reduced. An application of Lemma 2.3 would then force $\mathrm{Hom}(\bigoplus_{\lambda} Q, H_1) = 0$, contrary to Lemma 3.2. So we conclude that H_1 does not have a summand isomorphic to Q/R and it then follows that $H = D' \oplus H_1$ with $D' \cong D$ and H_1 is reduced with $E(H_1) \cong E(G_1)$.

(b) If $D = \bigoplus_{\mu} Q/R$ then the argument used in part (a) gives us that $H = D' \oplus H_1$ with $E(D) \cong E(D')$ and $E(H_1) \cong E(G_1)$; note that as G_1 is non-zero, so too is H_1. In this case an application of Proposition 3.1 yields two possibilities: either $D' \cong D$ or $D' \cong \widehat{\bigoplus_{\mu} R}$. Now $\mathrm{Hom}(D, G_1) = 0$ so that by Lemma 2.3 we must have $\mathrm{Hom}(D', H_1) = 0$. This, however, is impossible if $D' \cong \widehat{\bigoplus_{\mu} R}$, since the last term has a summand isomorphic to R. Thus we must have that $G = \bigoplus_{\mu} Q/R \oplus G_1, H = \bigoplus_{\mu} Q/R \oplus H_1$ and $E(G_1) \cong E(H_1)$. We claim that in this case H_1 is also reduced.

Clearly the standard argument shows that H_1 cannot have a summand isomorphic to Q since G_1 is reduced. Furthermore, as $\mathrm{Hom}(\bigoplus_{\mu} Q/R, G_1) = 0$ it follows from Lemma 2.3 that $\mathrm{Hom}(\bigoplus_{\mu} Q/R, H_1) = 0$ and this is impossible if H_1 contains a summand isomorphic to Q/R. So H_1 is reduced as claimed, and $G = \bigoplus_{\mu} Q/R \oplus G_1, H = \bigoplus_{\mu} Q/R \oplus H_1$ and $E(G_1) \cong E(H_1)$.

The final reduction we need relates to the following situation:

(c) If $D = \bigoplus_{\lambda} Q \oplus \bigoplus_{\mu} Q/R$ with both λ, μ non-zero, then applying the standard argument to the projection of G onto D along G_1, we get that H decomposes as $H = D' \oplus H_1$ with $E(D) \cong E(D')$ and $E(G_1) \cong E(H_1)$. It follows from Proposition 3.1 (iii) that $D' \cong D$. An identical argument to that in part (b) shows that H_1 is reduced.

In summary then we have established the following:

Theorem 3.3. *Let $G = D \oplus G_1$ with D non-zero divisible and G_1 non-zero reduced and suppose H is an R-module with $E(G) \cong E(H)$, then $H = D' \oplus H_1$ with $D' \cong D$ and H_1 is reduced with $E(H_1) \cong E(G_1)$.*

So we may restrict our considerations to reduced modules in the sequel.

4. Reduced modules

Suppose now that G, H are reduced R-modules with $E(G) \cong E(H)$. We examine the various possibilities; firstly we consider the situation in which G has a direct summand isomorphic to R. Since H is also reduced we see from our previous arguments that H must also have such a summand and now it follows from Corollary 2.6 that G and H are isomorphic.

Now consider the situation where G, and hence of course H, does not have a summand isomorphic to R. This means that a basic submodule of G is torsion and similarly for H. However, the fact that a basic submodule is torsion does not necessarily mean that the module itself is torsion. In the situation where $E(G) \cong E(H)$, we can derive some additional information: a torsion basic submodule is of the form $B = \bigoplus_{n=1}^{\infty} B_n$ where each B_n is a direct sum of cyclic R-modules $R(p^n)$ and the B_n are summands of the module. Consequently our standard argument using projections onto these summands in G will yield corresponding summands in H which are isomorphic and it is then easy to see that G, H may be regarded as having a common basic submodule. In fact we can say a good deal more as evidenced by our next result, a proof of which can be found in [9, Proposition 3.2].

Proposition 4.1. *If G is a reduced R-module which does not have a direct summand isomorphic to R and B is a basic submodule of G, then (i) G/tG is divisible;*
(ii) $G/B \cong (tG/B) \oplus D$ where tG/B is torsion divisible and D is torsion-free divisible and isomorphic to G/tG.

There are two possibilities to consider: G is torsion or G is mixed. This in turn leads to two cases which we will consider separately: (a) both G and H are torsion or (b) at least one of G, H is mixed.

In case (a) we can appeal to the celebrated theorem of Kaplansky - see [11, Theorem 28] or [7, Theorem 16.2.5] - mentioned in the introduction which tells us that $G \cong H$.

In case (b) we have just seen that G, H are extensions of a common R-module which is a direct sum of cyclic modules by a divisible R-module. Despite this, the possibility of obtaining a complete answer to our original question is totally dashed by the next example. This example has already appeared in [2, Theorem 3.2] and we paraphrase below the relevant part of that result.

Theorem 4.2. *There exists a mixed R-module G and a family of submodules $G(\psi)$ indexed by a real parameter ψ, $0 < \psi \leq 1$ such that*
(a) $G(\psi) \not\cong G(\psi')$ if $0 < \psi < \psi' \leq 1$;
(b) $E(G(\psi)) \cong E(G)$ for $0 < \psi \leq 1$;
(c) $G(1)$ is the torsion-completion of a basic submodule of G.

Summarising all of the above we have established:

Theorem 4.3. *If G, H are R-modules with $E(G) \cong E(H)$ then $G \cong H$* **unless**
(i) $G = \bigoplus_\mu Q/R$ and $H = \widehat{\bigoplus_\mu R}$ for some cardinal μ **or**
(ii) G, H have isomorphic torsion basic submodules B and at least one of G, H is a non-torsion extension of B by a divisible R-module.

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Remark 4.4. : The conditions on G, H in Theorem 4.3 above are necessary conditions but condition(i) is also sufficient in the sense that if $G = \bigoplus_\mu Q/R$ and $H = \widehat{\bigoplus_\mu R}$, then $E(G) \cong E(H)$ - see, for example the discussion of the Matlis Category Equivalence in [8, Section VIII.2] or [13].

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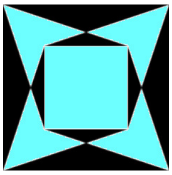

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Four Group-theoretic Proofs of Wedderburn's Little Theorem

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Abstract. Wedderburn proved in 1905 that a finite division ring is always a field. His result has intrigued generations of mathematicians, spurring generalizations and alternative proofs. The shortest, most elegant proof is surely Witt's from 1931, now the standard textbook treatment. Following a strategy of Zassenhaus, we present four overlapping group-theoretic proofs. The first uses a counting argument; the others hinge on properties of special classes of finite groups.

1. Introduction

Wedderburn's Little Theorem says that a finite division ring is a field. That is, if each nonzero element of a finite ring with identity has a multiplicative inverse then the ring is commutative. In the words of I. N. Herstein, the theorem "has caught the imagination of mathematicians because it is so unexpected, interrelating two seemingly unrelated things, the number of elements in a certain algebraic system and the multiplication of that system" [9]. In the words of Emil Artin, "this result of Wedderburn has fascinated most algebraists to a very high degree" [2]. As often in mathematics, the first proofs were somewhat clumsy. Quoting Artin again, Wedderburn [17] makes use of "divisibility properties which are hard to establish" and "several attempts were made to simplify the proofs." In 1931, Witt gave a short, elegant proof. It uses only elementary group theory in the form of the class equation (which Wedderburn also used) and easy divisibility properties. It is indeed a proof from THE BOOK (see [1, Chap. 6]).

Witt's argument, however, was far from the last word. There have been many other proofs—ones like Witt's that use essentially elementary methods and ones that stem from a broader perspective. As an example of the latter, note that Wedderburn's Big Theorem [18] puts the Little Theorem in a wider setting—the theory of central simple algebras and the Brauer group. This is a profound theory concerned, in its classical incarnation, with describing division algebras that are finite dimensional over their centres. Various results in the overall theory yield proofs of the Little Theorem. For instance, by properties of what are called cyclic algebras the theorem reduces to an easy calculation—checking surjectivity of the norm map for a finite extension of finite fields (see, for example, [10, Section 8.4]).

Our starting point is a technical lemma due to Zassenhaus [20]. It says that if $\mathbb{E}$ is a subfield of a finite division ring $\mathbb{D}$ then the normalizer and centralizer of $\mathbb{E}^\times$ in $\mathbb{D}^\times$ coincide. Zassenhaus goes on to deduce the Little Theorem by a purely group-theoretic argument (see Remark 1 below). We follow the same path: with Zassenhaus's lemma in hand, we apply tools from finite group theory to obtain the Little Theorem—four times. Out of a perverse sense of symmetry, we also include four proofs of Zassenhaus's lemma.

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The first proof of the Little Theorem is direct. Beyond the technical lemma, it uses only elementary group theory and simple counting. The other proofs rely on more advanced material: Burnside's Normal Complement Theorem, Carter subgroups, a property of Frobenius groups. For anyone new to these topics, we've tried to fill in enough background so that the bulk of the paper still makes sense. More precisely, a reader on friendly terms with (a) finite groups up to the Sylow theorems plus the notion of a nilpotent group and (b) fields and basic Galois theory should be able to follow the main thread of argument.

Three of our proofs share a common strategy: they proceed by showing that the maximal fields in a noncommutative finite division ring $\mathbb{D}$ (with $\mathbb{D}$ sometimes of minimal order) form a single conjugacy class (under the action of $\mathbb{D}^\times$). Since each element of $\mathbb{D}$ is contained in a maximal field, it follows that if $\mathbb{E}$ is a fixed maximal field in $\mathbb{D}$ then the group $\mathbb{D}^\times$ is a union of conjugates of $\mathbb{E}^\times$. A finite group, however, is never a union of conjugates of a proper subgroup (see Lemma 2 below), and so a noncommutative finite division ring cannot exist. The same strategy underlies the classic proof of the Little Theorem in van der Waerden's *Moderne Algebra*, the influential text based on lectures by Artin and Noether, first published in 1930–31. Van der Waerden's argument, which is due to Noether, uses early results in the theory of central simple algebras. It is probably the most reproduced proof after Witt's of the Little Theorem.[1]

Except for the first, our proofs are more artificial than Zassenhaus's. We could be accused (we have been) of using a series of sledgehammers to crack the Little Theorem. But they are such beautiful sledgehammers. The actual arguments, once the necessary background is in place, are short, requiring just a tap from any sledgehammers we wield. The proofs are meant as mathematical entertainments—an amusing application, we hope, of parts of finite group theory.

Acknowledgements. The treatment of Zassenhaus's lemma in Section 7 owes much to comments provided by a reviewer of a previous (pre-Bulletin) version of the paper. In addition, I've adopted several suggestions made by the referee of the current (Bulletin) version.

I would also like to note an older, more personal debt to two master expositors of mathematics, T. J. Laffey and D. L. McQuillan, under whose stimulating guidance I first learned of the world containing Wedderburn's theorem.

Notation

Throughout $\mathbb{D}$ denotes a finite divison ring. The centre of $\mathbb{D}$ is a finite field which we write always as $\mathbb{F}$. For any ring R (with identity), $R^\times$ denotes the group of units of R.

We write q for the cardinality of $\mathbb{F}$ and q^N for the cardinality of $\mathbb{D}$, so that N is the dimension of $\mathbb{D}$ as an $\mathbb{F}$-vector space.

For H a subgroup of a group G, the normalizer and centralizer of H in G are written as $N(H)$ and $C(H)$. The ambient group G should be clear from context. In fact, we work mostly with a subfield $\mathbb{E}$ of $\mathbb{D}$. Then $N(\mathbb{E}^\times)$ and $C(\mathbb{E}^\times)$ always mean the normalizer and centralizer in $\mathbb{D}^\times$ (that is, the ambient group is invariably $\mathbb{D}^\times$).

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[1]Van der Waerden, Witt and Zassenhaus were part of the remarkable flowering of abstract algebra in Germany in the 1920s and early 1930s in which Artin at Hamburg and Noether at Göttingen were leading figures. Witt was Noether's doctoral student and Zassenhaus was Artin's. Van der Waerden studied at Göttingen and was in the group (or ring) of young mathematicians centred around Noether; he also spent a year at Hamburg where he worked closely with Artin [16]. And then the cataclysm. In March 1933 the Nazis took power. In April a decree dismissed "non-Aryans" from government institutions including universities. Within weeks, a vibrant mathematical culture was destroyed [12]. Hilbert, when asked in 1934 by the new minister of education about mathematics at Göttingen "now that it was freed of the Jewish influence," replied: "Mathematics at Göttingen? There is really none anymore" [13, p. 205].

For S a finite set, $|S|$ denotes the number of elements in S. Given a subset T of S, we write $S \setminus T$ for the difference of S and T, that is, the set of elements of S that do not belong to T.

2. Zassenhaus's Lemma

The following technical result, due to Zassenhaus ([20, p. 59]), underpins each of our approaches to Wedderburn's theorem.

Lemma 1. *If $\mathbb{E}$ is a subfield of a finite division ring $\mathbb{D}$, then the normalizer and centralizer of $\mathbb{E}^\times$ in $\mathbb{D}^\times$ coincide.*

We only need the statement for now and so defer the task of proving the lemma to Section 7.

Remark 1. The formulation in [20] is superficially different: it says that if A is an abelian subgroup of $\mathbb{D}^\times$ then $N(A) = C(A)$. To obtain this version for a given abelian subgroup A of $\mathbb{D}^\times$, simply apply Lemma 1 to $\mathbb{F}(A)$, the subfield of $\mathbb{D}$ generated over $\mathbb{F}$ by A. Zassenhaus goes on to prove the Little Theorem by establishing a pleasant result: if a finite group G has the property that $N(A) = C(A)$ for every abelian subgroup A then G is abelian ([20, Theorem 7]).

Remark 2. Let $\mathbb{E}$ be a subfield of $\mathbb{D}$ that strictly contains the centre $\mathbb{F}$, so that the automorphism group $\mathrm{Aut}(\mathbb{E}/\mathbb{F})$ is nontrivial. A foundational result in the theory of central simple algebras—the Skolem–Noether Theorem—implies that each element of $\mathrm{Aut}(\mathbb{E}/\mathbb{F})$ is implemented by conjugation by some element of $\mathbb{D}^\times$ (see, for example, [5, Theorem 3.14]). The action of $N(\mathbb{E}^\times)$ on $\mathbb{E}$ by conjugation therefore induces an isomorphism $N(\mathbb{E}^\times)/C(\mathbb{E}^\times) \simeq \mathrm{Aut}(\mathbb{E}/\mathbb{F})$, and so $N(\mathbb{E}^\times) \neq C(\mathbb{E}^\times)$. Thus, if we were willing to make use of the theory of algebras, Lemma 1 yields a one-line (or one-paragraph) proof of the Little Theorem.

3. First Proof: Counting Maximal Fields

We assume that $\mathbb{D}$ has minimal order among noncommutative finite division rings and hope to find a contradiction.

Let $\mathbb{E}$ and $\mathbb{E}'$ be distinct maximal fields in $\mathbb{D}$. Then the subring $\langle \mathbb{E}, \mathbb{E}' \rangle$ generated by $\mathbb{E}$ and $\mathbb{E}'$ is noncommutative. It's also a division ring: in fact, each nonzero subring of $\mathbb{D}$ is a division ring (since each element of $\mathbb{D}^\times$ has finite order). By minimality of $|\mathbb{D}|$, it follows that $\langle \mathbb{E}, \mathbb{E}' \rangle = \mathbb{D}$. Thus $\mathbb{E} \cap \mathbb{E}'$ lies in the centre of $\mathbb{D}$, and so

$$\mathbb{E} \cap \mathbb{E}' = \mathbb{F}. \tag{1}$$

Recall our notation: $|\mathbb{F}| = q$ and $|\mathbb{D}| = q^N$. Let $\mathbb{E}_1, \dots, \mathbb{E}_r$ be representatives of the distinct conjugacy classes of maximal fields in $\mathbb{D}$. We write $|\mathbb{E}_i| = q^{m_i}$ for integers $m_i \geq 2$ (for $i = 1, \dots, r$). We use some counting and a simple estimate to show that $r = 1$, that is, the maximal subfields of $\mathbb{D}$ form a single conjugacy class.

Each element of $\mathbb{D}$ is contained in a maximal field. Using (1), we see that $\mathbb{D} \setminus \mathbb{F}$ is the disjoint union of the sets $\mathbb{E} \setminus \mathbb{F}$ as $\mathbb{E}$ varies through the maximal fields in $\mathbb{D}$. The number of distinct fields $x\mathbb{E}_i x^{-1}$ as x varies through $\mathbb{D}^\times$ is $[\mathbb{D}^\times : N(\mathbb{E}_i^\times)]$ (for $i = 1, \dots, r$). Thus, by Lemma 1, the field $\mathbb{E}_i$ has $\dfrac{q^N - 1}{q^{m_i} - 1}$ conjugates, and so the conjugates of $\mathbb{E}_i \setminus \mathbb{F}$ account for $\dfrac{q^N - 1}{q^{m_i} - 1}\,(q^{m_i} - q)$ elements in $\mathbb{D} \setminus \mathbb{F}$ (for $i = 1, \dots, r$). Therefore

$$q^N - q \;=\; \frac{q^N - 1}{q^{m_1} - 1}\,(q^{m_1} - q) + \cdots + \frac{q^N - 1}{q^{m_r} - 1}\,(q^{m_r} - q).$$

Rearranging, we obtain

$$\frac{q^N - q}{q^N - 1} = \frac{q^{m_1} - q}{q^{m_1} - 1} + \cdots + \frac{q^{m_r} - q}{q^{m_r} - 1}. \tag{2}$$

The left side is less than 1. Further, each term on the right side is greater than $\frac{1}{2}$. Indeed, for $m \geq 2$,

$$\begin{aligned} \frac{q^m - q}{q^m - 1} > \frac{1}{2} &\iff 2(q^m - q) > q^m - 1 \\ &\iff q^m - 2q + 1 > 0, \end{aligned}$$

and $q^m - 2q + 1 \geq q^2 - 2q + 1 = (q-1)^2 > 0$. Hence

$$\begin{aligned} 1 &> \frac{1}{2} + \cdots + \frac{1}{2} \quad (r \text{ terms}) \\ &= \frac{r}{2}, \end{aligned}$$

so $2 > r$ and $r = 1$.

From here, there are two ways to complete the argument.

Method 1. Since $r = 1$, the equality (2) now says that there is a positive integer $m < N$ such that

$$\frac{q^N - q}{q^N - 1} = \frac{q^m - q}{q^m - 1}. \tag{3}$$

Clearing denominators, we have

$$(q^m - 1)(q^N - q) = (q^m - q)(q^N - 1).$$

Expanding each side, cancelling common terms and rearranging then gives

$$q^{N+1} - q^N = q^{m+1} - q^m.$$

Thus $q^N(q-1) = q^m(q-1)$ and $N = m$—a contradiction.

Alternatively, the real function

$$\frac{q^x - q}{q^x - 1} = 1 - \frac{q-1}{q^x - 1}$$

is increasing (since q^x is increasing). Again, we see from (3) that $N = m$ which is absurd.

Method 2. We've proved that if $\mathbb{E}$ is a maximal subfield of $\mathbb{D}$ then

$$\mathbb{D}^\times = \bigcup_{x \in \mathbb{D}^\times} x\, \mathbb{E}^\times x^{-1}.$$

This is impossible: a finite group is never a union of conjugates of a proper subgroup.

Lemma 2. *Let H be a proper subgroup of a finite group G. Then*

$$\bigcup_{x \in G} xHx^{-1} \neq G.$$

Proof. The subgroup H has at most $[G : H]$ distinct conjugates in G. Indeed, for $x \in G$, the conjugate xHx^{-1} depends only on the left coset xH of H in G. Moreover, each conjugate of H contains the identity element. Hence

$$\begin{aligned} |\bigcup_{x\in G} xHx^{-1}| &\le \frac{|G|}{|H|}(|H|-1)+1 \\ &= |G| - \frac{|G|}{|H|} + 1 \\ &< |G|, \end{aligned}$$

which gives the result. □

Remark 3. Lemma 2 features in several proofs of the Little Theorem. As noted in the introduction, it provides the final step in the classic proof recounted by van der Waerden. Further, the group-theoretic principle that underlies Zassenhaus's proof (recalled in Remark 1) rests ultimately on Lemma 2. The lemma also plays a role in our third and fourth proofs.

4. Second Proof: Cyclic Sylow Subgroups

Next we rework an argument from 1964 due to T. J. Kaczynski [11][2]. The key observation is the following which we derive from Lemma 1 (the first several paragraphs of [11] give another route).

Proposition 1. *Every Sylow subgroup of* $\mathbb{D}^\times$ *is cyclic.*

Proof. We use induction on $|\mathbb{D}|$. The base case $|\mathbb{D}| = 2$ is trivial. Assuming that each Sylow subgroup of the multiplicative group of a division ring of order less than $|\mathbb{D}|$ is cyclic, we wish to show that $\mathbb{D}$ has the same property.

Let S be a Sylow l-subgroup of $\mathbb{D}^\times$ for some prime l. If $S \subset \mathbb{F}^\times$ then S is cyclic (by cyclicity of $\mathbb{F}^\times$). If S is not contained in $\mathbb{F}^\times$, then $S\mathbb{F}^\times/\mathbb{F}^\times$ is a nontrivial Sylow l-subgroup of $\mathbb{D}^\times/\mathbb{F}^\times$. In this case, we choose a nontrivial element $\beta\mathbb{F}^\times$ in the center of $S\mathbb{F}^\times/\mathbb{F}^\times$. Then, for each $\alpha \in S$, we have $\beta\alpha\beta^{-1} = \lambda\alpha$ for some $\lambda \in \mathbb{F}^\times$. Thus conjugation by β takes the field $\mathbb{F}(\alpha)$ to itself. Using Lemma 1, it follows that α and β commute. Hence $S \subset C(\beta)$, the centralizer of β in $\mathbb{D}$. Observe that $C(\beta) \neq \mathbb{D}$ as $\beta \notin \mathbb{F}$. Using our inductive hypothesis, we conclude that S is cyclic. □

For use in this section and the next, we list some properties of finite groups in which all Sylow subgroups are cyclic. More can be said: in fact, a finite group has cyclic Sylow subgroups if and only if it's a semidirect product of cyclic groups whose orders are relatively prime ([8, Theorem 5.16]).

Proposition 2. *Let G be a finite group in which every Sylow subgroup is cyclic and write l for the largest prime divisor of $|G|$. Then:*

(a) *a Sylow l-subgroup of G is normal;*
(b) *the group G is solvable;*
(c) *if G is nonabelian, it contains a normal abelian subgroup that is strictly larger than the centre.*

The proposition is a consequence of a famous result of Burnside ([8, Theorem 5.13]).

Burnside's Normal Complement Theorem. *Suppose a finite group G admits a Sylow subgroup S such that $N(S) = C(S)$. Then S has a normal complement in G. That is, there is a normal subgroup N of G such that $G = NS$ and $N \cap S = \{1\}$.*

[2]The author of the argument is better known today for activities outside mathematics.

The result is also known as Burnside's Transfer Theorem as it follows from properties of the transfer map—a natural homomorphism from a group G to the commutator group H/H' of a subgroup H of finite index . Isaacs' book [8] contains a cogent account of this map and several of its applications including Burnside's result.

Proof of Proposition 2. If G has prime-power order then (a) is obvious and (b) and (c) are well known. For the remainder of the proof, we assume that the order of G is divisible by at least two primes.

To establish part (a), we argue by induction on $|G|$. Write l_1 for the least prime divisor of G and let S_1 be a Sylow l_1-subgroup of G. First, we note that Burnside's Theorem implies that S_1 has a normal complement in G. To this end, observe that $S_1 \subset C(S_1)$, so that $[N(S_1) : C(S_1)]$ is not divisible by l_1. The action of $N(S_1)$ on S_1 by conjugation induces an embedding from $N(S_1)/C(S_1)$ into $\mathrm{Aut}(S_1)$. We have $|S_1| = l_1^e$ for some positive integer e. Since S_1 is cyclic, $\mathrm{Aut}(S_1)$ has order $\phi(l_1^e) = l_1^{e-1}(l_1 - 1)$. Thus $[N(S_1) : C(S_1)]$ divides $l_1 - 1$, and so $N(S_1) = C(S_1)$ (as l_1 is the least prime divisor of $|G|$). Hence, by Burnside's Theorem, $G = NS_1$ for a normal subgroup N of G such that $N \cap S_1 = \{1\}$.

Every Sylow subgroup of N is again cyclic. Our inductive hypothesis therefore implies that a Sylow l-subgroup S of N is normal in N, and thus is the unique Sylow l-subgroup of N. Now S is also a Sylow l-subgroup of G. Further, for $g \in G$, the group gSg^{-1} is a Sylow l-subgroup of N (by normality of N). Using uniqueness of S, we see that S is normal in G.

For part (b), we also use induction on $|G|$. With notation as in the proof of part (a), the quotient G/S inherits from G the property that each of its Sylow subgroups is cyclic. Thus, by our inductive hypothesis, G/S is solvable. As S is certainly solvable, it follows that G is solvable.

For part (c), suppose G is nonabelian and write Z for the centre of G. Then the nontrivial group G/Z has cyclic Sylow subgroups. Using part (a), we see that G/Z admits a nontrivial cyclic normal subgroup, say H/Z. Note that H is abelian (since H/Z is cyclic), normal in G and strictly contains Z. We've proved part (c). □

We now use Proposition 1 and Proposition 2 (c) to prove Wedderburn's theorem.

Proof. We assume that $\mathbb{D}$ is noncommutative and derive a contradiction.

By Proposition 1 and Proposition 2 (c), the group $\mathbb{D}^\times$ contains a normal abelian subgroup S that is strictly larger than $\mathbb{F}^\times$. We set $\mathbb{E} = \mathbb{F}(S)$, the subfield of $\mathbb{D}$ generated over $\mathbb{F}$ by S. Since S is normal in $\mathbb{D}^\times$, we see that $\mathbb{E}^\times$ is also normal in $\mathbb{D}^\times$.

As in the first proof, we write down two ways to complete the argument (see the last two paragraphs of [11] for yet another way).

Method 1. Since $N(\mathbb{E}^\times) = \mathbb{D}^\times$, Lemma 1 says that $C(\mathbb{E}^\times) = \mathbb{D}^\times$, and so $\mathbb{E}$ is contained in the centre $\mathbb{F}$ of $\mathbb{D}$—a contradiction.

Method 2. Recall that $|\mathbb{F}| = q$ and $|\mathbb{D}| = q^N$. As $\mathbb{E}$ strictly contains $\mathbb{F}$, we have $C(\mathbb{E}) \neq \mathbb{D}$. Thus

$$|\mathbb{E}| = q^m \text{ and } |C(\mathbb{E})| = q^n$$

for integers m and n with $m \leq n < N$. Note that n divides N: in fact, $N = ne$ where e is the dimension of $\mathbb{D}$ as a $C(\mathbb{E})$-vector space.

The action of $\mathbb{D}^\times$ on $\mathbb{E}$ by conjugation induces an embedding of groups

$$\mathbb{D}^\times/C(\mathbb{E})^\times \hookrightarrow \mathrm{Aut}(\mathbb{E}/\mathbb{F}).$$

In particular, $[\mathbb{D}^\times : C(\mathbb{E}^\times)]$ is at most $|\mathrm{Aut}(\mathbb{E}/\mathbb{F})| = m$, that is,

$$\frac{q^{ne} - 1}{q^n - 1} \leq m.$$

The left side is $q^{n(e-1)} + \cdots + q^n + 1$. Certainly $n < q^n$, and thus *a fortiori*

$$n < q^{n(e-1)} + \cdots + q^n + 1 \leq m,$$

in contradiction to $m \leq n$. □

5. Third Proof: Carter Subgroups

In 1961, R. W. Carter published a striking result [4].

Carter's Theorem. *Let G be a finite solvable group. Then G contains a nilpotent subgroup C such that $N(C) = C$. The subgroup C is unique up to conjugacy in G.*

There is an analogous statement in the theory of Lie algebras which prompted Carter's discovery. A Lie subalgebra of a Lie algebra that is nilpotent and coincides with its normalizer is called a Cartan subalgebra. Under suitable hypotheses, Cartan subalgebras exist and are unique up to a natural notion of conjugacy.[3]

Carter's Theorem, once formulated, is not difficult to prove. Indeed, it's an exercise without hints in Isaacs' text [8] (see p. 91) though the author does acknowledge that the problem may be "a bit harder than most of the problems in this book."

A nilpotent subgroup C of a group G such that $N(C) = C$ is now called a Carter subgroup of G. Non-solvable finite groups need not have Carter subgroups: for example, A_5, the smallest non-solvable group, has none. However, a Carter subgroup of an arbitrary finite group—when it exists—*is* unique up to conjugacy. This was put forward as a conjecture in 1976 and stood for around thirty years (see [15] for some of the history). It was finally proved by E. P. Vdovin as a culmination of a series of reductions and calculations based around the classification of finite simple groups—in particular, detailed properties of finite groups of Lie type [19].

Using Lemma 1 and the discussion in Section 4, we can quickly deduce the Little Theorem from Carter's result.

Proof. Let $\mathbb{D}$ be a noncommutative finite divsion ring. By Proposition 1 and Proposition 2 (b), the group $\mathbb{D}^\times$ is solvable. Let $\mathbb{E}$ be a maximal field in $\mathbb{D}$, so that $C(\mathbb{E}^\times) = \mathbb{E}^\times$. Then $N(\mathbb{E}^\times) = \mathbb{E}^\times$ by Lemma 1, and so $\mathbb{E}^\times$ is a Carter subgroup of $\mathbb{D}^\times$. Carter's theorem therefore says that the maximal subfields of $\mathbb{D}$ form a single conjugacy class (under the action of $\mathbb{D}^\times$). Since each element of $\mathbb{D}$ lies in a maximal field, it follows that $\mathbb{D}^\times$ is a union of conjugates of the proper subgroup $\mathbb{E}^\times$. Lemma 2, however, tells us that this is impossible. Thus a finite division ring is a field. □

Remark 4. We only used solvability of $\mathbb{D}^\times$ to ensure that the Carter subgroups of $\mathbb{D}^\times$ are conjugate. Can we instead appeal to conjugacy of Carter subgroups in a general finite group [19]? If so, this would yield an outrageous proof of the Little Theorem given that Vdovin's paper and its antecedents rest on the classification of finite simple groups. This approach, however, is not only wildly inefficient—it's circular: Wedderburn's theorem is a step, a tiny one, in the classification results that Vdovin uses.

6. Fourth Proof: Frobenius Groups

The final proof uses a uniqueness property of Frobenius groups. It relies on a powerful result of J. G. Thompson—we are indeed wielding a sledgehammer. For a more elementary approach to the Little Theorem via properties of Frobenius groups, see [6, 3].

Definition. A finite group G is a *Frobenius group* if it admits a nontrivial proper subgroup H such that

$$gHg^{-1} \cap H = \{1\} \text{ for all } g \in G \setminus H. \tag{4}$$

[3]Thanks to T. J. Laffey for confirming the influence on Carter of the theory of Cartan subalgebras.

Following [14], we say in this case that (G, H) is a *Frobenius pair.*

To draw out the definition, let's rephrase it in terms of the action of G by left multiplication on the space G/H of left cosets of H in G. The defining property (4) says exactly that each element of $G \setminus \{1\}$ fixes at most one point in G/H. Moreover, since H is nontrivial, some non-identity element of G fixes some point in G/H—each element of $H \setminus \{1\}$, for example, fixes the coset H. Conversely, suppose a finite group G acts transitively on a set X with $|X| > 1$ so that these properties hold, that is,

(a) no element of $G \setminus \{1\}$ fixes more than one element of X,

(b) some element of $G \setminus \{1\}$ fixes some element of X.

Let $x \in X$ and write H for the stabilizer in G of x. Then H satisfies (4) by (a), is nontrivial by (b), and is proper since $|X| > 1$. In all, (G, H) is a Frobenius pair.

In other words, a Frobenius group is a transitive permutation group on a finite set such that each nonidentity element fixes at most one point and some point *is* fixed by some nonidentity element (so that the action is not just the regular action).

There is a fascinating structure theory of Frobenius groups. For compelling accounts of part of this theory, see [8, Chap. 6] and [14, Chap. 6]. The first main result is due to Frobenius in 1901. It says that in any Frobenius pair (G, H) the elements of G that act without fixed points on G/H plus the identity element form a normal complement N to H in G, that is, $G = H \ltimes N$. The proof was an application and early triumph of the emerging theory of characters of finite groups, itself initiated by Frobenius in 1896. The group N is now called a *Frobenius kernel.*

Special properties of Frobenius groups gave rise to a conjecture that Frobenius kernels are always nilpotent. Thompson proved this conjecture in his celebrated 1959 doctoral thesis. The proof drew as much attention as the result: it hinged on a new, powerful criterion for an odd Sylow subgroup to admit a normal complement. Isaacs' book [8] includes a thorough discussion of Thompson's criterion (a slight modification of a simpler but still involved version that Thompson arrived at after his thesis) and its connection with Frobenius kernels (see Theorem 6.24 and Chapter 7). Another source, a more condensed one, is Feit's classic book [7] (see 22.2 and 25.10).

Nilpotence of Frobenius kernels leads to the following uniqueness property of Frobenius groups—the key to our final proof of the Little Theorem.

Uniqueness of Frobenius Structures. *Let (G, H_1) and (G, H_2) be Frobenius pairs in the same group G. Then H_1 and H_2 are conjugate in G.*

This is Exercise 6.6.6 in Serre's book [14] (see p. 86). Despite its label, the exercise is not in the least diabolical. In fact, Serre gives a generous hint that's effectively a solution.

We've sketched (more than) enough background to write down the fourth proof.

Proof. Let $\mathbb{D}$ be a noncommutative division ring of minimal order and let $\mathbb{E}$ be a maximal subfield of $\mathbb{D}$, so that $C(\mathbb{E}^\times) = \mathbb{E}^\times$. Then, by Lemma 1, $x\mathbb{E}^\times x^{-1} \neq \mathbb{E}^\times$ for $x \in \mathbb{D}^\times \setminus \mathbb{E}^\times$. Using minimality of $|\mathbb{D}|$, we noted at the start of Section 3 that a pair of maximal fields in $\mathbb{D}$ must intersect in the centre $\mathbb{F}$ (see equation (1) and the preceding discussion). In particular,

$$x\mathbb{E}^\times x^{-1} \cap \mathbb{E}^\times = \mathbb{F}^\times, \quad \text{for all } x \in \mathbb{D}^\times \setminus \mathbb{E}^\times.$$

Working in the quotient group $\mathbb{D}^\times/\mathbb{F}^\times$, it follows that

$$\begin{aligned} x\mathbb{F}^\times \cdot \big(\mathbb{E}^\times/\mathbb{F}^\times\big) \cdot (x\mathbb{F}^\times)^{-1} \cap \mathbb{E}^\times/\mathbb{F}^\times &= (x\mathbb{E}^\times x^{-1} \cap \mathbb{E}^\times)/\mathbb{F}^\times \\ &= \{1\}, \quad \text{for all } x \in \mathbb{D}^\times \setminus \mathbb{E}^\times. \end{aligned}$$

That is, $(\mathbb{D}^\times/\mathbb{F}^\times, \mathbb{E}^\times/\mathbb{F}^\times)$ is a Frobenius pair. Using uniqueness of Frobenius structures, we see that the maximal fields in $\mathbb{D}$ form a single conjugacy class.

To finish, we argue as in Method 1 or Method 2 of our first proof. □

7. Four Proofs of Zassenhaus's Lemma

It remains to prove Zassenhaus's lemma. As with the Little Theorem, we record four overlapping proofs. All share the same first step which is borrowed from [20]. It makes crucial use of surjectivity of the norm map for finite fields. From the theory of cyclic algebras, the Little Theorem is equivalent to this surjectivity property, so it's not a surprise that it lies at the base of the arguments below.

As noted earlier, this section owes much to the suggestions of a reviewer of a pre-Bulletin version of the paper. In particular, the fourth proof is a variant of an argument provided by this reviewer. The third proof is close to Zassenhaus's original argument.

For convenience, we recall the statement of the lemma.

Lemma 1. *If $\mathbb{E}$ is a subfield of a finite division ring $\mathbb{D}$, then the normalizer and centralizer of $\mathbb{E}^\times$ in $\mathbb{D}^\times$ coincide.*

Proof. Suppose $N(\mathbb{E}^\times) \neq C(\mathbb{E}^\times)$ for some subfield $\mathbb{E}$ of the finite division ring $\mathbb{D}$. Choose $\beta \in N(\mathbb{E}^\times)$ with $\beta \notin C(\mathbb{E}^\times)$.

Let l be the least positive integer such that $\beta^l \in C(\mathbb{E}^\times)$ and set $\mathbb{K} = \langle \mathbb{E}, \beta^l \rangle$, the subring of $\mathbb{D}$ generated by $\mathbb{E}$ and β^l. Note that $\mathbb{K}$ is a field since $\beta^l \in C(\mathbb{E}^\times)$. We write σ for the automorphism of $\mathbb{K}$ given by conjugation by β, that is,

$$\sigma(\lambda) = \beta\lambda\beta^{-1}, \ \ \lambda \in \mathbb{K}.$$

Observe that σ has order l.

Let $\mathbb{D}_1 = \langle \mathbb{K}, \beta \rangle$, the subring of $\mathbb{D}$ generated by $\mathbb{K}$ and β. For later use, note that

$$\beta^i\lambda = \sigma^i(\lambda)\beta^i, \quad i \in \mathbb{Z}, \, \lambda \in \mathbb{K}. \tag{5}$$

We put $\mathbb{K}_1 = \mathbb{K}^\sigma$, the fixed field of σ. Thus the automorphism group of the extension $\mathbb{K}/\mathbb{K}_1$ is generated by σ and $[\mathbb{K} : \mathbb{K}_1] = l$. Let N denote the norm map from $\mathbb{K}$ to $\mathbb{K}_1$, so that

$$N(\alpha) = \alpha\sigma(\alpha)\sigma^2(\alpha)\cdots\sigma^{l-1}(\alpha), \quad \alpha \in \mathbb{K}.$$

Now, for $\alpha \in \mathbb{K}$,

$$\begin{aligned}(\alpha\beta)^l &= \alpha\beta \cdot \alpha\beta \cdots \alpha\beta \quad (l \text{ terms})\\ &= \alpha \cdot \beta\alpha\beta^{-1} \cdot \beta^2\alpha\beta^{-2} \cdots \beta^{l-1}\alpha\beta^{-(l-1)} \cdot \beta^l\\ &= \alpha\sigma(\alpha)\sigma^2(\alpha)\cdots\sigma^{l-1}(\alpha)\beta^l\\ &= N(\alpha)\beta^l.\end{aligned}$$

The element $\beta^l \in \mathbb{K}$ is visibly fixed under conjugation by β, that is, $\beta^l \in \mathbb{K}_1$. Since norm maps are surjective for finite fields, we can choose $\alpha \in \mathbb{K}$ such that $N(\alpha) = \beta^{-l}$, equivalently $(\alpha\beta)^l = 1$. Thus we can and *do* adjust the element β so that $\beta^l = 1$. We still have $\mathbb{D}_1 = \langle \mathbb{K}, \beta \rangle$ and the automorphism σ of $\mathbb{K}$ is still given by conjugation by β.

We rewrite $\beta^l = 1$ as $(1-\beta)(1+\beta+\cdots+\beta^{l-1}) = 0$. Since $\beta \neq 1$, we have

$$1 + \beta + \cdots + \beta^{l-1} = 0, \tag{6}$$

and so

$$\lambda + \beta\lambda + \cdots + \beta^{l-1}\lambda = 0, \quad \text{for all } \lambda \in \mathbb{K}.$$

Equivalently, via (5),

$$\lambda + \sigma(\lambda)\beta + \cdots + \sigma^{l-1}(\lambda)\beta^{l-1} = 0, \quad \text{for all } \lambda \in \mathbb{K}. \tag{7}$$

From here, we write down four ways to complete the proof.

Method 1. Fix $\zeta \in \mathbb{K}$ such that $\mathbb{K} = \mathbb{K}_1(\zeta)$. Substituting $\zeta^0 = 1, \zeta, \ldots, \zeta^{l-1}$ in (7) gives

$$\zeta^i + \sigma(\zeta)^i\beta + \cdots + \sigma^{l-1}(\zeta)^i\beta^{l-1} = 0, \quad \text{for } i = 0, 1, \ldots, l-1. \tag{8}$$

We can rewrite these l equations as a single matrix equation

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \zeta & \sigma(\zeta) & \cdots & \sigma^{l-1}(\zeta) \\ \vdots & \vdots & \ddots & \vdots \\ \zeta^{l-1} & \sigma(\zeta)^{l-1} & \cdots & \sigma^{l-1}(\zeta)^{l-1} \end{bmatrix} \begin{bmatrix} 1 \\ \beta \\ \vdots \\ \beta^{l-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{9}$$

Thus the square matrix, say M, is singular. On the other hand, it's of Vandermonde form, so

$$\det M = \prod_{0 \le i < j < l} \big(\sigma^j(\zeta) - \sigma^i(\zeta)\big)$$

is nonzero. This contradiction proves the lemma.

Method 2. We derive a contradiction from (7) by a slightly different matrix argument.

Write $\mathrm{tr} : \mathbb{K} \to \mathbb{K}_1$ for the trace map of the extension $\mathbb{K}/\mathbb{K}_1$, so that

$$\mathrm{tr}(\lambda) = \lambda + \sigma(\lambda) + \cdots + \sigma^{l-1}(\lambda), \quad \lambda \in \mathbb{K}.$$

Since $\mathbb{K}/\mathbb{K}_1$ is separable, the symmetric bilinear form

$$(\lambda, \mu) \mapsto \mathrm{tr}(\lambda\mu) : \mathbb{K} \times \mathbb{K} \to \mathbb{K}_1$$

is nondegenerate. Fixing a $\mathbb{K}_1$-basis $\lambda_1, \ldots, \lambda_l$ of $\mathbb{K}$, the matrix $\big[\mathrm{tr}(\lambda_i\lambda_j)\big]$ is therefore invertible.

Substituting the basis elements $\lambda_1, \ldots, \lambda_l$ in (7), we have

$$\lambda_i + \sigma(\lambda_i)\beta + \cdots + \sigma^{l-1}(\lambda_i)\beta^{l-1} = 0, \quad \text{for } i = 1, \ldots, l. \tag{10}$$

As in Method 1, we gather these l equations into a single matrix equation

$$\begin{bmatrix} \lambda_1 & \sigma(\lambda_1) & \cdots & \sigma^{l-1}(\lambda_1) \\ \lambda_2 & \sigma(\lambda_2) & \cdots & \sigma^{l-1}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_l & \sigma(\lambda_l) & \cdots & \sigma^{l-1}(\lambda_l) \end{bmatrix} \begin{bmatrix} 1 \\ \beta \\ \vdots \\ \beta^{l-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{11}$$

Writing M for the given square matrix and $M^\top$ for its transpose, the product $MM^\top$ has ij entry

$$\lambda_i\lambda_j + \sigma(\lambda_i\lambda_j) + \cdots + \sigma^{l-1}(\lambda_i\lambda_j) = \mathrm{tr}(\lambda_i\lambda_j).$$

Thus $MM^\top$ is invertible, so M is invertible, in contradiction to (11).

Method 3. Using (5) and that $\beta^l \in \mathbb{K}$, we see that

$$\mathbb{D}_1 = \langle \mathbb{K}, \beta \rangle = \sum_{i=0}^{l-1} \mathbb{K}\beta^i. \tag{12}$$

We claim that the sum is direct. Note that our claim means that (6) cannot hold and thus establishes the lemma.

The claim follows from a powerful technique that goes back to Dedekind, often called "linear independence of characters." In detail, suppose we have a dependence relation

$$\lambda_1\beta^{i_1} + \cdots + \lambda_r\beta^{i_r} = 0 \tag{13}$$

for $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$ and $0 \le i_1 < \cdots < i_r < l$. In search of a contradiction, we assume that r is minimal, so that each $\lambda_i \neq 0$.

Multiplying (13) on the right by $\gamma \in \mathbb{K}$ and rewriting via (5), we have

$$\lambda_1\sigma^{i_1}(\gamma)\beta^{i_1} + \cdots + \lambda_r\sigma^{i_r}(\gamma)\beta^{i_r} = 0, \quad \gamma \in \mathbb{K}. \tag{14}$$

Choose $\gamma_1 \in \mathbb{K}$ such that $\sigma^{i_1}(\gamma_1) \neq \sigma^{i_2}(\gamma_1)$. Replacing γ by $\gamma_1\gamma$ in (14) gives

$$\lambda_1\sigma^{i_1}(\gamma_1)\sigma^{i_1}(\gamma)\beta^{i_1} + \lambda_2\sigma^{i_2}(\gamma_1)\sigma^{i_2}(\gamma)\beta^{i_2} + \cdots + \lambda_r\sigma^{i_r}(\gamma_1)\sigma^{i_r}(\gamma)\beta^{i_r} = 0.$$

On the other hand, multiplying (14) on the left by $\sigma^{i_1}(\gamma_1)$ gives

$$\lambda_1\sigma^{i_1}(\gamma_1)\sigma^{i_1}(\gamma)\beta^{i_1} + \lambda_2\sigma^{i_1}(\gamma_1)\sigma^{i_2}(\gamma)\beta^{i_2} + \cdots + \lambda_r\sigma^{i_1}(\gamma_1)\sigma^{i_r}(\gamma)\beta^{i_r} = 0.$$

Subtracting the last equation from the previous one, we see that

$$\lambda_2\left(\sigma^{i_2}(\gamma_1) - \sigma^{i_1}(\gamma_1)\right)\sigma^{i_2}(\gamma)\beta^{i_2} + \cdots + \lambda_r\left(\sigma^{i_r}(\gamma_1) - \sigma^{i_1}(\gamma_1)\right)\sigma^{i_r}(\gamma)\beta^{i_r} = 0.$$

By our choice of γ_1, the coefficient of β^{i_2} is nonzero (for γ nonzero) which contradicts minimality of r in (13). Therefore (12) is a direct sum and we've proved the lemma once more.

Method 4. Consider $\mathbb{K}$ as a $\mathbb{K}_1$-vector space. The field $\mathbb{K}$ embeds in $\mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$ via $m : \mathbb{K} \to \mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$ where $m(\lambda)(\mu) = \lambda\mu$ (for $\lambda, \mu \in \mathbb{K}$). We identify $\mathbb{K}$ with its image under m. That is, for $\lambda \in \mathbb{K}$, we simply write λ for the map $m(\lambda) \in \mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$. We also have $\sigma \in \mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$. In parallel to (5), these various elements of $\mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$ satisfy

$$\sigma^i\lambda = \sigma^i(\lambda)\sigma^i, \quad i \in \mathbb{Z},\ \lambda \in \mathbb{K}, \tag{15}$$

Now, by Dedekind's lemma (linear independence of characters), the sum

$$\sum_{i=0}^{l-1} \mathbb{K}\sigma^i \subseteq \mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$$

is direct, and hence has dimension l^2 over $\mathbb{K}_1$. The containment is therefore an equality. That is, each element of $\mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$ admits a unique expression as $\sum_{i=0}^{l-1}\lambda_i\sigma^i$ with each $\lambda_i \in \mathbb{K}$. Using the same notation, we see that there is a well-defined map of $\mathbb{K}$-vector spaces

$$\sum_{i=0}^{l-1}\lambda_i\sigma^i \longmapsto \sum_{i=0}^{l-1}\lambda_i\beta^i : \mathrm{End}_{\mathbb{K}_1}(\mathbb{K}) \longrightarrow \mathbb{D}_1. \tag{16}$$

We claim that the map is a ring homomorphism. Note that the lemma follows. Indeed, given the claim, the kernel of (16) is a two-sided ideal in the simple ring $\mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$ and hence is trivial. This means that (16) is injective and so the division ring $\mathbb{D}_1$ contains a copy of the matrix ring $\mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$—a contradiction. Alternatively, the map is surjective (its image, for example, contains $\mathbb{K}$ and β). Using simplicity of $\mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$ again, the rings $\mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$ and $\mathbb{D}_1$ must be isomorphic which is absurd.

To establish the claim, we appeal to the following principle whose proof is immediate.

(α) *Let R and S be rings and let $\phi : R \to S$ be a homomorphism of abelian groups. Suppose R is generated as an abelian group by a subset Γ and that*

$$\phi(\gamma_1\gamma_2) = \phi(\gamma_1)\phi(\gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. Then ϕ is a homomorphism of rings.

From our discussion, $\mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$ is generated as an abelian group by the elements $\lambda\sigma^i$ for $\lambda \in \mathbb{K}$ and $i \in \mathbb{Z}$. Writing ϕ for the map in (16) and using (α), we see that we only have to check one family of relations:

$$\phi(\lambda\sigma^i\mu\sigma^j) = \phi(\lambda\sigma^i)\phi(\mu\sigma^j), \qquad \lambda, \mu \in \mathbb{K},\ i, j \in \mathbb{Z}.$$

By (15), $\lambda\sigma^i\mu\sigma^j = \lambda\sigma^i(\mu)\sigma^{i+j}$, and so the left side is $\lambda\sigma^i(\mu)\beta^{i+j}$. By (5), the right side is

$$\lambda\beta^i\mu\beta^j = \lambda\sigma^i(\mu)\beta^{i+j}.$$

Thus (16) is a ring homomorphism and the (fourth) proof is complete. □

Remark 5. Dedekind's lemma (linear independence of characters) is at the heart of Zassenhaus's lemma. Its central role in Methods 3 and 4 is evident. For Methods 1 and 2, the key is that the square matrices in equations (9) and (11) are invertible—and invertibility of these matrices is a quick consequence of Dedekind's lemma (a preferable approach perhaps to the more computational one used above). To check the implication, write C_i for the ith column of either matrix and suppose there exist $\gamma_1, \ldots, \gamma_l \in \mathbb{K}$ such that

$$\gamma_1 C_1 + \cdots + \gamma_l C_l = 0,$$

the zero column vector. In each case, the relation says that $\sum_{i=0}^{l-1} \gamma_i \sigma^i \in \mathrm{End}_{\mathbb{K}_1}(\mathbb{K})$ vanishes on a $\mathbb{K}_1$-basis of $\mathbb{K}$. Thus $\sum_{i=0}^{l-1} \gamma_i \sigma^i = 0$, so that $\gamma_i = 0$ for all i, and the matrices are indeed invertible via Dedekind's lemma. Noether's motto, quoted in [16], was "it is already all in Dedekind."

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On Groups whose Squares are Subgroups

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Abstract. Let G be a group. The square of G is the set G^2 consisting of elements of the form g^2, where $g \in G$. If G^2 is a subgroup of G, we say that G has the square subgroup property. In this article, we study several conditions related to the square subgroup property, including the order of G when it is a finite group. We also provide several examples of groups with or without the square subgroup property.

1. Introduction

Let G be a group. We define the square of G by

$$G^2 = \{x^2 \ ; \ x \in G\}.$$

A natural question is whether G^2 is a subgroup of G. If this happens, we say that G has the square subgroup property. This problem has been studied in [2][4]. In particular, for G finite, [4] provides a sufficient condition as follows. Let $|G|$ be its order.

Theorem 1.1. ([4], Thms.1.1,2.1,2.5)
If $|G|$ is odd, or G is abelian or a dihedral group, then G^2 is a subgroup of G.

Theorem 1.1 does not cover the cases where $|G|$ is even, and does not provide examples where G^2 is not a subgroup. We address these issues later in Theorems 1.3 and 1.4.

A property of G^2 is provided by its comparison with the commutator subgroup, as given by the next theorem. Recall that the commutator subgroup of G is defined by

$$G' = \text{subgroup of } G \text{ generated by } \{xyx^{-1}y^{-1} \ ; \ x, y \in G\}.$$

Theorem 1.2. *If G^2 is a subgroup of G, then $G' \subset G^2$.*

While Theorem 1.1 provides examples of groups with the square subgroup property, we shall construct several examples which do not have this property. Let S_n denote the symmetric group. Let $A_n \subset S_n$ denote the alternating group, namely A_n consists of all even permutations. If q is a power of a prime number (for example $2^2, 2^3, 3^2$), up to isomorphism there is a unique finite field with q elements, and we denote it by $\mathbb{F}_q$. Let $\mathrm{SL}_n(\mathbb{F}_q)$ denote the $n \times n$ matrices with entries in $\mathbb{F}_q$ and with determinant 1.

Theorem 1.3. *If G is one of the following, then G^2 is not a subgroup of G: non-abelian simple group, $A_n (n \geq 4)$, $S_n (n \geq 6)$, $\mathrm{SL}_2(\mathbb{F}_q)(q > 3)$, $\mathrm{SL}_n(\mathbb{F}_q)(n \geq 3)$.*

Since Theorem 1.1 says that all groups of odd orders have the square subgroup property, it remains to consider the groups of even orders. This is answered by the next theorem.

Theorem 1.4. *Let $n \in \mathbb{N}$. The following conditions are equivalent:*
(a) *Every finite group of order n has the square subgroup property.*
(b) *$n \leq 8$ or 4 does not divide n.*

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For any $n > 8$ divisible by 4, the proof of Theorem 1.4 constructs a group of order n which does not have the square subgroup property. However, it is still far from exhausting all such groups. Therefore, it leads to an interesting problem for future study: For any $n > 8$ divisible by 4, find all the groups of order n which do not have the square subgroup property.

While the above discussions focus on finite groups, it is natural to study this problem for infinite groups as well, and yet there has been no existing work on it. We now study some examples of infinite groups. If $G = H \times K$ is a direct product of groups, then G^2 is a subgroup of G if and only if H^2 and K^2 are respectively subgroups of H and K. With this in mind, we obtain many examples of infinite groups G such that G^2 is not a subgroup. For instance if H is any one of the groups in Theorem 1.3, and $G = H \times \mathbb{R}^n$ is the direct product, then G^2 is not a subgroup of G. Conversely, the next theorem provides an example where G^2 is a subgroup. Let $\mathrm{GL}_n(\mathbb{C})$ be the multiplicative group of all $n \times n$ invertible complex matrices.

Theorem 1.5. *Let* $G = \mathrm{GL}_n(\mathbb{C})$. *Then* $G^2 = G$.

The main theorems are proved in the sections as follows.
Section 2: Theorem 1.2
Section 3: Theorem 1.3
Section 4: Theorem 1.4
Section 5: Theorem 1.5
We also prove several other results along the way, including Proposition 4.1, Proposition 4.5, and Theorem 4.6.

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2. Commutator Subgroups

In this section, we prove Theorem 1.2. Let $C = \{xyx^{-1}y^{-1}\ ;\ x, y \in G\}$ consists of the commutators of G, and it generates the commutator subgroup $G' = \langle C\rangle$. The conditions for $C = \langle C\rangle$ is more complicated than $G^2 = \langle G^2\rangle$. Nevertheless, Theorem 1.2 says that when the latter occurs, then G' is a subgroup of G^2.

Proof of Theorem 1.2:

Suppose that G^2 is a subgroup of G. For given $g \in G$ and $x \in G^2$, we write $x = a^2$ for some $a \in G$. Then $gxg^{-1} = ga^2g^{-1} = (gag^{-1})^2 \in G^2$. This implies that G^2 is a normal subgroup of G, so G/G^2 is a group.

For $g \in G$, we let $\overline{g} \in G/G^2$ denote the corresponding quotient element. For any $\overline{g_1}, \overline{g_2} \in G/G^2$, we have $\overline{g_1}^2\overline{g_2}^2 = \overline{e}\overline{e} = (\overline{g_1g_2})^2 = \overline{g_1g_2g_1g_2}$. So $\overline{g_1g_2} = \overline{g_2g_1}$, which implies that G/G^2 is abelian.

Let $g_1, g_2 \in G$. Since G/G^2 is abelian, $\overline{g_1g_2} = \overline{g_2g_1}$, so $\overline{g_1g_2}(\overline{g_2g_1})^{-1} = \overline{e}$. It follows that $g_1g_2g_1^{-1}g_2^{-1} \in G^2$. Hence G^2 contains all the commutators of G, namely $G' \subset G^2$. □

3. Square Subgroup Property

Let G be a finite group. If the square set G^2 is a subgroup of G, we say that G has the square subgroup property. In this section, we prove Theorem 1.3, which provides several families of examples where G does not have the square subgroup property.

The next proposition shows that the square subgroup property is preserved when taking quotients and direct sums.

Proposition 3.1. [4, Thm.2.2]
(a) *Suppose that* G^2 *is a subgroup of* G*, and* N *is a normal subgroup of* G*. Then* $(G/N)^2$ *is a subgroup of* G/N*.*
(b) *Let* $G = \oplus G_\alpha$ *be a direct sum of groups. Then* G^2 *is a subgroup of* G *if and only if each* G_α^2 *is a subgroup of* G_α*.*

Contrary to Proposition 3.1, the square subgroup property of G is not preserved when taking a subgroup H. This is illustrated by the following example. Recall that A_n consists of the even permutations of the symmetric group S_n.

Example 3.2. Let $G = S_4$ and $H = A_4$. We shall show that G has the square subgroup property, but H does not have the square subgroup property. We first claim that

$$G^2 = H. \tag{3.1}$$

It is clear that $G^2 \subset H$. Conversely, we consider

$$H = \{e\} \cup \{(a\ b)(c\ d)\} \cup \{(a\ b\ c)\}, \tag{3.2}$$

where $\{a, b, c, d\} = \{1, 2, 3, 4\}$. If $\sigma = (a\ b)(c\ d)$, then $\sigma = (a\ c\ b\ d)^2 \in G^2$. If $\sigma = (a\ b\ c)$, then $\sigma = (\sigma^2)^2 \in G^2$. Hence $H \subset G^2$, which proves (3.1) as claimed. So G^2 is a subgroup of G.

If $\sigma = (a\ b)(c\ d)$, then $\sigma^2 = e$. So (3.2) implies that $H^2 = \{e\} \cup \{(a\ b\ c)^2\}$, namely $|H^2| = 1 + 8 = 9$. This is not a factor of $|H| = 12$, so by Lagrange's theorem, H^2 is not a subgroup of H.

Recall that G is said to be solvable if there exists a chain of subgroups $\{e\} = H_1 \subset ... \subset H_n = G$ such that each H_i is a normal subgroup of H_{i+1}, and H_{i+1}/H_i is abelian. We will need the following lemma.

Lemma 3.3. *Every non-trivial finite solvable simple group is a cyclic group of prime order.*

Proof. Let G be a finite solvable simple group. Let $\{e\} = H_1 \subset ... \subset H_n = G$ be the chain of subgroups described above. Replacing n by a smaller number if necessary, we may assume that $H_{n-1} \neq G$. Since G is simple, it implies that $H_{n-1} = \{e\}$, so $G \cong G/H_{n-1}$ is abelian. Since a finite abelian simple group is a cyclic group of prime order, the lemma follows. □

The next theorem illustrates many examples where G do not have the square subgroup property. They include most of the finite simple groups and alternating groups.

Theorem 3.4.
(a) *Let* G *be a finite simple group, and suppose that* G^2 *is a subgroup of* G*. Then* G *is a cyclic group of prime order.*
(b) $(A_n)^2$ *is a subgroup of* A_n *if and only if* $n \leq 3$*.*

Proof. Let G be a finite simple group, and let G^2 be a subgroup of G. For all $x = a^2 \in G^2$ and $g \in G$, we have $gxg^{-1} = ga^2g^{-1} = (gag^{-1})^2 \in G^2$, hence G^2 is a normal subgroup of G. Since G is simple, we have $G^2 = \{e\}$ or $G^2 = G$, and we discuss them separately.

Suppose that $G^2 = \{e\}$. For all $a, b \in G$, $a^2b^2 = e \cdot e = e = (ab)^2$, so $ab = ba$. Therefore, G is abelian. By Lemma 3.3, G is a cyclic group of prime order.

Next suppose that $G^2 = G$. Then the mapping $f : G \longrightarrow G$ given by $f(g) = g^2$ is surjective. Since G is finite, f is bijective. If $|G|$ is even, then by Cauchy's theorem, G has an element of order 2, which contradicts the fact that f is bijective. So $|G|$ is odd. By the Feit-Thompson theorem (see for instance [1, p.104-106 Exercise12]), this implies that G is solvable. By Lemma 3.3, G is a cyclic group of prime order. This proves Theorem 3.4(a).

Next we prove Theorem 3.4(b). For $n \geq 5$, A_n is a non-abelian finite simple group. By Theorem 3.4(a), $(A_n)^2$ is not a subgroup of A_n. For $n = 4$, by Example 3.2, $(A_4)^2$ is not a subgroup of A_4. For $n \leq 3$, it is straightforward to check that $(A_n)^2$ is a subgroup of A_n. This proves Theorem 3.4(b). □

Our next objective is to show that for n large enough, $(S_n)^2$ is not a subgroup of S_n. We first state a useful lemma. We omit its proof, which is straightforward.

Lemma 3.5. *Let $\sigma = (a_1 \dots a_m) \in S_n$ be an m-cycle. Then*

$$\sigma^2 = \begin{cases} (a_1\, a_3 \dots a_m\, a_2\, a_4 \dots a_{m-1}) & \text{if } m \text{ is odd,} \\ (a_1\, a_3 \dots a_{m-1})(a_2\, a_4 \dots a_m) & \text{if } m \text{ is even.} \end{cases}$$

For $x \in \mathbb{R}$, let $\lfloor x \rfloor \in \mathbb{Z}$ denote the largest integer such that $\lfloor x \rfloor \leq x$.

Theorem 3.6. *$(S_n)^2$ is a subgroup of S_n if and only if $n \leq 5$.*

Proof. Let $n \geq 6$. Assume that $(S_n)^2$ is a subgroup of S_n, and we shall derive a contradiction. There exists $k \in \mathbb{N}$ such that $\lfloor \frac{n}{2} \rfloor < 2k \leq n - 2$. Since $(S_n)^2$ is a subgroup of S_n, it is a normal subgroup, and hence it is one of $\{e\}$, A_n or S_n. But clearly $(S_n)^2$ cannot be $\{e\}$ or S_n. We obtain the remaining possibility

$$(S_n)^2 = A_n. \tag{3.3}$$

Consider $\sigma = (1\ 2 \dots 2k)(2k+1\ 2k+2) \in S_n$. Note that $\sigma \in A_n$, so by (3.3), $\sigma = \tau^2$ for some $\tau \in S_n$. Let $\tau = \tau_1\tau_2 \dots \tau_s$ be a cyclic decomposition, where τ_i has length l_i. We have $(1\ 2 \dots 2k)(2k+1\ 2k+2) = \sigma = \tau_1^2 \dots \tau_s^2$. By Lemma 3.5, there exists j such that $l_j > 2k$, for otherwise there is no $(2k)$-cycle in τ^2, a contradiction. By Lemma 3.5, $l_j = 4k \geq 2(\lfloor \frac{n}{2} \rfloor + 1) > n$, which is a contradiction. We have shown that $(S_n)^2$ is not a subgroup of S_n for $n \geq 6$.

Next we consider $n \leq 5$. By direct computations, $(S_1)^2 = (S_2)^2 = \{e\}$, and $(S_3)^2 = A_3$. Also, Example 3.2 shows that $(S_4)^2 = A_4$. It remains to show that

$$(S_5)^2 = A_5. \tag{3.4}$$

Clearly $(S_5)^2 \subset A_5$. Conversely, pick $e \neq \sigma \in A_5$. Then σ is one of $(a\ b\ c)$, $(a\ b)(c\ d)$, $(a\ b\ c\ d\ e)$. The cases $(a\ b\ c), (a\ b)(c\ d)$ are treated in Example 3.2. If $\sigma = (a\ b\ c\ d\ e)$, then $\sigma = (\sigma^3)^2 \in (S_5)^2$. This proves (3.4). We have completed the proof of Theorem 3.6. □

Let F be a finite field. Let $\mathrm{SL}_n(F)$ be the $n \times n$ matrices with entries in F and with determinant 1, let $Z(\mathrm{SL}_n(F))$ be its center, and let

$$\mathrm{PSL}_n(F) = \mathrm{SL}_n(F)/Z(\mathrm{SL}_n(F)).$$

Recall that if q is a power of a prime number, we let $\mathbb{F}_q$ denote the unique finite field with q elements.

Lemma 3.7.
(a) $\mathrm{PSL}_2(\mathbb{F}_q)$ *is simple if and only if $q > 3$.*
(b) *If $n \geq 3$ and F is a finite field, then* $\mathrm{PSL}_n(F)$ *is simple.*

Proof. Lemma 3.7(a) is due to Jordan-Moore [3, Thm.8.13], and Lemma 3.7(b) is due to Jordan-Dickson [3, Thm.8.23]. □

The above lemma enables us to construct more examples of finite groups which do not have the square subgroup property.

Theorem 3.8. *Let q be a power of a prime number.*
(a) $(\mathrm{SL}_2(\mathbb{F}_q))^2$ *is not a subgroup of* $\mathrm{SL}_2(\mathbb{F}_q)$ *if* $q > 3$.
(b) *If* $n \geq 3$, *then* $(\mathrm{SL}_n(\mathbb{F}_q))^2$ *is not a subgroup of* $\mathrm{SL}_n(\mathbb{F}_q)$.
(c) $(\mathrm{GL}_n(\mathbb{F}_2))^2$ *is a subgroup of* $\mathrm{GL}_n(\mathbb{F}_2)$ *if and only if* $n \leq 2$.

Proof. Assume that $(\mathrm{SL}_2(\mathbb{F}_q))^2$ is a subgroup of $\mathrm{SL}_2(\mathbb{F}_q)$ for some $q > 3$. By Proposition 3.1(a), $(\mathrm{PSL}_2(\mathbb{F}_q))^2$ is a subgroup of $\mathrm{PSL}_2(\mathbb{F}_q)$. But by Lemma 3.7(a), $\mathrm{PSL}_2(\mathbb{F}_q)$ is a finite simple group, so by Theorem 3.4(a), it is cyclic of prime order. This is a contradiction, and we have proved Theorem 3.8(a).

Assume that $(\mathrm{SL}_n(\mathbb{F}_q))^2$ is a subgroup of $\mathrm{SL}_n(\mathbb{F}_q)$ for some $n \geq 3$. By Proposition 3.1(a), $(\mathrm{PSL}_n(\mathbb{F}_q))^2$ is a subgroup of $\mathrm{PSL}_n(\mathbb{F}_q)$. But by Lemma 3.7(b), $\mathrm{PSL}_n(\mathbb{F}_q)$ is a finite simple group, so by Theorem 3.4(a), it is cyclic of prime order. This is a contradiction, and we have proved Theorem 3.8(b).

Clearly $\mathrm{GL}_n(\mathbb{F}_2) = \mathrm{SL}_n(\mathbb{F}_2)$. For $n \geq 3$, $(\mathrm{SL}_n(\mathbb{F}_2))^2$ is not a subgroup of $\mathrm{SL}_n(\mathbb{F}_2)$ by Theorem 3.8(b). For $n = 2$, $\mathrm{SL}_2(\mathbb{F}_2) \cong S_3$, so $(\mathrm{SL}_2(\mathbb{F}_2))^2$ is a subgroup of $\mathrm{SL}_2(\mathbb{F}_2)$ by Theorem 3.6. The case of $n = 1$ is trivial. We have proved Theorem 3.8(c). □

Proof of Theorem 1.3:
We have shown that the following finite groups do not have square subgroup property.
Non-abelian simple groups: Theorem 3.4(a),
$A_n (n \geq 4)$: Theorem 3.4(b),
$S_n (n \geq 6)$: Theorem 3.6,
$\mathrm{SL}_2(\mathbb{F}_q)(q > 3)$, $\mathrm{SL}_n(\mathbb{F}_q)(n \geq 3)$: Theorem 3.8(a,b). □

4. Orders of Groups

In this section, we prove Theorem 1.4. Recall that G is said to have the square subgroup property if G^2 is a subgroup of G.

Proposition 4.1. *If $|G| = 2k$ where k is odd, then G^2 is a subgroup of G.*

Proof. Suppose that $|G| = 2k$, where k is odd. Then G contains a subgroup H of index 2 (see p.122, Ex.13 of [1]). We claim that

$$H = H^2 \subset G^2. \tag{4.1}$$

To obtain (4.1), the only thing to prove is $H \subset H^2$. Since $|H| = k$ is odd, we have $k + 1 = 2r$ for some $r \in \mathbb{N}$. Pick $x \in H$. We have $x = xx^k = (x^r)^2 \in H^2$. This proves that $H \subset H^2$, which implies (4.1) as claimed.

Conversely, we claim that

$$G^2 \subset H. \tag{4.2}$$

Pick $x = g^2 \in G^2$, where $g \in G$. Since H is of index 2 in G, H is normal in G, so G/H is a group. Now $xH = g^2H = (gH)^2 = eH$, so $x \in H$. This proves (4.2) as claimed. By (4.1) and (4.2), we have $G^2 = H$, so G^2 is a subgroup of G. □

Theorem 4.2. *The smallest finite group which does not have the square subgroup property has order 12.*

Proof. Suppose that G is a group, and $|G| < 12$. We consider the following two cases.
Case 1: $|G|$ is a multiple of 4.

If $|G| = 4$, then G is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $|G| = 8$, then G is isomorphic to one of the following,

$$\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, Q_8, \tag{4.3}$$

where D_8 is the dihedral group and Q_8 is the quaternion group. Each of the above groups has the square subgroup property.

Case 2: $|G|$ is not a multiple of 4.

In this case by Theorem 1.1 and Proposition 4.1, G has the square subgroup property.

We have shown that if $|G| < 12$, then G has the square subgroup property. On the other hand $|A_4| = 12$, and by Example 3.2, A_4 does not have the square subgroup property. This proves the theorem. $\square$

As an application of our study on square subgroup property, it leads to the following well-known result in finite group theory.

Corollary 4.3. *If G is a non-abelian finite simple group, then $|G|$ is a multiple of 4.*

Proof. Since G is nonabelian, it cannot be cyclic. By Theorem 4.1(a), G does not have the square subgroup property. By Theorem 1.1 and Proposition 3.4, 4 divides $|G|$. $\square$

In view of Theorem 1.1 and Proposition 3.4, it remains to consider groups of order divisible by 4. We focus on such groups for the rest of this section.

Proposition 4.4. *There exists a finite group G of order 16 which does not have the square subgroup property.*

Proof. Let $G = \langle a, b \,;\, a^4 = b^4 = e, ba = a^{-1}b\rangle$. Here G is a nonabelian group of order 16, and $G \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_4$.

We claim that

$$G^2 = \{e, a^2, b^2\}. \tag{4.4}$$

It is clear that $\{e, a^2, b^2\} \subset G^2$, and it remains to prove the opposite inclusion. From $ba = a^{-1}b$, we have $ba^i = a^{-i}b$ for all $i \in \mathbb{Z}$. By induction,

$$b^j a^i = a^{(-1)^j i} b^j \tag{4.5}$$

for all $i, j \in \mathbb{Z}$. Suppose that $x \in G^2$. Then $x = g^2$ for some $g \in G$. Assume that $g = a^r b^s$ for some $r, s \in \mathbb{Z}$. By (4.5),

$$x = g^2 = (a^r b^s)^2 = a^r b^s a^r b^s = a^r a^{(-1)^s r} b^s b^s = a^{r+(-1)^s r} b^{2s} = \begin{cases} a^{2r} & \text{if } s \text{ is even,} \\ b^{2s} & \text{if } s \text{ is odd.} \end{cases}$$

Hence $x \in \{e, a^2, b^2\}$, which proves (4.4). By (4.4), $|G^2| = 3$, which is not a factor of 16. So G^2 is not a subgroup of G. $\square$

The finite 2-groups form a family of groups whose orders are divisible by 4. The next proposition constructs finite 2-groups that do not have the square subgroup property.

Proposition 4.5. *For any $n \geq 4$, there exists a finite group G of order 2^n such that G^2 is not a subgroup of G.*

Proof. Let $n \geq 4$. Let H be the group of order 16 constructed in Proposition 4.4. Let $G = H \times \mathbb{Z}_{2^{n-4}}$. By Proposition 3.1(b), G^2 is not a subgroup of G. $\square$

We note that Proposition 4.5 fails for $n = 3$. Up to isomorphism, there are exactly five groups of order $2^3 = 8$ (see (4.3)), and they all have the square subgroup property. Therefore, the smallest 2-group which does not have the square subgroup property has order 16.

Let $n \geq 2$. We define the dicyclic group of order $4n$ by

$$\mathrm{Dic}_n = \langle a, x \,;\, a^{2n} = e, x^2 = a^n, x^{-1}ax = a^{-1}\rangle.$$

Every element of Dic_n is uniquely of the form $a^k x^j$, where $k = 0, 1, ...2n-1$ and $j = 0, 1$.

Theorem 4.6. *Let $n \geq 2$. Then Dic_n has the square subgroup property if and only if n is even.*

Proof. Let $G = \mathrm{Dic}_n$. We first claim that

$$G^2 = \{a^{2m}\ ;\ m = 0, 1, ..., n-1\} \cup \{a^n\}. \tag{4.6}$$

Let $T = \{a^{2m}\ ;\ m = 0, 1, ..., n-1\} \cup \{a^n\}$. Clearly $a^{2m} \in G^2$ for all $m = 0, 1, ..., n-1$. Also, $a^n = x^2 \in G^2$. Hence $T \subset G^2$. Conversely, we consider $g^2 \in G^2$. Write $g = a^k x^j$, where $k = 0, 1, ..., 2n-1$ and $j = 0, 1$. If $j = 0$, then $g^2 = a^{2k} \in T$. If $j = 1$, then $g^2 = a^k x a^k x = a^k a^{-k} x^2 = x^2 = a^n \in T$. Hence $G^2 \subset T$. This proves (4.6) as claimed.

If n is even, then by (4.6), $(\mathrm{Dic}_n)^2 = \langle a^2 \rangle$, so $(\mathrm{Dic}_n)^2$ is a subgroup of Dic_n.

Suppose that n is odd. Let $n = 2k+1$. By (4.6), $a^{2k}, a^n \in (\mathrm{Dic}_n)^2$, but $a^n(a^{2k})^{-1} = a \notin (\mathrm{Dic}_n)^2$. Hence $(\mathrm{Dic}_n)^2$ is not a subgroup of Dic_n. □

Proof of Theorem 1.4:

We want to show that conditions (a) and (b) of Theorem 1.4 are equivalent. We first show that (b) implies (a). Suppose that $n \le 8$ or 4 does not divide n. If $n \le 8$, then by Theorem 4.2, every group of order n has the square subgroup property. If 4 does not divide n, then by Theorems 1.1 and 4.1, every group of order n has the square subgroup property.

Next we show that (a) implies (b). Suppose that $n > 8$ and 4 divides n. Then $n = 4k$ for some $k \ge 3$. We want to construct a group G of order $4k$ and does not have the square subgroup property. For $k = 3$, we let $G = A_4$, see Example 3.2. For $k = 4$, we let G be the group constructed in Proposition 4.4. For $k \ge 5$ and odd, we take $G = \mathrm{Dic}_k$ and apply Theorem 4.6. For $k \ge 5$ and even, we write

$$k = 2^t r\ ,\ t, r \in \mathbb{N} \text{ and } r \text{ is odd.}$$

If $r = 1$, we let G be the group in Proposition 4.5. If $r > 1$, we let $G = \mathrm{Dic}_r \times \mathbb{Z}_{2^t}$. By Proposition 3.1(b) and Theorem 4.6, $G = \mathrm{Dic}_r \times \mathbb{Z}_{2^t}$ does not have the square subgroup property. This proves Theorem 1.4. □

5. General Linear Groups

In this section, we prove Theorem 1.5. For any square matrix A, we let A_{ij} denote its (i,j)-th entry. We make the convention that empty spots in a matrix denote the entry 0.

Lemma 5.1. *Let A be an $n \times n$ upper-triangular complex matrix with entries 1 along the diagonal, and $A_{ij} = x_{j-i}$ for all $i < j$, namely*

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ & x_0 & x_1 & \dots & x_{n-3} & x_{n-2} \\ & & x_0 & & \vdots & \vdots \\ & & & \ddots & x_1 & x_2 \\ & & & & x_0 & x_1 \\ & & & & & x_0 \end{pmatrix}. \tag{5.1}$$

Then $(A^2)_{ij} = \begin{cases} \sum_{k=i}^{j} x_{k-i} x_{j-k} & \textit{for } i \le j \\ 0 & \textit{for } i > j. \end{cases}$

Proof. For $i > j$, it is clear that $(A^2)_{ij} = 0$. For $i \le j$, we have $(A^2)_{ij} = \sum_{k=1}^{n} A_{ik} A_{kj} = \sum_{k=i}^{j} x_{k-i} x_{j-k}$. □

Recall that a Jordan block is a square matrix with diagonal entries $\lambda \in \mathbb{C}$, entries 1 above the diagonal, and 0 elsewhere. Let $\mathbb{C}^\times$ denote the multiplicative group consisting of nonzero complex numbers.

Lemma 5.2. *Let $\lambda \in \mathbb{C}^\times$, and consider the $n \times n$ Jordan block*

$$J = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}.$$

There exists an $n \times n$ complex matrix B such that $B^2 = J$.

Proof. Pick $\alpha \in \mathbb{C}$ such that $\alpha^2 = \lambda$. Then $J = \alpha^2 M$, where

$$M = \begin{pmatrix} 1 & \frac{1}{\lambda} & & & \\ & 1 & \frac{1}{\lambda} & & \\ & & 1 & \ddots & \\ & & & \ddots & \frac{1}{\lambda} \\ & & & & 1 \end{pmatrix}.$$

Let A be the matrix in (5.1). By Lemma 5.1, $A^2 = M$ holds if

$$2x_1 = \frac{1}{\lambda} \text{ and } \sum_{k=0}^{j} x_k x_{j-k} = 0 \text{ for all } j = 2, ..., n-1. \tag{5.2}$$

We note that (5.2) has a unique solution for $x_1, x_2, ..., x_{n-1}$. It starts with $x_1 = \frac{1}{2\lambda}$, then inductively with $x_2 = -\frac{1}{2}x_1^2 = -\frac{1}{8\lambda^2}$, $x_3 = -\frac{1}{2}(x_1x_2 + x_2x_1) = \frac{1}{16\lambda^3}$, and more generally

$$x_j = -\frac{1}{2}\sum_{k=1}^{j-1} x_k x_{j-k} \text{ for all } j = 2, ..., n-1.$$

In this way, $A^2 = M$.

Let $B = \alpha A$. Then $J = \alpha^2 M = \alpha^2 A^2 = B^2$. This proves the lemma. □

Recall that $\mathrm{GL}_n(\mathbb{C})$ is the multiplicative group of all $n \times n$ nonsingular complex matrices. We now show that $(\mathrm{GL}_n(\mathbb{C}))^2 = \mathrm{GL}_n(\mathbb{C})$.

Proof of Theorem 1.5:

Let $X \in \mathrm{GL}_n(\mathbb{C})$. There exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that $J = PXP^{-1}$ is the Jordan form of X, namely

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix}$$

where each J_i is a Jordan block. By Lemma 5.2, there exist $B_1, ..., B_r$ such that $B_i^2 = J_i$ for all $i = 1, ..., r$. Let

$$B = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{pmatrix}$$

Clearly $B^2 = J$. Then

$$X = P^{-1}JP = P^{-1}B^2P = (P^{-1}BP)^2 \in (\mathrm{GL}_n(\mathbb{C}))^2.$$

This proves the theorem. □

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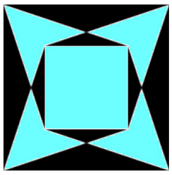

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Steven H. Strogatz: Infinite Powers: The Story of Calculus, Atlantic Books, 2019. ISBN:978-1-78649-295-1, GBP 15.99, 360+xxiii pp.

REVIEWED BY PJ O'KANE

In James Brennan's review [1] of Donald Marshall's book on Complex Analysis in the summer 2019 Bulletin, he discussed how there may be many different opinions on the best starting point for introducing students to a new subject. For complex analysis, they ranged from Cauchy's to Riemann's views before Weierstrass' position that it should be power series prevailed. In this book, Steven Strogatz argues that the concept of infinity should be used when introducing them to calculus. That doing so is not the norm is exemplified by Serge Lang's classic text, *A First Course in Calculus* where, while infinite decimals are mentioned in the first chapter, the word "infinity" and its fuller import for calculus does not appear until page 184, and only four times thereafter.

This is not a textbook that demonstrates that new approach, nor does it claim to be. In fact, it is not a book of mathematics. Rather, it is a book about mathematics that appeals to those, like the reviewer, who may have a reasonable grounding in the subject but not enough to be a practicing mathematician. It takes the reader through the evolution of calculus, from the why to the how, weaving in the motivations, challenges, successes and sometimes pettiness along the way, before talking about its everyday use and future direction. They are all interesting — especially the pettiness.

At the outset, he distinguishes between the concepts of "completed infinity" and "potential infinity" using the repeating decimal representation of $\frac{1}{3}$ as an example. Interpreting $0.333\ldots$ as an infinite succession of threes is an example of the former, while interpreting it as a limit that you can get progressively closer to by just adding further digits is an example of the latter. Potential infinity, he says, is more elegant and easier to work with than completed infinity because it stays in the world of the finite. He illustrates this with a quick explanation of how it neatly resolves Zeno's paradox of Achilles and the Tortoise. The trick is using limits: "With limits and infinity, the discrete and the continuous become one", he remarks. (He also resolves the paradox using straightforward algebra, by the way.)

An interesting trait of the author is that he sometimes pursues a line of thought, seemingly ignoring a potential challenge that may trouble the reader, only to address it later and instantly remove that nagging concern in a very convincing way. The effect is to achieve full closure, not only on the point being discussed but also on all the intervening points where the concentration may have been slightly distracted. It makes an interesting form of argument. He does it when emphasising the power and success of calculus based on the principle that everything can be continuously subdivided and subsequently reassembled. For a number of pages the background concern is, how is that compatible with the concept of the quantum, the antithesis of continuous, but then

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he gets to it, citing infinite decimal representation, while still but an approximation, as the reconciliation. The same clinching argument on approximations is presented later when discussing the accuracy of Newton's conclusions on orbits despite having imagined entire planets condensed to single points using infinitesimals. He sums it up nicely: "How's that for a lie that reveals the truth?"

It's when talking about the evolution of calculus that the human dimension arises. We are inclined to think of calculus as beginning with Newton and Leibniz in their two different flavours, but Strogatz goes back as far as Archimedes for its roots in his attempts to quantify Pi through successive division and reassembly, according to what became the so-called "Method". He notes that Archimedes wrote to Eratosthenes around this time saying that he used the technique more often as a means of verification than of discovery. He then makes a passing but interesting comment on how common that is in mathematical discovery: "First comes intuition. Rigor comes later." It's a phenomenon that should be familiar to former Maynooth graduates who benefited from an open invitation to bring in previously unseen problems and watch them being solved by a practicing mathematician. It was always a revelation to see how an approach could be taken, abandoned, revisited and then combined with another to eventually get to a solution whose final, succinct write-up belied the longer path to first reaching it.

Back to the history of calculus, he explains its origins as integration coming first, and differentiation following, the reverse of how it is typically taught. He explains how differentiation progressed through a contemporaneous but independent linking of algebra and geometry by the imperious Descartes and the more genial Fermat. The greats, it would seem, were not all, or at least not always, quite so noble. Descartes certainly wasn't. He considered Fermat a provincial upstart and took umbrage at his questioning of his results. Worst of all he became vengeful on discovering that what he thought was his creation, Analytic Geometry, had been formulated ten years earlier by Fermat. Unfortunately, though, it would seem that arrogance prevails on occasion, as what people are introduced to today as Cartesian coordinates were actually first developed by Fermat.

The insights into the practical applications of calculus are interesting, and one, virus treatment, is particularly topical, as he talks about how it was used in the battle against HIV. The early days of its treatment was marked by three phases: initial infection, a period of apparent remission and then, unfortunately, the terminal impact. "Apparent" is the operative word here as during what was thought to be an asymptomatic phase, sometimes lasting up to ten years, an ongoing battle was raging between the body's self-defence and the mutation of the virus. What appeared outwardly calm disguised a delicate balance between the rate at which the virus was growing and the rate at which it was being killed off by the immune system, something in the order of ten billion virus particles per day. That discovery and its modelling using calculus led to a complete revision of its medical treatment and the adoption of a three-drug regime that provided a better defence against the ever-mutating virus, turning a once-fatal disease into a manageable one. At the time of writing a major concern is the delta strain of Coronavirus. If the same approach is being taken today, how ironic that calculus may be trying to get rid of delta!

Other applications of calculus he discusses include how the FBI used it to shrink the size of fingerprint file storage by a factor of twenty, how it explains the rule of 72 used in finance as a ready reckoner for how long it takes for money to double in value (if the rate multiplied by the term equals 72), how Thomas Jefferson employed it to make a better plough, how CAT scans work and more bizarrely how, using just some shredded cheese, a ruler and a microwave oven, you can calculate the speed of light (don't bother, just Google it).

The numerous tidbits of information he discloses are wonderful. For example, related to the last point above, he casually remarks that the idea of using microwaves to heat food arose from an engineer who noticed that after an extended period beside a magnetron the peanut bar in his pocket had begun to melt. He doesn't mention what else might have been affected.

On the future of calculus, or at least on future applications, he becomes more philosophical. He talks about its role in the investigation into the incredible efficiency of DNA packaging using supercoiling even though it is definitely based on the discrete rather than the continuous; about its use along with machine learning and AI in creating a possibly unbeatable chess player and the implications of the apparent intuition involved; the Gödel-like implications for complex nonlinear systems arising from a proof by Kovalevskaya that some outcomes are fundamentally unsolvable, leading to the jarring conclusion that "determinism does not imply predictability."

It is said that John Nash was inspired to be a mathematician by reading Bell's *Men of Mathematics*, and Andrew Wiles cites stumbling across a book in a library as the catalyst for devoting his life to proving Fermat's Last Theorem. The author speaks fondly of Ms Santon who introduced him to the concept of infinity and ultimately his love of mathematics. Everyone deserves a Ms Stanton, someone who kindles a youthful fascination in a subject, any subject. Mine was Seamus McTague, our headmaster, who introduced magic squares to a group of 10-year olds and challenged them to fill one in overnight (and because it was a challenge, not homework, it was all the more inviting). It fostered a life-long interest in the subject and shows how small actions, passing remarks and casual comments by educators, especially early ones, can have profound effects that sadly, they may never be aware of.

Today we have much more at our disposal for seeding an interest in any topic or activity. Giving access to books like this may be one such catalyst for the next generation of mathematical creators, thinkers - and reviewers.

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Desmond MacHale and Yvonne Cohen: New Light On George Boole, Atrium, Cork University Press, 2018. ISBN:978-1-78205-290-6, €19.95, 476+xvi pp.

REVIEWED BY HELENA ŠMIGOC

The book is too detailed to be a casual read. It is not suitable for someone seeking a quick recap of the life of George Boole. Rather, it presents a varied collection of snippets from Boole's life, ideal for someone already familiar with this influential mathematician, and interested in learning about his personal life. From excerpts of letters and comments, an intimate picture of a man emerges. We get to know George Boole as a loving son and brother, thoughtful friend, deep thinker, mathematician, hard worker, and more. We get a glimpse into his insecurities, celebrations, and values. The further we get into the book more we feel like we know George Boole personally, beyond dry biographical facts. It almost seems like the authors were following the words of Boole's friend Joseph Hill (taken from the book): "I feel that a memoir, like a portrait, may require many little strokes, insignificant in themselves, but collectively useful for filling up the picture with suitable shading."

Well-researched commentary on quotations from letters helps to transport the reader back in time. We are connected with historical events and circumstances, ranging from politics and religion to pregnancy apparel, and extending to the first hippopotamus in Central Europe and the medical practices of the time.

The opening chapter, *Boole Family History Notebook*, gives a reproduction of the text most likely written by Mary Ellen (Boole's eldest daughter) followed by an informative commentary. We learn about Boole's ancestry: a family that valued education. A picture of George's Boole father John Boole emerges as a science enthusiast with strong personality. Ups and downs of the family are outlined, painting a picture of the circumstances that formed George Boole's early life.

The second chapter is dedicated to a reproduction of a biographical memoir written by George Boole's sister MaryAnn. This memoir is appearing in print for the first time in this book. The reproduction is accompanied with a short commentary on the text that allows itself to be slightly critical of MaryAnn for not even mentioning Boole's wife in her account. Through the memoir we again meet the strong figure of John Boole, and follow the life of George Boole through MaryAnn's retelling.

In the third chapter, *1849–1855: Home and Work*, an intimate insight into the relationship between siblings emerges through excerpts from Boole's letters to his sisters. These are accompanied by comments providing historical setting and other relevant supplementary information. A variety of topics is discussed, from family affairs and every-day routines, to politics and university affairs.

After providing a historical backdrop, the fourth chapter assumes a similar format to the previous one, this time focusing on Boole's social interactions and travels. Through his letters, we get a first-hand account of life in Cork at the time. Historical events

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are connected to the letters. We again see how close Boole was to his sister, letting her know how he felt, about people he met, details of his trips, and sights he has seen. Several prominent Irish figures and families are seen to have crossed paths with George Boole.

Chapters 5–9 are dedicated to close friends of Boole's. Friendships of mutual admiration and respect emerge in snippets of letters that Boole wrote to Dr Bury. Boole clearly held his friend's opinion in high regard. He turned to him for advice about his wife's pregnancy, as well as asking for comments on chapters of the book he was writing. The chapter on John Bury concludes with his biographical letter on George Boole.

Joseph Hill was a school friend of George Boole, and the two men sustained their friendship of intellectual prowess. Through letters we see Boole as a man with an impressive span of interests: languages, literature, even biblical analysis. Letters written by Hill solidify the impression that the two men found intellectual companionship together. Among snippets from the life of Boole in this chapter, the one where Boole and Hill meet Charles Babbage is particularly significant.

Reverend E. R. Larken is another lifelong friend and correspondent of Boole's that we meet. Through excepts from letters and the accompanying commentary we get a peek into Boole's considerations around the publication of his first book, *A Mathematical Analysis of Logic*, apprehension over his application to an Irish professorship at Queen's College Cork (QCC), and religious beliefs.

In his letters to William Brooke, Boole allows himself to bluntly criticise Sir Robert Kane and the administration in QCC at the time. This is contrasted with affectionate passages about his family revealing gentleness and care towards his wife and young daughters. We learn about his travel in both Europe and Ireland. The reader is transported to a beautiful Wicklow scenery through enthusiastic Boole's account only to be brought ruthlessly to reality with snippets of his letters that refer to ill health written shortly before his untimely death. The chapter ends on a sombre note, with Brooke's sorrowful writing shortly after the death of his friend.

Cooper was a schoolmaster and related to Boole by marriage, while Clarke was a pupil both of Cooper and Boole. Cooper and Boole were acquaintances, but did not become close friends. Nevertheless, Cooper is another character in the book that admired Boole, and was influenced by him. Clark, who built a political career in Canada, had fond memories of Boole as a teacher. A few anecdotes help us to build a picture of Boole in the classroom: a bit eccentric, enthusiastic about the subject matter, and close to his pupils. Chapter 10 further fills out our impression of Boole, with memories of him from a list of friends and former students.

Chapter 11 revolves around correspondence with mathematicians. Encouragement, support, and thirst for mathematical discoveries emanates from correspondence with Duncan Gregory, Robert Leslie Ellis, Issac Todhunter, and others. We get a look into how mathematicians corresponded at the time, and the kind language and supportive attitude they used in commenting on each others' work.

Chapter 12 discusses circumstances surrounding the tragic and premature death of George Boole. He and his wife believed in the healing powers of "hydropathy," which was introduced in Cork by Richard Barter. It seems that Boole at least partly relied on this treatment when he was very ill, which did not help his recovery. The chapter includes a distressing-to-read letter written by Booles' neighbour Annie Gibson detailing the progression of Boole's decline in health during the last days of his life.

The 13th chapter, titled *Booleana*, provides an eclectic mix of topics connected to Boole. We learn about poetry written by Boole's daughter Mary Ellen Hinton, scientists that were contemporaries with Boole, residences of Boole in Cork, and of accomplishments of some of Boole's descendants. The chapter concludes with a list of various examples that memorialise Boole: from stained glass windows to a crater on the moon, and even a George Boole Google^{TM} Doodle.

The last chapter explores the hypothesis that Sir Arthur Conan Doyle based his character of Professor James Moriarty, arch-nemesis of Sherlock Holmes, on George Boole. Some highly suggestive evidence is presented, but (as authors themselves note) it is unlikely that the villain is based on any single person. Following a story that connects Boole and his wife with H. G. Wells, who knew Sir Arthur Conan Doyle well, makes it seem very likely Sir Arthur Conan Doyle knew of Boole. Nevertheless, we get a sense of Boole's reputation, which seems to have reached outside mathematical circles after the publication of Boole's book *The Laws of Thought.* The arguments presented make it easy to imagine Boole inspiring a fictional figure who embodies logic, deduction, and proof.

The organisation of the book is clever, in that it allows each chapter to be read in isolation, and in any order. Rather than a tome which must be read from start to finish, this work can be approached from any direction, with the reader dipping their toe into whatever topic or chapter strikes their fancy. As such, it is not just a reference book which belongs in any library, but also an ideal book for browsing by anyone who wants to transport themselves to a fascinating age and explore the life of a fascinating man: a product of his time whose work lives on and whose influence continues to be felt.

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PROBLEMS

IAN SHORT

Problems

The first problem this issue was posed by Anthony O'Farrell, editor of this Bulletin.

Problem 88.1. Consider the sequence $x_0, x_1, \dots$ defined by $x_0 = \sqrt{5}$ and $x_n = \sqrt{2 + x_{n-1}}$, for $n = 1, 2, \dots$. Prove that

$$\prod_{n=1}^{\infty} \frac{2}{x_n} = 2\log\left(\frac{1+\sqrt{5}}{2}\right).$$

The second problem is courtesy of J.P. McCarthy of Munster Technological University.

Problem 88.2. Let P be a 3-by-3 matrix each entry of which is an n-by-n complex Hermitian matrix; that is, each entry P_{ij} is an n-by-n complex matrix equal to its own conjugate transpose P_{ij}^*. Suppose that the sum along any row or column of P is the n-by-n identity matrix I_n:

$$\sum_{k=1}^{3} P_{kj} = \sum_{k=1}^{3} P_{ik} = I_n.$$

Suppose also that the entries of P along rows and columns satisfy

$$P_{ik}P_{il} = \delta_{kl}P_{ik} \quad \text{and} \quad P_{kj}P_{lj} = \delta_{kl}P_{kj},$$

where δ_{kl} is 1 if k and l are equal and otherwise it is 0 (and no summation convention should be applied). Prove that the entries of P commute with one another.

The third problem comes from Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia.

Problem 88.3. Prove that

$$\sum_{n=2}^{\infty} \frac{(-1)^n H_{\lfloor n/2 \rfloor}}{n} = (\log 2)^2,$$

where H_n denotes the nth harmonic number

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

and $\lfloor \cdot \rfloor$ denotes the floor function.

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Solutions

Here are solutions to the problems from *Bulletin* Number 86.

The first problem was solved by Eugene Gath of the University of Limerick, Daniel Văcaru of Pitești, Romania, the North Kildare Mathematics Problem Club, and the proposer, Yagub Aliyev of ADA University, Azerbaijan. We present the solution of Eugene Gath.

Problem 86.1. Find the nearest integer to

$$10^{2021} - \sqrt{(10^{2021})^2 - 10^{2021}}.$$

Solution 86.1. Let $f(n) = n - \sqrt{n^2 - n}$, for each positive integer n. Observe that

$$n - \sqrt{n^2 - n} = \frac{n}{n + \sqrt{n^2 - n}}.$$

Hence

$$f(n) > \frac{n}{n+n} = \frac{1}{2} \quad \text{and} \quad f(n) \leqslant \frac{n}{n} = 1.$$

Consequently, the nearest integer to $f(n)$ is 1 for all values of n including $n = 10^{2021}$. □

Yagub observes that the continuous function $g(x) = 10^x - \sqrt{10^{2x} - 10^x}$ $(x > 0)$ behind this question tests the limits of graphing software, with most software unable to plot the graph of g accurately beyond about $x = 15$.

The next problem was solved by Prithwijit De of HBCSE, Mumbai, India, Eugene Gath, the North Kildare Mathematics Problem Club and the proposer, Seán Stewart. The solution we present is that of Prithwijit De.

Problem 86.2. Evaluate

$$\int_0^1 \frac{1}{x} \arctan\left(\frac{2rx}{1+x^2}\right) dx,$$

where r is a real constant.

Solution 86.2. Define $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(r) = \int_0^1 \frac{1}{x} \arctan\left(\frac{2rx}{1+x^2}\right) dx.$$

Then $f(0) = 0$ and f is differentiable on $\mathbb{R}$ with derivative

$$f'(r) = \int_0^1 \frac{2(1+x^2)}{(1+x^2)^2 + (2rx)^2}\, dx.$$

Substituting $x = \tan(\theta/2)$ gives

$$f'(r) = \int_0^{\pi/2} \frac{1}{1 + r^2 \sin^2 \theta}\, d\theta = \frac{\pi}{2\sqrt{1+r^2}}.$$

Integrating with respect to r gives

$$f(r) = \frac{\pi}{2} \ln\left(r + \sqrt{1+r^2}\right),$$

where we have used $f(0) = 0$ to find the constant of integration. □

The third problem was solved by Henry Ricardo of the Westchester Area Math Circle, NY, USA, Ángel Plaza of the Universidad de Las Palmas de Gran Canaria, Spain, Seán Stewart, Eugene Gath, Daniel Văcaru, the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland of University College Cork. Choosing between the variety of cunning solutions submitted is tricky! As usual we present only one solution, in this case the concise solution of Eugene Gath.

Problem 85.3. Prove that

$$\sum_{n=0}^{\infty} \frac{9n+5}{9n^3+18n^2+11n+2} = 3\log 3.$$

Solution 86.3. Observe that

$$\frac{9n+5}{9n^3+18n^2+11n+2} = 3\left(\frac{1}{3n+1}+\frac{1}{3n+2}+\frac{1}{3n+3}-\frac{1}{n+1}\right).$$

Let $f(x) = 3\log(1+x+x^2)$, for $x>0$. Observe that

$$f(x) = 3(\log(1-x^3)-\log(1-x)),$$

for $0<x<1$. By substituting suitably into the Maclaurin series for $\log(1+y)$ we obtain

$$f(x) = 3\sum_{n=0}^{\infty}\left(\frac{x^{3n+1}}{3n+1}+\frac{x^{3n+2}}{3n+2}+\frac{x^{3n+3}}{3n+3}-\frac{x^{3n+3}}{n+1}\right),$$

for $0<x<1$. This series converges at $x=1$. It follows from Abel's theorem that the series expansion for f is valid when $x=1$. Consequently,

$$3\sum_{n=0}^{\infty}\left(\frac{1}{3n+1}+\frac{1}{3n+2}+\frac{1}{3n+3}-\frac{1}{n+1}\right) = f(1) = 3\log 3.$$

The result now follows from the partial fraction expansion obtained at the start. □

Finally this issue we thank Tom Barry, Chairman of New Ireland Assurance, for pointing out an error in Problem 85.2 and its solution. That problem required us to show that $2^{13} = 8192$ is the only integer between 4129 and 9985 that cannot be expressed as the sum of two or more consecutive integers. However, as Tom points out, every positive integer can be expressed as the sum of two or more consecutive integers; for example,

$$8192 = (-8191) + (-8190) + \cdots + 8192.$$

Really the problem should have asked for the sum of two or more consecutive *positive* integers. With that addition, the solution published last issue needs some adjustments.

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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