Irish Mathematical Society
Cumann Matamaitice na hÉireann

Bulletin

Number 87 Summer 2021

ISSN 0791-5578

Irish Mathematical Society Bulletin

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

Correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

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and only if not possible in electronic form to the address

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Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

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EDITORIAL

This issue is predominantly about mathematicians. We mark the recent loss of Paddy Barry and Brian Murdoch, two of the leaders of Irish academic life for a substantial part of the past century. In Barry's case, he was taken by COVID, which has also deprived the mathematical world of luminaries such as John Conway and Isadore Singer. We also include a short article about the career of John O'Sullivan, a little-known but influential member of our diaspora. I hope that fans of dissections will enjoy learning more about Henry Pergigal in Seán Stewart's account of that remarkable Victorian eccentric. This issue also includes a solution to a dissection problem from Bulletin 85. I would like to acknowledge the signal contribution of Ian Short, who has now edited the problem page for over a decade.

Discriminating persons will rejoice to learn that Tom Lehrer has put all his songs into the public domain. The lyrics of all his songs, including many that are not on the records, may be downloaded from TomLehrerSongs.com until the end of 2024.

I recommend the article by Michael Schmitz, *A plea for finite calculus*, in College Math J 52, no 2, March 2021, 94–105. It is entertaining, well-written, provocative, and balanced. He has thought deeply about the real problems with secondary maths education, and has read widely among the masters. His comments on the history of calculus and of instruction in calculus are sound and striking.

The 2021 Annual Scientific Meeting of the Society (aka the "September Meeting") will be hosted by UCC and MTU on 2–3 September 2021, with one day of activites at UCC and the second on the Cork campus of MTU.

We remind organisers and other contributors that the normal deadline for submissions is 15 December for the Winter issue and 15 May for the Summer issue.

Until now it has been editorial policy to limit rather severely the size of the files that make up the Bulletin articles, in order to avoid problems for readers having limited download speeds. We reviewed this policy recently, and realized that we have been somewhat behind the times. I am now adjusting our limits, and hope this does not cause difficulty. Readers are asked to let me know if they cannot download the material.

Note that the website serves up pdf files of the individual articles, as well as the pdf file of the whole Bulletin. I want to acknowledge the efficient and unfailing support of our website manager, Michael Mackey.

For a limited time, beginning as soon as possible after the online publication of this Bulletin, a printed and bound copy may be ordered online on a print-on-demand basis at a minimal price[1].

Links for Postgraduate Study

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http://web.am.qub.ac.uk/wp/msrc/msrc-home-page/postgrad_opportunities/postgrad_opportunities_2020_additional/
TCD: http://www.maths.tcd.ie/postgraduate/
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The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor.

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(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€200
Ordinary member	€30
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The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

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The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

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(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

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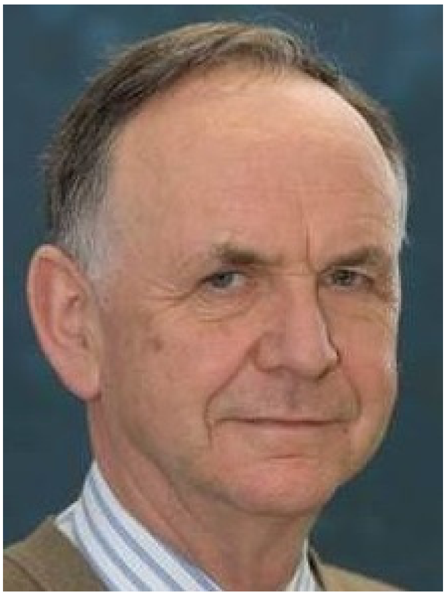
Irish Math. Soc. Bulletin
Number 87, Summer 2021, 3–14
ISSN 0791-5578

Patrick Denis Barry: 1934–2021

TOM CARROLL, FINBARR HOLLAND, DONAL HURLEY, TONY O'FARRELL, PHIL RIPPON



P.D. Barry

Preamble *by* Tom Carroll

Paddy Barry held the Chair in Mathematics at UCC from 1964 until his retirement in 1999. As a student at UCC in the early 1980s, I was fortunate to have many excellent lecturers. First Year Honours mathematics was taught in 1980–81 by Tony Seda who taught us analysis, Paddy Barry who taught us abstract algebra and number systems, and Finbarr Holland who taught us matrices and linear algebra. I should say that 'us' here includes Stephen Buckley and Pat McCarthy, both now at Maynooth, and Jerry Murphy now at DCU. In our third year, Paddy taught a course on differential equations. Paddy's lectures were meticulously prepared and each covered a lot of ground. Some academics take the course they've been teaching for many years and in time turn their lecture notes into a book; in Paddy's case it felt like the book was already written and, indeed, the notes on the board came with chapter headings!

When I had the good fortune to return to Cork in 1990 as a member of staff, Paddy was no longer my lecturer but my Head of Department. Departmental meetings were held in his office in Aras na Laoi. Everyone got to have his or her say and Paddy listened patiently, even to the new lecturer who had very little of value to add. Paddy set the tone for the department and led by example. Only gradually, as I got to know

2010 *Mathematics Subject Classification.* 01A70, 30D15, 34M03, 97G10.
Key words and phrases. Barry, entire functions, Fatou set, differential equations, plane geometry.
Received on 10-5-2021; revised 11-6-2021.

other lecturers across the university, did I realise how fortunate and even exceptional it was to work in such a supportive and collegial department.

Paddy passed away peacefully on 2nd January 2021. He was a true gentleman. This tribute, written by his colleagues, covers both his life and his mathematics - I have simply compiled and edited the result. *Ar dheis Dé go raibh a anam dílis.*

Paddy's life and times *by* Finbarr Holland

In 1964, when he was aged thirty, Patrick Barry (Paddy to everybody who knew him) was appointed Professor of Mathematics at UCC in succession to Paddy Kennedy who had moved to take up the Chair of Mathematics at the recently established University of York.

As it happened, 1964 coincided with the centennial of the death of George Boole, Cork's first Professor of Mathematics, and on 25th May that year, UCC hosted a meeting of members of the Royal Irish Academy (RIA) in his honour, at which J. L. Synge gave the commemorative lecture. Later on, Cork University Press issued *George Boole: A Miscellany*, a booklet of essays edited by Paddy which *inter alia* contains Synge's lecture. In it, also, Paddy draws a comparison between what the department was like in Boole's day and his own.

Three years earlier Paddy had returned home to Cork from Stanford University, where he had spent a year as an Instructor of Mathematics, to begin his teaching career at an institution that had changed very little since his undergraduate days. At the time, only a very few members of the teaching staff had individual offices, and secretarial support was limited to say the least, though some departments had the luxury of sharing secretarial support and – legend has it – the same typewriter which had to be carried across campus from office to office! In my final degree year, along with about ten other students, I took a course on differential equations from him. The following year he taught Lebesgue Integration à la Burkill's Cambridge Tract to a small number of postgraduates, including myself. As well, he prepared me for the 1962 Travelling Studentship Examination, teaching me the theory of Entire Functions on a one-to-one basis in the College Rest!

Paddy took up his professorship at a time when he had the assistance of only one full-time staff member, namely the late Siobhán Vernon, and his teaching and examining load would have been heavy. Between them, Siobhán and himself would have delivered lectures at various levels to students in the Faculties of Arts, Commerce, Engineering, Medicine and Science. But following the launching of Sputnik by the Soviets, times were changing: the student population began to swell with students taking STEM subjects, leading especially to an increase in numbers studying Electrical Engineering, Chemistry and Mathematical Science.

UCC in the 1960s was a vibrant and carefree place for those privileged enough to be there. A more liberal attitude emerged with staff and students mingling more freely outside the classroom especially after meetings of Clubs and Societies. The wearing of gowns by students was no longer a requirement for attendance at class, roll calls were taken only sporadically, and student members of religious orders were no longer easily identifiable by their clothes. Scholarship became a much respected attribute to possess: poets, dramatists and musicians, whose works were widely acclaimed, emerged from the ranks of the staff; more students began to pursue postgraduate studies and, between 1960 and 1970, about 30 UCC students secured NUI Travelling Studentships in a variety of subjects, eight in Mathematical Science. Students also excelled on the sporting field: for instance, the UCC Hurling Club contested the Cork County Final about five times in that period, winning it in 1962 and 1970.

The pace of change was accelerated in 1967 with the appointment of President Donal McCarthy (1967–78), who set about modernising the College. Under McCarthy's reforming zeal, students flocked to UCC from the Munster region, new degree programmes were introduced, young scholars with newly minted PhDs from abroad were appointed to teach and examine them, library holdings were expanded, subscriptions were taken out for research journals, and new buildings were erected to accommodate an increase in student and staff numbers and library stock.

Paddy was central to McCarthy's ambitious plans. Known for his acuity and probity, following a sabbatical at Imperial College, London, his peers elected him to UCC's Governing Body, and he was made its first Vice-president (1974–76). As such, with his analytical skills, eye for detail and fairness, he developed objective criteria that led the way in streamlining the appointments system, served on numerous interview boards and helped to acquire degree-awarding status for Mary Immaculate College, Limerick, which trained primary school teachers.

His tenure as Head of Department (1964–1999) was one of harmony and collegiality. Being even-tempered, thoughtful and non-confrontational by nature, he always managed to coax consensual decisions at meetings he chaired. Having to cater for a broad range of student ability, interest and class size, and deliver a large amount of service teaching, he appointed staff with a diverse range of specialisms, and gave each the freedom to develop his or her own courses and research. At the beginning of each academic year, teaching duties were equitably assigned subject to timetable constraints and individual preferences, to the mutual satisfaction of all concerned. Towards the end of his role as HoD, he introduced an innovative part-time two-year postgraduate degree course in Mathematical Education for secondary-school teachers of Mathematics. This was offered on two occasions in a ten year period, and was availed of by a total of about forty teachers from the Munster area; It raised these teachers' profiles, earned them an extra salary increment and was hugely beneficial to their students.

Paddy had a life-long passion for classical geometry and loved to teach it. In 2001, as Professor Emeritus, he published *Geometry with Trigonometry* [1], a rigorous account of Euclid's geometry suitable for teachers of school mathematics and based on Birkhoff's approach. His treatment is the foundation of the geometry section of the current Leaving Certificate mathematics syllabus. A second edition of the book appeared in 2015 [2]; cf. Anca Mustaţă's thought-provoking review [30]. In his declining years, he wrote up extensive notes under the heading 'Some Generalization in Geometry', which form the basis of his third book *Advanced Plane Geometry*, published in 2019 by Logic Press [3]. This last is accessible to anyone who has mastered [2] and is one of the few texts published in Ireland since Mac Niocaill's [29] that is accessible to teachers, and guides them skilfully beyond the basics.

Patrick Denis Barry was born in Ballynacargy, Co. Westmeath, on 20$^{\text{th}}$ October 1934. His father, also called Patrick Denis, was a Garda sergeant; his mother a National School teacher. He was the fifth in a family of three boys and four girls. When Paddy was two the family moved to Co. Cork, first to the village of Glenville before taking up permanent residence in Mallow in 1945, a town then linked by road and rail to the main Irish cities. There he received his secondary education at the Patrician Academy, sitting the Leaving Certificate examination in 1952, his performance winning him a university scholarship from Cork County Council. In the same year, he achieved first place in the UCC Entrance Scholarship examination, and second place in the examination for the Irish Civil Service.

During his schooldays, Paddy played cricket, soccer, tennis and badminton, the latter a sport at which he was particularly skilful and which he continued to play late in life. Incidentally, by playing such sports at a time when the GAA operated its infamous ban on 'foreign' games, he showed early signs of having an independent mindset, and a steely determination to follow his own inclination, something that was characteristic of him.

Having won two scholarships, he became the first pupil from his school to go to university when, in October 1952, he enrolled in UCC to study Mathematical Science, commuting to the College by train. He graduated in 1955 with a First Class Honours BSc, majoring in Mathematics and Mathematical Physics. That same year he represented Ireland at badminton at Under–21 level.

He continued to study in UCC until 1957 when he obtained his master's degree and a Travelling Studentship from the National University of Ireland.

But having already accepted the position of Research Assistant to Walter Hayman FRS at Imperial College, London, and being scrupulous, he declined this award, which passed to Diarmuid Ó Mathúna, a contemporary of his at UCC, who used it to obtain his PhD at MIT. Hayman, whose first PhD student had been Paddy Kennedy, now directed Paddy's doctoral studies at I.C. He earned a Diploma from I.C., and a PhD from the University of London, for a thesis entitled *On the minimum modulus of integral functions of small order*, which was an outgrowth of his first research paper 'The minimum modulus of certain integral functions', published in the Journal of the LMS in 1958 [22]. Indeed, in his autobiography *My Life and Functions* [26, Chapter 5], Hayman writes that Paddy Barry was 'the only student I ever had who came to me with a PhD problem already prepared. It was on the minimum modulus of small integral and subharmonic functions [20] a subject on which Barry became the world expert.'

On receipt of his doctorate he spent a year at Stanford University as a Mathematics Instructor before returning to his *alma mater* in 1961 where he was appointed first a lecturer in the Mathematics Department and, in 1964, Professor and Head of Department, positions he occupied until his mandatory retirement in 1999, when he became Professor Emeritus of his subject.

Shortly after returning to Cork, Paddy met and married Frances King, a vivacious young woman from Belfast whose sister's boyfriend had a post in UCC's English Department, and was Paddy's flat-mate! Fran became a secondary teacher of Mathematics, and later acquired a PhD in group theory for a thesis written under the guidance of Des MacHale, to become one of the few Irish secondary teachers with a doctorate.

Paddy loved cooking and liked to show off his culinary skills at dinner parties hosted by Fran and himself, producing a variety of exotic dishes, made – according to his children – with mathematical precision! His speciality was a delicious cinnamon-infused apple pie.

Paddy died in a Dublin nursing home, from Covid-19, Fran having pre-deceased him by fifteen years. They are survived by their children: Conor, a film maker in Dublin; Una, who practises general medicine in Calgary, Canada; and Brian, a surgeon in Cork.

Paddy's early research on entire (integral) functions
by Finbarr Holland

Over a span of about 40 years, beginning in 1957, when he was still a postgraduate student in UCC, Paddy wrote 11 research papers on growth problems associated with either slowly growing entire or subharmonic functions. In this section we review some of his main results that have as common theme the growth of the ratio of the minimum and maximum modulus of an entire function of small order. The maximum modulus on the circle of radius r of an entire function f is defined by $M(r, f) = \max\{|f(z)| : |z| = r\}$ while the minimum modulus is defined, as function of r, by $m(r, f) = \min\{|f(z)| : |z| =$

$r\}$. The (upper) order $\rho(f)$ of an entire function f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

It's familiar that the function $r \to \log M(e^r, f)$ belongs to the class Ψ of continuous non-decreasing functions that are convex on $(-\infty,\infty)$; members of Ψ feature in the hypotheses of many of the theorems enunciated in Paddy's papers. The main object of interest in Paddy's work on entire functions is the relationship between $m(r,f)$ and $M(r,f)$. Subject to various restrictions on the size of f, he gives corresponding estimates for the distribution function of this ratio with respect to logarithmic measure μ defined on Lebesgue measurable sets of $(0,\infty)$ by $\mu(E) = \int_E \frac{1}{t}\,dt$. The key idea he needs to obtain such estimates is an extension of the elementary pigeonhole principle according to which, if A is the average of a finite number of positive numbers and $0 < \lambda < 1$, then the proportion of numbers greater than or equal to λA doesn't exceed $1/\lambda$. Paddy achieves his objectives by first defining a majorant ϕ of $\log\big(M(r,f)/m(r,f)\big)$, and identifying the integral $\int_{[0,r]} \phi\,d\mu$ with the image of the counting function of the zeros of f, viz., $n(t) = \#\{z : f(z) = 0,\ |z| \le t,\}$, under a certain linear operator. Paddy carries this strategy through to a successful conclusion for entire functions of genus 0 that are not polynomials.

His work is of a very general nature, applying not merely to a single function, but to members of a class of functions satisfying similar conditions at infinity. It clearly demonstrates his appetite and aptitude for meticulous attention to detail, which was his forte. He loved to examine the subject of his interest in minute detail at every step of his analysis. He sets out to obtain best-possible results at every opportunity and, by employing intricate reasoning and skilful manipulations, obtains sharp estimates and exact constants wherever possible. He produces explicit examples to show that the results he obtains, subject to the underlying assumptions, are best possible. In his long paper [20], for instance, he manages to eschew rough estimates – big Oh hardly sees the light of day – which is surprising in a paper on classical analysis. His work has an air of finality about it; his was the last word on the subject he analysed, it would appear. But he did leave something unfinished: this was a conjecture that remained open for about 20 years, which I'll come to below. A summary of his main results is presented in [21]; the detail is given in [20].

To give some idea of his achievements, suppose that f is an entire function of genus zero, that $f(0) = 1$, and that $\{z_n\}$ are its zeros arranged in order of increasing moduli so that $\sum_{n=1}^{\infty} \frac{1}{|z_n|} < \infty$. Then, by Hadamard's factorization theorem,

$$f(z) = \prod_{n=1}^{\infty} \Big(1 - \frac{z}{z_n}\Big).$$

If n is the associated counting function then, by Jensen's formula,

$$\int_{[0,r]} n\,d\mu = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta \le \log M(r), \quad 0 < r < \infty.$$

Paddy supplements this with the following attractive identity (cf. [21, Lemma 2]) which is fundamental to his purpose. He chooses a convenient majorant $\phi(r)$ for $\log\big(M(r,f)/m(r,f)\big)$, viz.

$$\phi(r) = \log\Big(\prod_{n=1}^{\infty} \frac{|1 + r/r_n|}{|1 - r/r_n|}\Big), \quad r_n = |z_n|.$$

The fundamental identity that he obtains can be stated as a linear integral equation connecting n and ϕ. It involves the following nonnegative kernel function

$$K(s,t) = \log \frac{1+\kappa(t/s)}{1-\kappa(t/s)}, \quad 0 < s,\, t < \infty,$$

where, for $u > 0$, $\kappa(u) = \min(u, 1/u)$. This generates a linear operator – which we'll call the Barry transform – under which the image G of a function g is given by $G(s) = \int_0^\infty K(s,t) g(t)\, d\mu(t)$.

Lemma 0.1. *If $0 < r < \infty$, then*

$$\int_{[0,r]} \phi\, d\mu = \int_{[0,\infty)} K(r,\cdot)\, n\, d\mu.$$

Using delicate estimations and intricate reasoning, this result is employed to obtain an array of interesting theorems that are shown to be sharp in some respects. The following results may give a flavour of what interested him and what he achieved.

Theorem 0.2. [21, Theorem 5] *Suppose that f is an entire function such that*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\log^2 r} = \sigma < \infty. \tag{1}$$

If $0 < \delta < 1$ and $\epsilon > 0$ then, for $r \geq r_0(\epsilon)$,

$$\mu\Big([0,r] \cap \Big\{t : \log \frac{M(t,f)}{m(t,f)} \geq \delta^{-1} 2\pi^2 (\sigma+\epsilon)\Big\}\Big) \leq \delta \log r.$$

We sketch the proof. First of all, if r is sufficiently large,

$$n(r) \leq \frac{\log M(r^2,f)}{\log r} \leq \frac{(\sigma+\epsilon)(\log(r^2))^2}{\log r} = 4(\sigma+\epsilon)\log r.$$

Next, since the Barry transform of $\log t$ is equal to $(\pi^2/2)\log s$, an application of the lemma shows that

$$\int_0^r \phi(t)\, d\mu(t) < 2\pi^2(\sigma+\epsilon)\log r, \quad r \geq r_0(\epsilon),$$

whence the result follows readily.

(Defining the upper logarithmic density of a Lebesgue measurable set $E \subset (0,\infty)$ as

$$\text{upper log-dens}(E) = \limsup_{r\to\infty} \frac{\mu\big((0,r] \cap E\big)}{\log r},$$

the conclusion of the theorem tells us that the upper logarithmic density of the set $\{t : \log \frac{M(t,f)}{m(t,f)} \geq \delta^{-1}2\pi^2(\sigma+\epsilon)\}$ doesn't exceed δ. In fact, as a reading of his work shows, the conclusions of Paddy's theorems are generally expressed in terms of logarithmic densities of one kind or another.)

This result applies, in particular, to the functions considered by Paddy in his first paper [22].

Paddy also proved in [20] the following result for entire functions satisfying (1): if $\log M(r,f) = \mathrm{o}(\log^2 r)$ as $r \to \infty$, and $\epsilon > 0$, then

$$\lim_{r\to\infty} \frac{\mu\Big((0,r] \cap \big\{r : m(r,f) > (1-\epsilon)M(r,f)\big\}\Big)}{\log r} = 1.$$

He also sought the best possible lower bound for

$$\limsup_{r\to\infty} \frac{m(r,f)}{M(r,f)}$$

for the class of entire functions satisfying the hypothesis (1). He proved in [20] that it is not less than any of the numbers

$$e^{-\pi^2\sigma}, \quad \prod_{k=1}^{\infty} \tanh^2\left(\frac{2k-1}{8\sigma}\right), \quad \frac{e^{1/4\sigma}-3}{e^{1/4\sigma}+1},$$

and conjectured that it was in fact equal to

$$C = \prod_{k=1}^{\infty} \tanh^2\left(\frac{2k-1}{8\sigma}\right).$$

In support of this he showed that

$$\limsup_{r\to\infty} \frac{m(r, f_0)}{M(r, f_0)} \leq C,$$

where

$$f_0(z) = \prod_{k=1}^{\infty} \left(1 - ze^{-k/2\sigma}\right),$$

the latter confirming that the constant $e^{-\pi^2\sigma}$ is sharp for large σ. This conjecture lasted for the best part of twenty years. It was settled in the affirmative in 1979 by A.A. Gol′dberg [25], and independently by P.C. Fenton [24] two years later.

Paddy later revisited this topic in a long paper [13] published in the Proceedings of the Royal Irish Academy. There he replaces the condition (1) by the stronger condition

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{\psi(r)} = \sigma < \infty, \tag{2}$$

where $\psi(r) = o(\log^2 r)$ as $r \to \infty$, and obtains a variety of results on the size of the set of r where $m(r, f)/M(r, f)$ is close to 1.

The papers *On a theorem of Besicovitch* and *On a theorem of Kjellberg* *by* Phil Rippon

Several of Paddy's papers are widely cited and continue to be highly influential. For example, the paper [19] includes an extremely strong $\cos\pi\rho$-type theorem. The origins of such types of result lie in the classical work of Wiman and of Valiron, in particular their result that if the order ρ of an entire function is less than 1, then

$$\limsup_{r\to\infty} \frac{\log m(r, f)}{\log M(r, f)} \geq \cos\pi\rho.$$

This statement shows, for example, that for any function of order $\rho < 1/2$, there is an unbounded sequence of values of r for which the minimum modulus $m(r, f)$ is greater than $M(r, f)$ raised to a fixed *positive* power, and in particular that $m(r, f)$ must be unbounded for such functions.

Paddy's remarkable work in [19] *On a theorem of Besicovitch* strengthens the above result to show that if $0 \leq \rho < 1$ and $\rho < \alpha < 1$, then the inequality

$$\frac{\log m(r, f)}{\log M(r, f)} \geq \cos\pi\alpha$$

holds for all values of r in a set E that has lower logarithmic density at least $1 - \rho/\alpha$. Besicovitch had earlier proved a weaker result of this type with E a set of upper linear density at least $1 - \rho/\alpha$. Though just ten pages long, Paddy's paper [19] is highly technical, highly ingenious, and as ever beautifully explained. His result immediately implies the above theorem of Wiman and Valiron, and it shows moreover that for functions of order $\rho < 1/2$, there is a number $\sigma \geq 2$ (for example $\sigma = 1/(1/2 - \rho)$)

such that $m(r,f)$ is greater than $M(r,f)$ raised to a fixed positive power for at least one value of r in each interval of the form $[R,R^\sigma]$ for R sufficiently large.

The latter result has made Paddy's $\cos\pi\rho$-type theorem a favourite tool in complex dynamics for studying a conjecture of Noel Baker that an entire function f of order less than 1/2 cannot have unbounded components of its Fatou set, the open set where the iterates of f form a normal family; see [27] for a survey on the history of Baker's conjecture and [31] for more recent developments.

Baker's conjecture is still open but there are many partial results, frequently building on the fact that whenever f has order less than 1/2 and $M(r,f)$ behaves in a fairly regular manner, then Paddy's theorem tells us that any curve which stretches sufficiently far radially must have an image that also stretches about the same amount radially, in a certain precise sense, and this repeated radial stretching under iteration of f is incompatible with the curve lying in a component of the Fatou set.

I recall his modest surprise, sometime around 2013 on a visit to UCC as external examiner, when Paddy learnt that his paper [19] was often cited by complex dynamics authors. Paddy then took advice on how to look up citations and was excited to find the large number on MathSciNet (currently 50 citations for his paper [19]) and then a day later the larger number on Google Scholar (currently 132 citations)!

The paper [18] *On a theorem of Kjellberg* is a partner to [19] and is also still widely cited, though less so in complex dynamics. Here Paddy combined his techniques from [19] with earlier techniques of Kjellberg to prove a result in which the hypothesis that the order of f is less than 1 is replaced by the weaker hypothesis that the lower order of f is less than 1, and in the conclusion lower logarithmic density is replaced by the weaker conclusion of upper logarithmic density. Once again the proofs are ingenious and elegantly expressed, with meticulous credit given to other authors.

Paddy's work on classification of functions *by* Donal Hurley

Paddy's work on differential equations [10–12], joint with D.H., was motivated by a wish to classify functions that arise in analysis and enable properties to be developed. He considered functions satisfying homogeneous linear differential equations of operator format. The operator format equations were based on two operators introduced by George Boole, and are defined as follows; for any function $w(z)$

$$\pi w(z) = zw'(z) \quad \text{and} \quad \rho w(z) = zw(z).$$

Operator format differential equations are then formed by equating to zero finite sums of the form

$$\sum_l \sum_m a_{l+1,m+1}\pi^l \rho^m w(z)$$

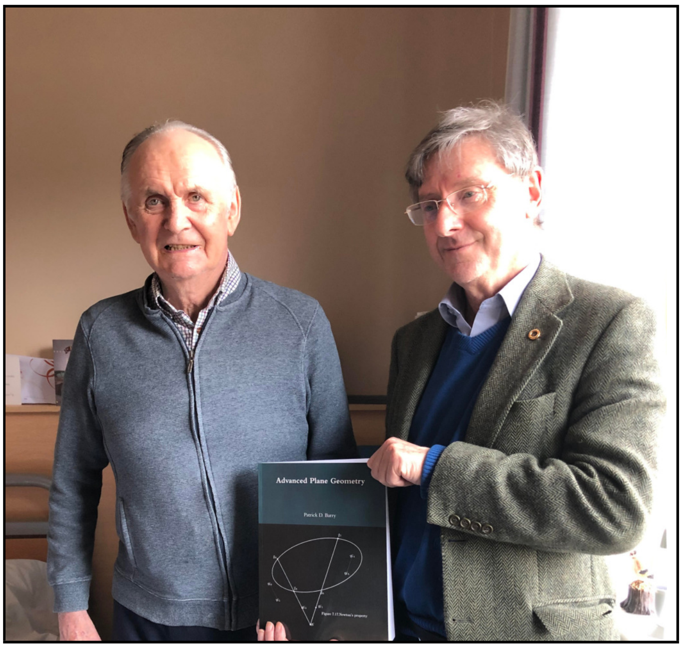
where $a_{l+1,m+1}$ are independent of z. Assuming that the solutions are of the form

$$w(z) = \sum_{n=-\infty}^{\infty} k_n\left(\frac{z}{\lambda}\right)^n$$

where λ is a constant, on substituting this expression for $w(z)$ into the operator format differential equation, one arrives at a homogeneous linear difference equation for the coefficients $\{k_n\}$.

A basic differential classification of functions, generated by the coefficients $\{k_n\}$, is determined by the order of the differential equation, the order of the difference equation for the $\{k_n\}$, and the number of non-zero coefficients k_n in the difference equation. Many familiar functions were encountered in his work. However, computations rapidly become very complicated which necessitated using computer software packages.

The publication in Proceedings of the AMS [12] is a result obtained as a byproduct of this work on classification of functions.



Paddy Barry and Tony O'Farrell with a copy of *Advanced Plane Geometry* (first published in 2019)

Paddy Barry and School Geometry *by* Tony O'Farrell

MacDonald [28] quotes Plato, who said of Education: *If it ever leaves its proper path and can be restored to it again, to this end everyone should always labour throughout his life with all his powers.* No-one ever took up this challenge with such determination and energy as Paddy Barry.

There have been six main revisions of the Irish school geometry syllabus in the past sixty years. Syllabus I was in force 1934-1968, and the revisions came into junior-level exam papers as follows: II:1969, III:1976, IV:1990, V:1995, VI:2003, VII:2015 (affecting first-year students three years earlier in each case). All this change took place during Paddy's active career.

Possessed of deep learning, and a strong sense of duty, he took seriously the responsibility of university mathematicians to monitor and assist with developments in the schools' programme. He was particularly concerned about changes to the geometry syllabus that took place in the nineteen-sixties. These changes were seriously misguided. The whole sorry story is almost unbelievable, and is documented in Susan MacDonald's PhD thesis [28]. Paddy was tireless and relentless over a long period in his efforts to correct the problem. Of his writings about school geometry, the most significant is his book *Geometry with Trigonometry* [1, 2]. This text was eventually adopted by the National Committee for Curriculum and Assessment (NCCA) as the bedrock underlying the geometry programme in the Irish secondary Mathematics syllabus. It is a fully rigorous text on Euclidean geometry, goes substantially beyond the schools' programme, and is suitable for study by university undergraduates and practising teachers.

Paddy's contributions to school geometry went on throughout his career. The fact that Leaving Certificate (LC) mathematics could substitute for the Matriculation Examination gave the universities some leverage over the Department of Education examiners – viz., the chief inspector until the creation of the State Examination Commission. The NUI Senate could, in principle, refuse to accept a pass in the leaving certificate examination as satisfying the matriculation requirement in Mathematics. As a result the draft leaving certificate papers were sent in advance to NUI (statutory) professors for comment, and Paddy always paid attention.

The view in the Department was that there was no reason to consult university people about the junior cycle programme. This reflected a failure to understand that catastrophe could result from tinkering with the logical structure underpinning the geometry programme. As a result, university people had no power in relation to syllabus revision, and enthusiastic engagement with new ideas combined with inadequate understanding on the part of those who did have the power landed us in trouble.

Paddy used every opportunity available to him to sound the alarm in advance, and to press for corrective action. He was not inclined to work with a megaphone. He tried to bring pressure through Royal Irish Academy committees, the Irish Mathematics Teachers' Association (IMTA), and the NUI Senate, and tried to advise the syllabus committees and inspectors. His efforts were blocked for a long time.

Paddy worked in parallel to provide the ingredients for a return to a sound programme, and to educate anyone who would listen, by means of his geometry book [1, 2] and his own course materials intended for classroom use.

The incoherent hybrid Syllabus II did terrible damage to geometrical teaching and study, and we are still some way from recovering. It continued to taint all succeeding versions, until Syllabus VI, an outcome of the NCCA'a Maths Project. In this last Paddy's case was finally accepted, and the geometry programme is again coherent, based on the foundation [2], the Level 2 account [32], and the syllabus [23]. This acceptance was assisted by the fact that some people educated or infuenced by Paddy were serving on relevant committees. However the geometry content at higher-level remains impoverished, compared with the best international standards, there is a persistent issue with textbooks, and it remains to be seen how long it may take us to get back to a stable position.

The adoption of Paddy's *Geometry with Trigonometry* as the Level 1 account underpinning school geometry led to the sale of all copies remaining in print, a second printing, and publication of a new second edition by a subsidiary of Elsevier.

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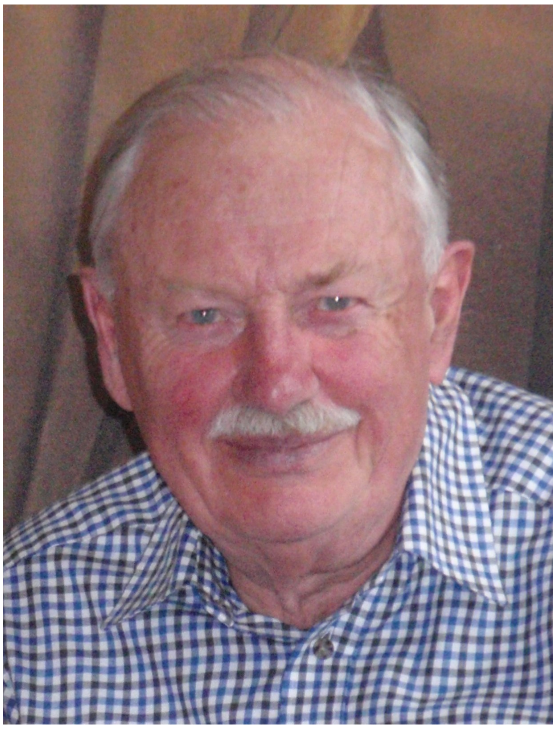
Irish Math. Soc. Bulletin
Number 87, Summer 2021, 15–17
ISSN 0791-5578

Brian Hughes Murdoch (1930-2020)

ELIZABETH OLDHAM

Brian Murdoch, who died on 9 December 2020 at the age of 90, spent most of his long mathematical career in Trinity College Dublin. As Erasmus Smith's Professor of Mathematics from 1966 to 1989, he contributed greatly to the growth and diversification of the School of Mathematics, and he made many other contributions to mathematical education. My own path crossed his in several ways, and those personal experiences have inevitably shaped this obituary. I am grateful to other people for helping me to flesh out aspects of Brian's life with which I was less familiar.



Brian was born in England on 3 April 1930, but he grew up in Dublin. As a primary school pupil, he attended Kingstown School in Dun Laoghaire; he went on to Newtown School in Waterford, and later to the High School in Dublin. In 1947, he entered the School of Mathematics in Trinity, the place to which so much of his life was devoted. He was an outstanding student. This was reflected, for example, in his being awarded a Foundation Scholarship during his first undergraduate year. (Scholars of Trinity are selected on the basis of a special examination, typically taken during students' second of the four undergraduate years; in times past, some very able students achieved "Schol"in their first year.) A further mark of his ability is that, after graduating in 1951 with a Gold Medal, he went to Princeton and studied with William Feller, writing his PhD

2020 *Mathematics Subject Classification.* 01A70.
Key words and phrases. Brian Murdoch.
Received on 1-6-2021.

dissertation on preharmonic functions. Following that, he held posts at what is now Newcastle University and at Queen's University Belfast; he then returned to his alma mater as a Junior Lecturer in 1957. The rest of his career was spent in Trinity. Over the years, he lectured on various aspects of pure mathematics, notably analysis, probability and geometry. Among the topics addressed in his published papers were preharmonic functions and random walks. He gave much time and thought to his teaching and examining, and former students remember his kind and supportive approach. (I was one such student, and I recall him telling us, when we were preparing for Schol, how he worked on his examination papers — which, when drafts were revisited, sometimes looked too easy, so he would lengthen them or insert extra challenges. We decided that he had worked very diligently that year on his Schol paper, which was exceptionally long; however, it became obvious that he took this into account punctiliously when marking our scripts.) He was elected a Fellow of the College in 1965 — at the end of my own time as a mathematics undergraduate — and became Erasmus Smith's Professor of Mathematics in spring 1966, holding that post until his retirement in 1989.

The hallmark of Brian's long period in the chair was dedicated service. When first appointed, he brought welcome stability after each of his two predecessors had come and gone within a couple of years. In autumn 1966, he was joined in the School of Mathematics by David Spearman, who had been appointed as University Professor of Natural Philosophy. Working as a team, they transformed the School of Mathematics from one that focused on the development of a small number of outstanding mathematicians to one that was more accessible and had a wider vision: producing considerably more graduates — many of whom who would take their sound mathematical knowledge productively into aspects of life other than academic — while still catering for specifically dedicated mathematicians. The programme was broadened by giving students access to modules in computer science and statistics. A further initiative was the inception of joint degrees in theoretical physics, in mathematics and economics, and in mathematics and philosophy. These were pioneering individually designed courses which were introduced long before "2-subject Moderatorships"were adopted into the University curriculum.

Brian was a member of the Religious Society of Friends (Quakers), and his Quaker principles underpinned his life. This was reflected, not only in his selfless dedication to the School of Mathematics and its students, but also in his leadership style — he strove tirelessly for consensus — and in his work for various bodies outside the college at local and national level. He served on the management committees of Rathgar Junior School and Newtown School, both of which are Quaker schools: founded by Quakers and providing a Quaker ethos, while being open to students of all religions and none. Brian also represented Quakers on the Secondary Education Committee, the body established in 1968 to administer the "Protestant Block Grant Scheme"agreed with the Department of Education when free secondary education was introduced. (The scheme assists children of the Protestant community in accessing post-primary education in a school that accords with their faith tradition, and was devised as a means of extending free education to groups that would otherwise be unable to benefit suitably.)

Some of Brian's contributions to the world of school education were more specifically concerned with mathematics. A personal reminiscence dates back to my own early days in the Trinity School of Education, when the uptake of post-primary education in Ireland was expanding at the same time as the School of Mathematics was developing and diversifying as described above. Brian told me of the developments and identified the potential for producing more specialist teachers of mathematics. He gave much as an individual also. For years, representing Trinity, he worked with the Department of Education in checking Leaving Certificate papers. He played a role too for the National

Council for Curriculum and Assessment; he was a member of the Course Committee for Leaving Certificate Mathematics that drew up the courses introduced in the 1990s, making inputs especially with regard to probability. I served as Education Officer to that committee, and so had the pleasure of working alongside Brian and recognising at first hand his emphasis on seeking consensus. The other university mathematician on the committee was Paddy Barry, sadly also recently deceased; the two third-level representatives were notable contributors to the development of the courses. Brian was also actively engaged with the Young Scientist Exhibition, which has done so much to provide a stimulus for creative student work in the STEM area.

Brian's interests outside mathematics included travel and camping, initially on his own but later with his wife Winifred (Winnie), née Bewley. Subsequently, family holidays were often spent by the sea in Co. Galway and in France. He enjoyed classical music, and supported the Choral Society concerts in Trinity. Former students remember another aspect of his support for college activities; he was the contact for obtaining Bewley's brack and buns, greatly enjoyed features at the Dublin University Mathematical Society's annual Opening Meetings! Brian was a kind and friendly man and is fondly remembered by his colleagues. He is survived by Winnie, their children Hazel, Peter and Fiona, and grandchildren Amy, Zoe, Niamh, Katie, Lucy and Ruari.

My thanks are due to several people who helped in the production of this obituary, in particular to Winnie Murdoch, David Spearman, David Malone and Colm Mulcahy.

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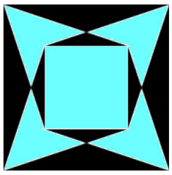

Irish Math. Soc. Bulletin
Number 87, Summer 2021, 19–22
ISSN 0791-5578

John J. O'Sullivan, 1948-2006
A career that remained under the radar

SÉAMUS HANLEY

Abstract. A brief account is given of John O'Sullivan's short but high-impact career. A link is provided to the video-recording of the ceremony at which he was posthumously granted the Pioneer Technology Award from the US Missile Defence Agency. The citations attest to his important technical and societal contributions.

The Terminal High Altitude Area Defense (THAAD) anti-ballistic missile defense system was developed by the USA and is now in use in Hawaii, the Middle East and South Korea. The THAAD interceptor system does not carry a warhead; its kinetic energy of impact destroys the incoming missile, and minimizes the risk of exploding conventional missiles, or detonating nuclear tipped ones.

A UCC-educated academic-turned-scientist played a central rôle — until now unknown to his peers — in developing the THAAD system.

John J. O'Sullivan was one of 13 children of a gardener who worked at Garnish Island, Glengarriff, West Cork. He was born in 1948. After receiving his PhD in the USA, he did a stint as an academic, but switched to defence work in the Washington DC area at age 32. He died at age 58 in 2006, two years before the Terminal High Altitude Area Defense system was first deployed.

Because of the under-the-radar nature of his work, conventional measures of John's 'footprint' do not apply. This note is meant to make his impact better known, and

2020 *Mathematics Subject Classification.* 01A70.
Key words and phrases. Obituary, Missile Defense.
Received on 20-4-2021; revised 10-5-2021.

(belatedly, with the help of today's information technology) to celebrate his far-too-short life.

John obtained his secondary education at St Augustine's College, Dungarvan, and won a Cork county-council scholarship to UCC in 1965. There, he quickly left the rest of us in the science programme in the dust: in the first year, he was one of just 12 students allowed into Professor Fahy's honours physics class, and he quickly caught the attention of Professors Paddy Barry, Finbarr Holland and Siobhán O'Shea in mathematics, and Professor Paddy Quinlan in mathematical physics. John earned his BSc in 1968 and his MSc in 1969.

In 1969, John won an NUI Travelling Studentship Prize, which paid for one year abroad. He used this, and a funding package he had already secured, to pursue a PhD in mathematics at the University of Notre Dame.

I do not remember why he took this one offer over the many other attractive offers he had. For our three undergraduate years, both John and I were lodged in the Honan Hostel at UCC; Professor Seán Teegan was the Warden. I do remember that Teegan — who had held a research fellowship at Notre Dame — invited John (and me because I was also considering post-graduate studies in North America) to watch travel-slides from his year at Notre Dame.

John earned his PhD in mathematics in 1973 and did post-doctoral work at the Institute for Advanced Studies in Princeton, and at Bonn University before taking up an academic position in the department of mathematics at Penn State in 1976. His academic research ([1, 2, 3, 4, 5] are examples) focused on differential geometry.

I was best man at his wedding in South Bend, Indiana and kept in contact with him while I was working in Buffalo and Boston. I visited him in Princeton and at State College, but lost contact with him after I moved back to Canada in 1980.

I was not surprized that John moved to applied work in the US defence domain. In an early project he led the mathematical modeling effort and was a key member of the software design team that developed the enlisted manpower forecasting system for the US Army. Later he managed the strategic defense technology division at another not-for-profit corporation, before joining the also not-for-profit Aerospace Corporation in 1989.

He did visit my family and me once in Montreal in the 1990s. But he tended to stay off-the-grid and under the radar. Unfortunately, the next I heard of him was in 2006, when I got a phone call from his brother (and godchild) Denis, who told me the sad news of John's untimely death.

How he died is ironic, and a reminder that in 2006, and even more so in 2019-2021, enemies can so easily penetrate our own personal defense systems, some of them even aided by our current medical environment. While leaving his office in the Pentagon, John spotted a colleague who was to give a briefing on Missile Defence the following week; John wanted to advise him on one or two things. In turning to brief his colleague, John twisted his ankle. The following morning his ankle had swollen to the extent that he could barely put on his shoe. He went to a hospital, where he was x-rayed, given crutches and painkillers, and sent home. That evening, he worked late into the night on a presentation he was giving the following day. During the night his partner Elaine went to check on him as he hadn't come to bed. She found him at his desk, unresponsive. He was rushed to hospital where he subsequently died. The post mortem examination revealed that John had contracted a Methicillin-resistant Staphylococcus aureus (MRSA) infection while attending that hospital: the bacteria had entered his body via a bullous blister on his sprained shin or ankle. John had never been out of work sick a single day in his life, and his family had often heard him say that his name was never on a prescription.

John's funeral was accorded the highest military honour that a civilian can receive in the USA. It was attended by the highest ranking members of the USA Army and Air Force. Also in attendance were many senior politicians from Congress. In a gesture accorded to very few, the American flag was flown at half mast on both the Pentagon and on Capital Hill during his funeral service. One of these flags now takes pride of place in his brother Denis' home in Midleton, Co. Cork.

At a 2007 meeting where several of John's American and Irish family were the guests of the Agency, John was posthumously given the Pioneer Technology Award from the US Missile Defence Agency.

The video-recording of the proceedings, prepared by the Agency, can now be viewed at `http://www.biostat.mcgill.ca/hanley/JohnOSullivanPioneerTechnologyAward2007.mp4`. The portion concerning John begins about six minutes in.

A two-minute video was played just before the award was presented by Lieutenant Obering, Director of the US Missile Defence Agency. It tells how, as one of the first members of the Phase One Engineering Team, John's technical expertise was recognized and respected. John later served as the director of the team, exhibiting superior leadership in his oversight of numerous activities focused on the resolution of high priority missile defense issues. The video narrative continues:

> Dr O'Sullivan's signature contribution to the development of missile defense technology, however, was his remarkable work on the Terminal High Altitude Area Defense program. He led the study that first identified the need for an upper-tier missile defense capability to support the PATRIOT system in defending U.S. deployed forces. He later established the framework for the development of the Terminal High Altitude Area Defense program and directed the engineering team support for the program office. He led the effort to identify and assess existing technologies that would provide the program with the capability to intercept missiles both inside and outside the earth's atmosphere. He also served as the chief technical advisor in the process of selecting the prime contractor for the program and was instrumental in the program's transition from research and development to testing.
>
> Dr O'Sullivan's tireless commitment to the Terminal High Altitude Area Defense program in particular, and his selfless dedication to the Phase One Engineering Team in general, have been essential to the development of a missile defense capability for the United States. His legacy lives or in the technologies that are now ensuring the safety and security of our nation.

Brigadier General Patrick O'Reilly, at the time Deputy Director, and later Director of the US Missile Defence Agency, prefaced the video with his own words, in which he spoke about John's 'sheer brilliance' in the missile defense technology, and how he was able to relate so well to people of all levels:

> That was a period of time where it was easy to get a room of this size of PhDs (> 200) to talk to you *ad nauseam* about why it wouldn't work. John was so brilliant he could answer their questions, and would adapt his answers to the audience he was talking to. He could base his answers on layman's terms, on linear terms, calculus-base answers, or tensor algebra. [...] In the 18 years I had the pleasure to work with him, I didn't ever see him not come back with an answer that required the smartest minds in our country who were saying it couldn't be done, and John was saying it could, and he was proved ultimately right over and over again.

Brigadier O'Reilly referred specifically to:

> the studies John did in the late 1980s, even before the Gulf War, on whether or not you could fly a space-designed missile system in the atmosphere. He was the one who truly saw, based on mathematical calculations, not just feelings

> or intuition, how the system could work, both in the 'endo' atmosphere (at an altitude below 100 Km) and the 'exo' atmosphere.

He also described how John was a pioneer in 'seeing the need for a cadre and generation of missile defense experts,' and how he spent 'countless hours' mentoring them.

At the opening of its new facilities in Huntsville, Alabama in 2008, Aerospace paid tribute to this 'legendary figure in ballistic missile defense' [6, page 53], naming the conference room in John's honour.

In his incoming interview with the Irish Examiner in 2017, the UCC President — a physicist who also contributed to work on the USA defence system, and who also has strong links to Glengarriff in West Cork — spoke of the Irish imagination and attitude. It is these that lead to our reputation 'as poets and raconteurs' but they can also 'add value to those in the fields of science, technology, engineering and mathematics.' The glowing testimonials from the 'top brass' in the US defence department are not limited to John's technical talents, but also his leadership, integrity and communication skills.

John's career has points in common with that of another USA-based but Irish-educated teacher-turned-inventor from a century earlier. John Philip Holland was born in the coastguard's residence in Liscannor, Co. Clare in 1841, finished his formal education at 13, and began thinking about submarines at age 17 when he began teaching at the North Monastery CBS in Cork. He quit teaching at age 32, and moved to the USA. He lived to age 73 and saw his submarine system through to its implementation and adoption by the US Navy.

As evidenced by Holland and O'Sullivan, the Irish imagination is perhaps even stronger in those who spent their youth close to the coastline, and had the time to contemplate both the oceans and the heavens. Such people not only dream big, they make Ireland punch well above its weight.

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Irish Math. Soc. Bulletin
Number 87, Summer 2021, 23–33
ISSN 0791-5578

On an inverse tangent problem

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Abstract. Even before the beginnings of Calculus a variety of methods for constructing tangents to plane curves were known. But what about the converse problem—raised by Debaune in the seventeenth century: under what conditions will a given collection of straight lines be tangents to the same curve? Utilizing Hermite's interpolation theorem, we show in Section 2 that the members of any *finite* collection of lines are tangents to infinitely many differentiable plane curves. After first developing a prescient observation of Descartes in Section 3, we state and prove our main theorem in Section 4,. This describes sufficient conditions for a one-parameter collection of lines in complex form to be the family of tangents of a differentiable curve in the complex plane. As an application, we derive Jakob Steiner's nineteenth century result that, all save three members of the collection of Wallace-Simson lines of a triangle, are tangents to a deltoid whose incircle is the nine-point circle of the triangle.

1. Introduction

Since the time of Descartes (1596–1650) and Fermat (1601–1665)—and indeed long before [1]—a variety of methods have been developed for constructing tangents to plane curves whose equations were known in different coordinate systems, explicitly or implicitly. But what about the inverse problem? Knowing the tangents to a curve, is it possible to determine its equation? This problem appears to have been first raised by Florimond Debaune (1601–1652) ([2], p. 351), but mathematicians of the day were unable to solve it. While Descartes made a pertinent observation about the problem, which we develop in Section 3, it was left to Leibniz (1646–1716) to provide a satisfactory answer several decades later ([2], p. 426), one that ultimately led to the study of differential equations.

It is Debaune's inverse tangent problem that motivates the topic discussed here, but we treat a slightly different question. Precisely, we ask: under what conditions are members of a collection of straight lines in the complex plane $\mathbb{C}$ tangents to a differentiable curve? We begin by showing that a finite number of lines in $\mathbb{C}$ are tangents to infinitely many polynomials, and, guided by an observation made by Descartes in response to Debuane's question, proceed to present sufficient conditions under which members of a one-parameter collection of lines in $\mathbb{C}$ are tangents to a differentiable curve. We illustrate our methods by showing that all (save three) members of the collection of Wallace-Simson lines of a triangle are tangents to the deltoid that encloses the nine-point circle of the triangle, a result due to Jakob Steiner [8].

2. A finite number of lines

To keep the algebra to a minimum, we'll work throughout with the complex form of the equation of a straight line, rather than with its cartesian form. If L is a straight

2020 *Mathematics Subject Classification.* 14H50, 51N25.

Key words and phrases. Tangent, curve, Hermite's interpolation formula, one-parameter family, Wallace-Simson lines, deltoid.

Received on 7-10-2020; revised 11-5-2021.

line in the complex plane, its characteristic feature is that the unimodular expression

$$t = \frac{a-b}{\bar{a}-\bar{b}}$$

is the same for every pair of distinct points a, b belonging to L. This invariant is called the *clinant*[1] of L—a useful term coined in 1890 by F. Franklin [3] and often cited in the work of Frank Morley ([5],[6]). Hence, if z and a belong to L, and are distinct, then $z - t\bar{z} = a - t\bar{a}$. Accordingly, the equation of a straight line in $\mathbb{C}$ can be described as the set of complex numbers z that satisfy an equation of the form $z + \tau\bar{z} = c$, where τ and c are constants, τ is a *turn*, i.e., a member of the unit circle T, and $\tau\bar{c} = c$; in which case $-\tau$ is the clinant of the line.

Given n such lines in $\mathbb{C}$ with equations $z + \tau_i\bar{z} = c_i,\ i = 1, 2, \ldots, n$, where, for each subscript i, $|\tau_i| = 1$, and $\tau_i\bar{c}_i = c_i$, we'll proceed to show that they are tangents to an analytic polynomial of degree $2n-1$. Before doing so, however, it's convenient to recall Hermite's interpolation problem, which calls for a polynomial to have preassigned values and derivatives at specified places. To set the scene, select n distinct (real or complex) numbers $x_1, x_2, \ldots, x_n$, and consider the problem of finding a polynomial p such that $p(x_i) = a_i$, $p'(x_i) = b_i$, $i = 1, 2, \ldots, n$, for preassigned real or complex numbers a_i, b_i, $i = 1, 2, \ldots, n$. Viewing this as a system of linear equations in the coefficients of p, and examining the matrix M of coefficients, which is of Vandermonde's type, it's not too difficult to show that $|\det M| = \prod_{1\le i<j\le n} |x_i - x_j|^4 > 0$. Hence, there is a unique polynomial p of degree $2n-1$ that interpolates the data. While this existence argument is sufficient for our purposes, it is useful to know Hermite's explicit formula for p. According to this, as can be readily verified,

$$p(x) = \sum_{i=1}^{n} \Big(a_i\big((1 - 2\pi_i'(x_i)(x - x_i)\big) + b_i(x - x_i)\Big)\pi_i(x)^2,$$

where $\pi(x) = \prod_{i=1}^{n}(x - x_i)$, and

$$\pi_i(x) = \frac{\prod_{j\neq i}(x - x_j)}{\pi'(x_i)},\ i = 1, 2, \ldots, n.$$

In fact, this is a special case of a more general interpolation formula due to Spitzbart[7].

Returning to our tangent problem: for $i = 1, 2, \ldots, n$, denote by u_i a square root of $-\tau_i$, take $a_i = c_i/2$, $b_i = u_i$, and apply Hermite's result to obtain an analytic polynomial p of degree $2n-1$ such that

$$p(x_i) = \frac{c_i}{2},\ p'(x_i) = u_i,\ i = 1, 2, \ldots, n$$

Consider now the tangent to p at the point $p(x_i)$; since $p'(x_i) \neq 0$ it has equation $0 = \Im\{(z - p(x_i))\overline{p'(x_i)}\}$. Inserting the values of p and p' at x_i, this equation reduces to

$$0 = z - \frac{c_i}{2} - \Big(\bar{z} - \frac{\bar{c}_i}{2}\Big)\frac{u_i}{\bar{u}_i}.$$

But, by hypothesis, $|\tau_i| = 1$, $\tau_i\bar{c}_i = c_i$, and $u_i^2 = -\tau_i$, by choice. Hence, the latter form of the equation of the tangent becomes $z + \tau_i\bar{z} = c_i$, as required.

By way of illustration, we'll apply Hermite's formula to derive the equation of a quintic polynomial $p : \mathbb{R} \to \mathbb{C}$ that touches the three lines

$$L_1 : z + \bar{z} = -2;\ L_2 : z + \bar{z} = 0;\ L_3 : z + \bar{z} = 2;$$

[1]Two lines are parallel iff they have the same clinant, and perpendicular, iff the sum of their clinants is zero,

at the points $p(-1)$, $p(0)$, and $p(1)$, respectively. These lines are parallel to the imaginary axis having the same clinant, viz., -1, the square of i. Also, $c_1 = -2, c_2 = 0$, and $c_3 = 2$. In addition, $x_1 = -1, x_2 = 0$, and $x_3 = 1$, whence $\pi(x) = x(x^2 - 1)$ and

$$\pi_1(x) = \frac{1}{2}x(x-1),\ \pi_2(x) = 1 - x^2,\ \pi_3(x) = \frac{1}{2}x(x+1).$$

Applying the formula, with $a_1 = c_1/2 = -1$, $a_2 = c_2/2 = 0$, $a_3 = c_3/3 = 1$, and $b_1 = b_2 = b_3 = i$, after some tedious calculation we get

$$p(z) = \frac{1}{2}(3(i-1)z^5 - 5(i-1)z^3 + 2iz),\ z \in \mathbb{R},$$

as the desired polynomial, which can, of course, be verified directly.

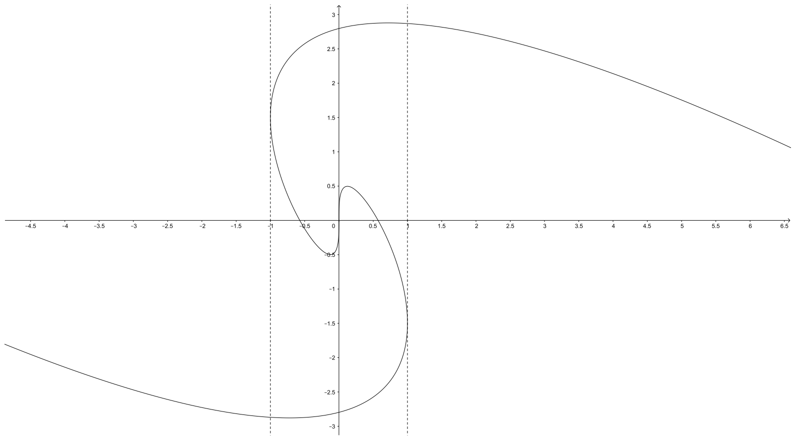

FIGURE 1. Quintic polynomial touching 3 parallel lines; $x = 0, x = \pm 1$

It can also be easily verified that the concurrent lines

$$M_1 : z - i\bar{z} = 0; M_2 : z + \bar{z} = 0; M_3 : z + i\bar{z} = 0;$$

are tangents to the quintic

$$\frac{1}{4}z(z^2-1)\Big(uz(z-1) - 4i(1-z^2) + \bar{u}z(z+1)\Big),\ z \in \mathbb{R},$$

where $u := (1+i)/\sqrt{2}$ is a square root of i, the clinant of M_1.

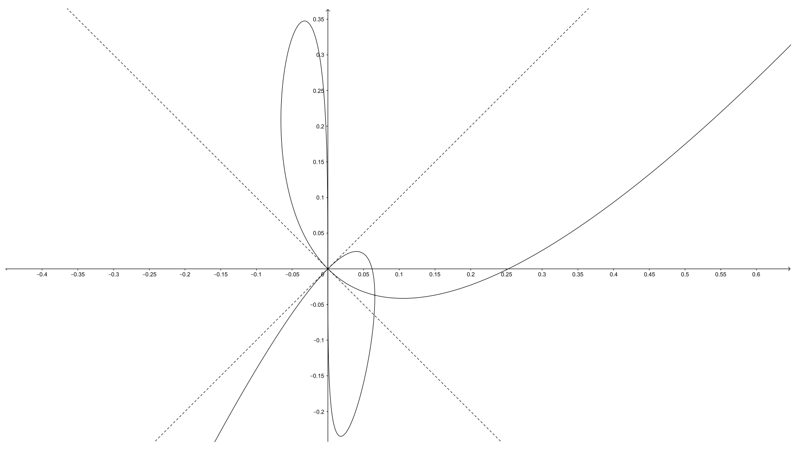

FIGURE 2. Quintic polynomial touching 3 concurrent lines; $x = 0, y = \pm x$

A slight modification of the argument preceding these illustrative examples shows that the given lines $z + \tau_i \bar{z} = c_i, i = 1, 2, \ldots, n$, are also tangent to a trigonometric polynomial. To see this, select n real numbers $\theta_1, \theta_2, \ldots, \theta_n$, so that the turns $x_k = e^{i\theta_k}$, $k = 1, 2, \ldots, n$, are distinct. Keeping the same notation as before, determine the analytic polynomial p of degree $2n - 1$ that satisfies the conditions

$$p(x_k) = \frac{c_k}{2},\ p'(x_k) = i\bar{x}_k u_k,\ k = 1, 2, \ldots, n.$$

Define the trigonometric polynomial f on $(-\infty, \infty)$ by $f(x) = p(e^{ix})$. If $f'(x) \neq 0$, the equation of the tangent at $f(x)$ is given by

$$z - \frac{f'(x)}{\overline{f'(x)}}\bar{z} = f(x) - \frac{f'(x)}{\overline{f'(x)}}\overline{f(x)}.$$

In particular, since $f(\theta_k) = p(x_k) = \frac{c_k}{2}$, and $f'(\theta_k) = ix_k p'(x_k) = -u_k \neq 0$, the equation of the tangent at $f(\theta_k)$ is the set of z such that

$$\begin{aligned}
0 &= z - \frac{u_k}{\bar{u}_k}\bar{z} - \frac{c_k}{2} + \frac{u_k}{\bar{u}_k}\frac{\bar{c}_k}{2} \\
&= z - u_k^2\bar{z} - \frac{c_k}{2} + u_k^2\frac{\bar{c}_k}{2} \\
&= z + \tau_k\bar{z} - \frac{c_k + \tau_k\bar{c}_k}{2} \\
&= z + \tau_k\bar{z} - c_k,
\end{aligned}$$

since $\tau_k\bar{c}_k = c_k$, by hypothesis. Thus the family of lines $z + \tau_i\bar{z} = c_i$, $i = 1, 2, \ldots, n$, are tangents to f, a 2π-periodic function.

It follows from this that any n straight lines are tangents to infinitely many analytic polynomials of degree $2n - 1$, and also to infinitely many trigonometric polynomials of degree $2n - 1$. The latter means, in particular, that the lines are tangents to many closed curves in $\mathbb{C}$.

Contrast this statement with the fact that three non-concurrent lines, no two of which are parallel, are tangents to precisely four circles, namely, the incircle and the three excircles of the triangle determined by the lines, something we learn in secondary school. For instance, the lines

$$z - \bar{z} = 0; z + i\bar{z} = 1 + i; z - i\bar{z} = -1 + i;$$

are tangents to the four circles

$$C_1 : |z - (\sqrt{2} - 1)i| = \sqrt{2} - 1; C_2 : |z - \sqrt{2} - i)| = 1; C_3 : |z + \sqrt{2} - i| = 1;$$

and $C_4 : |z + (\sqrt{2} + 1)i| = \sqrt{2} + 1$. Of these, C_1 is the incircle of the triangle with vertices $-1, 1$, and i, and C_2, C_3 and C_4 are its excircles.

This raises the possibility, that, by imposing suitable incidence relations on a set of n lines, it may be possible to produce a finite number of closed curves to which some or all of the lines are tangents. Our intention is to explore this possibility in a future publication, whose purpose is to complement the approach taken in [5], where it is shown that finitely many curves of a certain kind touch n lines.

3. Descartes' insight

We learn from ([2], p. 426), that Descartes gave the following response to Debeaune about the latter's inverse tangent problem mentioned in Section 1: "I do not believe that it is in general possible to find the converse to my rule of tangents, nor of that which M. Fermat uses, ...". But, on the same page, he leaves the following insightful remark to posterity: "There is indeed another method that is more general and a priori, namely, by the intersection of two tangents, which should always intersect between the points at which they touch the curve, as near one another as you can imagine; for in considering what the curve ought to be, in order that this intersection may occur between the two points, and not on that side or the other, the construction for it may be found."

This statement appears to apply in particular to the graphs of real convex (or concave) functions defined on subintervals of the real axis, and one can present sufficient conditions for it to hold for parametrically defined functions. What follows is our interpretation of what we believe Descartes may have had in mind.

Theorem 3.1. *Suppose I is a subinterval of $(-\infty, \infty)$ and $\gamma : I \to \mathbb{C}$ is twice continuously differentiable on I, and determines a curve Γ with non-zero curvature at a point $u \in I$. Then there exists a neighbourhood N of u such that if $s, t \in N$ and $s \neq t$, the tangents L_t and L_s to Γ at $\gamma(t)$ and $\gamma(s)$, respectively, intersect at a unique point $z(t, s)$, say, and*

$$\lim_{t\to u} z(t, u) = \gamma(u).$$

Proof. By hypothesis, γ is differentiable on I and its derivative doesn't vanish there. Therefore the equation of the tangent to Γ at any point $\gamma(t)$ is given by the set of z such that $\Im\{(z - \gamma(t))\overline{\gamma'(t)}\} = 0$, equivalently, $z + \tau(t)\bar{z} = c(t)$, where

$$\tau(t) = -\frac{\gamma'(t)}{\overline{\gamma'(t)}}, \text{ and } c(t) = \gamma(t) + \tau(t)\bar{\gamma}(t).$$

Hence

$$-\tau'(t) = \frac{\gamma''(t)\overline{\gamma'(t)} - \gamma'(t)\overline{\gamma''(t)}}{\overline{\gamma'(t)^2}} = \frac{2i\Im\big(\gamma''(t)\overline{\gamma'(t)}\big)}{\overline{\gamma'(t)^2}}.$$

This expression is continuous and non-zero at u, by assumption. Hence, by continuity, at least one of $\Re\tau', \Im\tau'$ is non-zero on some neighbourhood N of u. Hence, by the Mean Value Theorem, at least one of $\Re\tau, \Im\tau$ is one-one on N, whence τ is one-one on N. Consequently, if $t, s \in N$, and $t \neq s$, the corresponding tangents L_t, L_s intersect and their point of intersection $z(t, s)$ is given by

$$\bar{z}(t, s) = \frac{c(t) - c(s)}{\tau(t) - \tau(s)}.$$

Clearly,

$$\begin{aligned}
\lim_{t\to u} \bar{z}(t, u) &= \frac{c'(u)}{\tau'(u)} \\
&= \frac{\gamma'(u) + \tau(u)\overline{\gamma'(u)} + \tau'(u)\bar{\gamma}(u)}{\tau'(u)} \\
&= \frac{\tau'(u)\bar{\gamma}(u)}{\tau'(u)} \\
&= \bar{\gamma}(u).
\end{aligned}$$

In other words, the claim is true. $\square$

This result seems to be the basis for the recipe utilized by various authors who seek to determine the equation of a curve from a one-parameter set of lines they assume are its tangents.

4. One-parameter families of lines

As we've seen, the members of any *finite* collection of straight lines are tangents to infinitely many curves. However, this no longer holds if the collection is infinite. For instance, it's easy to see that the members of the one-parameter family of lines indexed by t on $[0,\infty)$, with cartesian equations $y+tx=1$, are not all tangents to the same planar curve defined on $[0,\infty)$. In this section, we prescribe sufficient conditions for all or some members of a one-parameter family of lines in $\mathbb{C}$ to be tangents to the same curve.

Definition 4.1. We call a pair of functions ϕ,ψ defined on an interval I of the real line *compatible on* I, if, for all $t\in I$, $|\phi(t)|=1$ and $\phi(t)\overline{\psi(t)}=\psi(t)$.

For instance, the members of each of the ordered pairs $(1,\cos x)$, $((ix-1)/(ix+1), i/(ix+1))$, and $(e^{inx},(1+e^{ix})^n)$, where n is a nonnegative integer, are compatible on any subinterval of $(-\infty,\infty)$.

Such a pair of compatible functions defines a one-parameter family of lines indexed on I, with equations $z+\phi(t)\bar{z}=\psi(t),\ t\in I$. Under what circumstances will such a pair generate lines some or all of which are tangents to a differentiable curve parameterised on I?

To get a handle on this problem, notice as before that if $f:I\to\mathbb{C}$ is differentiable and $t\in I$, then the curve $\Gamma_f=f(I)$, has a tangent T_t at $f(t)$ as long as $f'(t)\neq 0$, in which case its equation is the set of z such that $\Im\{(z-f(t))\overline{f'(t)}\}=0$, equivalently, $z+\tau_f(t)\bar{z}=c_f(t)$, where

$$\tau_f(t)=-\frac{f'(t)}{\overline{f'(t)}}\ \text{and}\ c_f(t)=f(t)+\tau_f(t)\overline{f(t)}.$$

Clearly, τ_f and c_f are compatible on I. Consequently, if, for some compatible pair ϕ,ψ on I, and some $t\in I$, the equation $z+\phi(t)\bar{z}=\psi(t)$ coincides with that for T_t, then $\phi(t)=\tau_f(t)$ and $\psi(t)=c_f(t)$, so that

$$\overline{f'(t)}\phi(t)+f'(t)=0,\ \text{and}\ f(t)+\overline{f(t)}\phi(t)=\psi(t).$$

Conversely, if for a given pair of compatible functions these functional equations are satisfied by an appropriate function f at some point $t\in I$, the set of z such that $z+\phi(t)\bar{z}=\psi(t)$ is the tangent to Γ_f at $f(t)$.

To consider further the solution f of these last displayed equations, assume ϕ,ψ are differentiable on I. Then, by differentiation of the second equation, and using the first, we see that

$$\begin{aligned}\psi'(t) &= \frac{d}{dt}\big(f(t)+\overline{f(t)}\phi(t)\big)\\ &= f'(t)+\overline{f'(t)}\phi(t)+\overline{f(t)}\phi'(t)\\ &= \overline{f(t)}\phi'(t).\end{aligned}$$

In other words, at least formally,

$$f(t)=\frac{\overline{\psi'(t)}}{\overline{\phi'(t)}}.$$

This formula, backed up by Theorem 3.1, suggests a means of recovering the equation of a curve some or all of whose tangents are assumed to be of the form $z+\bar{z}\phi(t)=\psi(t)$, for some $t\in I$, where ϕ,ψ are at least compatible and possess certain differentiability properties, as yet unstated. The next theorem supports this statement.

Theorem 4.2. *Suppose ϕ, ψ are compatible on I, twice differentiable there and such that ϕ' and $\psi''\phi' - \psi'\phi''$ are both non-zero on I. Then the collection of lines L_t : $z + \phi(t)\bar{z} = \psi(t)$, $t \in I$, coincides with the family of tangents to a differentiable curve C parameterised on I by the complex conjugate of ψ'/ϕ'.*

Proof. Define f to be the complex conjugate of ψ'/ϕ'. Let $C = f(I)$. Since by hypothesis, $\phi' \neq 0$, f is well-defined on I and differentiable there with derivative given by

$$\bar{f}' = \frac{\psi''\phi' - \psi'\phi''}{(\phi')^2}, \tag{1}$$

which is non-zero on I by assumption. Hence, the tangent T to C at $f(t)$ has equation $z + \tau(t)\bar{z} = c(t)$, where $\tau(t) = -\frac{f'(t)}{\overline{f'(t)}}$ and $c(t) = f(t) + \tau(t)\overline{f(t)}$.

We claim that T coincides with L_t. Since $\phi\bar{\psi} = \psi$, and $\frac{d}{dx}\bar{g} = \overline{g'}$ for any differentiable function g on $(-\infty, \infty)$, we have that $\phi'\bar{\psi} + \phi\bar{\psi}' = \psi'$, whence

$$\bar{\psi} + \frac{\phi\bar{\psi}'}{\phi'} = \bar{f}, \text{ and so } \phi\bar{f} = \phi\bar{\psi} + \frac{\phi^2\bar{\psi}'}{\phi'} = \psi + \frac{\phi^2\bar{\phi}' f}{\phi'}.$$

Hence

$$f + \phi\bar{f} = \psi + (1 + \frac{\phi^2\bar{\phi}'}{\phi'})f = \psi,$$

because $1 = \phi\bar{\phi}$ and so $0 = \phi'\bar{\phi} + \phi\overline{\phi'}$. Hence, in particular, $f(t) + \phi(t)\overline{f(t)} = \psi(t)$, which means that $f(t) \in L_t$. Next, we prove that T and L_t have the same clinants. The claim is that $f'(t) + \phi(t)\overline{f'(t)} = 0$. But, as we've just seen, $f + \phi\bar{f} = \psi$, hence $f' + \phi\overline{f'} + \phi'\bar{f} = \psi' = \phi'\bar{f}$, which means that $f' + \phi\overline{f'} = 0$, and so, in particular, the claim is true. Thus, L_t and T are parallel, and so coincident, since they share the point $f(t)$.

□

Example 4.3. All but one of the lines

$$z + e^{3ix}\bar{z} = (1 + e^{ix})^3,$$

parameterised on $[0, 2\pi]$, is a tangent to the cardioid curve $z(x) = (1 + e^{ix})^2$.

Proof. The family of given lines is generated by the compatible functions $\phi(x) = e^{3ix}$, and $\psi(x) = (1 + e^{ix})^3$. Also, $\phi' \neq 0$ and

$$\frac{\psi'(x)}{\phi'(x)} = \frac{3ie^{ix}(1 + e^{ix})^2}{3ie^{3ix}} = (1 + e^{-ix})^2,$$

so that

$$\psi''(x)\phi'(x) - \psi'(x)\phi''(x) = \phi'(x)^2\Big(\frac{\psi'(x)}{\phi'(x)}\Big)' = -2i\phi'(x)^2 e^{-ix}(1 + e^{-ix}).$$

Hence $\psi''(x)\phi'(x) - \psi'(x)\phi''(x)$ is non-zero save at $x = \pi$. Hence Theorem 4.2 applies on each of the intervals $[0, \pi)$, $(\pi, 2\pi]$, and the stated result follows. □

Example 4.4. Suppose $|\alpha| \neq 1$. All the lines

$$z + \frac{e^{2ix} - \alpha}{1 - \bar{\alpha}e^{2ix}}\bar{z} = \frac{e^{ix}}{1 - \bar{\alpha}e^{2ix}},$$

parameterised on $[0, 2\pi]$, are tangents to an ellipse.

Proof. To begin with, it's not too difficult to see that the differentiable functions ϕ, ψ defined on $[0, 2\pi]$ by

$$\phi(x) = \frac{e^{2ix} - \alpha}{1 - \bar{\alpha}e^{2ix}}, \ \psi(x) = \frac{e^{ix}}{1 - \bar{\alpha}e^{2ix}},$$

are compatible, satisfy the conditions of Theorem 4.2, and that

$$\phi'(x) = \frac{2e^{ix}(1 - |\alpha|^2)}{(1 - \bar{\alpha}e^{2ix})^2}, \ \psi'(x) = \frac{1 + \bar{\alpha}e^{2ix}}{(1 - \bar{\alpha}e^{2ix})^2},$$

Hence the given lines are tangents to the curve defined by

$$f(x) = \frac{e^{ix} + \alpha e^{-ix}}{2(1 - |\alpha|^2)}.$$

which describes an ellipse. □

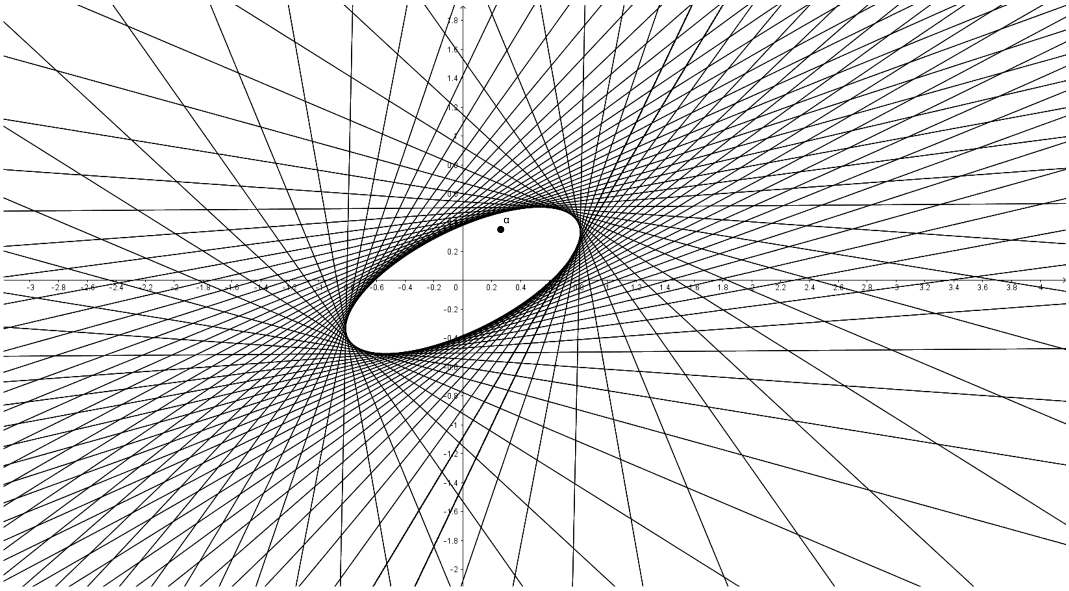

FIGURE 3. A family of lines that touch an ellipse

5. WALLACE-SIMSON LINES OF A TRIANGLE

Suppose the numbers t_1, t_2, and t_3 are distinct turns. Consider them to be the vertices of a triangle ABC inscribed in the unit circle T, and denote by s_1, s_2, and s_3 their corresponding symmetric polynomials. The three equations

$$z + t_1t_2\bar{z} = t_1 + t_2, \ z + t_2t_3\bar{z} = t_2 + t_3, \ z + t_3t_1\bar{z} = t_3 + t_1$$

are those of the sides of ABC. If $|t| = 1$, it's easy to verify that the numbers p, q, r, defined by

$$2pt = t^2 + (t_1 + t_2)t - t_1t_2, \ 2qt = t^2 + (t_2 + t_3)t - t_2t_3, \ 2rt = t^2 + (t_3 + t_1)t - t_3t_1,$$

are the projections from t onto these lines. (For instance, p is on the line $z + t_1t_2\bar{z} = t_1 + t_2$, and t_1t_2 is the clinant of the line joining t and p.) Also,

$$2t(p - q) = (t_1 - t_3)t - t_1t_2 + t_2t_3 = (t_1 - t_3)(t - t_2),$$

so that

$$\frac{p - q}{\bar{p} - \bar{q}} = \bar{t}t_1t_2t_3 = \bar{t}s_3.$$

Hence, $\bar{t}s_3$ is the clinant of the line through p and q. By symmetry, this is also the clinant of the line through q and r. Hence, the points p, q, and r are collinear, and lie on the line $WS(t)$, one form of whose equation is

$$z - \bar{t}s_3\bar{z} = p - \bar{t}s_3\bar{p}.$$

This line is called the Wallace-Simson line of ABC associated with t.

5.1. **The Steiner deltoid of ABC.** Inserting the value of p given above, an alternative form of the equation for $WS(t)$ follows, namely,

$$z - \bar{t}s_3\bar{z} = \frac{1}{2}(t + s_1 - s_2\bar{t} - s_3\bar{t}^2).$$

Writing $\phi(t) = -\bar{t}s_3$ and $\psi(t) = p - \bar{t}s_3\bar{p} = p + \phi(t)\bar{p}$, observe that $|\phi(t)| = 1$, and so

$$\phi(t)\overline{\psi(t)} = \phi(t)\bar{p} + \phi(t)\overline{\phi(t)}p = |\phi(t)|^2 p + \phi(t)\bar{p} = p + \phi(t)\bar{p} = \psi(t).$$

Hence, the infinitely differentiable functions $\phi(e^{ix})$, $\psi(e^{ix})$ are compatible on $(-\infty, \infty)$. The equations

$$z + \phi(x)\bar{z} = \psi(x),\ 0 \le x \le 2\pi,$$

therefore determine the one-parameter family of Wallace-Simson lines of ABC. Utilizing Theorem 4.2 we'll show that all but three members of this family are tangents to a three-cusped hypocycloid. To this end, note that

$$\phi(x) = -e^{-ix}s_3,\ \psi(x) = \frac{1}{2}(e^{ix} + s_1 - s_2e^{-ix} - s_3e^{-2ix}),$$

and so

$$\phi'(x) = ie^{-ix}s_3,\ \psi'(x) = \frac{1}{2}(ie^{ix} + is_2e^{-ix} + 2s_3ie^{-2ix}).$$

Hence

$$\frac{\psi'(x)}{\phi'(x)} = \frac{e^{2ix} + s_2 + 2s_3e^{-ix}}{2s_3} = \frac{1}{2}(\bar{s}_3e^{2ix} + \bar{s}_1 + 2e^{-ix}),$$

since $s_2 = \bar{s}_1s_3$. According to Theorem 4.2, the function f whose tangents are among those of the given family of Wallace-Simson lines is given by

$$f(x) = \frac{1}{2}(s_3e^{-2ix} + s_1 + 2e^{ix}),\ 0 \le x \le 2\pi,$$

which is the equation of a deltoid, a closed curve with three cusps, resembling a curvilinear equilateral triangle. Alternatively, it can be viewed as the image of the unit circle under the map

$$z_S(t) = \frac{1}{2}(s_1 + 2t + s_3\bar{t}^2) = \frac{t_1 + t_2 + t_3}{2} + t + \frac{t_1t_2t_3}{2}\bar{t}^2,\ |t| = 1.$$

Since

$$f'(x) = -ie^{-2ix}(e^{3ix} - s_3) = -ie^{-2ix}(e^{3ix} - t_1t_2t_3),$$

f fails to have tangents at only its three cusp points, namely the points $z_S(\alpha), z_S(\beta), z_S(\gamma)$, where the turns α, β, γ are the distinct cube roots of $t^3 - s_3$.

Notice that the constant term $\frac{1}{2}s_1 = \frac{1}{2}(t_1 + t_2 + t_3)$ in the equation of f is the centre of the nine-point circle associated with ABC, whose equation is $|z - \frac{s_1}{2}| = \frac{1}{2}$. Hence, if $|t| = 1$, the point $(s_1 + 2t + s_3\bar{t}^2)/2$ on the deltoid lies on the nine-point circle iff

$$|2t + s_3\bar{t}^2| = 1, \text{ i.e., } |2t^3 + s_3| = 1,$$

the solutions of which satisfy $t^3 = -s_3$. This is so because if $|u| = 1$, then $|2u + 1| = 1$ iff $u = -1$. Thus, the nine-point circle of ABC touches the deltoid at three points. Moreover, for all $t \in T$,

$$|z_S(t) - \frac{s_1}{2}| = \frac{1}{2}|2t + s_3\bar{t}^2| \ge \frac{1}{2}|2|t| - |s_3\bar{t}^2|| = \frac{1}{2}.$$

Therefore the nine-point circle of a triangle is inscribed in the deltoid generated by its family of Wallace-Simson lines. We refer to this deltoid as the *Steiner* deltoid of ABC in honour of Jakob Steiner who discovered these results in the nineteenth century [8]; and denote it by δ_S.

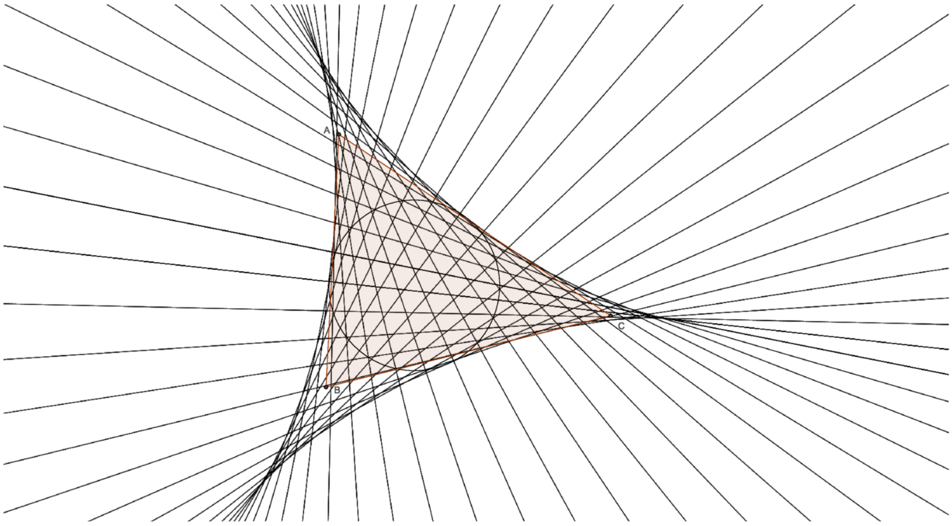

FIGURE 4. Triangle, 9 point circle, and Steiner deltoid

5.2. **An outer deltoid of ABC.** This section was motivated by the article [4].

Consider the line through a turn t that is parallel to $WS(t)$, its Wallace-Simson line with respect to the same triangle ABC. Since the clinant of $WS(t)$ is $\bar{t}s_3$, the equation of the line under discussion has equation $z - \bar{t}s_3\bar{z} = t - s_3\bar{t}^2$. Therefore, as t varies in T, another one-parameter family of such lines is generated, some or all of which are tangent lines to a curve whose equation can be found by appealing to Theorem 4.2 once more. Indeed, setting $\phi(x) = -e^{-ix}s_3, \psi(x) = e^{ix} - s_3e^{-2ix}$, the curve in question can be seen to have equation $f(x) = 2e^{ix} + s_3e^{-2ix} \equiv z_o(e^{ix})$, where z_o is defined on T by $z_o(t) = 2t + s_3\bar{t}^2$, $|t| = 1$. This curve is another deltoid, δ_o, say, whose cusps occur at the roots on T of the cubic $t^3 - s_3$. Observe also that

$$|z_o(t)| \geq ||2t| - |s_3||t^2|| = |2 - 1| = 1,$$

for all $t \in T$. Hence, δ_o encloses T and touches it at some point $t \in T$, iff $|2t^3 + s_3| = 1$, i.e., iff $t^3 = -s_3$. In other words, T is the inscribed circle of δ_o, which touches it at precisely three points. Thus, δ_o encloses ABC, and touches its circumscribing circle T at three points. We refer to δ_o as an outer deltoid associated with ABC. Since $2z_S = s_1 + z_0$, it's evident that δ_S is a translate of a scaled version of δ_o. Clearly, tangents to δ_S are parellel to those of δ_o.

For example, if $t_1 = -1, t_2 = 1$, and $t_3 = i$, so that $s_1 = i, s_2 = -1$, and $s_3 = -i$, ABC is a right-angled triangle, whose nine-point circle is $|z - \frac{i}{2}| = \frac{1}{2}$, whose Wallace-Simson lines have equations $z + i\bar{t}\bar{z} = \frac{1}{2}(i + t + \bar{t} + i\bar{t}^2)$, $t \in T$, and whose Steiner deltoid and outer deltoid, respectively, shown in the accompanying diagram, have equations $z_S(t) = \frac{1}{2}(i + 2t - i\bar{t}^2)$ and $z_o(t) = 2t - i\bar{t}^2$, respectively.

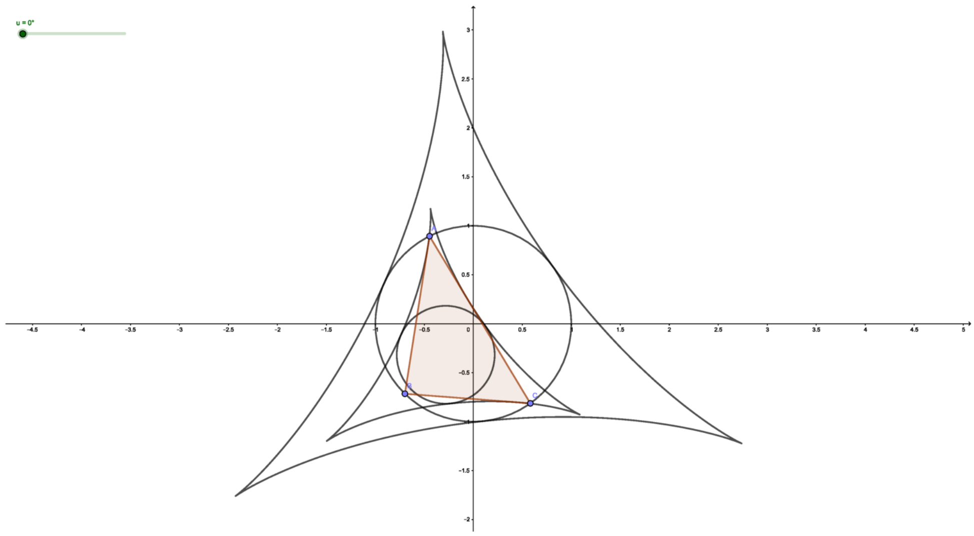

FIGURE 5. The two deltoids

ACKNOWLEDGEMENT

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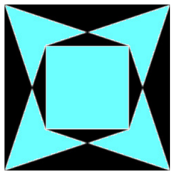

Irish Math. Soc. Bulletin
Number 87, Summer 2021, 35–43
ISSN 0791-5578

Weighted Sylvester sums on the Frobenius set

TAKAO KOMATSU AND YUAN ZHANG

Abstract. Let a and b be relatively prime positive integers. In this paper the weighted sum $\sum_{n\in\mathrm{NR}(a,b)}\lambda^{n-1}n^m$ is given explicitly or in terms of the Apostol-Bernoulli numbers, where m is a nonnegative integer, and $\mathrm{NR}(a,b)$ denotes the set of positive integers nonrepresentable in terms of a and b.

1. Introduction

The *Frobenius Problem* is to determine the largest positive integer that is NOT representable as a nonnegative integer combination of given positive integers that are coprime (see [13] for general references).

Given positive integers $a_1,\dots,a_m$ with $\gcd(a_1,\dots,a_m)=1$, it is well-known that for all sufficiently large n the equation

$$a_1x_1+\cdots+a_mx_m=n \tag{1}$$

has a solution with nonnegative integers $x_1,\dots,x_m$.

The *Frobenius number* $F(a_1,\dots,a_m)$ is the LARGEST integer n such that (1) has no solution in nonnegative integers. For $m=2$, we have

$$F(a,b)=(a-1)(b-1)-1$$

(Sylvester (1884) [17]). For $m\ge 3$, exact determination of the Frobenius number is difficult. The Frobenius number cannot be given by closed formulas of a certain type (Curtis (1990) [6]), the problem of determining $F(a_1,\dots,a_m)$ is NP-hard under Turing reduction (see, e.g., Ramírez Alfonsín [13]). Nevertheless, the Frobenius numbers for some special cases are calculated (e.g., [12, 14, 16]). One convenient formula is by Johnson [9]. An analytic approach to the Frobenius number can be seen in [4, 10]. Some formulae for the Frobenius number in three variables can be seen in [19].

For given a and b with $\gcd(a,b)=1$, let $\mathrm{NR}(a,b)$ denote the set of nonnegative integers nonrepresentable in term of a and b, namely the set of all those nonnegative integers n which cannot be expressed in the form $n=ax+by$, where x and y are nonnegative integers.

There are many kinds of problem related to the Frobenius problem. The problems of the number of solutions (e.g., [18]), and the sum of integer powers of the gaps values in numerical semigroups (e.g., [5, 8, 7]) are popular. Another famous problems is about the so-called *Sylvester sums* $\sum_{n\in\mathrm{NR}(a,b)}n^m$, where m is a nonnegative integer (see, e.g., [20] and references therein). Recently in [3], a more general case is considered, involving the largest integer, the number of integers and the sum of integers whose number of representation is exactly equal to a given number k, and is tackled using similar power sums.

2020 *Mathematics Subject Classification.* 05A15, 11D07, 11B68, 11P81, 05A17.

Key words and phrases. Frobenius problem, weighted sums, Sylvester sums, Apostol-Bernoulli numbers.

Received on 28-2-2021; revised 13-5-2021.

In this paper, we consider the weighted sum

$$S_m^{(\lambda)}(a,b) := \sum_{n\in\mathrm{NR}(a,b)} \lambda^{n-1}n^m \quad (\lambda \neq 0)\,.$$

Sylvester [17] showed that $S_0^{(1)}(a,b) = (a-1)(b-1)/2$, and Brown and Shuie showed [5] that

$$S_1^{(1)}(a,b) = \frac{1}{12}(a-1)(b-1)(2ab-a-b-1)\,.$$

Rødseth [15] obtained a general formula for $S_m^{(1)}$ in terms of Bernoulli numbers and deduced

$$S_2^{(1)}(a,b) = \frac{1}{12}(a-1)(b-1)ab(ab-a-b)\,.$$

Tuenter [20] also investigated $S_m^{(1)}$ by taking a different approach. He established relations between Sylvester sums and the power sums over the natural numbers. Wang and Wang [21] considered the alternating Sylvester sums

$$T_m(a,b) = \sum_{n\in\mathrm{NR}(a,b)} (-1)^n n^m$$

by using Bernoulli and Euler numbers.

The purpose of this paper is to give an explicit expression for $S_m^{(\lambda)}(a,b)$. For $m=1$, we can give the following formula.

Theorem 1.1. *For $\lambda \neq 0$ with $\lambda^a \neq 1$ and $\lambda^b \neq 1$,*

$$S_1^{(\lambda)}(a,b) = \frac{1}{(\lambda-1)^2} + \frac{ab\lambda^{ab-1}}{(\lambda^a-1)(\lambda^b-1)} - \frac{(\lambda^{ab}-1)\big((a+b)\lambda^{a+b}-a\lambda^a-b\lambda^b\big)}{\lambda(\lambda^a-1)^2(\lambda^b-1)^2}\,.$$

We also give a general expression for $S_m^{(\lambda)}(a,b)$ in terms of the Apostol-Bernoulli numbers. The alternating Sylvester sums in [21] can be also expressed as $T_m(a,b) = -S_m^{(-1)}(a,b)$.

The main new results (Theorems 4.1 and 4.3 below) cover all values of m and λ, and express $S_m^{(\lambda}(a,b)$ in terms of the Apostol-Bernoulli numbers. In case $m=1$ and $\lambda^a \neq 1$ the expressions reduce to those given explicitly in Theorem 1.1.

2. An explicit expression for $m=1$

As in [5], define

$$f(x) = \sum_{n=0}^{ab-a-b} \big(1-r(n)\big)x^n\,,$$

where $r(n)$ denotes the number of representations of n in the form $n = sa+tb$, where s and t are nonnegative integers. Since $r(n) = 0$ or 1 for $0 \le n \le ab-1$, we have

$$\begin{aligned} f'(\lambda) &= \sum_{n=1}^{ab-a-b} n\big(1-r(n)\big)\lambda^{n-1} = \sum_{\substack{1\le n\le ab-a-b\\ r(n)=0}} n\lambda^{n-1} \\ &= \sum_{n\in\mathrm{NR}(a,b)} \lambda^{n-1}n = S_1^{(\lambda)}(a,b)\,. \end{aligned}$$

We use the following fact from [5].

Lemma 2.1.

$$f(x) = \frac{g(x)}{h(x)}\,,$$

where

$$g(x) = \sum_{k=1}^{b-1} \frac{x^{ak} - x^k}{x-1} \quad \textit{and} \quad h(x) = \sum_{k=0}^{b-1} x^k\,.$$

Suppose that $\lambda \ne 1 \ne \lambda^a$. Then

$$h(\lambda) = \frac{\lambda^b - 1}{\lambda - 1}$$

and

$$h'(\lambda) = \sum_{k=0}^{b-1} k\lambda^{k-1} = \frac{b\lambda^{b-1}}{\lambda - 1} - \frac{\lambda^b - 1}{(\lambda-1)^2}\,.$$

Also, we have

$$g(\lambda) = \frac{(\lambda^{ab}-1)(\lambda-1) - (\lambda^a - 1)(\lambda^b - 1)}{(\lambda^a - 1)(\lambda-1)^2}$$

and

$$\begin{aligned} g'(\lambda) = & \frac{(ab+1)\lambda^{ab} - ab\lambda^{ab-1} - (a+b)\lambda^{a+b+1} + a\lambda^{a-1} + b\lambda^{b-1} - 1}{(\lambda^a - 1)(\lambda - 1)^2} \\ & - \frac{a\lambda^{a-1}}{\lambda^a - 1} g(\lambda) - \frac{2}{\lambda - 1} g(\lambda)\,. \end{aligned}$$

Hence, we finally get

$$\begin{aligned} S_1^{(\lambda)}(a,b) = f'(\lambda) &= \frac{g'(\lambda)h(\lambda) - g(\lambda)h'(\lambda)}{\big(h(\lambda)\big)^2} \\ &= \frac{1}{(\lambda-1)^2} + \frac{ab\lambda^{ab-1}}{(\lambda^a - 1)(\lambda^b - 1)} - \frac{(\lambda^{ab}-1)\big((a+b)\lambda^{a+b} - a\lambda^a - b\lambda^b\big)}{\lambda(\lambda^a-1)^2(\lambda^b-1)^2}\,. \end{aligned}$$

In particular, for $\lambda = 2$, we have the following.

Corollary 2.2.

$$\begin{aligned} \sum_{n \in \mathrm{NR}(a,b)} 2^{n-1} n = 1 &+ \frac{ab2^{ab-1}}{(2^a-1)(2^b-1)} \\ &- \frac{(2^{ab}-1)\big((a+b)2^{a+b} - 2^a a - 2^b b\big)}{2(2^a-1)^2(2^b-1)^2}\,. \end{aligned}$$

For example, for $a = 3$ and $b = 17$,

$$\begin{aligned} S_1^{(2)}(3,17) &= 2^0 \cdot 1 + 2^1 \cdot 2 + 2^3 \cdot 4 + 2^4 \cdot 5 + 2^6 \cdot 7 + 2^7 \cdot 8 + 2^9 \cdot 10 \\ &\quad + 2^{10} \cdot 11 + 2^{12} \cdot 13 + 2^{13} \cdot 14 + 2^{15} \cdot 16 + 2^{18} \cdot 19 + 2^{21} \cdot 22 \\ &\quad + 2^{24} \cdot 25 + 2^{27} \cdot 28 + 2^{30} \cdot 31 \\ &= 37515351605\,. \end{aligned}$$

From Theorem 1.1 (or the above Corollary),

$$\begin{aligned} S_1^{(2)}(3,17) = & \frac{1}{(2-1)^2} + \frac{3 \cdot 17 \cdot 2^{3\cdot 17 - 1}}{(2^3-1)(2^{17}-1)} \\ & - \frac{(2^{3\cdot 17}-1)\big((3+17)2^{3+17} - 3 \cdot 2^3 - 17 \cdot 2^{17}\big)}{2(2^3-1)^2(2^{17}-1)^2} \end{aligned}$$

$$= 37515351605\,.$$

Similarly, by replacing 2 by another value, we can obtain that

$$\begin{aligned}
S_1^{(5)}(3,17) &= 900879734470832437423896\,,\\
S_1^{(1/2)}(3,17) &= \frac{8822132865}{1073741824}\,,\\
S_1^{(-1)}(3,17) &= 408\,,\\
S_1^{(-5/3)}(3,17) &= \frac{7605085294789029411198 64}{205891132094649}\,,\\
S_1^{(\pm\sqrt{2})}(3,17) &= 34250061 \pm 6965604\sqrt{2}\,.
\end{aligned}$$

3. Weighted sums of higher power

Since

$$\begin{aligned}
f''(x) &= \frac{g''(x)}{h(x)} - \frac{2g'(x)h'(x) + h(x)''(x)}{\big(h(x)\big)^2} + \frac{2g(x)\big(h'(x)\big)^2}{\big(h(x)\big)^3}\\
&= \sum_{n=2}^{ab-a-b} n(n-1)\big(1-r(n)\big)x^{n-2}\,,
\end{aligned}$$

we get

$$xf''(x) + f'(x) = \sum_{n=0}^{ab-a-b} n^2\big(1-r(n)\big)x^{n-1}\,.$$

Hence,

$$S_2^{(\lambda)}(a,b) = \lambda f''(\lambda) + f'(\lambda)\,.$$

For simplicity, put $X_1 = (a+b)\lambda^{a+b} - a\lambda^a - b\lambda^b$ and $X_2 = (a+b)^2\lambda^{a+b} - a^2\lambda^a - b^2\lambda^b$. Since

$$f'(\lambda) = \frac{1}{(\lambda-1)^2} + \frac{ab\lambda^{ab-1}}{(\lambda^a-1)(\lambda^b-1)} - \frac{(\lambda^{ab}-1)X_1}{\lambda(\lambda^a-1)^2(\lambda^b-1)^2}\,,$$

we get

$$\begin{aligned}
f''(\lambda) = &-\frac{2}{(\lambda-1)^3} + \frac{ab(ab-1)\lambda^{ab-2}}{(\lambda^a-1)(\lambda^b-1)} - \frac{2ab\lambda^{ab-2}X_1}{(\lambda^a-1)^2(\lambda^b-1)^2}\\
&- \frac{(\lambda^{ab}-1)(X_2-X_1)}{\lambda^2(\lambda^a-1)^2(\lambda^b-1)^2} + \frac{2(\lambda^{ab}-1)X_1}{\lambda^3(\lambda^a-1)^3(\lambda^b-1)^3}\,.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
S_2^{(\lambda)}(a,b) = &-\frac{\lambda+1}{(\lambda-1)^2} + \frac{a^2b^2\lambda^{ab-1}}{(\lambda^a-1)(\lambda^b-1)} - \frac{2ab\lambda^{ab}X_1 + (\lambda^{ab}-1)X_2}{\lambda(\lambda^a-1)^2(\lambda^b-1)^2}\\
&+ \frac{2(\lambda^{ab}-1)X_1}{\lambda^2(\lambda^a-1)^3(\lambda^b-1)^3}\,.
\end{aligned}$$

Similarly, we see that

$$\begin{aligned}
S_3^{(\lambda)}(a,b) &= \lambda^2 f'''(\lambda) + 3\lambda f''(\lambda) + f'(\lambda)\,,\\
S_4^{(\lambda)}(a,b) &= \lambda^3 f^{(4)}(\lambda) + 6\lambda^2 f'''(\lambda) + 7\lambda f''(\lambda) + f'(\lambda)\,,\\
S_5^{(\lambda)}(a,b) &= \lambda^4 f^{(5)}(\lambda) + 10\lambda^3 f^{(4)}(\lambda) + 25\lambda^2 f'''(\lambda) + 15\lambda f''(\lambda) + f'(\lambda)\,.
\end{aligned}$$

4. Apostol-Bernoulli numbers

Though one may obtain explicit expressions of $S_m^{(\lambda)}(a,b)$ for small positive integers m, it is harder to obtain the formulas for large m. In this section, using the so-called Apostol-Bernoulli numbers, we give an expression of $S_m^{(\lambda)}(a,b)$ for general positive integral m.

The Apostol-Bernoulli polynomials $\mathcal{B}_n(x,\lambda)$ are defined by the generating function [1, p.165, (3.1)]:

$$\frac{ze^{xz}}{\lambda e^z-1}=\sum_{n=0}^\infty \mathcal{B}_n(x,\lambda)\frac{z^n}{n!}\quad (|z+\log\lambda|<2\pi)\,. \tag{2}$$

When $\lambda=1$ in (2), $B_n(x)=\mathcal{B}_n(x,1)$ are the classical Bernoulli numbers. When $x=0$ in (2), $\mathcal{B}_n(\lambda)=\mathcal{B}_n(0,\lambda)$ are *Apostol-Bernoulli numbers* [11, Definition 1.2], defined by

$$\frac{z}{\lambda e^z-1}=\sum_{n=0}^\infty \mathcal{B}_n(\lambda)\frac{z^n}{n!}\quad (|z+\log\lambda|<2\pi)\,. \tag{3}$$

They seem to be also called λ-Bernoulli numbers. When $\lambda=1$, the generating function of the left-hand side in (3) is exactly the same as that of the classical Bernoulli numbers B_n. But it does not imply that $\mathcal{B}_n(1)=B_n$ on the right-hand side though quite a few authors misunderstand. In fact, as seen in [1, p.165], the first several values are given by

$$\mathcal{B}_0(\lambda)=0,\quad \mathcal{B}_1(\lambda)=\frac{1}{\lambda-1},\quad \mathcal{B}_2(\lambda)=-\frac{2\lambda}{(\lambda-1)^2},\quad \mathcal{B}_3(\lambda)=\frac{3\lambda(\lambda+1)}{(\lambda-1)^3},$$
$$\mathcal{B}_4(\lambda)=-\frac{4\lambda(\lambda^2+4\lambda+1)}{(\lambda-1)^4},\quad \mathcal{B}_5(\lambda)=\frac{5\lambda(\lambda^3+11\lambda^2+11\lambda+1)}{(\lambda-1)^5}\,.$$

But,

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\ B_5=0,\ B_6=\frac{1}{42},\ \dots\,.$$

For $\lambda\ne 1$, Apostol-Bernoulli polynomials $\mathcal{B}_n(x,\lambda)$ can be expressed explicitly by

$$\mathcal{B}_n(x,\lambda)=\sum_{k=1}k\binom{n}{k}\sum_{j=0}^{k-1}(-1)^j\lambda^j(\lambda-1)^{-j-1}j!\left\{ {k-1 \atop j} \right\}x^{n-k}\quad (n\ge 0) \tag{4}$$

[11, Remark 2.6], where the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ are given by

$$\left\{ {n \atop k} \right\}=\frac{1}{k!}\sum_{j=0}^k(-1)^{k-j}\binom{k}{j}j^n\,.$$

When $x=0$ in (4), Apostol-Bernoulli numbers $\mathcal{B}_n(\lambda)$ have an explicit expression in terms of the Stirling numbers of the second kind [1, p.166, (3.7)],[11, p.510, (3)][1].

$$\mathcal{B}_n(\lambda)=n\sum_{j=0}^{n-1}(-1)^j\lambda^j(\lambda-1)^{-j-1}j!\left\{ {n-1 \atop j} \right\}\quad (n\ge 0) \tag{5}$$

We use a similar approach to Rødseth in [15]. Let n, r and s be integers with

$$r\equiv n\pmod a\quad (0\le r<a),\qquad bs\equiv r\pmod a\quad (0\le s<a)\,.$$

Notice that

$$n\in\mathrm{NR}(a,b)\Longleftrightarrow \exists t\in\mathbb{Z}\ (1\le t\le\lfloor bs/a\rfloor),\ n=-at+bs$$

[1] In both references, the sum begins from $j=1$. However, the value for $n=1$ does not match the correct one $\mathcal{B}_1(\lambda)=1/(\lambda-1)$.

$$\Longleftrightarrow \exists k \in \mathbb{Z}\ (0 \le k \le (bs-r)/a-1),\ n = ak+r\,.$$

Note that the case $\lambda = 1$ is discussed in [15]. Since

$$S_m^{(\lambda)}(a,b) = \sum_{r=0}^{a-1}\sum_{k=0}^{\frac{bs-r}{a}-1}\lambda^{ak+r-1}(ak+r)^m\,,$$

for $\lambda \ne 1$, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} S_m^{(\lambda)}(a,b)\frac{z^m}{m!} &= \frac{1}{\lambda}\sum_{r=0}^{a-1}\sum_{k=0}^{\frac{bs-r}{a}-1}(\lambda e^z)^{ak+r}\\
&= \frac{1}{\lambda}\frac{1}{(\lambda e^z)^a-1}\left(\sum_{r=0}^{a-1}(\lambda e^z)^{bs} - \sum_{r=0}^{a-1}(\lambda e^z)^r\right)\\
&= \frac{1}{\lambda}\frac{1}{(\lambda e^z)^a-1}\left(\sum_{s=0}^{a-1}(\lambda e^z)^{bs} - \sum_{r=0}^{a-1}(\lambda e^z)^r\right)\\
&= \frac{1}{\lambda}\frac{az}{(\lambda e^z)^a-1}\frac{bz}{(\lambda e^z)^b-1}\frac{(\lambda e^z)^{ab}-1}{abz^2} - \frac{1}{\lambda}\frac{1}{\lambda e^z-1}\,. \qquad (6)
\end{aligned}$$

Assume that $\lambda^a \ne 1$ and $\lambda^b \ne 1$. The second term (without sign) of the right-hand side is equal to

$$\begin{aligned}
\frac{1}{\lambda}\frac{1}{\lambda e^z-1} &= \frac{1}{\lambda z}\sum_{m=0}^{\infty}\mathcal{B}_m(\lambda)\frac{z^m}{m!}\\
&= \frac{1}{\lambda}\sum_{m=0}^{\infty}\frac{\mathcal{B}_m(\lambda)}{m}\frac{z^{m-1}}{(m-1)!}\\
&= \frac{1}{\lambda}\sum_{m=0}^{\infty}\frac{\mathcal{B}_{m+1}(\lambda)}{m+1}\frac{z^m}{m!} \quad (\mathcal{B}_0(\lambda)=0)\,.
\end{aligned}$$

The first term is divided into two parts. One part (without sign) is given as

$$\begin{aligned}
&\frac{1}{\lambda}\frac{1}{abz^2}\frac{az}{(\lambda e^z)^a-1}\frac{bz}{(\lambda e^z)^b-1}\\
&= \frac{1}{\lambda}\frac{1}{abz^2}\left(\sum_{i=0}^{\infty}\mathcal{B}_i(\lambda^a)a^i\frac{z^i}{i!}\right)\left(\sum_{j=0}^{\infty}\mathcal{B}_j(\lambda^b)b^i\frac{z^j}{j!}\right)\\
&= \frac{1}{\lambda}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}a^{i-1}b^{m-i-1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{m-i}(\lambda^b)\frac{z^{m-2}}{m!}\\
&= \frac{1}{\lambda}\sum_{m=0}^{\infty}\frac{1}{(m+1)(m+2)}\sum_{i=0}^{m+2}\binom{m+2}{i}a^{i-1}b^{m-i+1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{m-i+2}(\lambda^b)\frac{z^m}{m!}\,.
\end{aligned}$$

Another part is given as

$$\begin{aligned}
&\frac{\lambda^{ab-1}}{abz^2}\frac{az}{(\lambda e^z)^a-1}\frac{bz}{(\lambda e^z)^b-1}e^{abz}\\
&= \lambda^{ab-1}\left(\sum_{k=0}^{\infty}a^k b^k\frac{z^k}{k!}\right)
\end{aligned}$$

$$\times\left(\sum_{\ell=0}^\infty \frac{1}{(\ell+1)(\ell+2)}\sum_{i=0}^{\ell+2}\binom{\ell+2}{i}a^{i-1}b^{\ell-i+1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{\ell-i+2}(\lambda^b)\frac{z^\ell}{\ell!}\right)$$
$$=\lambda^{ab-1}\sum_{m=0}^\infty\sum_{\ell=0}^m\binom{m}{\ell}\frac{1}{(\ell+1)(\ell+2)}$$
$$\times\sum_{i=0}^{\ell+2}\binom{\ell+2}{i}a^{m-\ell+i-1}b^{m-i+1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{\ell-i+2}(\lambda^b)\frac{z^m}{m!}\,.$$

Comparing the coefficients on both sides of (6), we get the following expression.

Theorem 4.1. *For $\lambda\ne 0$ with $\lambda^a\ne 1$ and $\lambda^b\ne 1$, and a nonnegative integer m,*

$$S_m^{(\lambda)}(a,b)=\lambda^{ab-1}\sum_{\ell=0}^m\sum_{i=0}^{\ell+2}\binom{\ell+2}{i}\binom{m}{\ell}\frac{a^{m-\ell+i-1}b^{m-i+1}}{(\ell+1)(\ell+2)}\mathcal{B}_i(\lambda^a)\mathcal{B}_{\ell-i+2}(\lambda^b)$$
$$-\frac{1}{(m+1)(m+2)\lambda}\sum_{i=0}^{m+2}\binom{m+2}{i}a^{i-1}b^{m-i+1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{m-i+2}(\lambda^b)$$
$$-\frac{\mathcal{B}_{m+1}(\lambda)}{(m+1)\lambda}\,.$$

Remark 4.2. When $m=1$ in the expression of Theorem 4.1, that of Theorem 1.1 is obtained.

If $\lambda^a=1$ or $\lambda^b=1$ in (6), without loss of generality, we can assume that $\lambda^a=1$ and $\lambda^b\ne 1$. Because $\gcd(a,b)=1$, $\lambda^a=\lambda^b=1$ is impossible for $\lambda\ne 1$. Then, the first term of the right-hand side of (6) is equal to

$$\frac{1}{\lambda}\frac{az}{e^{az}-1}\frac{bz}{\lambda^b e^{bz}-1}\frac{e^{abz}-1}{abz^2}$$
$$=\frac{1}{\lambda z}\left(\sum_{k=0}^\infty\frac{a^kb^k}{k+1}\frac{z^k}{k!}\right)\left(\sum_{i=0}^\infty B_ia^i\frac{z^i}{i!}\right)\left(\sum_{j=0}^\infty\mathcal{B}_j(\lambda^b)b^j\frac{z^j}{j!}\right)$$
$$=\frac{1}{\lambda z}\left(\sum_{k=0}^\infty\frac{a^kb^k}{k+1}\frac{z^k}{k!}\right)\left(\sum_{\ell=0}^\infty\sum_{i=0}^\ell\binom{\ell}{i}a^ib^{\ell-i}B_i\mathcal{B}_{\ell-i}(\lambda^b)\frac{z^\ell}{\ell!}\right)$$
$$=\frac{1}{\lambda z}\sum_{m=0}^\infty\sum_{\ell=0}^m\sum_{i=0}^\ell\binom{m}{\ell}\binom{\ell}{i}\frac{a^{m-l+i}b^{m-i}}{m-\ell+1}B_i\mathcal{B}_{\ell-i}(\lambda^b)\frac{z^m}{m!}$$
$$=\frac{1}{\lambda}\sum_{m=0}^\infty\sum_{\ell=0}^{m+1}\sum_{i=0}^\ell\binom{m+1}{\ell}\binom{\ell}{i}\frac{a^{m-l+i+1}b^{m-i+1}}{(m-\ell+2)(m+1)}B_i\mathcal{B}_{\ell-i}(\lambda^b)\frac{z^m}{m!}\,.$$

Comparing the coefficients on both sides of (6), we get the following expression.

Theorem 4.3. *For $\lambda\ne 0$ with $\lambda^a=1$ and $\lambda^b\ne 1$, and a nonnegative integer m,*

$$S_m^{(\lambda)}(a,b)=\sum_{\ell=0}^{m+1}\sum_{i=0}^\ell\binom{m+1}{\ell}\binom{\ell}{i}\frac{a^{m-l+i+1}b^{m-i+1}}{(m-\ell+2)(m+1)\lambda}B_i\mathcal{B}_{\ell-i}(\lambda^b)$$
$$-\frac{\mathcal{B}_{m+1}(\lambda)}{(m+1)\lambda}\,.$$

Remark 4.4. When $\lambda = -1$ in Theorem 4.1 or Theorem 4.3, formulas for Sylvester sums (5.11)–(5.14) in [21] are obtained. For, when a is odd, $\mathcal{B}_n((-1)^a) = -nE_{n-1}(0)/2$ $(n \geq 0)$, where $E_n(x)$ are Euler polynomials defined by

$$\frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \quad (|z| < \pi)\,.$$

In particular, when $\lambda = -1$ and $m = 1, 2$ in Theorem 4.3, we have the following formulas. The first relation is not included in the formula in Theorem 1.1.

Corollary 4.5. *When a is even and b is odd,*

$$S_1^{(-1)}(a,b) = \frac{b(ab - a - b) + 1}{4}\,,$$
$$S_2^{(-1)}(a,b) = \frac{ab(b-1)(2ab - a - 3b)}{12}\,.$$

For example, for $a = 4$ and $b = 11$, we get

$$\begin{aligned} &S_1^{(-1)}(4,11)\\ &= (-1)^0 \cdot 1 + (-1)^1 \cdot 2 + (-1)^2 \cdot 3 + (-1)^4 \cdot 5 + (-1)^5 \cdot 6 + (-1)^6 \cdot 7\\ &\quad + (-1)^8 \cdot 9 + (-1)^9 \cdot 10 + (-1)^{12} \cdot 13 + (-1)^{13} \cdot 14 + (-1)^{16} \cdot 17\\ &\quad + (-1)^{17} \cdot 18 + (-1)^{20} \cdot 21 + (-1)^{24} \cdot 25 + (-1)^{28} \cdot 29\\ &= 80\,. \end{aligned}$$

From Corollary 4.5, we also get

$$S_1^{(-1)}(4,11) = \frac{11(4 \cdot 11 - 4 - 11) + 1}{4} = 80\,.$$

Similarly, $S_2^{(-1)}(4,11) = 1870$.

Acknowledgement

The authors thank the anonymous referees for useful comments and suggestions.

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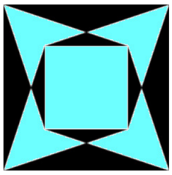

Irish Math. Soc. Bulletin
Number 87, Summer 2021, 45–49
ISSN 0791-5578

Derangements and Continued Fractions for e

PETER LYNCH

Abstract. Several continued fraction expansions for e have been produced by an automated conjecture generator (ACG) called *The Ramanujan Machine*. Some of these were already known, some have recently been proved and some remain unproven. While an ACG can produce interesting putative results, it gives very limited insight into their significance. In this paper, we derive an elegant continued fraction expansion, equivalent to a result from the Ramanujan Machine, using the sequence of ratios of factorials to subfactorials or derangement numbers.

Six students entering an examination hall place their cell-phones in a box. After the exam, they each grab a phone at random as they rush out. What is the likelihood that none of them gets their own phone? The surprising answer — about 37% whatever the number of students — emerges from the theory of derangements.

We may call any permutation of the elements of a set an arrangement. A *derangement* is an arrangement for which every element is moved from its original position. Thus, a derangement is a permutation that has *no fixed points.* The number of derangements of a set of n elements is also called the *subfactorial* of n. Various notations are used for subfactorials: $!n$, d_n and $n¡$ are common; we will use $!n$ (read as 'bang-en').

Dougherty-Bliss and Zeilberger (2020) proved a generalized continued fraction expansion involving Euler's number. They described the occurrence of derangement numbers in the expansion as a "remarkable coincidence", and further commented that "There does not seem to be any immediate combinatorial reason for the derangement numbers to appear." Our derivation in this paper of an expansion for e — equation (7) below — starting from the ratio of factorials to subfactorials, makes the connection clear.

Properties of Derangements

Derangements were first considered by Pierre Reymond de Montmort. In 1713, with help from Nicholas Bernoulli, he managed to find an expression for the connection between $!n$ and $n!$. The answer, which he obtained using the inclusion-exclusion principle (Zeilberger, 2008, pg. 560), is

$$!n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \pm \frac{1}{n!}\right) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}\,. \tag{1}$$

Of course, we see from this that $\lim_{n\to\infty}(!n) = n!/e$. In fact, we can write a more precise connection between derangements and arrangements:

$$!n = \left\lfloor \frac{n! + \frac{1}{2}}{e} \right\rfloor .$$

This implies that $!n$ is the nearest whole number to $n!/e$.

2010 *Mathematics Subject Classification.* 05A05, 11A55.

Key words and phrases. Continued fractions, Permutations, Derangements.
Received on 22-12-2021; revised 26-5-2021.

The number $!n$ of derangements of an n-element set may be calculated using a second-order recurrence relation:

$$!n = (n-1)\big(!(n-1)+!(n-2)\big)$$

with $!0 = 1$ and $!1 = 0$. The subfactorials also satisfy a first-order recurrence relation,

$$!n = n\times!(n-1) + (-1)^n$$

with initial condition $!0 = 1$, which may be compared to $n! = n \times (n-1)!$ with initial condition $0! = 1$. The first eight values of $!n$ are 1, 0, 1, 2, 9, 44, 265 and 1 854 (for further values, see sequence A000166 in the Online Encyclopedia of Integer Sequences).

The factorial function may be espressed in the familiar integral form:

$$n! = \int_0^\infty x^n e^{-x}\mathrm{d}x\,.$$

There is a corresponding integral expression for the subfactorial:

$$!n = \int_0^\infty (x-1)^n e^{-x}\mathrm{d}x\,, \tag{2}$$

Expansion of (2) yields de Montmort's result (1). It also allows extension of the subfactorial function to non-integral arguments and analytic continuation to the complex plane.

Continued Fractions and Convergents

A continued fraction expansion of an irrational number x is written, in expanded form (centre) and concise form (right), as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_0; a_1, a_2, a_3, \dots]\,,$$

where a_n are integers. If a_n is positive for $n \geq 1$ this is called the *simple* continued fraction expansion of x, and this expansion is unique.

A *generalized* continued fraction expansion is written

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\,\frac{a_4}{b_4+}\,\cdots\,,$$

where a_n and b_n are integers and $a_n \neq 0$. By truncating the expansion, we obtain the *convergents*

$$r_n = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\,\frac{a_4}{b_4+}\,\cdots\,\frac{a_n}{b_n}\,.$$

We write $r_n = p_n/q_n$, with p_n and q_n coprime integers and define the starting values

$$p_{-1} = 1\,, \qquad q_{-1} = 0\,, \qquad p_0 = b_0\,, \qquad q_0 = 1\,.$$

Then, p_k and q_k for $k \geq 1$ are given by recurrence relations:

$$p_k = b_k p_{k-1} + a_k p_{k-2}\,, \qquad q_k = b_k q_{k-1} + a_k q_{k-2}\,, \tag{3}$$

which may be proved by induction (Jones & Thron, 1980, pg. 20).

This process can be inverted: given a sequence of numerators p_n and denominators q_n (or just their ratios, the convergents $r_n = p_n/q_n$), we can solve (3) for a_n and b_n:

$$a_n = \frac{p_{n-1}q_n - p_n q_{n-1}}{p_{n-1}q_{n-2} - p_{n-2}q_{n-1}}\,, \qquad b_n = \frac{p_n q_{n-2} - p_{n-2}q_n}{p_{n-1}q_{n-2} - p_{n-2}q_{n-1}} \tag{4}$$

together with the starting values $b_0 = p_0$, $a_1 = (p_1 - b_0 q_1)$ and $b_1 = q_1$.

Continued Fractions for e

From the limit expression $e = \lim_{n\to\infty}(1 + 1/n)^n$, Euler's number is the limit of the sequence

$$\frac{2^1}{1^1}, \frac{3^2}{2^2}, \frac{4^3}{3^3}, \ldots, \frac{(n+1)^n}{n^n}, \ldots$$

The terms may be regarded as the convergents of a continued fraction,

$$r_n = \frac{p_n}{q_n}, \qquad \text{where} \qquad p_n = (n+1)^n \quad \text{and} \quad q_n = n^n .$$

We can generate a continued fraction by using (4). It begins as

$$1 + \frac{1}{1-}\frac{1}{5-}\frac{13}{10-}\frac{491}{196-}\frac{487903}{9952-}\frac{2384329879}{958144-}\cdots . \tag{5}$$

The error of this expansion ($\log_{10}|r_n - e|$) as a function of truncation is shown in Fig. 1 (dashed line). It is clear that the convergence is very slow.

Euler made extensive studies of continued fractions. For example, his 50-page paper *Observations on continued fractions* (Euler, 1750) contains numerous original results. One of his best-known expansions is

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots] . \tag{6}$$

The error of Euler's expansion is shown in Fig. 1 (dotted line). It converges much faster than (5). There is a clear signal of period 3, consistent with the recurring pattern $(1, 1, n)$ in (6).

Continued fraction from derangement numbers

A beautiful continued fraction emerges from the relationship between arrangements and derangements. We saw above that

$$\frac{\text{Arrangements of } n \text{ elements}}{\text{Derangements of } n \text{ elements}} = \frac{n!}{!n} \to e .$$

If we define the numerators and denominators of convergents to be

$$p_n = n! \quad \text{and} \quad q_n = !n ,$$

we can solve for the factors a_n and b_n. The starting values $p_0 = 1, p_1 = 1, q_0 = 1, q_1 = 0$ yield $a_0 = 0, b_0 = 1, a_1 = 1, b_1 = 0$. Then (4) may be solved to yield $a_n = b_n = n - 1$ for $n \geq 2$. Thus we get the expansion

$$e = 1 + \frac{1}{0+}\frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\frac{4}{4+}\cdots .$$

A small adjustment enables us to write this in the elegant form

$$e = 2 + \frac{2}{2+}\frac{3}{3+}\frac{4}{4+}\frac{5}{5+}\frac{6}{6+}\cdots . \tag{7}$$

The error of (7) is shown in Fig. 1 (solid line). Convergence is more rapid than for the other two expansions.

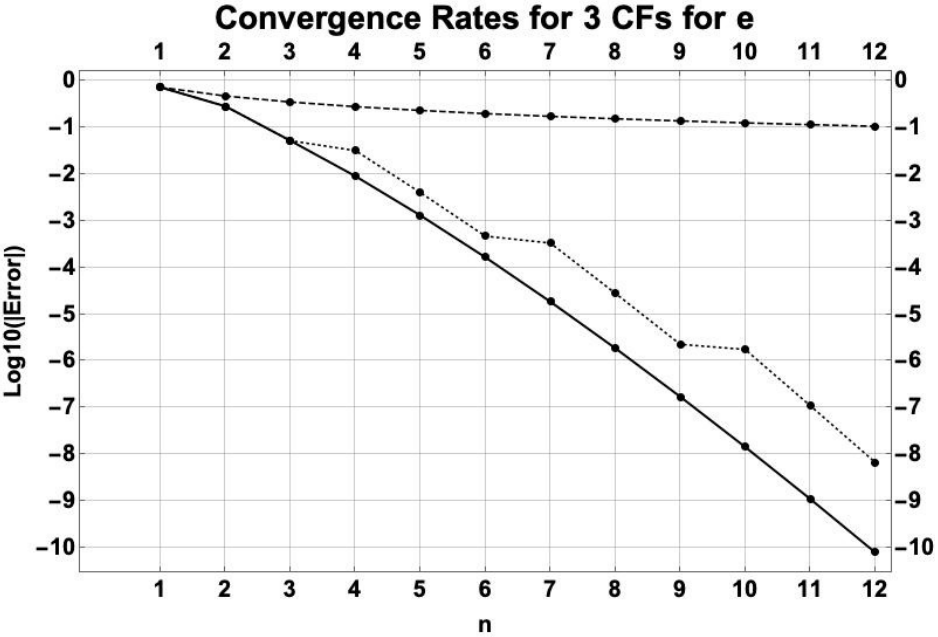

FIGURE 1. Logarithm of the error $\log_{10}|r_n - e|$ in the continued fraction expansions for e. Dashed line: $r_n = (1+1/n)^n$, Eq. (5) . Dotted line: Convergents of Euler's expansion (6). Solid line: $r_n = (n+1)!/!(n+1)$, Eq. (7).

THE RAMANUJAN MACHINE

An Automated Conjecture Generator (ACG) called *The Ramanujan Machine*[1] has been implemented by a team of mathematicians at the Israel Institute of Technology. This ACG system is capable of producing conjectures about mathematical (and physical) constants, expressed in the form of continued fractions, using only numerical data as input. A paper describing the system is available on the arXiv preprint server (Raayoni, et al., 2020).

The Ramanujan Machine comprises algorithms designed to discover new conjectures, running on a network of computers. The goal of the project is to formulate conjectures that may then be proved mathematically. The ACG has already generated a number of very interesting new conjectures, as well as reproducing several results that were already well known. The website (`http://www.ramanujanmachine.com/`) enables researchers to submit proofs of conjectures, code new algorithms and (if they wish) allow access to their computers for distributed computation.

While the Ramanujan Machine generates conjectures but not proofs, it has inspired a complementary project using *symbolic* rather than numerical computation. Dougherty-Bliss and Zeilberger (2020) describe a system that generates automatic proofs of continued fraction expansions. Their system produced some infinite families of expansions together with rigorous proofs of their validity.

One of the continued fractions discovered by the Ramanujan Machine is

$$\frac{1}{e-1} = \frac{1}{1+}\,\frac{2}{2+}\,\frac{3}{3+}\,\frac{4}{4+}\,\frac{5}{5+}\,\frac{6}{6+}\cdots, \tag{8}$$

[1] G. H. Hardy, in his Introduction to Ramanujan's *Collected Papers* (1927), wrote that Ramanujan's mastery of continued fractions was "beyond that of any mathematician in the world".

which is easily seen to be equivalent to (7) above. This is indicated in Raayoni, et al. (2020) as a "known" result. A proof was presented by Kadyrov and Mashurov (2019). Lu (2019) gave elementary proofs of other generalized continued fraction formulae for e.

The connection with derangement numbers was not made by any of these authors. However, Balof and Jenne (2014) analysed the continued fraction

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{2}{3+}\,\frac{3}{4+}\,\frac{4}{5+}\cdots,$$

which was first derived by Euler, and they presented a combinational interpretation of the expansion in terms of derangements.

Acknowledgement

I am grateful to an anonymous reviewer, who informed me about the paper of Balof and Jenne (2014) and whose comments helped me to improve the presentation of this work.

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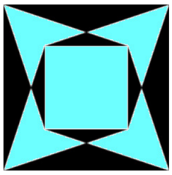

Irish Math. Soc. Bulletin
Number 87, Summer 2021, 51–85
ISSN 0791-5578

Henry Perigal. Dissector, paradoxer, habitué of learned societies, and ornamental lathe turner extraordinaire

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Abstract. The life and times of the little-known English amateur mathematician Henry Perigal and the role he played in British scientific society during the second half of nineteenth century makes for a very curious tale. Today best remember for his graceful and ingenious dissection proof for Pythagoras' theorem, his interests were principally geometric. These ranged from dissection methods, curves formed from compound circular motion, the ornamental arts enabling the mechanical realisation of geometric curves, to the obdurate belief he carried with him to the end concerning the motion of the moon and its lack of rotation on its axis as it orbited about the earth. In this paper we intent to throw light on this under-appreciated character who mixed at the very highest levels of Victorian England's scientific establishment but whose own achievements were far more modest.

1. Introduction

Friday, May 10, 1895 must have been a joyous occasion for Henry Perigal, the venerable old man of the London scientific establishment. For decades he had been a regular attendee at the Royal Institution's Friday Evening Discourses, but always as a 'visitor.' That evening would be his first as a Member, having earlier in the week been elected to the position [136, p. 564]. By then he was ninety-four years old — he is possibly the oldest person ever to have been elected a member — but despite his great age he continued to spend his evenings attending scientific meetings and lectures held throughout London.

So who was Henry Perigal, and what is he today best remembered for? Perigal was your archetypal Victorian scientific amateur. With his hobbies and interests being broadly geometric, he knew and moved amongst some of Britain's leading scientific figures of the second half of the nineteenth century. Even though his own scientific contributions were more modest — he was more curious philosopher than serious man of science — he played an important rôle over a period of several decades in the functioning of London's scientific establishment. Always most regular in his attendance at many of the scientific societies and learned institutions found throughout London, Perigal could be said to be the perfect embodiment of Victorian England's self-improvement ethos. Holding heterodox views about the motion of the moon, he spent a large part of his life attempting to convince others of the error of their ways. Many an ingenious device was built by him in an attempt to support his mistaken notion concerning the non-rotation of the moon. By the time of his death at the close of the nineteenth century, science, that once found a place for interested amateurs like Perigal, had moved on towards ever increasing specialisation dominated by professionals. In this article we sketch the life story of this most intriguing and colourful character.

2010 *Mathematics Subject Classification.* 01A55, 01A70.

Key words and phrases. Henry Perigal, dissection, Pythagoras' theorem, paradoxer, moon controversy, kinematic curves, geometric chuck.

Received on 26-10-2020.

2. Brief biographical sketch

Born on the 1st of April, 1801, in Surrey, by the time he died on June 6, 1898 at the grand old age of 97 Perigal's life had spanned almost the entire nineteenth century. The eldest son of six children, he was named after his father, who lived for an impressive 99 years. As a result our Henry was known as Henry Perigal *Junior* until he himself was well into old age.

Little is known about Henry's early life and the type of education he received but it is thought to be consistent with that received by boys at the time from middle class families. It is not known if he attended university but it appears as though he did not as it is not mentioned in the account given for Henry in the Perigals' family history [88, pp. 40–41]. As a young man he worked for two years as a clerk in the Privy Council and afterwards in the old Victualling Office at Somerset House, rising to the level of chief clerk. After being pensioned off at an early age, he joined in 1844 the stockbroking firm of Messrs Henry Tudor and Sons at 29 Threadneedle Street, London, where the senior partner at the time was a personal friend of his. Here he worked as a clerk for many years before retiring in November 1888 at age 87 [40, pp. 387–388]. Though modestly employed, Perigal's long years of service at the same firm suggest he must have found the work agreeable. It was said he was a great believer in regular work, and friends found it difficult to induce him on occasion to leave the office for half a day [23]. He rarely left London and is thought never to have travelled abroad during the second half of his life, though in his younger days he did make a voyage to Madeira [76]. Starting from his time at Threadneedle Street, for the next forty odd years his life was one of routine, where he divided his time between days spent at the office and his evenings attending scientific meetings or lectures. He never married.

Perigal's scholarly output was sustained, though atypical for the time. He preferred to publish his work in pamphlet form instead of the more usual approach common in his day of submitting papers for publication in standard academic journals. These pamphlets were published mostly for private distribution among his friends and acquaintances. Occasionally they were also given as gifts to various scientific societies to which he belonged. This is unfortunate, as it makes locating and accessing Perigal's complete body of work difficult. The paucity in conventionally-published material is all the more unusual particularly for someone who belonged and engaged with so many leading scientific societies as Perigal did. We know of at least twenty pamphlets published by Perigal but as he was in the habit of publishing short single leaf communications the total number written is no doubt higher. One of his earliest known pamphlets was entitled 'Bow-pen drawings' [90]. A bow-pen is an instrument for drawing lines with ink. The work contained 91 drawings and was published in 1832. His last pamphlet appeared in 1894 [128].

Perigal's interest in geometric curves and mechanical devices used in their creation saw him producing many loose sheets filled with curves. The curves he was particularly drawn to were those resulting from compound circular motions [99]. These he published in profusion, and in his day they were what he was best known for. Tracking all these down is difficult, though we do know Perigal deposited at least three volumes of autographic copies of some of these printed sheets containing many of his curves with the Royal Society, Royal Institution, and Royal Astronomical Society [49].

Perigal is said to have been in good health right up until about 12 to 18 months before the end of his life. For example, on the occasion of the Astronomer Royal's 90th birthday celebration the 90-year-old Perigal was observed walking up the steep hill to the reception with 'almost a jaunty step.' [12]. At age 94 he visited the offices of the monthly popular-science magazine *Science-Gossip*, and the proprietor and then editor John T. Carrington was amazed when Perigal avoided the passenger lift with scorn, preferring instead to make his ascent to Carrington's office by mounting several flights of stairs [22].

Later in life Perigal allowed his beard and hair to grow long and in appearance looked the quintessential gentleman philosopher. Despite the somewhat heterodoxical views he held regarding the motion of the moon, his gentle demeanour, unwavering commitment to service, and endearing personal character won him a place in the highest echelons of nineteenth century British scientific society. As testament of this his passing was a great loss and noted by many [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 76, 28, 138, 142, 134].

3. Learned societies of London

Henry Perigal was very much the embodiment of the Victorian era with its particular attention to self-improvement. In his case his own attempts at self-improvement were channelled into his sustained association with numerous scientific societies located in London over the course of his long life. His unwavering commitment and constancy in attending meetings was legendary. This he maintained throughout the course of his life right up until about 12 months before his death. Later in his life people wrote how they found it quite extraordinary that for someone of such a great age it was possible to maintain his strict, regimental attendance given it required him to be up to quite late on most evenings of the week [13]. Such was the regularity of his attendance, in his later years he would be universally referred to as the 'venerable' old figure of London's scientific scene. Indeed, so regular was his attendance at such meets over the course of so many years that his sudden absence just before his ninety-sixth birthday brought great cause for concern [129]. His death was said to have left a terrible blank among the regular attendants at the many scientific societies to which he belonged [28]. Many years later the Scottish geographer and meteorologist Hugh Robert Mill (1861–1950) recalled how when he was young, on the occasion of reading his first paper before the Royal Meteorological Society the patriarchal figure of Perigal was quietly seated in the audience. He writes he was then in his eighty-seventh year and '. . . had long been one of the landmarks of the evening meetings.' [84, p. 323].

It is difficult to count exactly the many learned societies to which Perigal belonged. Table 1 gives a list of those known. For three of these societies he was a Fellow, the highest rank any society can bestow upon its members. As a Fellow he would have been entitled to use three different sets of post-nominal letters after his name: FRAS (Fellow of the Royal Astronomical Society), FRMetSoc (Fellow of the Royal Meteorological Society), and FRMS (Fellow of the Royal Microscopical Society). Of the three, he is found using the first on almost all occasions, the second occasionally, and the third only rarely. However, as alluded to in our opening introduction, the membership he was most proud of was that of the Royal Institution, attained after attending loyally for many years the Institution's Friday Evening Discourses. On his election this would have further entitled him to use the designatory letters MRI (Member of the Royal Institution).

Of all the societies to which Perigal belonged he had the closest association with the Royal Meteorological Society. After joining what was then the British Meteorological Society a few months after its foundation on 3 April 1850, he was appointed its Treasurer on May 24, 1853, a position he held continuously for just over 45 years until the time of his death [141]. By the time of his death the society had changed its name twice. The first change was to The Meteorological Society in 1866, when it was incorporated by Royal Charter, and then to the Royal Meteorological Society in 1883 when the privilege of adding 'Royal' to the title was granted. Indeed, longevity amongst meteorologist seemed to be a prerogative of theirs and was often the subject of comment [187, 36].

Name of learned society	Date joined/elected
Society for the Encouragement of Arts, Manufactures and Commerce	April 17, 1823
Royal Astronomical Society	Joined, date unknown Fellow, February 8, 1850
Royal Meteorological Society	June 4, 1850 Fellow, December 22, 1851
Royal Microscopical Society	1852 Fellow, date unknown
Royal Aeronautical Society	12 February, 1866
London Mathematical Society	January 23, 1868
Amateur Mechanical Society	January 1, 1869
Physical Society of London	February, 1874
Association for the Improvement of Geometric Teaching	January, 1874
Quekett Microscopical Club	July 22, 1881
British Astronomical Association	November 15, 1890
Royal Institution	Member, May 6, 1895
Royal Photographic Society	Date unknown
Camera Club	Date unknown

Table 1. A list of those learned societies to which Henry Perigal is known to have belonged and the date when he joined, if known. To those societies where he was a Fellow, his date of election, if known, is indicated.

For example, the father of English meteorology and 'godfather of clouds,' Luke Howard, lived to 92, and Perigal's close friend the English meteorologist James Glaisher lived to 94.

The gratitude the Society felt to Perigal for his lengthy term of service can be seen in the recognition it bestowed upon him. In addition to being a Fellow, on April 15, 1893, a complimentary dinner was held by the Society in his honour marking the occasion of his ninety-second birthday and his forty years of service as Treasurer [139, 140]. A week after his death at the June 15 meeting a resolution was moved and unanimously adopted that in part recorded the Society's desire to recognise the valuable service Perigal had rendered to the Society over so many years and the keen interest 'which he took in the discharge of his duties.' [141]. The following year a photograph of Perigal appeared as a frontispiece in the July 1899 issue of the society's journal, the *Quarterly Journal of the Royal Meteorological Society* [143]. This is reproduced in figure 1. It is a photograph taken by the German-born English engineer and photographer John Matthias Augustus Stroh (1828–1914) in 1890 when Perigal was aged 89. Also in the same year, as part of welcoming the Fellows to the new offices of the Society, the President held what was then known as an 'At Home.' This was a Victorian British social custom where the host would be available to receive visitors on a specific day of the week, hence they were 'at home.' In addition to two demonstrations of the optical lantern being held and refreshments served, a number of rooms exhibiting various instruments, photographs, awards, and other items of memorabilia had been set up. Of the total of 93 items on display that evening one was a photograph of Perigal, another a portrait of him, and the third a plaque in his honour [144]. It was a fitting tribute to a man who had given such a large part of his life in service to the society.

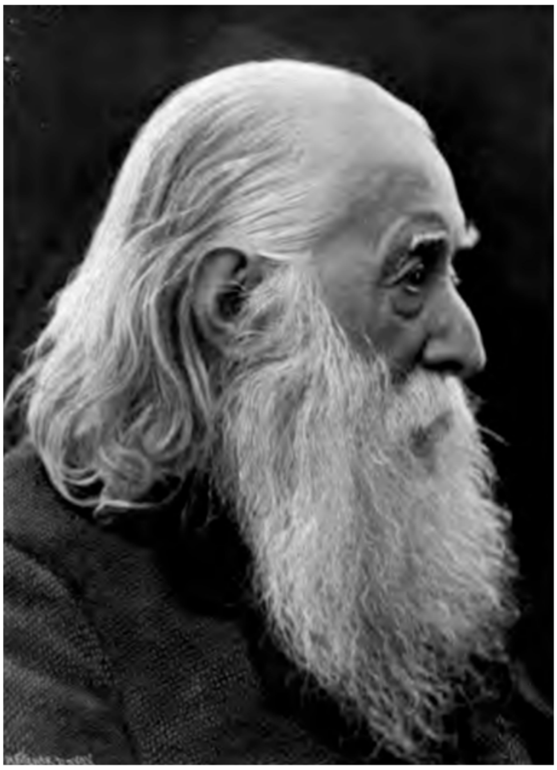

FIGURE 1. A photograph of Henry Perigal which appeared as the frontispiece to the July 1899 issue of the *Quarterly Journal of the Royal Meteorological Society* [143].

Despite the large number of learned and scientific societies to which Perigal belonged the one he most coveted was only available to members through election. It was the oldest and most revered of all the learned societies in the country. It was of course the Royal Society. Founded in 1660 its members, all of whom are Fellows, must nominate for election into the Society. Perigal put his name forward on three separate occasions starting in 1853 [147]. Unsuccessful, he followed this up in 1854 and again in 1855 without success [148]. Many years later in his recollections about his older brother, Frederick Perigal (1812–1905) wrote that while his brother believed his investigations into compound circular motion, the many mechanical devices he built, together with his numerous other scientific accomplishments merited election, he was only too aware he would be judged harshly considering the unorthodox views he held regarding the motion of the moon [89]. We shall return below to these apsects of Perigal's life.

Perigal did manage to gain election to the Royal Astronomical Society Club, an inner circle of the Royal Astronomical Society. In this case the unorthodox views Perigal held about the motion of the moon were, oddly, not enough to prevent his election. Numbers to the Club were strictly limited and not all Fellows of the Society, to which Perigal had been elected on February 8, 1850, were Club members. A number of benefits open to members of the Club not available to others included dining together on the days when the Society met [185]. Perigal was elected to the Club on June 17, 1853, and in what would later prove to be an ironic twist, his proposer was the English mathematician Augustus de Morgan (1806–1871). I say ironic as it would be de Morgan who a decade later called attention to Perigal's denial of the moon's motion on her axis.

Like all the societies to which Perigal belonged he was a regular attendee at the London Mathematical Society meetings. He was elected a member on January 23, 1868. On his passing, the then president of the society, Edwin Bailey Elliott (1851–1937), wrote that even though very little flowed from his pen into the society's journal of record Perigal's frequent attendance had helped make their gatherings occasions for the informal exchange of ideas and acted as a source of stimulation for those who had the 'cause of mathematics at their heart.' [51]. Though this had been one of the stated aims of the Society in its younger days Elliott lamented that the place for the curious amateur at their meetings for people like Perigal had by the dawn of the twentieth century largely been lost.

The London Mathematical Society had been established in January 1865. When it formed Perigal would have very much liked to have joined immediately but feared his heterodoxical views on the motion of the moon would disqualify him. He was much relieved when three years later de Morgan, who was not only the first president of the Society but was coming to his rescue for a second time, offered to put his name forward and signed his nomination form [76].

Perigal's long association with the Royal Microscopical Society is an interesting one. Though the microscopy had never been so great an interest of his compared to astronomy or geometry, Perigal always took the keenest interest in the proceedings of the society, even more so once he stop regularly attending the society's meetings due to his failing health. His good friend, wharfinger, amateur scientist, and fellow member of the Royal Microscopical Society, John Jewell Vezey (1844–1906) was in the habit of paying his friend Perigal a weekly visit for many years later in his life. Vezey recounts how Perigal was always most eager to hear about the latest developments and goings on in the scientific world [146]. Though in late 1896 Perigal had recently complained to his friend that his memory for details had started to fail him unless it concerned particular areas of interest where he had worked [145], his intellectual curiosity never ceased to disappear despite his advancing age and fading mind.

4. A dissection for the ages

Today if Perigal is remember at all it is for the elegant dissection proof he gave for Pythagoras' theorem. Nowadays dissections are a beautiful area of recreational mathematics largely inhabited by amateur mathematicians. Dissection problems call for the cutting of one or more figures into pieces that can be rearranged to form other figures in a way areas are preserved. Dissections are often cast as puzzles, in which case the object is usually to find as few pieces as possible, or used as elementary ocular demonstrations proving results about the equivalence of areas. Perigal's Pythagorean dissection is ostensibly a dissection problem about how two smaller unequal squares can be cut and reassembled using all pieces to form a single larger square. Finding such a dissection immediately gives one a proof for Proposition 47 found in Book 1 of Euclid's *Elements*, the most famous theorem in perhaps all of mathematics, that of Pythagoras' theorem.

That this is the case, consider a right-angled triangle whose hypotenuse has length c with the other two sides of lengths a and b. Forming squares of equal lengths on each side of the right-angled triangle we see the area of each square will be a^2, b^2, and c^2 such that for the three areas one must have $a^2 + b^2 = c^2$ in accordance with Pythagoras' theorem. While many such dissections are possible, what makes Perigal's dissection so special is the simplicity of its construction and the sheer elegance of its final form. It consists of five pieces, four coming from the larger of the two squares with each of the pieces congruent to each other while the fifth piece comes from the smaller square and is not divided. When formed into a single larger square the positions of the five pieces possess four-fold symmetry. It is truly a dissection for the ages.

Perigal tells us he came across his dissection in 1830 while attempting to square the circle [121, p. 103]. Squaring the circle is impossible since π is transcendental, as was proved in 1882 by the German mathematician Ferdinand von Lindemann (1852–1939) [73] several decades after Perigal first starting working on the problem. Even after Lindemann's result had shown the impossibility of squaring the circle, it seems Perigal remained unconvinced. In the minutes of a meeting held by the London Mathematical Society on April 12, 1894, we are told that, quite extraordinarily, Perigal had presented some diagrams illustrating circle squaring by dissection [75].

What has today become known as Perigal's dissection is given in figure 2. What should be immediately obvious is the overall charm of the dissection. Using as many identical pieces that are as few in number as possible the resultant high degree of symmetry is what gives the dissection its elegance. Seeing this dissection for the first time is for many sure to be a memorable occasion. It certainly was for this author.

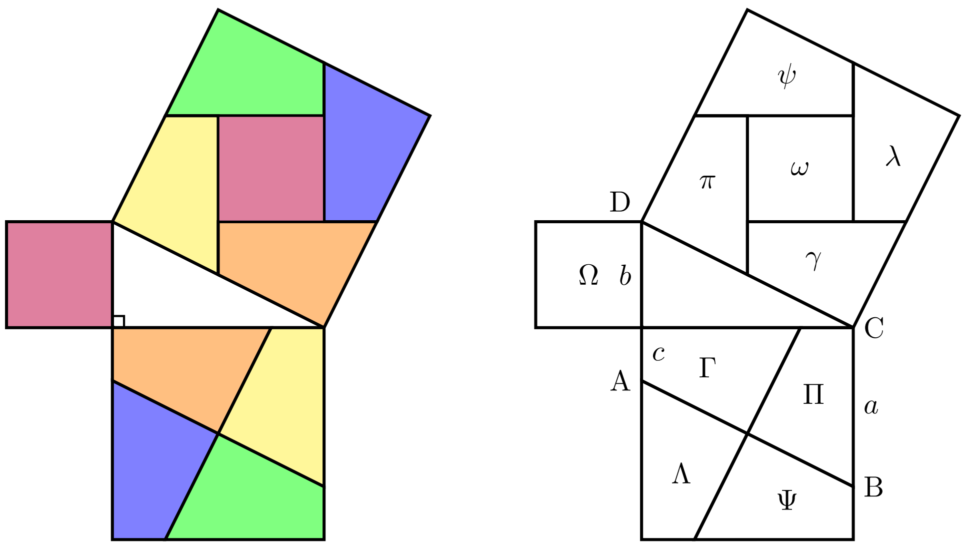

Figure 2. Left: Perigal's famous dissection that gives a proof of Pythagoras' theorem. Right: Perigal's dissection now labelled to be used in the proof showing the congruence of areas.

The construction used for the dissection is quite simple. For a given right-angled triangle, on each of its sides draw a square. For the square sharing the longer side of the triangle which is not the hypotenuse locate its centre, the point where the two diagonals of the square intersect. Draw a line which is parallel to the hypotenuse of the triangle passing through the centre of the square just found. Next draw a line that passes through the same centre of the square that is perpendicular to the previous line just drawn. The square is now divided into four equal quadrilaterals as shown in figure 2. The square appearing on the shortest side of the triangle in not dissected. The five pieces can then be translated, without rotating or reflecting any piece, and fitted exactly into the largest square appearing on the hypotenuse of the right-angled triangle, as indicated in the top square appearing in the figure.

For a proof of this one needs to show the dissection of the square into four congruent pieces together with the smallest square can be assembled to make up the largest square.

The proof we present follows very closely that originally given by Perigal. For an algebraic proof, see [44]. We will use the labelled diagram appearing on the right in figure 2. By construction AB is parallel to the hypotenuse CD and AD is parallel to BC. Thus ABCD is a parallelogram with AD = BC or $a = b + c$. The two construction lines that pass through the centre of the square on the major leg of the triangle are equal in length to the hypotenuse. Since the two construction lines passing through the centre intersect one another at right angles they are bisected. Two of the four sides of quadrilaterals Γ, Π, Ψ, and Λ are therefore equal to half the hypotenuse. As all sides of quadrilaterals Γ and γ are parallel and as two of the sides of the latter quadrilateral are equal to half the side of the square located on the hypotenuse, quadrilaterals Γ and γ are congruent. In a similar manner the congruence between quadrilaterals Λ and λ, Π and π, and Ψ and ψ can be shown. Now it remains to prove the congruence between the quadrilateral ω and the square Ω. The longest side of quadrilateral γ is equal to a. The shortest side in quadrilateral λ is equal to c. Thus the length of one side of quadrilateral ω is equal to $a - c = b$. In a similar manner it can be shown all sides of quadrilateral ω are equal to b. Finally, as the angle between the longest and shortest sides in quadrilateral γ is a right angle, the adjacent angle in quadrilateral ω is also a right angle as the two angles are supplementary. In a similar manner it can be shown all internal angles within quadrilateral ω are equal to right angles. Thus ω is a square congruent with Ω and completes the proof.

Perigal was obviously well pleased with his discovery. So much so that for a time he used a picture of the dissection on the front of his visiting cards (see figure 3). Work on this and other dissections he had found were drawn up and published in 1835 as a pamphlet for private circulation among his friends [91]. This was unfortunate as its limited distribution meant not many people were aware of this most exquisite dissection, despite Perigal's best efforts — he gave hundreds away as a curious dissection puzzle [121].

FIGURE 3. Some visiting cards of Henry Perigal. Photographs courtesy of the author.

Perigal does not tell us exactly how the discovery of his dissection came to be other than it was somehow found during the course of his attempts to square the circle. Perhaps it was one of serendipity. What is certain is it would be his greatest scientific accomplishment and is the reason why his name has come down to us today. Not realised by Perigal at the time is the fact that his dissection can be readily obtained by taking a tessellation of the two smaller squares and overlying it with a tessellation of the largest square that is formed on the hypotenuse of the right-angled triangle. This is shown in figure 4. If one tessellation is moved relative to the other a continuous family of dissection proofs for Pythagoras' theorem emerge. This was apparently first pointed out by the German mathematician Friedrich Paul Mahlo (1883–1971) in 1908 as part of his doctoral dissertation [80] and later independently by others including the English mathematician Major Percy Alexander MacMahon (1854–1929) [79] and the English mathematician and educationist Arthur Warry Siddons (1876–1959) following a suggestion sent to him by a sixteen year old girl named M. Charlesworth [155]. Also not realised by Perigal was his dissection is completely hingeable, something not recognised until the late 1980s by the mathematician David Singmaster [58, pp. 33–34]. This is shown in figure 5. Swinging the square appearing on the right in an anticlockwise direction, two unequal squares are formed with the two formed squares obviously attached to each other at a single hinge point while swinging the square in a clockwise direction forms one single larger square.

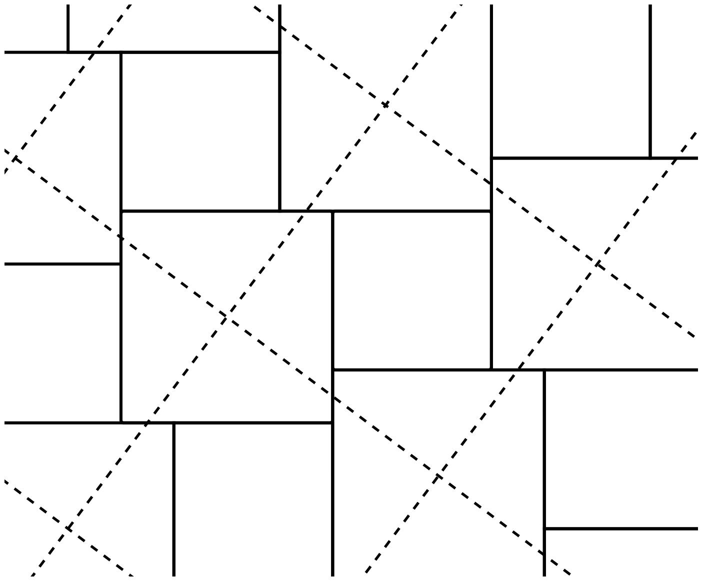

Figure 4. Perigal's Pythagorean dissection resulting from a superposition between two tessellations.

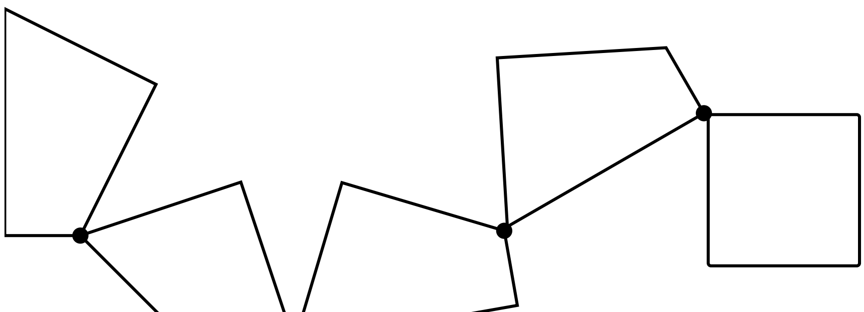

Figure 5. Perigal's Pythagorean dissection in hingeable form.

A close friend of Perigal's for many years was the English meteorologist, aeronaut, and astronomer James Glaisher (1809–1903). They first met in 1855 [76, p. 735]. It was Glaisher's son, the mathematician James Whitbread Lee Glaisher (1848–1928), who some forty years later as editor-in-chief of the journal *Messenger of Mathematics*, on seeing a selection of Perigal's geometric dissections encouraged him to publish some of these in the *Messenger*. This Perigal duly did with his first paper appearing in the November 1872 issue of the *Messenger* [121]. As the journal was only in its second year of publication it is likely Glaisher had been looking out for suitable material to publish in his fledgling journal, and father must have mentioned to son that his old friend Perigal may have something of interest. With Perigal's first paper on geometric dissections Glaisher would not be disappointed. It contained what Perigal describes as his best geometric dissection. At the conclusion of the paper, in an editorial, Glaisher commented how struck he was by the elegance of Perigal's dissection, and while he had seen other dissection proofs for Pythagoras' theorem before, none were as simple nor contained the symmetrical division of one square only. Glaisher remarked that Perigal had perhaps four or five other original dissections that he hoped to see published in up and coming issues of the *Messenger*. Only one other paper two years later ever appeared [123]. It was a dissection showing how a square could be converted into a rectangle of equal area with one of its sides given, perhaps not one of his best efforts. However, Perigal still had one more important dissection up his sleeve.

What is today regarded as Perigal's second most important geometric dissection is his six-piece trisection of the square. Here the problem is to dissever a single square into pieces that can be rearranged to form three smaller squares all identical in size. A nine-piece dissection was discovered as early as the tenth century by the Persian mathematician and astronomer Abū al-Wafā' who worked in Baghdad [57, pp. 31–32] but it was not minimal in terms of number of pieces. Perigal proposed the first six-piece solution to the square trisection problem. Working in reverse he started with three equal smaller squares aligned in a row which when dissected into six pieces and rearranged formed a single larger square. His solution is believed to have been found around the same time as his more famous namesake dissection. The dissection Perigal found for the trisection of the square is shown in figure 6.

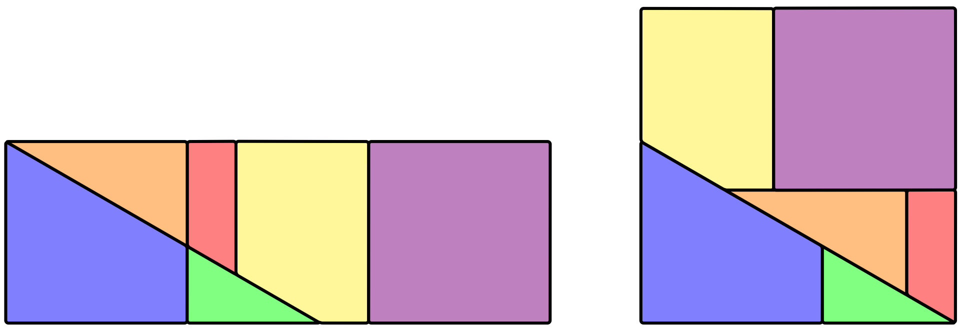

FIGURE 6. Perigal's six piece dissection for the square trisection problem.

In its construction it is very simple. Starting with the three smaller squares lined up in a row as shown in the left of figure 6, placing a compass at the top left corner of the square positioned on the left, extend it open to the top right corner of the middle square. Swinging it down to where it meets the base this point is then joined using a line to the top left corner of the square positioned on the left. Next, placing a compass on the bottom right corner of the square positioned on the right, extend it open to the bottom left corner of the middle square. Swing it up to where it intersects the top. A perpendicular from the top formed by the row of squares at this point is then dropped until it reaches the previous line just drawn. The row of three squares is now divided into six pieces which may be rearranged without rotation or reflection to form the larger square shown on the right of figure 6. That the two figures are congruent in area can be proved using elementary trigonometry. Once again the beauty of this dissection lay in its simplicity, and at six pieces it is thought to be minimal.

A collection of fifteen dissections of Perigal's including his Pythagorean and his square trisection dissections finally appeared in a short booklet published in 1891 under the auspices of the Association for the Improvement of Geometrical Teaching (later renamed The Mathematical Association) [127]. It begins with an extract from his 1872 paper published in the *Messenger*. The fifteen dissection figures then follow without comment. The Association for the Improvement of Geometrical Teaching was founded in 1871 as a teachers' subject association concerned with developing alternative approaches to the standard treatment of geometry then taught in schools in Britain. Perigal's dissections not only fitted nicely within the association's remit but made available his work to a far wider audience beyond the world of learned academics and Perigal's immediate circle of friends. For teachers, methods showing how two unequal squares could be divided and reassembled so as to form a third larger square was undoubtedly the simplest and neatest ocular proof that could be used when introducing Pythagoras' theorem and it was not long before Perigal's dissection started appearing in elementary texts on geometry [78, pp. 93–94], [70, p. 93], [60, p. 189, Ex. 1022], [61, pp. 278–279].

Perigal's first paper on his most famous dissection, as already noted, appeared in the November 1872 issue of the *Messenger*. In an interesting twist, six months earlier his dissection had already appeared in print. Given by his good friend the English mathematician and astronomer Solomon Moses Drach (1815–1879) it can be found tucked away in the May 31 issue of the *English Mechanic and World of Science*, a popular weekly science magazine [50]. In response to a previous correspondent's supposedly new proof of Pythagoras' theorem Drach writes that his friend Perigal had many years ago given a truly new proof for this theorem using what he described as a 'mathematico-mechanical' proof; namely a proof by geometric dissection. Sadly the beauty of Perigal's dissection is partially lost as the figure of the dissection accompanying the text is poorly drawn. Drach's account seems to have gone largely unnoticed.

An interesting question to ask is how widely known or how well-regarded by contemporaries were Perigal's two greatest scientific accomplishments – his dissection proof for Pythagoras' theorem and his six piece square trisection? If an obituary is a summation of one's life work listing all one's greatest achievements for posterity, then the answer would seem not very much. Of the fifteen obituaries I have managed to find for Perigal only one mentions his geometric dissection work. Fittingly, it is the one written for the London Mathematical Society [76].

Unknown to Perigal, and indeed to many for a long time after Perigal, is a striking resemblance between Perigal's dissection and one found in an anonymous Persian manuscript on ornamental geometry written around the turn of the fourteenth century. Entitled *Fī tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa* (On similar and complementary interlocking figures), appearing in the bottom half of the page of folio 182v is a square interlocked with an irregular octagram [87]. This is shown in figure 7. It shows how a square can be transformed into an octagram. No text accompanies the figure, the text seen in the top left corner is for the figure that appears above it on the same page. Here the dissection uses eight pieces but it is obvious that if the smaller central square were not divided into four identical right-angled isosceles triangles one would have exactly Perigal's dissection.

As an enduring legacy of Perigal's dissection, in February of 1979 a US patent was awarded for the design of an apparatus to be used as a didactical aid in the teaching of Pythagoras' theorem that visually presented his celebrated dissection in mechanical form [137]. Here the device contained a knob that when turned varied the size of the acute angle in a right-angled triangle formed between its base and hypotenuse. In this

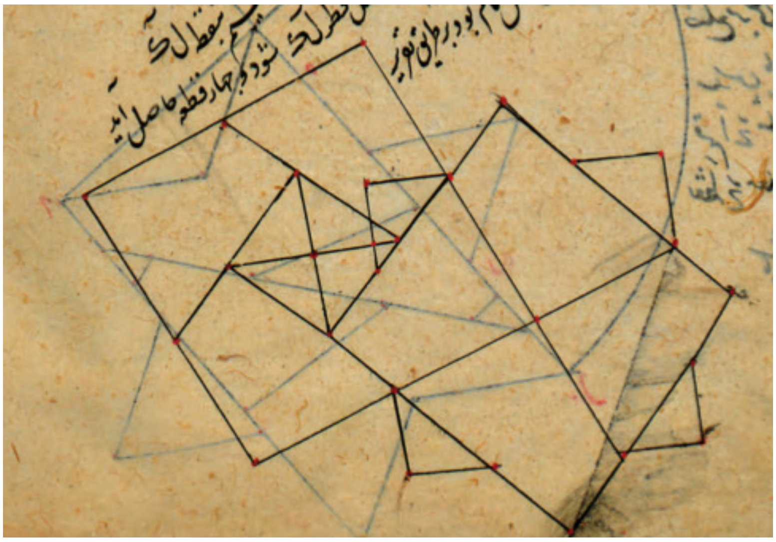

Figure 7. A striking dissection from a fourteenth century anonymous manuscript resembling Perigal's dissection. Image courtesy of Bibliothèque nationale de France.

way, from Perigal's dissection, the learner saw Pythagoras' theorem was true for all acute angles rather than just one particular case. It is not known if such a device was ever made. Of course nowadays such a demonstration can be readily achieved using a simple computer animation.

5. The Moon controversy and Perigal the paradoxer

Perigal's view on the moon and his belief that it did not rotate on its own axis as it revolved about the earth is particularly curious. By the mid-1800s it was generally accepted by all within the scientific community that the fact that the moon always presented the same face when viewed from earth was the result of a compound circular motion. In this case it is a double motion which consists of a revolution of the moon around the earth combined with a rotation of the moon about its own axis. Perigal's unorthodox view on this matter led him to being labelled a 'paradoxer,' a neologism of de Morgan's. Between the years 1863 to 1866 a column called 'A Budget of Paradoxes' started appearing from time to time within the pages of *The Athenæum*, a weekly periodical published in London that covered a wide range of topics including literature, fine arts, music, theatre, politics, and popular science. Authored by de Morgan, this column focused on the exploits of people he called paradoxers. According to him a paradoxer was one who held views contrary to the generally excepted mainstream view. To de Morgan the term was not necessarily a pejorative term. It was only when the paradoxer was clearly wrong, or worst still – deluded – did they become what we would today call a crank.

Perigal's view on the motion of the moon remained unshaken throughout the course of his life. It led to him becoming embroiled in the so-called 'moon controversy' that periodically flared up throughout the course of the nineteenth century and saw him writing an ever-increasing number of pamphlets stating and defending his position. He also invented and built numerous mechanical models which he believed demonstrated his point regarding the non-rotation of the moon about its axis as it orbited round the earth. That the moon was thought not to rotate about its own axis relative to the fix stars was not new with Perigal. It was disputed on and off for at least decades before him. What seems to have led him down this path towards his final conviction was his extensive work with the lathe, and the geometric chuck in particular. Relying as it does for its operation on compound circular motion, the geometric chuck allowed Perigal to gain intimate familiarity with the workings of double and triple circular motions. As much as being a serious scientific tool for the investigation of complex, compound circular motion it unfortunately sustained his mistaken view.

Perigal seems to have fixed upon his ill-founded notion of a non-rotating moon sometime in the early 1840s. We find him first expressing his belief in one of his early private pamphlets he wrote dating from 1846–1849 [95]. For Perigal an orbiting moon rotating about its own axis ought to show all its sides to the earth, just as the earth, which clearly rotates, shows all its sides to the sun as it orbits about the latter. He also saw it as one of semantics stemming from the ambiguity between the terms *rotation* and *revolution* [98]. For Perigal, if rotation was movement of a body about its own axis and revolution was movement of a body round some exterior, usually distance, centre then revolution was an extension of rotation while rotation was a limit of revolution. In the Perigalian system of astronomy it was as if an invisible rod was attached between the earth and moon that was responsible for holding the face of the moon in its fixed position, preventing it from rotating about its own axis as it orbited round the earth.

The distribution of the two pamphlets Perigal had written outlining his views on the non-rotation of the moon was not very wide. Privately distributed, their small print runs ensured only close friends, acquaintances, and perhaps a few members of learned societies actually saw his work. And there they would have stayed, politely dismissed as a quaint idea from a man who busied himself with harmless eccentricities, destined to fade away as quickly as they came. Unfortunately this was not so. There were many others who shared Perigal's pet obsession with the moon. A few years later it burst forth into the most vociferous of public debates. Starting at the Philosophical Institution in Birmingham by one Jelinger C. Symons (1809–1860), who at the time was one of Her Majesty's permanent inspectors of schools [37], what began as a local discussion at one of the regular evening weekly meetings, so excited had the contest become it spilt over into the local papers before finally finding itself a few column inches of space in the hallowed pages of *The Times* of London. On Tuesday 8 April, 1856, a letter to the editor penned by Symons appeared in *The Times* under the heading 'The moon has no rotary motion' [158]. In the letter, Symons questioned why almost all school astronomy texts asserted the moon rotated about its own axis and suggested this was clearly not the case as for the moon to rotate all its sides ought to be visible from here on earth. A small glowing ember had just been dropped into a tinderbox. By the following day a full blown wildfire was raging out of control and would take years before finally exhausting itself.

The response was immediate. The following day *The Times* reported a vast postbag in reply to Symons' letter of which they chose to publish seven as a representative sample of all those received [166]. All were quick to impugn his non-rotatory thesis and admonished him for having the temerity to think it necessary to make his heterodox views a subject for *The Times*. Letters continued to tumble in at a rate of knots throughout the month of April. Symons doubled down and wrote again on April 14 still denying the rotation of the moon [159].

Symons' at times braggadocio tone had not helped either. It galvanised opposition against him and ensured the controversy continued to rumble on in the letters column of *The Times* until year's end when on December 13 Symons finally took leave from the debate by writing that he had decided not to take any further part in its discussion before concluding 'Common sense is with us already, and I dare say philosophers will come round to it in due time.' [160]. By this time, Symons had moved the debate to the pages of *The English Journal of Education* where he found a more sympathetic audience for his unconventional views. Here many of Symons' acolytes who had sent letters to *The Times* in support of him but failed in gaining publication, and others who had written to him directly supporting his view, found a ready outlet [167]. As the year wore on, and early into the following year, many letters in support for Symons' and a few against him appeared [68, 53, 168, 162, 85, 62, 156, 63].

By and large it appears Perigal was not one of Symons' public votaries as he himself did not take part in the unfolding controversy surrounding the moon debate of 1856. If he did, at least no letter to either *The Times* or elsewhere appearing under his name is to be found. Perigal did however contribute as an appendix to a pamphlet Symons wrote on lunar motion [161] a mathematical proof of the type of curves he thought the moon ought to trace out in space if rotation about it own axis while orbiting round the earth were to occur [103]. By year's end Perigal had also written, at first a short than in more expanded form, pamphlet on the moon controversy [104, 105]. Here his main arguments for the moon's non-rotation covered the same ground as his earlier work. New material critiquing the work of many of those who had written to *The Times* admonishing Symons' non-rotatory view of the moon was also included. As Perigal saw it the central issue was what grounds were there for continuing to assert the moon rotated on its axis? As he writes, for him [105, p. 10]

> The only grounds assigned are arbitrary definitions of rotation and revolution, which we say do not properly and strictly define either of them, and, moreover, are not definitions, but fallacious and untenable proportions.

The dispute was quickly narrowing down to one over the meaning of words. Wielding the semantic sword the intention seemed to be to confound common name motions for rotation and revolution as one and the same thing but which were really distinct from each other.

After 1856 things seemed to have settled down for a time until September 10, 1864, when de Morgan took aim at Perigal in his 'Budget of Paradoxes' column for *The Athenæum* [48]. At issue were Perigal's three pamphlets [95, 98, 104]. Unlike many of the other paradoxers he had encountered, de Morgan saw Perigal differently. He commenced his column by informing his readers that twenty years earlier he had had the good fortune of working with Perigal who had produced for him most of the diagrams that were used in his article 'Trochoidal curves' that appeared in the *Penny Cyclopædia* [47]. These Perigal had cut directly from the lathe. These curves were produced using a so-called geometric chuck that for its operation relied on compound circular motion, a device we will have more to say about shortly. For a person who was intimately familiar with compound circular motion de Morgan thought Perigal should have known better. For the moon to permanently show only one of its faces as it orbits about the Earth was possible only if the rate of rotation of the moon about its axis exactly matches the rate of revolution round the Earth. Perigal referred to this 'assumption' as the 'Dogma of the Moon's Rotation' [110, p. 8] and believed it was based on a sophism he traced back to Galileo [110]. Despite this, de Morgan lets Perigal off lightly. The former clearly had respect for the latter and praises Perigal as being the most able amongst all the rotation deniers who had turned out with Symons.

Ironically de Morgan's gentle attack on Perigal may have been brought about by Perigal himself. In the January 1864 issue of *The Astronomical Register* Perigal writes that 'A friend of ours has taken the trouble to *versify* "THE MOON CONTROVERSY"' [108]. Four octavo pages of the controversy in verse then followed under the name of 'Cyclops.' From the doctrines espoused and the turn of phase used, de Morgan suspected it was the work of Perigal, or at the very least, one of Perigal's very close supporters. Later we would learn it was Perigal himself, for Cyclops was a nickname of his given to him by a young girl who could not pronounce 'cycliod' correctly. Indeed, enchanted by his new name, Perigal later asked for it to be included on his tombstone (see section 8). Nor would it be the last time Perigal would commit his non-rotatory defence of the moon to verse. Further prose and poems of his defending the absence in rotation of the moon can be found in [72, p. 9], [27], [151].

The Astronomical Register was only in its second year of publication when Cyclops' poem appeared. The journal's remit was as 'a medium of communication for amateur observers, and all others interested in the science of astronomy.' As anodyne as Cyclops' poem may have seemed, by choosing to publish it the editors of the *Register* had unwittingly stepped into what would quickly become a maelstrom all of their own making. The *Register* was a monthly periodical. The following month, February, saw three letters responding to Cyclops poem and the moon controversy more generally [169]. To the first of these, by a certain 'Argus,' was reserved the most severe animadversion. As a taste of what was about to come, Argus opens with:

> As you have permitted Mr. Perigal to bore us again with his crotchets about the moon, I trust you will allow me to ask him a simple question.

In March five more letters were published [170] including a reply from Cyclops [109]. By April it had become a full blown *vexata quæstio* with the Editor pleading for mercy as a consequence of the enormous volume of letters received on the subject [171]. In the May issue fourteen letters appear, no doubt the tip of a very large iceberg [172]. At their conclusion the editor suggested 'we should be glad to see the "Moon Controversy" drawing to a close' and reminded correspondents that the inappropriate tone that had been used in some of the letters received would no longer be tolerated. It was all to no avail. Letters continued to pour in, and be published, unabated every month until February the following year [173, 174, 175, 176, 177, 178, 179, 180, 181, 182]. An editorial change in policy announced in April 1865 to start in May to try and stem the flow saw 'communications not of general interest' would no longer appear as letters but would instead be inserted in pamphlet form as an appendix at the end of each issue at the personal expense of the author [182]. But still they came. Indeed the first to appear in appendix form was from Perigal himself. He sent two letters. The first was a reply to de Morgan's A Budget of Paradoxes column from the previous year [111]. The second was a repeat of an earlier attempt of his to eliminate what Perigal saw as an equivoque arising from the misuse of the terms rotation and revolution [112].

In the first letter, after republishing de Morgan's column in its entirety, Perigal's reply carefully took some of the former's criticisms and responded to them in verse. A particularly entertaining exchange comes from a line towards the end of Cyclops' poem concerning those that would have you believe the moon's rotation about its axis exists. It reads 'But still it totters *proofless*!' to which de Morgan responses by writing 'Proof requires a person who can give and a person who can receive,' only to have Perigal in his reply amusingly retort:

> For *proofs* we need a *giver* and *receivers*;
> For *dogmas* mere *asserters* and *believers*!

After May 1865 a few more letters appeared as appendices [66, 188, 67, 33] in the *Register* but the change in policy had its desired effect in stemming the flow of letters and brought the debate to a final close. And there it remained for the best part of three years before Perigal again revived it, this time with a simple ball and compass model he hoped would help clarify the difference between rotation and revolution[117]. Several more letters trickled in over the coming months [183, 1, 30, 31, 38, 39, 54, 59, 150, 184]. By late 1870 the debate was all but over [42], finally disappearing from the pages of the *Register* once and for all.

Of course Perigal was not for turning. In the January 14, 1870 issue of *English Mechanic and Mirror of Science* a letter from Perigal appeared [118]. Renewing once more his emphatically held conviction of the impossibility of the moon rotating about its axis as it revolved round the earth he did so by again putting forth an incorrect model relying on the rigid connection between two bodies as one revolved round the other. The following week his error was pointed out by two correspondents. One of these letters was from the English astronomer and populariser of astronomy Richard Anthony Proctor (1837–1888) [130]. He wrote that there could not be too many occasions where he could recall the *English Mechanic* receiving a contribution from the august ranks of a Fellow of the Royal Astronomical Society and had heard a 'whisper' Perigal's views about rotation and revolution were a little heterodox. Writing rather tongue in cheek how pleased he was to find nothing to disagree with him in his letter the only point Proctor noticed was Perigal's letter seemed to have dropped a concluding paragraph. Some years later Proctor recalls asking Perigal if the absence of the moon turning on its axis supported a lunar heliocentric model of the universe [131, p. 179]. The latter admitted his objections to accepted views were by no means confined to the question of the moon's rotation and suggested the idea of a lunar heliocentric model was closer to the truth than many thought it to be. It seems Perigal was quite happy to hold more than one heretical view when it came to the moon.

Perigal continued to make diagrams and construct models that he hoped would convince others the moon did not rotate about its axis. Later in life it is said Perigal would try on many occasions to illustrate this fallacy using his walking stick as an experimental prop in the tea-room of the Royal Astronomical Society [28]. There was no hiding the fact Perigal held unorthodox views. When Perigal was described as a 'paradoxer pure and simple' [134] it was because the incorrect belief he clutched onto was widely known. As one anonymous writer observed, Perigal's idée fixe on the moon's lack of motion about its axis was '...familiar to everyone who was ever in his company.' [25, p. 480]. Because of this one may have thought he would have been *persona non grata* wherever he went, especially within astronomical circles. But surprisingly this was not the case and is a remarkable testament to his personal character. Almost unique among paradoxers he mixed and had long term friendships with some of Victorian England's leading scientific people. These included two Astronomer Royals; Sir George Biddell Airy (1801–1892) and Sir William Henry Mahoney Christie (1845–1922); the meteorologist, aeronaut, and astronomer James Glaisher (1809–1903), the physicist Silvanus Phillips Thompson (1851–1916), the three English mathematicians Arthur Cayley (1821–1895), Samuel Roberts (1827–1913), and Solomon Moses Drach (1815–1879), and the engineer and photographer Washington Teasdale (1830–1903). As he aged he was found to be a charming though somewhat eccentric old man who openly confessed his main astronomical aim in life was to convince others of their grave error in thinking the moon rotated on its axis, especially among the young whose views on the matter were the least firm [134]. Despite his sustained perseverance and avuncular manner it was all to no avail.

6. The model maker and instrument builder

Throughout his life Perigal was a prolific maker of models and builder of various ingenious contrivances. Most of these were made to show the effects of compound circular motion or to demonstrate the truth of the lunarian paradox. Some were however very simple and demonstrated his famous geometric dissection or other geometric dissections he had devised [81, 82].

In March [5] and April [6] of 1846 and again in February [7] and March [8] of the following year Perigal had the opportunity of displaying several of his apparatuses at a number of *soirées* put on by Spencer Joshua Alwyne Compton (1790–1851), Second Marquess of Northampton, who at the time was the President of the Royal Society. One of these was what he termed a *kinescope.* It consisted of a bright steel ball attached to a system of multiplying wheels. For double circular motion, selecting various gear ratios between the two wheels, turning a handle rapidly saw light being reflected off the steel ball as it moved. Due to a persistence in vision the curve it traced out could be seen. One particular curve Perigal was fond of producing was what he called a 'retrogressive parabola.' It was retrogressive in the sense that the steel ball moved back and forth along a finite portion of a 'parabola' either side of its vertex. Indeed Perigal's kinescopes were the mechanical embodiment of what today are known as Lissajous figures, Perigal's work antedating that of Lissajous by a decade [77]. These figures are described by the set of parametric equations

$$x(t) = a\sin(pt) \quad \text{and} \quad y(t) = b\sin(qt + \phi). \tag{1}$$

Here $a, b, p, q, t, \phi \in \mathbb{R}$. Physically, in terms of the moving steel ball found in Perigal's kinescope x and y give the position of the steel ball in the plane, t is the time, a and b the amplitudes of oscillation, p and q the frequencies of the rotating gear wheels, and ϕ a phase shift that depends on the initial starting position used for the steel ball.

Lissajous figures are closed if and only if q/p is rational. Perigal's retrogressive parabola can be obtained on setting $a = b$ such that $a \neq 0$, $p = 1$, $q = 2$, and $\phi = \frac{\pi}{2}$ in (1). Since $q/p = 2$, not only will the resulting Lissajous figure be closed its Cartesian equation is given by

$$y = \frac{a^2 - 2x^2}{a}, \quad -a \leqslant x \leqslant a, a \neq 0,$$

a parabola with vertex at $(0, a)$. Illustrations of two of Perigal's kinescopes are shown in figure 8.

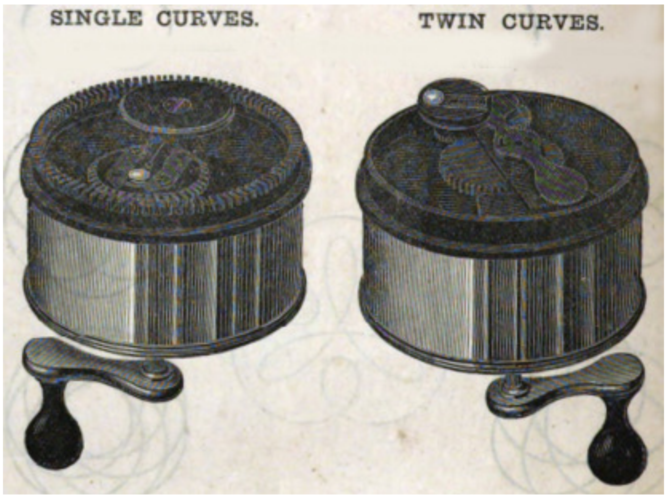

FIGURE 8. Two of Perigal's kinescopes that were made for sale [107]. These could be obtained from Messrs R. & J. Beck, 31 Cornhill, London [126].

To these retrogressive curves Perigal supposed the paths followed by comets may be ascribed. While it was generally agreed comets followed either highly elongated elliptic orbits, thereby periodically returning, while others followed hyperbolic trajectories and were destine never to return, Perigal's retrogressive curves could model the former case. Being finite, they were closed, and were periodic as an object could regress back and forth along the curve. For Perigal a comet moving along one of his retrogressive parabolae allowed for a periodic return [4]. The parabolic shape was no problem either as the apparent path of an object moving along very elongated ellipses resembles very closely that of a parabola. One problem was that Perigal's proposal would have comets returning to us from opposite directions with each passing, something that was not explained and seems to have been overlooked by him.

Another model displayed by Perigal at the February 1847 soirée was a selenescope [7]. It consisted of three ivory balls as models for the moon such that one of three different movements could be imparted to each ball. For the first, it received a motion that caused it to rotate about its own axis and orbit round the earth with the same period and in the same direction. The second was the same as the first except the two motions were in opposite directions. For the third the moon revolved round the earth but did not rotate about its axis. For Perigal, this model confirmed to him the moon did not rotate about its own axis. These he continued to demonstrate for many years. At a soirée held by the President of the Royal Astronomical Society in June of 1864 we are told that, judging by the crowd which had gathered around Perigal's table, his various apparatuses showing the movement of the moon continued to generate much attention and excitement [9]. Still later in July 1876 under the title 'Mr Perigal's kinematic models,' we find in addition to his kinescopes and selenescope Perigal demonstrating various other of his models at the exhibition of scientific apparatus held in the galleries at South Kensington Museum [45]. These included what Perigal termed a 'soldier experiment,' a model designed to demonstrate the relative effects of rotation and revolution, a 'compass experiment' whereby a magnestised compass needle was shown to, and an unmagnetised compass needle was shown not to maintain its parallelism while revolving in a circle, and two gyroscopes that demonstrated the combined effects of revolution and of rotation, and of revolution only [45, pp. 71–72]. In all these models it is clear Perigal's intention was to convince interested visitors how preposterous the notion of a moon rotating on its axis was.

Shortly after appearing as part of the special loan collection of scientific apparatus at South Kensington Museum the model of his selenscope together with several other of his apparatuses were donated to the Royal Astronomical Society [65]. These donations were made on June 1879. In addition to his selenscope he also presented to the Society a lunarium which consisted of two clock dials rotating on a board and two other instruments he termed 'rotameters.' These, he writes, were to 'assist the Fellows of the Society in studying the resultant effects of double circular motion ... with particular reference to the movements of the earth, moon, and planets.' [65, p. 232]. One of these rotameters is shown in figure 9.

In his ninetieth year in a programme for a Royal Society *conversazione* Perigal can still be found exhibiting one of his models [149]. Taking place on the evening of May 14, 1890, in the rooms at Burlington House the Royal Society's *conversazione* was one of the most important annual events on London's scientific calendar. As one of thirty-four exhibits and exhibitors that evening we find Perigal hard at work demonstrating what in the programme is listed as a 'kinematic paradox.' Described as a remarkable result concerning double circular motion leads us to suspect it involved one of Perigal's rotameters, a model he remained convinced was an accurate description for the earth–moon system. Depending on double circular motion for its operation, when used as a model for the motion of the moon about the earth the apparent paradox, as Perigal saw it, was it did not produce the rotation the moon was supposed to have about its axis as it revolved round the earth.

Simpler models made and given away by Perigal were his card models used for demonstrating his famous geometric dissections. In fact, a model for his Pythagorean dissection and a nine part trisection of a square whose parts consisted of three different shapes

Figure 9. One of Perigal's rotameters. The inscription around the rim reads: Presented to the Royal Astronomical Society by Henry Perigal F.R.A.S &c, &c. 13th June 1879 to assist the Fellows of the Society in studying the resultant effects of double circular motion. Photograph courtesy of Brady Haran.

and sizes are known to have been exhibited by Perigal at the *The Great Exhibition of the Works of Industry of All Nations* of 1851, an international exhibition that took place in Hyde Park, London, between the months of May and October [52, pp. 314–315]. As late as 1904 Edward Mann Langley (1851–1933) writing in a review of a geometric text noted card models had once been produced by Perigal for the purpose of demonstrating his most famous dissection but did not know if they could still be obtained [71].

His indomitable belief concerning the moon's lack of rotation, while unfortunate, only added to the eccentric character of this curious natural philosopher. Writing shortly after his death an anonymous writer observed [24]

> By the death of Mr. Henry Perigal . . . the world loses one of its most scientific paradoxers. In season and out of season did he proclaim that the moon might, could, would, should, and did not rotate on her axis, and we have among our instruments at the Royal Astronomical Society a model constructed by him with that brilliant mechanical ability which was his leading characteristic, to enforce his hypothesis.

There is no doubt Perigal was an inventive builder of models and mechanical contraptions. Kinescopes, rotameters, selenescopes, lunarians, or gyroscopes – it was devices that involved compound circular motion to which he was principally drawn. All were built with a view to convincing others the moon did not rotate as it orbited the earth. Misguided belief, perhaps, but his failed attempts to convince others of his own non-rotatory belief left behind a rich collection of instruments that flowed from a creative mind.

7. Ornamental lathe turner and courbes merveilleuse

Perigal was a consummate ornamental lathe tuner. His tool of choice was what was then, and still is, known as the *geometric chuck*. Invented by John Holt Ibbetson in the early part of the nineteenth century the geometric chuck is a mechanism consisting of a number of geared wheels of various sizes on arbors that could move either in the same or opposite directions attached to two or more foundation plates called stages designed to produce two or more circular motions in parallel planes. When attached to the head of a lathe and cranked by hand turning a handle, the geometric chuck was capable of producing a myriad array of intricate curves and patterns etched either into wood or drawn onto paper by attaching a fixed stylus. All curves produced were the result of the superposition of circular motions.

Perigal's father was good friends with Ibbetson [43]. As a young man Perigal and Ibbsetson lived a few doors apart from each other in Smith Street, Chelsea, and this is perhaps how Perigal's interest in the geometric chuck was initially piqued. It was an expensive hobby to take up as the chuck alone was thought to have cost fifty guineas [56, pp. 68–69], a considerable sum of money at the time, and this was the principal reason why its use among ornamental turners was never great. It is most likely Perigal in the first instant benefited from the help and advice received from Ibbetson in the use of the geometric chuck. It would not be long before the apprentice began to outshine his master. An illustration depicting a geometric chuck from Perigal's day together with Perigal's own geometric chuck are shown in figure 10.

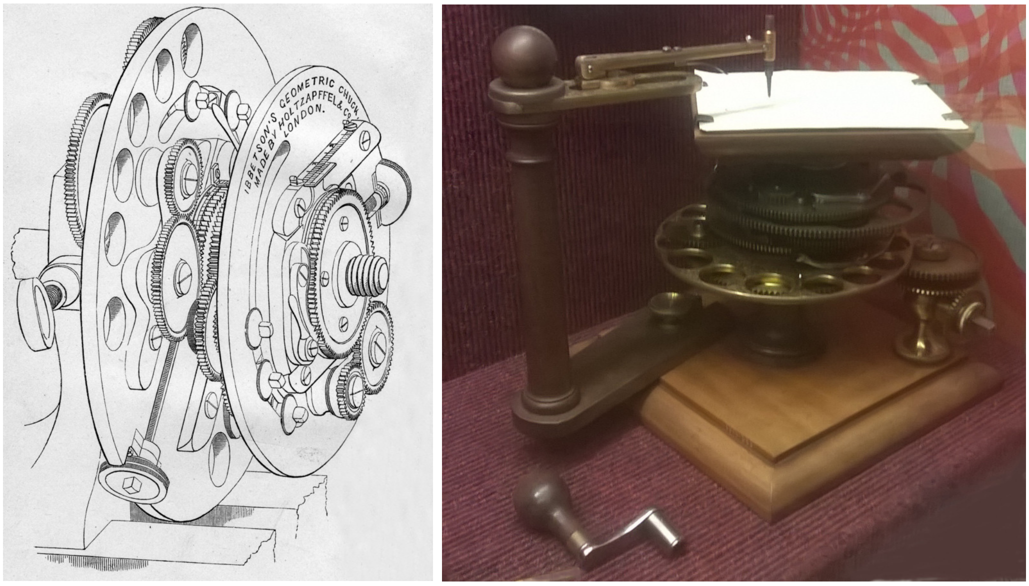

Figure 10. left: An illustration of an Ibbetson geometric chuck that appeared as the frontispiece of Thomas Sebastian Bazley's *Index to the Geometric Chuck* of 1875 [34]. Right: The geometric chuck used by Perigal. It is laid in the horizontal position with a small sheet of paper mounted above one of the stages. It rotated beneath a fixed stylus as the lathe was slowly turned by hand. Photograph courtesy of Laurence Scales.

For most of the figures appearing in Perigal's published work the curves drawn are white on a black background. These were obtained from wooden blocks cut in the lathe with the geometric chuck, with the finished blocks used directly as stamps in the printing process. For additional cost the curves in the blocks could be cut to a sufficient depth to allow casts to be taken from them in type metal. Black curves on a white background could then be obtained.

Of all his contemporaries Perigal devoted the greatest amount of time to the study and classification of curves produced by the geometric chuck. While he recognised that the geometric chuck was well-suited to the purposes of ornamental turning, he later wrote that to him the greater value of the geometric chuck was as a means of investigating curves produced by compound circular motion [122]. Perigal did this by mainly confining his work to the purely mechanical where he produced and published, mostly privately, hundreds of curves [92, 96, 97, 101, 106] though later on he did attempt to describe some of them mathematically [106, 108].

For Perigal his interest in curves of the type produced from compound circular motion was how they could 'exemplify and elucidate' the laws of motion [92]. As such, Perigal referred to the general class of curves of this type as 'kinematic' curves. We are told Perigal first traced out such curves geometrically by hand in 1835. It was not until 1840 that he was finally able to produce his kinematic curves mechanically by continuous circle motions using a geometric chuck [11, 125, 35]. For planets orbiting about the sun in nearly circular orbits the relative orbits for the different planets when seen from here on earth is the direct result of compound circular motion. Of the multitude of curves produced, Perigal's first aim was to identify those curves he thought could potentially account for the apparent motion of the planets. These he made and they can be found recorded in [95, 120].

The type of curves Perigal paid greatest attention to were those he termed *bicircloids*. These were a class of curves made up of two compound circular motions. These included well-known curves such as cycloids, epicycloids, hypocycloids, trochoids, epitrochoids, and hypotrochoids. Also included among these curves were the Lissajous figures, first developed by the American mathematician Nathaniel Bowditch (1773–1838) in relation to his study of two-dimensional motion of a pendulum when suspended from two points [41]. As already noted, Perigal was especially captivated by these figures. The Lissajous figure of the 'parabola' which he called the *retrogressive parabola* was a particular favourite [128]. Along with his dissection proof for Pythagoras' theorem he used it on his visiting cards (see figure 3) and would see to it eventually gracing one side of his tombstone (see figure 15). Though it is still not a widely used term today a bicircloid is defined as the locus of a point attached rigidly on a normal to the circumference of a circle rolling without slipping along a fixed circle [133, pp. 51–52]. A sample of some of Perigal's mechanically traced bicircloids produced using a geometric chuck are shown in figure 11.

Early on, Perigal is known to have sent copies of some of his mechanically drawn curves cut from the lathe using a geometric chuck to Sir John Frederick William Herschel (1792–1871) [46], the English polymath who at the time was widely regarded as the leading man of science in the kingdom. In a series of four letters dating from May 1839 until February 1840 Perigal sent to Herschel a selection of his curves together with an invitation to pay him a visit at his home in Chelsea to see how the curves were made. Though the offer does not appear to have been taken up by Herschel, in writing to a person of such statue it shows the importance Perigal placed in his work on kinematic curves.

Perigal took a keen interest in the types of terms used in describing the curves he drew. Initially he had called bicircloids *spiroeids* [92] but later switched this to bi-circloids [96, 97] before finally settling on bicircloids [101]. His sensibilities were occasionally rankled by terms in current usage such as the word *eccentric*, used to indicate circles not concentric with one another. To Perigal there was nothing 'odd' or

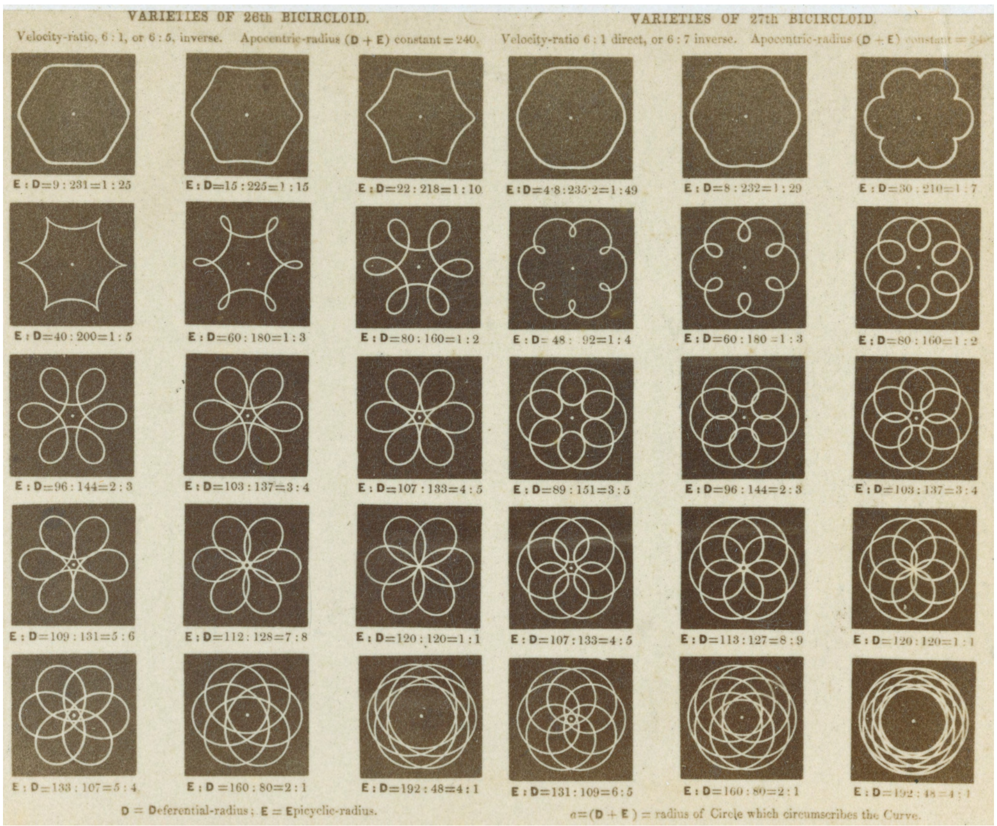

FIGURE 11. Example of a sheet of 'bicircloids' produced by Perigal. Photograph courtesy of Alexander Turnbull Library, Wellington, New Zealand (Ref: PA1-o-190-10).

'irregular' in a circle and he preferred the term *ex-centric*, meaning out of the centre, instead [119]. The term did not catch on.

It was of course the nature of the motion of the moon that was responsible for sustaining his interest in compound circular motion. He saw the bicircloids traced out from his lathe as an accurate description of the paths followed by heavenly bodies and it was just a matter of Perigal convincing his contemporaries of this truth. But in the end it was the faith he placed in his kinematic curves that ultimately led him astray in regard to the motion of the moon and other heavenly bodies. An interesting anecdote about the faith Perigal placed in the ability of the geometric chuck to act as an accurate description of the world around him comes to us from the Irish playwright George Bernard Shaw (1856–1950). He recalls [152, p. 12] that on a summer evening while standing on the pier at Broadstairs in Kent grazing up at the night sky an elderly gentleman from the Royal Astronomical Society came over and stood next to him. Seeing his sight was intently focused on the moon above, the elderly gentleman asked him how far off did he suppose the moon to be. Shaw professed that while he was not a scientific man he supposed perhaps it lay about forty miles away. Expecting to shock the gentleman by his answer as he knew the distance he had just quoted was somewhat less than the usual one, Shaw was surprise to discover how interested the elderly gentleman had become. Asked how he had arrived at his figure Shaw said he had made a guess based on looks. The excited elderly gentleman responded by saying he was a very good judge. Indeed, leaving fractions aside he confidently announced the distance to the moon was thirty-seven miles! He then proceeded to give Shaw a very elaborate and, at least to him, entirely convincing demonstration. He suggested that if astronomers cared to trace the actual orbit of the moon using a geometric chuck on a lathe as he had done, rather than basing their calculations of distance on the method of parallax, they would surely see the conclusion of thirty-seven miles to the distance of the moon was correct. The identification of the elderly gentleman from the Royal Astronomical Society who accosted Shaw on Broadstairs pier that evening as Perigal was not made by Shaw himself. Instead it was made the following year by an anonymous writer writing in *The Observatory* [26] and some years later by the American bibliographer Dan H. Laurence (1920–2008) [153, pp. 149–150], [154, pp. 213–214, 290], his comment on the geometric chuck having given away his identity. It seems Perigal the paradoxer was a lunar iconoclast in more ways than one.

Despite Perigal's shortcomings in his attempts to attach celestial significance to his kinematic curves, his skill and understanding in producing bicircloids from the geometric chuck was recognised among his peers. To help others understand how the generation of various different types of curves could be achieved he devised 39 rules in the use of the geometric chuck [86, pp. 276–280] and he was responsible for producing many figures used by others. For example his mathematician friend Solomon Moses Drach was in the habit of exhibiting collections of Perigal's various bicircloid drawings [74]. There was a time when his skillfully drawn curves had astonished everyone who saw them [69, p. 131]. By the late 1800s lantern slides were the cutting edge technology of their day. When Perigal's kinematic curves, in lantern slide form, were projected on a screen, a large audience found them a source of wonder when shown as part of a regular lecture Teasdale gave on scientific diagrams and the use of the lantern as an educational instrument [163, 164].

Perigal's curves also helped in embellishing the work of others. We have already seen how de Morgan was particularly grateful for the curves Perigal produced for the former's article 'trochoidal curves' that appeared in the *Penny Cycloædia* [47]. The English astronomer Richard Anthony Proctor (1837–1888), in his *Treatise on the Cycloid*, after having seen de Morgan's article in the *Penny Cycloædia* removed all the figures he had produced and replaced them with curves chiefly produced by Perigal [132]. And despite the curves of Perigal Proctor had included in his text he was only too willing to confide in his readers his fear of the inadequacy of the number chosen against the '...immense number, variety, and beauty of the sets of diagram published by Mr Perigal himself.' [132, p. xii]. Sadly not long after his death the valuable work of Perigal on his lathe was all but forgotten. One writer of a letter to the editor of the *Philosophical Magazine* urged some years later [64], that Perigal's method for drawing the ellipse or other Lissajous figures (see [100] and [102]) should be revived and recalled before being completely lost in the mist of time. This lone attempt was obviously unsuccessful as Perigal's work in this area is today largely unknown to most.

8. Pyramids, prizes, cabinets of curiosities, and tombstones

As noted already, during his life it was rare for Perigal to communicate his work to the scientific community via publication in academic journals. One exception to this is found early on in his career where a paper on the possible method used in the construction of the pyramids appeared in *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* [94]. The paper was the product of a piece of work Perigal had offered for communication to the British Association for the Advancement of Science a few months earlier on a probable mode of raising very large weights such as the stones used in the construction of the pyramids.

As the ancient Egyptians had no advanced lifting machinery, the question of how stones of great size and weight could have been raised into place in the building of the pyramids had always vexed scholars. Herodotus tells us the great blocks of stone were moved over the ground on wooden rollers and raised up the steps along the sides of the pyramid using short planks of wood, but gives no further details as to how the actual lifting was achieved. Perigal's suggestion was rather simple and involved nothing beyond what was available to the Egyptians at the time of the construction of the Great Pyramid of Giza, which is thought to have been built somewhere around 2500 BCE. Finished stones were brought up to the base of the pyramid on wooden rollers. The rollers were then removed one by one until only the roller beneath the centre of the stone remained. This left the block of stone in a tipped position. Using thin planks of wood these were then used to build a fulcrum next to the remaining roller that was a little higher than it, while additional planks were placed underneath the raised end of the stone to a height below that of the fulcrum. A person walking along the top of the stone would then walk to the end of the stone where the planks underneath the stone had been placed, causing it to tip up. The last of the rollers could now be removed and a second fulcrum built up to a height just above that of the first while planks at a lower height compared to the fulcrum just constructed were built up under the section of the stone that had been lifted up. The person on top of the stone would now walk to the opposite end of the block causing it to rock back the other way. From here the process would be repeated until the stone was finally raised up to a height of the next step. It was then rolled off onto the step using wooden rollers placed underneath it. From here the stone would continue to journey up the side of the pyramid, being raised from step to step by repeating the process just described until it was finally placed in one of the courses of masonry. The sequence of diagrams shown in figure 12 depicts how the method proposed by Perigal was undertaken.

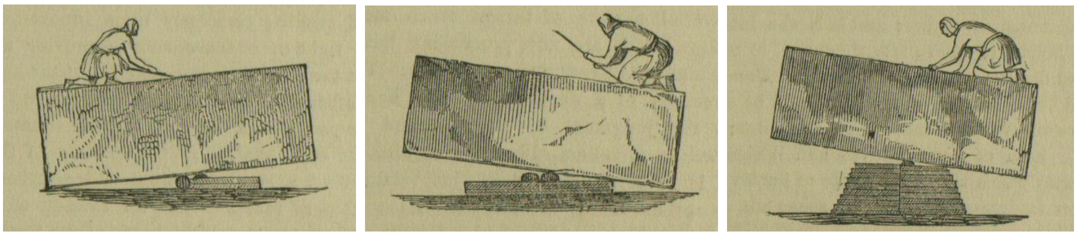

FIGURE 12. A sequence of diagrams depicting how the great stones of the pyramids could be raised using very simple technology based on a proposal first put forward by Perigal in 1844 at the fourteenth meeting of the British Association for the Advancement of Science [3].

The beauty in Perigal's proposal lay in its simplicity. As Perigal was fond of saying, it was as though 'the stone was made to raise itself by means of its own weight' [93]. Whether the method used by the ancient Egyptians in the construction of the pyramids was identical with the one put forward by Perigal is impossible to say.

The background story to Perigal's stone-raising paper is interesting. These are relayed to us in a series of three letters that prefaced his paper published in the *Philosophical Magazine* [94]. As mentioned already, in late 1844 Perigal had offered his paper for communication to the British Association at their up coming meeting to be held in York of that year. The paper was set to be delivered on September 27. For reasons that remain unclear, on September 18 Perigal had called upon the great English physicist Michael Faraday (1791–1867) to take charge of his paper. During the course of their discussions regarding how it could be possible to lift stone blocks of such great weight Faraday recalled a similar principle to the one being described by Perigal having been proposed some years earlier to lift heavy gunnery by one Lieutenant-Colonel Charles

Cornwallis Dansey of the Royal Artillery, a decorated veteran of the battle of Waterloo. It had not occurred to Perigal that his proposed method for the raising of very large and heavy objects could have had applications to present day military purposes. By this stage Perigal had already had the explanatory diagrams that he intended to accompany his paper engraved, these having been made at some personal expense on his part (examples of these are shown in figure 12). Fearing allegations of plagiarism, Perigal wrote to Lieut-Col Dansey the following day outlining what he was planning to present to the British Association while at the same time enquiring if his probable mode of raising large objects had been anticipated by him. By reply post Dansey informed Perigal his proposed method had indeed been long practiced in the artillery drills. Dansey informed Perigal his proposed method had been partially used in the raising of a ship in dock and the technique had even been used to move Younger Memnon, a colossal ancient Egyptian statue made from granite, when installed onto a raised plinth in the British Museum in 1834. Despite all this Dansey felt the time and trouble Perigal had bestowed on the subject deserved to be better known by the general public beyond a small circle of artillerymen and encouraged him to proceed with his account.

All this was very fortunate. As Dansey suspected, learned men of science were unfamiliar with such a technique. It created considerable attention and excitement when first presented before the Mechanical Science section of the meeting. Usually these meetings were rather highbrowed and staid affairs. On this occasion it set tongues wagging. The following day it was reported in the local and London press [2, 3]. Models based on his proposal were exhibited in the museum of the Royal Institution some years later [135], and the proposal even found a place in books of compilations containing popular accounts of science [165, 186].

Perigal's interest in kinematic curves has now been established. He continued to study and think about these curves deeply throughout his life. In early December of 1867 an interesting letter from Perigal first appeared in *The Athenæum*, a week later in *The Mechanics' Magazine*, and a week after this in the *English Mechanic and Mirror of Science* [113, 114, 115]. Dated the 4th of November Perigal proceeds to tell us he had recently made a very interesting discovery concerning one of his kinematic curves and its intersection with a circle. What he found was a finite kinematic curve that cuts a circle in twelve places such that five of these points are equidistant from each other while the other remaining seven points were also equidistant from each other. The circle was thus divided into five equal arcs by the first set of points and seven equal arcs by the second set of points. Though the curve that satisfied this requirement was not new to Perigal he was not aware if such a 'singular' result was more generally known. Suspecting a general class of curves that divide a circle into equal number of arcs must be possible, so enamored had he become by this result he offered it as a challenge with a prize of £5, a not insignificant sum of money at the time, to any person who could demonstrate analytically, geometrically, or mechanically in a new and original manner the following. Find three plane curves that satisfy the following properties:

1. A finite curve that cuts a given circle at exactly five equidistant points.
2. A finite curve that cuts a given circle at exactly seven equidistant points.
3. A finite curve that cuts a given circle at exactly five equidistant points and at exactly seven equidistant points. The points of intersection between the two sets need not be all distinct.
4. Each curve must be continuous such that at the points of intersection with the circle the curve does not self-intersect itself nor is it cusped at all the points of intersection.

Solutions would be accepted up until the end of the year. Perigal referred to this problem as the 'polygonal sectioning of a circle' as joining adjacent points of intersection using a line segment a regular pentagon from the first curve will be formed while doing the same for the second curve gives a regular heptagon. In the third, one has a regular pentagon and heptagon which may or may not share a common point for one of their vertices. As Perigal was explicitly asking for finite curves to be found he clearly had his bicircloids in mind.

A week after the challenge was published in *The Mechanics' Magazine*, and exactly four days before year's end when the time to claim the prize would have expired, a solution by a certain R. B. appeared in the same magazine [29]. The curves Perigal sought are by no means unique. R. B. chose to give his or her solution for possible curves satisfying the stipulated conditions in polar form. Three curves in polar form (r, θ) similar to those given by R. B. that satisfy the four conditions given by Perigal are:

1. $r = 2 + \sin\left(\frac{5\theta}{2}\right)$ and $r = 1$;
2. $r = 2 + \sin\left(\frac{7\theta}{2}\right)$ and $r = 1$;
3. $r = 2 + \sin\left(\frac{5\theta}{2}\right)\sin\left(\frac{7\theta}{2}\right)$ and $r = 2$.

Plots of these curves are shown in figures 13 and 14.

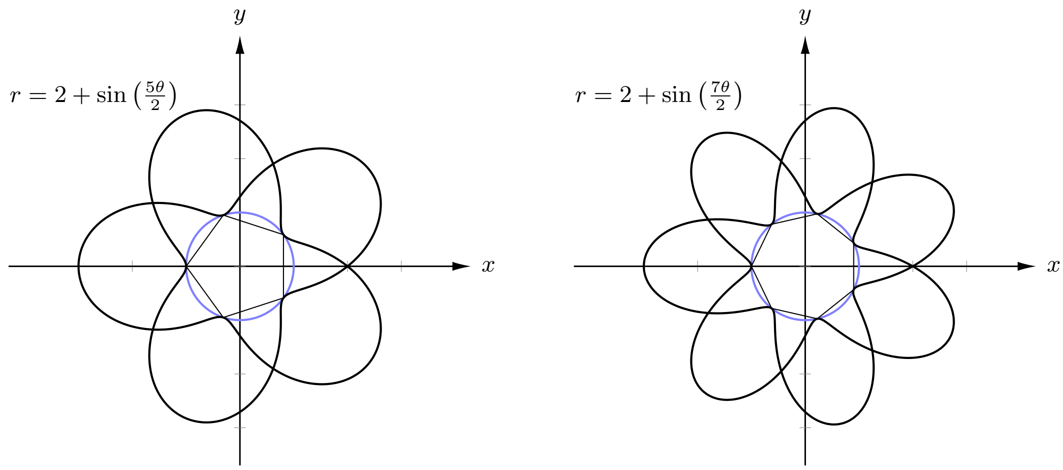

FIGURE 13. Plots of the first two curves that satisfy Perigal's conditions. The circles $r = 1$ appear in blue. Inside each circle, on the left is drawn a regular pentagon while on the right is drawn a regular heptagon.

News the challenge had been accepted and the prize won was duly announced the following month [32, 116] though it appears the curves found by R. B. were not those initial sought by Perigal. What these curves were remains a mystery as Perigal does not tell us what they were. Indeed this was not the only time Perigal has left us wondering. A decade later saw Perigal communicate to the London Mathematical Society a paper entitled 'On a kinematic paradox,' [124] but nothing more is ever said about this paradox leaving one intrigued and left to ponder. The paradox we believe concerned the rotameter [10].

Later in his life Perigal's study at his home had become a private cabinet of curiosities [157]. As the place where all his work with the lathe was performed, where various instruments of his had been designed and built, and where everything connected to his scientific interests collected, the clutter of the room overflowed most of the space. A fortunate visitor to his home once observed [55, p. 25]:

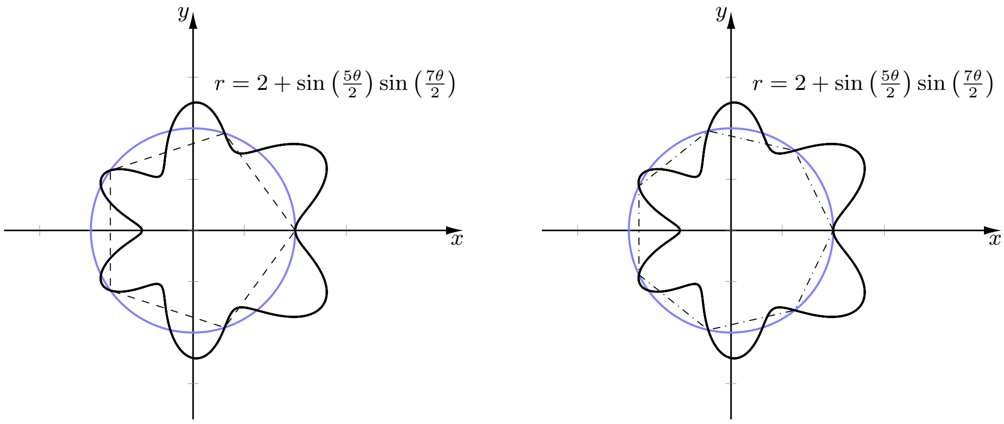

FIGURE 14. Plot of the third curve that satisfy Perigal's conditions. The circle $r = 2$ appears in blue. Inside the circle, on the left is drawn a regular pentagon while on the right is drawn a regular heptagon.

> What a scene it was, that labyrinth of strange relics of science, the marvels of bow-pen lacework, the instruments covered up to keep the dust off, the Philosopher's simple couch in the corner all in view of these quaint things, and the Philosopher himself indefatigably squaring the circle or trisecting an angle, or proving that the world is all wrong about the moon. I don't know what it was that he was at then, but it was all like a leaf out of a book, wonderful and almost incredible. And the birthday album laid there with the autographs of all the high priests of science. What has become of it I wonder, and of the bow-pen work, and all the odd things strewn about in such profusion? I must write an account of it some day. It was exquisite.

It must have been a magical place.

As in life, Perigal took his eccentric streak to the grave. The tombstone over his final resting place is noteworthy for the collection of his greatest deeds inscribed on it. The epitaph he left us points to what he considered to be his greatest scientific achievements. After being cremated at Woking in Surrey his ashes were interned in the churchyard of St. Mary and St. Peter Church in Wennington that was then located in Essex but is now part of Greater London (the London Borough of Haverin). Over his ashes was erected a square column tombstone of approximately 1.5 metres in height. On the front face is a very long inscription which reads:

(on column)
SACRED TO THE MEMORY OF
HENRY PERIGAL
(CYCLOPS)
F.R.A.S. F.R.M.S. M.R.I.
40 YEARS TREASURER OF R. MET. S. &C
BORN 1ST APRIL 1801.
DIED 6TH JUNE 1898.
CREMATED AT WOKING
HIS ASHES LIE BENEATH
DESCENDED FROM A HUGUENOT FAMILY
WHO ESCAPED FROM FRANCE TO ENGLAND
AFTER THE REVOCATION OF THE
EDICT OF NANTES IN 1688.

(on upper plinth)
A LEARNED AND INGENIOUS GEOMETRICIAN
HE INVESTIGATED AND ILLUSTRATED
THE LAWS OF
COMPOUND CIRCULAR MOTION.
(on lower plinth)
GREATLY BELOVED AND HIGHLY ESTEEMED
BY A LARGE CIRCLE OF RELATIVES AND FRIENDS.
"WHEN I CONSIDER THY HEAVENS THE WORK OF
THY FINGERS, THE MOON AND THE STARS WHICH THOU
HAST ORDAINED, WHAT IS MAN THAT THOU ART
MINDFUL OF HIM?" PS. VIII., 3. 4.

The inscription appearing on the lower plinth, except for the last three letters of the first line, is now lost. An inscription also appears at the back. Containing a single quotation located on the central column, it reads:

"ONE OF THOSE UNWELCOME
PREACHERS WHO THANKLESSLY
RETEACH THEIR TEACHERS."

A decade after it was erected it was asked in *Notes and Queries*, a periodical devoted to reader questions, if the source of the quotation was known [83]. It was not, and therefore appears to be one of Perigal's own. It seems to me Perigal saw himself as the 'unwelcomed preacher,' preaching the moon did not rotate about its axis, who tirelessly tried to 'reteach his teachers,' namely all and sundry, of such a fallacy. A photograph of the tombstone as it appeared shortly after it was erected is shown in figure 15 to the left, and as it did in 2006 to the right. As can be seen the initial black engraving used for the lettering and diagrams has all but disappeared.

FIGURE 15. LEFT: Perigal's tombstone as it looked shortly after his death. The figure seen on the right side is that of his 'retrogressive parabola.' His more famous Pythagorean dissection appears out of view to the left. Photograph taken from [89]. RIGHT: As his tombstone appeared in 2006. Photograph courtesy of John Attfield.

More interesting are the inscriptions that appear on either side of the main inscription of the tombstone. On the south side is Perigal's retrogressive parabola. It is the diagram that can be seen on the right side of the figure shown on the left. The inscription below the diagram reads: DISCOVERED BY H. P. / 1835. On the north side appears a diagram showing his now celebrated dissection proof for Pythagoras' theorem. Below the diagram the inscription reads: DISCOVERED BY H. P. / 1830. Today each of these depictions are barely visible, having lost their original black outlining. A photograph together with a retracing showing how Perigal's dissection appears more recently can be seen in [43]. It is a fascinating self-styled tribute left behind by a man who thought the value of his work often went unappreciated by his peers.

9. Conclusion

Henry Perigal – amateur mathematician, pamphleteerist, master ornamental lathe turner, doyen of London's scientific establishment who dabbled in astronomy, meteorology, microscopy, and photography. A man whose contributions lay at the periphery of scientific concerns throughout the mid to late nineteenth century, he was nonetheless destined to be remembered for the dissection that now bears his name. Highly regarded and held with affectionate esteem Perigal was welcomed wherever he went. Moving and interacting within the highest circles of Victorian England's scientific establishment it was all the more curious to see Perigal labelled a paradoxer during his own life. Mulishly staunch in his view on the moon's lack of rotation, that was the quiddity of Perigal. History has bequeathed to us an intriguing, peculiar, and at times exasperating character who today deserves to be better known. We have told the story of this respected amateur, and in doing so, hope to have lifted him out of obscurity.

Acknowledgements

The author is grateful to Lance Putnam for providing copies of the two Perigal documents [92] and [101]. Permission from Brady Haran, Laurence Scales, and John Attfield for the photographs used in Figures 9, 10, and 15 are gratefully acknowledged.

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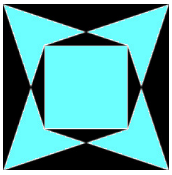

Irish Math. Soc. Bulletin
Number 87, Summer 2021, 87–89
ISSN 0791-5578

Ronald S.Calinger: Leonhard Euler: Mathematical genius in the Enlightenment, Princeton University Press, 2016. ISBN:978-0-691-11927-4, GBP 37.95, 669+xvii pp.

REVIEWED BY ANTHONY G. O'FARRELL

This book was published five years ago, so this review is far from timely, and I apologise to the publishers for this abuse of the usual convention about review copies. I also declare my own unsuitability as reviewer. My decision to keep this to review myself was the result of a lifelong fascination with the achievements of Leonhard Euler, and was not motivated by any confidence that I was up to the task. My only excuse for this is my belief that any attempt to grapple with the scope of Euler's work is beyond the reach of anyone but another Euler. The only reasonable attitude that ordinary mortals can adopt in relation to him is profound and abject humility. How else to consider this sober, pious, kindly and generous Swiss family man, who made Newtonian Physics actually useful, whose name came up in every course in Mathematics and Mathematical Physics in my undergraduate days, whose differential equations were just beginning to revolutionise practical weather-forecasting in my youthful days in the Meteorological Service, whose techniques and notation are standard, and whose textbooks are a primary influence on the programme for elementary, secondary, and university mathematical and engineering education?

Francis Horner, in a brief memoir on the life and character of Euler included in Hewlett's 1822 translation to English (of Johann Bernoulli III's translation to French of the St Petersburg German edition) of Euler's Elements of Algebra, had this to say of Euler's mind:

> An object of such magnitude, so far elevated above the ordinary range of human intellect, cannot be approached without reverence, nor nearly inspected, perhaps, without some degree of presumption.

In considering Euler, one is dealing with a man who wrote in German, Latin, French, and Russian and read many more languages, who had an eidetic memory, who could visualise and manipulate arbitrarily complex expressions in his head, who stood for three decades (1745-1775) at the summit of European science, who was interested in everything, who served singlehandedly as the Google Scholar of his day, who touched no subject that he did not adorn, who produced on average a paper every twelve days even after his blindness became total, and who is still cited in the experimental scientific and mathematical research literature hundreds of times per annum.

Ronald Calinger frankly acknowledges the problem, and suggests that a definitive account of Euler's thought would have to be the work of many people, each expert in one of the various relevant areas, and in a position to utilise primary sources in all the languages used by Euler, as well as secondary sources in English, Italian, Spanish, Chinese and Japanese. Calinger tries, in 650 pages, to give a 'comprehensive biography', 'decribing, explaining and summarizing what Euler achieved', and 'by paying more attention to Euler's correspondence and academic records than did earlier concise biographies' to remove some myths and clarify his relationships with other luminaries

Received on 15-5-2021.

of the Enlightenment. Calinger acknowledges that although the first three series of the *Opera omnia* have at last been published, the work of organising and publishing Euler's correspondence is still in progress — many of Euler's letters are extensive scientific papers, *de facto*.

It is striking that Calinger can say that Euler, whose 'public' life was almost co-extensive with the so-called 'short eighteenth century', receives little or no attention in conventional histories of the period. If it is the business of historians to illuminate the causes of major change in the affairs of men, then it is actually bizarre that they seem unaware of his personal impact. In the last analysis, his effect on our world has been far greater than that of the sovereigns and savants on whom they focus, and the most enduring legacy of Frederick the Great and Catherine the Great is due to their patronage of Euler.

The book follows the chronology of Euler's life (1707–1783), fleshing out and correcting the kind of detail that can be found elsewhere, and providing some background data about the social, political and religious structures of the period. Having read it through in 2016, I've been dipping into it ever since, with profit. Recently I read it through again, in order to conclude this review.

The great strength of this book is its use of the vast Euler correspondence and other contemporary records to piece together the events of Euler's life. It a monumental work of scholarship.

The author's style is rather ponderous. He explains *everything*, and repeats himself. When he discusses an Euler publication, he gives the original title (except when in Russian) and an English translation, and he *may repeat both* when the same publication is discussed in another chapter. This is a bit irritating when the book is read through, but I must admit the advantages: (1) the reader need not understand Latin, German or French, and (2) it is possible to dip into any chapter without reading what went before.

For people of mathematical bent, the act of opening the book at random is liable to provoke an hour, or a day or more, of exploration or calculation, because one is drawn to the questions and controversies mentioned. Euler's enormous range means this can take you in almost any direction. (I should mention that on points of detail this text has very occasional errors, ranging from possible misprints to mere nonsense. The author does not pretend to provide a treatise on the substance of Euler's work, and given the range of topics it is excusable that there are a handful of lapses in the account of technical matters. Anyone who aspired to read all Euler's letters and actually digest the technical content would first need to arrange an antedeluvian lifespan.) Euler was ready to take on any challenge: pensions, annuities and tontines, lotteries, a bridge puzzle in Königsberg and bridge models in St Petersburg, strength of beams and columns, the motion of the Moon, the rings of Saturn, comet trajectories, navigation, the design of ships and sails, alternative means of ship propulsion, surveying the Russian empire, map projections, rigid body dynamics, elasticity, fluid mechanics, pneumatics, hydrostatics, solid mechanics, values of the zeta function, Pell's equation, quadratic, cubic and biquadratic residues, sums of squares, continued fractions, pentagonal numbers, magic squares, the knight's tour, combinatorics, polyhedra, elliptic integrals, tables of logarithms and trigonometric functions, tables for the almanacs whose (monopoly) sale was supposed to generate his academy's funds, telescopes, microsopes, reading glasses, thermometers, tautochrones in various media, music, harmony, geometry of triangles, differential geometry, developable surfaces, agriculture, catechesis, ancient chronologies, and, on his very last day, the differential equation for hot-air balloons.

Calinger does a very good job of painting the world of Euler's day, so different from ours: the domestic arrangements, the mechanics of international travel, the inordinate expense of postage (so vital to Euler), the gulf between nobles and commoners such as

Euler, the spectacle of a Prussian king who preferred to speak French, who appointed a President to his Academy who could not even read German, and who thought it proper to pay Voltaire twenty times as much as Euler. I did not know that Euler was rewarded, not only by his Russian and Prussian employers, but also by the British and the French (from the *secret account of the French navy*) in recognition of his innovations in military science: on navigation, longitude, shipbuilding and handling, masts, sails, speed and stability, ballistics and artillery. Just imagine what would happen to someone, in our age of nation-states, who was known to receive payments and salaries from the authorities on both sides of a major war. Imagine the Russian state employing in a high position of trust someone who had decrypted as needed, and translated to German, Russian military communications intercepted in wartime. People who lost everything in 1945 when the Red Army ground its way to Berlin might have been surprised to hear that when, during the Seven Years War, the Cossacks got out of control and sacked Euler's Charlottenburg estate, Euler was compensated for his loss by the Russians.

I was happy, in the past, that Maynooth's librarians were prepared to invest in the *Opera omnia*, and I recommend that every university librarian do likewise. There is no substitute for holding in your hands those beautifully-produced quarto volumes[1]. For youngsters beginning to read mathematics in German, I've always recommended Edmund Landau's profound texts in his famous telegraphic style, but for Latin it is hard to beat Euler, simple, clear, and beautiful. All the published work listed in the Eneström catalogue is now freely accessible on the internet at `http://eulerarchive.maa.org`, sometimes with links to translations. This should be pointed out to all students of Mathematics. However, they should be told to read critically. Euler published mistakes, cheerfully correcting them later, when detected, in line with his unwavering commitment to the search for truth, and in his applied work he was constantly on the lookout for experimental and observational evidence that might contradict received dogma (such as the inverse-square law of gravitation). Some of his Physics has been completely discarded. His aptitude for philosophy has been derided, perhaps unjustly, because of the influence of Voltaire and other French free-thinkers. I am no judge, but it is worth noting that he was an influence on Kant, was praised by Schopenhauer (and even Goethe), and that a firm Christian faith does not actually disqualify, even today. His willingness to devote serious effort to the education of women is also remarkable, as is his approach to pedagogy, characterized by experiment, attention to feedback, and flexible adaptation.

Calinger's book deserves a place in each library, and the price is reasonable enough to allow its ownership by gainfully-employed professionals and any of their favourite grandchildren who like Mathematics and Physics.

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[1]— except, perhaps, for the experience of holding the original papers. In the wonderful library of the Royal Irish Academy, in Dawson Street, one may read the *Novi Commentarii* of the St Petersburg Academy.

Irish Math. Soc. Bulletin
Number 87, Summer 2021, 90–92
ISSN 0791-5578

S. G. Dani and A. Papadopoulos (Eds.): Geometry in History, Springer, 2019. ISBN:978-3-030-13608-6, USD 139.99, 123+xiv pp.

REVIEWED BY BRENDAN O'SULLIVAN

This volume consists of nineteen chapters discussing different branches of geometry from a historical perspective. The intended audience is that of the general mathematical community with an interest in how certain geometrical ideas developed and evolved over time. It is important to note that each chapter is written by practising experts in the field as opposed to historians trying to capture developments in geometry from a layperson's perspective. In many of the essays, it is striking to see the impact that philosophy had on the development of mathematical ideas down through the ages.

As might be expected, the range of writing styles is diverse and this tends to undermine the overall coherency of the text. The chapter lengths vary from just ten pages to a hefty ninety-two pages which contributes to a disjointed effect for the reader. Despite these aesthetical concerns, there is an effort to create a logical progression between the essays in terms of content. The book has two separate sections, the old and the new. The first (spanning seven chapters) examines topics that have roots in Greek antiquity and the second (consisting of the remaining chapters) concerns itself with more modern material.

The first essay looks at Plato's theory of anthyphairesis, a topic from the fourth century BCE, which has its modern counterpart in the theory of continued fractions. Plato believed that mathematics was a means to gaining a better understanding of reality, he was convinced that geometry was the key to unlocking the secrets of the universe. The author contends that Plato's theory of knowledge of Forms is built upon the concept of periodic anthyphairesis. Plato was critical of the axiomatic method, believing that it led to an overreliance on hypotheses divorced from true knowledge. He argued for the sole use of Division and Collection, a philosophical version of the periodic anthyphairesis, to acquire any knowledge in geometry.

The second chapter looks at the work of the topologist Ren Thom's reevaluation of Aristotle's writings on science, in particular biology. Thom identified that much of Aristotle's assertions have a definite topological content, even if they were written over two thousand years before the field of topology was formally born. The author contends that Thom succeeded in providing a link between modern mathematics and science in Greek antiquity.

The ideas of Thomas Kuhn, concerning paradigm shifts, are considered in the third chapter. A paradigm shift is a fundamental change in the basic concepts and experimental practices of a scientific discipline. The author believes that this is not applicable in mathematics, arguing that mathematical thought evolves in a continuous manner rather than an abrupt change. To illustrate this belief, the reader is given an account of the development of one mathematical thought over time. It begins with Ptolemy's dynamical model of the solar system, used to explain the variations in speed and direction of the apparent motion of the Sun, Moon and planets. It then explains how his

Received on 2-3-2021.

thoughts in this area were considered by the likes of Fourier and Mendeleev before finishing with Schrodinger's work on quantum mechanics, described as 'a complexification of Ptolemy's epicycles'.

Convexity, in the mathematical writings of Greek antiquity, is considered in chapter four. Aristotle, Euclid, Apollonius, Archimedes and more are shown to have explored the area of convexity in their mathematical work before it became recognised formally in the twentieth century. It is possible to trace Aristotle's ideas around the topic of convexity to the more formal work of the likes of Minkowski and Caratheodory. This idea supports the contention of chapter three that mathematical ideas evolve over time rather than being the product of a paradigm shift.

In chapter five, the author considers the relationship between mathematics and art. Modernism in mathematics is described here as the algebraization of spatiality while maintaining geometrical and topological thinking. It looks at the work of Fermat and Descartes, where the notion of algebraization of geometry became necessary. This is developed further using the 'irrationality' of quantum mechanics as an example and how Heisenberg's work led to a 'rationalization'. This is linked to the problematic nature of the incommensurable that was considered in ancient Greek mathematics, before concluding with a consideration of John Tate's principle of 'Think geometrically, prove algebraically'.

The sixth chapter considers the evolution of the ancient Greek theory of curves up to the synthetic differential geometry of today. It looks at the work of several major mathematicians along this journey: Huygens on evolutes, Euclid on curvature of surfaces up to Busemann, Feller and Alexandrov on Carnot groups. The contribution of each mathematician is considered in great detail in an effort to show how the idea was shaped and developed over time.

In chapter seven, geometry is considered a means for describing the shape of the universe. The author explores the development of this idea from ancient times to the modern day. It looks at the areas of cosmology and the philosophy of space and time. Topology, set theory, differential and projective geometry are all employed to explore concepts like infinity, infinitesimal and curvature. This marks the end of the first section that considered geometry's origins in ancient times.

Chapters seven and eight focus on configuration theorems, these are theorems within projective geometry whose statements involve finite sets of points and arrangements of lines. Chapter eight looks at the importance of Pappus' and Desargues' configuration theorems, dating back to the fourth and seventeenth centuries respectively. The authors view the theorems as a bridge between geometry and algebra. These theorems did not gain recognition until the twentieth century when the area received increased consideration. Chapter nine explores the impact that configuration theorems have had and the many results that they have yielded in modern dynamics. The chapter closes with an examination of Richard Schwartz's work on the pentagram map and the theory of skewers.

The essay in chapter ten looks at the work of Henri Poincar in the area of topology. Poincar's philosophy is very much that of a scientist originating in his own daily practice of science and the scientific debates of his time. The author shows that he was also strongly influenced by contemporary philosophical doctrines, such as Kant and Althusser. Consideration is given to the influence of several philosophers such as Frege, Husserl and Russell in the area of geometry.

Chapter eleven looks at the applications of the study of the dynamics of the iterates of a map found by perturbing the germ at the origin of a planar rotation. The author explains how such work led to the development of the Andronov-Hopf-Neimark-Sacker bifurcation theory, Kolmogorov-Arnold-Moser theory and Poincar's theory of normal forms.

The next chapter looks at Gromov's h-principle, which gives conditions where a manifold carrying a geometric structure in a weak sense carries a genuine geometric structure. Other work relating to this h-principle completed by Thom, Smale, Eliashberg, Mishachev and Thurston is also discussed.

Chapter thirteen considers flexibility and rigidity phenomena in symplectic geometry. The chapter begins with an outline of Poincar's desire for the development of this branch of geometry. This came to pass years after his death when symplectic topology emerged through the work of Arnold and Gromov. The chapter concludes with an account of the most recent findings in this area.

A historical survey on the theory of locally homogenous geometric structures is presented in chapter fourteen. The theory is traced back to Charles Ehresmann in the early twentieth century. Further work led to the Ehresmann-Weil-Thurston holonomy principle, which identifies a relation between the classification of geometric structures on a manifold and the representation of its fundamental group into a Lie group. The rest of the chapter looks at discrete subgroups of Lie groups.

Chapter fifteen looks again at the work of Ehresmann, but this time in relation to the development of differential geometry. An Ehresmann connection and its importance for fibre bundles is explained. His work on jet bundles is also accounted for.

Finsler and Riemannian geometries are compared in chapter sixteen. The emphasis is on the asymmetry of the distance function associated with a Finsler manifold. References to several mathematicians working in this area are made.

The next two chapters look at the topology of 3- and 4- manifolds. Chapter seventeen looks at the work completed around the three-dimensional Poincar conjecture. It took over a hundred years before the first valid proof emerged. A lot of mathematics was generated in the search for such a proof. Chapter eighteen is much broader, describing the important problems surrounding four-dimensional manifolds and the work of many mathematicians. It also accounts for the proof of the Poincar conjecture in higher dimensions.

The final chapter provides an interesting contrast to the preceding essays. It is an autobiographical account by Valentin Ponaru concentrating on the period where he decided to become a mathematician. He describes the problems that he worked on and those that he corresponded with at the time. It gives a first-hand account of what life was like for a mathematician behind the iron curtain.

This book allows the reader to grasp just how large the area of geometry is. It is apparent that not all topics within geometry were covered, most likely due to a lack of suitable authors. The major contributors like Poincar, Ehresmann, Thom, Thurston and Gromov are referenced repeatedly. Most authors succeed in linking the mathematical work of today with its origins in the past. Due to the contrasting writing styles and varying chapter lengths, this book is not an easy read. This volume would be a useful reference text in a library, where readers could use it to research the historical origins of certain geometrical topics and see how they developed over time.

Brendan O'Sullivan Brendan has taught mathematics for over twenty years at post-primary level. He served as Chairperson of the Irish Mathematics Teachers' Association from 2015 to 2019.

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Irish Math. Soc. Bulletin
Number 87, Summer 2021, 93–95
ISSN 0791-5578

PROBLEMS

IAN SHORT

Problems

I learned the first problem this issue from a paper by Boris Springborn (*Enseign. Math.* 63, 2017, 333–373).

Problem 87.1. Determine the maximum distance between a straight line intersecting a triangle and the vertices of that triangle.

The second problem is courtesy of Des MacHale of University College Cork.

Problem 87.2. Prove that if each element x of a ring satisfies $x^4 + x = 2x^3$ then the ring is commutative.

Elementary answers only please (using basic properties of rings). Those of you who solve Problem 87.2 might like to tackle the following more challenging variant. Prove that if each pair of elements x and y of a ring satisfies $(x^4 - x)y = y(x^4 - x)$ then the ring is commutative. Des has kindly offered a prize of his recent book *The Poetry of George Boole* for the first correct, elementary solution to this more challenging problem!

The third problem comes from Finbarr Holland of University College Cork.

Problem 87.3. Determine the sums of the series

$$\sum_{m,n=1}^{\infty} \frac{1}{mn(m+n+1)} \quad \text{and} \quad \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{mn(m+n+1)}.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 85.

The first problem was solved by Seán Stewart of Bomaderry, Australia, and the proposer, Des MacHale. We present the solution of Stewart (using a different reference towards the end).

*Problem 85.1***.** Dissect an equilateral triangle into four pieces that can be reassembled, without flips, to form three equilateral triangles of different sizes. Can this be accomplished with just three pieces?

Solution 85.1. We start with an observation. An equilateral triangle with sides of length a has area $a^2\sqrt{3}/4$. If this triangle can be decomposed into three smaller equilateral triangles with sides of lengths x, y and z, then by equating areas we obtain

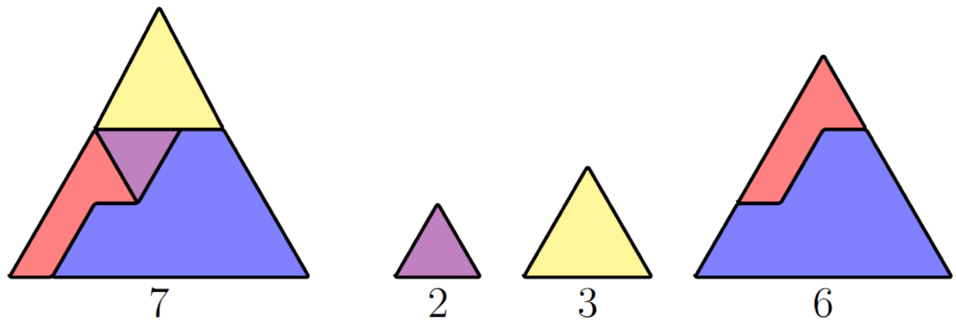
$$a^2 = x^2 + y^2 + z^2.$$

A simple positive integer solution of this equation is

$$7^2 = 2^2 + 3^2 + 6^2.$$

This motivates us to look for a four-piece dissection of an equilateral triangle of side length 7 into three smaller equilateral triangles of side lengths 2, 3, and 6. One such dissection is shown below.

Received 9-6-2021.



For the second part of the question, to dissect an equilateral triangle into three smaller unequal equilateral triangles using just three pieces entails that each piece must be an equilateral triangle. However, a result of Tutte (*Math. Proc. Cam. Phil. Soc.* 44, 1948, 463–482) says that it is impossible to dissect an equilateral triangle into unequal equilateral triangles. □

The next problem was solved by Daniel Văcaru of Pitești, Romania, and the North Kildare Mathematics Problem Club. The solution was also known to the proposer, Des MacHale. The solution we present follows that of Văcaru and the Problem Club

Problem 85.2. An absent-minded professor of mathematics cannot remember her debit card PIN. However, she remembers that the PIN lies between 4129 and 9985 and it cannot be expressed as the sum of two or more consecutive integers. Can you help her determine the PIN?

Solution 85.2. Let m be the pin, where $4129 < m < 9985$. Suppose that m is divisible by an odd prime p. Observe that, for any integer n, we have

$$(n+1)+(n+2)+\cdots+(n+p) = p\big(n+\tfrac{1}{2}(p+1)\big).$$

By choosing $n = m/p - \frac{1}{2}(p+1)$ we see that m is a sum of p consecutive integers, a contradiction.

Hence m is not divisible by an odd prime, so it is a power of 2. The only power of 2 between 4129 and 9985 is $2^{13} = 8192$, so $m = 8192$. □

The third problem was solved by the North Kildare Mathematics Problem Club. A solution was also offered in the magazine of the M500 Society, a mathematical society of the Open University. I learned of the problem from that magazine.

Problem 85.3. Arrange the integers 1 to 27 in a $3\times3\times3$ cube in such a way that any row of three integers (excluding diagonals) has sum 42.

Solution 85.3. The following solution was found computationally, along with 31 others. The three matrices represent layers of the cube.

$$\begin{pmatrix} 14 & 1 & 27 \\ 21 & 17 & 4 \\ 7 & 24 & 11 \end{pmatrix} \qquad \begin{pmatrix} 25 & 15 & 2 \\ 5 & 19 & 18 \\ 12 & 8 & 22 \end{pmatrix} \qquad \begin{pmatrix} 3 & 26 & 13 \\ 16 & 6 & 20 \\ 23 & 10 & 9 \end{pmatrix}$$

□

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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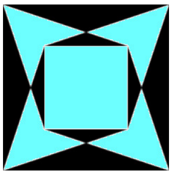

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