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Cornel Ioan Vălean: (Almost) Impossible Integrals, Sums, and Series, Springer, 2019. ISBN:978-3-030-02461-1, EUR 64.99, 539+xxxviii pp.

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On first opening this book it did not take me long to realise there was something Boschian about it. It is a vast store of results for definite integrals and infinite series that at first glance seem to defy what is possible to achieve analytically. Those drawn to the evaluation of integrals or series will find the present volume particularly difficult to pass by. Evaluating integrals has always had its coterie of dedicated admirers. G. H. Hardy once famously remarked 'he could never resist the challenge of a definite integral' [9, p. xi]. From the predilections of the current author it seems he would tend to have to agree.

Forming part of Springer's 'Problem Books in Mathematics' series the book is divided into two parts. The first three chapters on integrals forms the first half of the book (Chapter 1 the problems, Chapter 2 some hints, Chapter 3 their solutions), the last three chapters on series and a few finite sums forms the second half of the book (Chapter 4 the problems, Chapter 5 some hints, Chapter 6 their solutions). There are sixty problems in each part. Classical but tricky integrals such as (Problem 1.7 (i))

$$\int_0^1 \frac{\log^2(1+x)}{x}\,dx = \frac{1}{4}\zeta(3), \tag{1}$$

where ζ denotes the Riemann zeta function make an appearance but there is much more besides. As an example of the latter consider (Problem 1.33)

$$\int_0^1 \mathrm{Li}_2\left(\frac{x}{x-1}\right)\mathrm{Li}_2\left(\frac{x}{x+1}\right)\frac{dx}{x}.$$

Here Li_2 is the dilogarithm. It is problems like this that live up to the title of the book as being 'almost' impossible. Such an integral is sure to leave many scratching their head and wondering how a closed-form solution is even possible. That it is I leave to the interested reader.

Part of the fun and unusualness of the book is each of the 120 problems appear under their own headings. Here one finds, for example (Problem 1.41)

> A Little Integral-Beast from *Inside Interesting Integrals* Together with a Similar Version of It Tamed by Real Methods

or (Problem 4.37)

> Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 1st Episode

While the titles are delightfully curious and at times contain an interesting turn of phase, a small quibble is in other places the text could have benefited from tighter editing from the publisher given the first language of the author is not English.

Key words and phrases. Definite integrals, series, harmonic numbers, Euler sums, Riemann zeta function, polylogarithms.

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As expected from a problem book there is little given in the way of preliminaries. It is more a case of diving in head first and hoping you do not immediately hit the bottom. A course in integral calculus is a given and one in the elements of real analysis would be helpful. The reader is also expected to have a high degree of fluency with 'standard' special functions such as polylagarithms, the polygamma function, and the Riemann zeta function, and to a lesser extent the Dirichlet beta function, the inverse tangent integral, Lerch transcendent, and Legendre's chi function of order two. Without knowledge of these progress will be largely impossible. Some familiarity in the manipulation of infinite series is also expected.

The first half of the book is devoted to definite integrals. Most of these are single integrals with a handful of double integrals and the occasional triple integral. The solutions given are comprehensive. An interesting feature is only real methods are considered – there is no contour integration – and seems to be very much the preference of the author. Sometimes two different methods are given and alternative approaches are referenced. Occasionally interesting pieces of information about a particular integral or its method of solution is included. As an example of this, for the first of the integrals given in Problem 1.3 we learn it is believed to have first appeared in a book containing a collection of problems suitable for the Cambridge course published in 1867 by the English mathematician Joseph Wolstenholme (1829–1891) [12, p. 214]. Some of the integrals are classical. Several were proposed by the author, having first appeared in the problem sections of journals such as *The American Mathematical Monthly* and *La Gaceta de la Real Sociedad Matemática Española*. Others still are completely new and original.

The methods used in solving the integrals are varied. Some are well known, others are inventive and creative. These include differentiating under the integral sign after the introduction of a parameter; what is commonly known as 'Feynman's trick'; converting a single integral to a double integral, and the use of infinite series. Still other ways is by exploiting algebraic identities, employing symmetry, or by creating a system of relations involving the integrals I and J and finding $I+J$ and $I-J$ first. As an example of the first of these latter approaches, from the algebraic identity

$$(A+B)^2+(A-B)^2=2A^2+2B^2,$$

on setting $A=\log(1-x)$ and $B=\log(1+x)$ the integral in (1) can be found. Such an ingenious approach was extensively used by De Doelder to evaluate many interesting integrals [5]. On display is the author's considerable manipulative dexterity and even for those who may know how to solve a particular problem there is much one can learn from a close reading of the provided solutions.

The second half of the book is devoted to the evaluation of series but it is really a rumination by the author on a particular type of series known as *Euler sums*. These are infinite series involving the harmonic numbers. As their name suggests, these series are named after the great Swiss mathematician Leonhard Euler, who along with the German mathematician Christian Goldbach initiated their study in the mid-eighteenth century. The nth generalised harmonic number of order $p\in\mathbb{N}$ is defined by

$$H_n^{(p)}=\sum_{k=1}^{n}\frac{1}{k^p},$$

such that $H_0^{(p)}\equiv 0$. When $p=1$ we have $H_n\equiv H_n^{(1)}$, the nth harmonic number. If we let $\pi=(\pi_1,\pi_2,\ldots,\pi_k)$ be a partition of integer p into k summands so that $p=\pi_1+\cdots+\pi_k$, $\pi_1\leqslant\pi_2\leqslant\cdots\leqslant\pi_k$, and q is an integer such that $q\geqslant 2$, the classical (non-linear) Euler sum is defined by [6, p. 16]

$$S_{\pi,q} = \sum_{n=1}^{\infty} \frac{H_n^{(\pi_1)} H_n^{(\pi_2)} \cdots H_n^{(\pi_k)}}{n^q}. \tag{2}$$

One of Euler's famous early results is

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).$$

Indeed, thirty-two different proofs of this classic result can be found in [4]. Later Euler found a generalisation

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q} = \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{n=1}^{q-2} \zeta(n+1)\zeta(q-n),$$

but it is disappointing to find this is not one of the problems given in the text (it is however stated without proof; see Eq. (3.45) on page 87). Perhaps the author considered the linear case far too elementary? A multitude of mostly non-linear Euler sums or closely related series involving the harmonic numbers appear. Seven problems towards the end deal specifically with alternating Euler sums.

My favourite of all the Euler sums is what is today referred to as the series of Au-Yeung. It is (Problem 4.22)

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4) = \frac{17\pi^4}{360}. \tag{3}$$

Missed by Euler this series is largely responsible for the renewed fortunes of these sums. Starting in the mid-1990s with the work of Bailey, Borwein, and Girgensohn [1] and Borwein, Borwein, and Girgensohn [2] interest in series of this type was revived by an accidental discovery of (3). On this latter point the Borwein brothers write [3, p. 1191]

> This identity [namely (3)] was surprising and new to us when Enrico Au-Yeung (an undergraduate student in the Faculty of Mathematics in Waterloo) conjectured it on the basis of a computation of $500,000$ terms (five digit accuracy!); our first impulse was to perform a higher-order computation to show it to be false. It is not easy to naively compute the value of the sum to more than about eight places.

The brothers went on to prove the result with the literature on Euler sums now vast. Ironically the proof of (3) responsible for rejuvenating interest in Euler sums was itself a rediscovery having first appeared, surprisingly, as a problem in the September 1948 issue of *The American Mathematical Monthly* [8].

From (2) we see a seemingly endless variety of Euler sums are possible. Many of the problems found in the second half of the book are devoted to evaluating particular examples of such sums. Beyond the literature little in the way of Euler sums has so far found itself a place in modern texts. A handful of problems are given in [10, pp. 228–229] and a more substantial set, but at a level slightly easier than those found in the current text, can be found in [7, pp. 148–151], which makes the present collection a welcome addition. Having problems and their solutions for a large collection of Euler sums in one location means the text should serve as a future source of reference for sums of this type.

Not all the series problems appearing in the second half of the book are straight up evaluation of Euler sums. More general and unusual sums containing a product between the nth harmonic number and the tail of the Riemann zeta function can be found. One example of this is (Problem 4.45 (i))

$$\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(4) - 1 - \frac{1}{2^4} - \cdots - \frac{1}{n^4}\right) = \frac{5}{48}\zeta(6) = \frac{\pi^6}{9072}.$$

As was the case for the integrals presented in the first half of the book, solutions to the series are found using only real methods. The methods used in their evaluation are a combination of conversion to an integral (often one already found in the first half of the book), the use of generating functions, or through series manipulation, of which the author makes heavy use. Here Abel's summation, changing the order of double summations, and the author's own *Master Theorem of Series* [11] are among the various techniques used. In the hands of the present author these are a powerful armamentaria.

I love Euler sums and would like to think they are useful. The author gives no hint of their usefulness. They simply appear as distant peaks that need to be scaled and conquered. In terms of usefulness they can often serve in the evaluation of integrals. But must a use be found for everything? Who cannot help but marvel at a beautiful closed-form solution to a problem many think is not possible? Euler sums may be a minor but scenic tributary of modern day mathematics but is one deeply rooted in the past that can be considered interesting for their own sake. If they are to eventually find wide applicability the fact remains for this to be discovered by others.

For the most part the author's manipulations with series and integrals are very clever and a lot of fun to watch. At times an integral or series that seems impossible disappears under a sea of what initially seems to be unrelated calculations only for it to re-emerge several pages later with its victor clutching at it solution. At other times invoking a hidden symmetry of the problem sees the initial impasse or but melt away. Many times my reaction to this was one of wonderment. Where do people get such ideas from? On encountering this we marvel that there are those who can imagine such things and make such unexpected connections.

So what is one to make of the book? Some readers may feel they have been transported back in time, finding themselves negotiating some rich undiscovered vein of late eighteenth or early nineteenth century mathematics. Pure mathematicians will probably grumble at the cavalier approach taken to rigour such as interchanges made between infinite summations and integrations, but for the type of person this book is most likely to appeal to this is but a small cavil. The book can serve as a useful supply of difficult definite integrals and infinite series problems for undergraduates or as a useful starting point for those wishing to attempt similar types of problems that arise from time to time in the various journals with dedicated problem pages. And there are of course a small group of those for whom the challenge of a definite integral is difficult to resist. In this book they have found themselves the perfect antidote.

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