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PROBLEMS

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Problems

I learned the first problem this issue from a paper by Boris Springborn (*Enseign. Math.* 63, 2017, 333–373).

Problem 87.1. Determine the maximum distance between a straight line intersecting a triangle and the vertices of that triangle.

The second problem is courtesy of Des MacHale of University College Cork.

Problem 87.2. Prove that if each element x of a ring satisfies $x^4 + x = 2x^3$ then the ring is commutative.

Elementary answers only please (using basic properties of rings). Those of you who solve Problem 87.2 might like to tackle the following more challenging variant. Prove that if each pair of elements x and y of a ring satisfies $(x^4 - x)y = y(x^4 - x)$ then the ring is commutative. Des has kindly offered a prize of his recent book *The Poetry of George Boole* for the first correct, elementary solution to this more challenging problem!

The third problem comes from Finbarr Holland of University College Cork.

Problem 87.3. Determine the sums of the series

$$\sum_{m,n=1}^{\infty} \frac{1}{mn(m+n+1)} \quad \text{and} \quad \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{mn(m+n+1)}.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 85.

The first problem was solved by Seán Stewart of Bomaderry, Australia, and the proposer, Des MacHale. We present the solution of Stewart (using a different reference towards the end).

Problem 85.1. Dissect an equilateral triangle into four pieces that can be reassembled, without flips, to form three equilateral triangles of different sizes. Can this be accomplished with just three pieces?

Solution 85.1. We start with an observation. An equilateral triangle with sides of length a has area $a^2\sqrt{3}/4$. If this triangle can be decomposed into three smaller equilateral triangles with sides of lengths x, y and z, then by equating areas we obtain

$$a^2 = x^2 + y^2 + z^2.$$

A simple positive integer solution of this equation is

$$7^2 = 2^2 + 3^2 + 6^2.$$

This motivates us to look for a four-piece dissection of an equilateral triangle of side length 7 into three smaller equilateral triangles of side lengths 2, 3, and 6. One such dissection is shown below.

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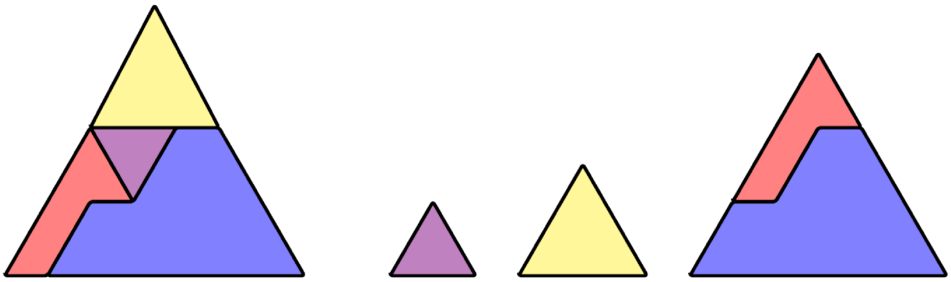

For the second part of the question, to dissect an equilateral triangle into three smaller unequal equilateral triangles using just three pieces entails that each piece must be an equilateral triangle. However, a result of Tutte (*Math. Proc. Cam. Phil. Soc.* 44, 1948, 463–482) says that it is impossible to dissect an equilateral triangle into unequal equilateral triangles. □

The next problem was solved by Daniel Văcaru of Pitești, Romania, and the North Kildare Mathematics Problem Club. The solution was also known to the proposer, Des MacHale. The solution we present follows that of Văcaru and the Problem Club

Problem 85.2. An absent-minded professor of mathematics cannot remember her debit card PIN. However, she remembers that the PIN lies between 4129 and 9985 and it cannot be expressed as the sum of two or more consecutive integers. Can you help her determine the PIN?

Solution 85.2. Let m be the pin, where $4129 < m < 9985$. Suppose that m is divisible by an odd prime p. Observe that, for any integer n, we have

$$(n+1)+(n+2)+\cdots+(n+p) = p\big(n+\tfrac{1}{2}(p+1)\big).$$

By choosing $n = m/p - \frac{1}{2}(p+1)$ we see that m is a sum of p consecutive integers, a contradiction.

Hence m is not divisible by an odd prime, so it is a power of 2. The only power of 2 between 4129 and 9985 is $2^{13} = 8192$, so $m = 8192$. □

The third problem was solved by the North Kildare Mathematics Problem Club. A solution was also offered in the magazine of the M500 Society, a mathematical society of the Open University. I learned of the problem from that magazine.

Problem 85.3. Arrange the integers 1 to 27 in a $3\times3\times3$ cube in such a way that any row of three integers (excluding diagonals) has sum 42.

Solution 85.3. The following solution was found computationally, along with 31 others. The three matrices represent layers of the cube.

$$\begin{pmatrix}14 & 1 & 27\\ 21 & 17 & 4\\ 7 & 24 & 11\end{pmatrix}\qquad\begin{pmatrix}25 & 15 & 2\\ 5 & 19 & 18\\ 12 & 8 & 22\end{pmatrix}\qquad\begin{pmatrix}3 & 26 & 13\\ 16 & 6 & 20\\ 23 & 10 & 9\end{pmatrix}$$

□

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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