Irish Math. Soc. Bulletin
Number 86, Winter 2020, 79–101
ISSN 0791-5578

The integral double Burnside ring of the symmetric group S_3

NORA KRAUSS

Abstract. The double Burnside R-algebra $\mathrm{B}_R(G, G)$ of a finite group G with coefficients in a commutative ring R has been introduced by S. Bouc. It is R-linearly generated by finite (G, G)-bisets, modulo a relation identifying disjoint union and sum. Its multiplication is induced by the tensor product. In his thesis at NUI Galway, B. Masterson described $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$ as a subalgebra of $\mathbf{Q}^{8\times 8}$. We give a variant of this description and continue to describe $\mathrm{B}_R(\mathrm{S}_3, \mathrm{S}_3)$ for $R \in \{\mathbf{Z}, \mathbf{Z}_{(2)}, \mathbf{F}_2, \mathbf{Z}_{(3)}, \mathbf{F}_3\}$ via congruences as suborders of certain R-orders respectively via path algebras over R.

1. Introduction

1.1. **Groups.** Groups describe symmetries of objects. That is to say, any mathematical object X has a symmetry group, called automorphism group $\mathrm{Aut}(X)$, consisting of isomorphisms from X to X. For instance, for a natural number n, the set $\{1, 2, \ldots, n\}$ has as automorphism group the symmetric group $\mathrm{Aut}(\{1, 2, \ldots, n\}) = \mathrm{S}_n$. This group consists of all bijections from $\{1, 2, \ldots, n\}$ to itself. For example, we obtain

$$\begin{aligned}\mathrm{S}_3 &= \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\right\} \\ &= \{\mathrm{id}, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\} \ .\end{aligned}$$

In the first row, $\begin{pmatrix}1&2&3\\a&b&c\end{pmatrix}$ is the map sending $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto c$.

In the second row, we have used the cycle notation, e.g. $\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = (1, 3, 2)$, the latter meaning

$$\begin{tikzcd} 1 \arrow[rr, bend left] & & 3 \arrow[dl, bend left] \\ & 2 \arrow[ul, bend left] & \end{tikzcd} .$$

We multiply by composition, e.g. $(1, 2) \bullet (1, 3) = (1, 2, 3)$.
By a theorem of Cayley, any finite group is isomorphic to a subgroup of S_n for some n.

1.2. **The Biset category and biset functors.** Suppose given finite groups H and G. An (H, G)-*biset* X is a finite set X together with a multiplication with elements of H on the left and a multiplication with elements of G on the right that commute with each other, i.e.

$$(h \cdot x) \cdot g = h \cdot (x \cdot g) =: h \cdot x \cdot g$$

for $h \in H$, $g \in G$ and $x \in X$.

As a first example, $M_1 := \mathrm{S}_3$ is a $(\mathrm{S}_2, \mathrm{S}_3)$-biset via multiplication in S_3. So for $h \in \mathrm{S}_2 = \{\mathrm{id}, (1, 2)\}$, $g \in \mathrm{S}_3$ and $x \in M_1$ we let $h \cdot x \cdot g := h \bullet x \bullet g$.

As a second example, consider the cyclic group $\mathrm{C}_3 = \{\mathrm{id}, (1, 2, 3), (1, 3, 2)\}$ and the group isomorphism $\alpha : \mathrm{C}_3 \to \mathrm{C}_3$, $x \mapsto x^2$. Then the set $M_2 := \mathrm{C}_3$ is a $(\mathrm{C}_3, \mathrm{C}_3)$-biset,

2020 *Mathematics Subject Classification.* 19A22.
Key words and phrases. double Burnside ring, symmetric group, path algebra.
Received on 3-9-2020; revised 15-10-2020.

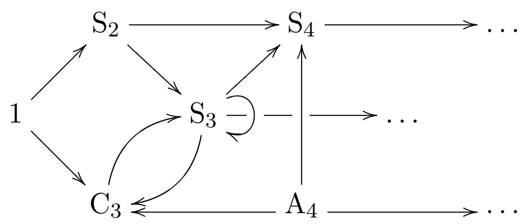
on the left via multiplication, on the right via application of α and then multiplication. E.g.

$$\begin{array}{rcll} (1,2,3)\cdot(1,3,2)\cdot(1,3,2) & = & (1,2,3)\bullet(1,3,2)\bullet\alpha((1,3,2)) & \\ & = & (1,2,3)\bullet(1,3,2)\bullet(1,2,3) & = \quad (1,2,3)\ . \end{array}$$

Suppose given a commutative ring R. S. Bouc introduced the *biset category* Biset_R, see [5, §3.1], see also the historical comments in [5, §1.4]. As objects, the category Biset_R has finite groups. The R-module of morphisms between two finite groups H and G is given by the double Burnside R-module $\mathrm{Biset}_R(H,G) = \mathrm{B}_R(H,G)$, which is R-linearly generated by finite (H,G)-bisets, modulo a relation identifying disjoint union and sum. In particular, each (H,G)-biset M yields a morphism $H \xrightarrow{[M]} G$ in Biset_R. Composition of morphisms in Biset_R is given by a tensor product operation on bisets that is similar to the tensor product of bimodules. Given an (H,G)-biset M and an (G,K)-biset N, we write $M \underset{G}{\times} N$ for their tensor product, which is an (H,K)-biset. So in Biset_R, we have the commutative triangle

$$\begin{tikzcd} H \arrow[rr, "{[M \underset{G}{\times} N]}"] \arrow[dr, "{[M]}"'] & & K \\ & G \arrow[ur, "{[N]}"'] & \end{tikzcd} \quad .$$

The category Biset_R may roughly be imagined by a picture like this.



Here, A_4 is the alternating group on 4 elements. Each biset yields an arrow, and so does each R-linear combination of bisets. Of course, there are many more objects in Biset_R – each finite group is an object there – and many more arrows between them that are not in our picture.

1.3. **Biset functors.** Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of finite groups closed under forming subgroups, factor groups and extensions. Following Bouc [3, §3.4.1], we say that an (H,G)-biset M is *$(\mathcal{X},\mathcal{Y})$-free* if for each $m \in M$ the left stabilizer of m in H is in $\mathcal{X}$ and the right stabilizer of m in G is in $\mathcal{Y}$. We have the subcategory $\mathrm{Biset}_R^{\mathcal{X},\mathcal{Y}}$ of Biset_R: As objects, it has finite groups. The R-module of morphisms in $\mathrm{Biset}_R^{\mathcal{X},\mathcal{Y}}$ between two finite groups H and G is given by the submodule of $\mathrm{B}_R(H,G)$ generated by the images of $(\mathcal{X},\mathcal{Y})$-free (H,G)-bisets, cf. [3, Lemme 4].

Certain classical theories may now be formulated as contravariant functors from $\mathrm{Biset}_R^{\mathcal{X},\mathcal{Y}}$ to the category of R-modules, called *biset functors* over R.

Consider a prime number p. Let $\mathcal{X}$ be the class of all finite groups. Let $\mathcal{Y}$ be the class of finite groups whose orders are not divisible by p. Then e.g. the $(\mathrm{S}_2,\mathrm{S}_3)$-biset M_1 and the $(\mathrm{C}_3,\mathrm{C}_3)$-biset M_2 from §1.2 yield morphisms in $\mathrm{Biset}_{\mathbf{Z}}^{\mathcal{X},\mathcal{Y}}$.

Suppose given an object of $\mathrm{Biset}_{\mathbf{Z}}^{\mathcal{X},\mathcal{Y}}$, i.e. a finite group G. Let

$$\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z} = \{0,\ldots,p-1\}\ ,$$

where we agree to calculate modulo p. An $\mathbf{F}_p$-representation of G is a finite dimensional $\mathbf{F}_p$-vectorspace V, together with a left multiplication with elements of G. Such a representation is called simple if it does not have a nontrivial subrepresentation. Each representation has a sequence of subrepresentations with simple steps, called composition factors.

Let $\mathrm{Rep}_{\mathbf{F}_p}(G)$ be the free abelian group on the set of isoclasses of simple representations. Each $\mathbf{F}_p$-representation V of G yields an element $[V]$ in $\mathrm{Rep}_{\mathbf{F}_p}(G)$, namely the formal sum of its composition factors. Given finite groups H and G and an (H,G)-biset M, we obtain the map

$$\begin{array}{rcl} \mathrm{Rep}_{\mathbf{F}_p}(G) & \xrightarrow{\mathrm{Rep}_{\mathbf{F}_p}([M])} & \mathrm{Rep}_{\mathbf{F}_p}(H) \\ {[V]} & \mapsto & [\mathbf{F}_p M \underset{\mathbf{F}_p G}{\otimes} V] \ , \end{array}$$

using the usual tensor product over rings.

These constructions furnish a contravariant $\mathbf{Z}$-linear functor $\mathrm{Rep}_{\mathbf{F}_p}$ from $\mathrm{Biset}_{\mathbf{Z}}^{\mathcal{X},\mathcal{Y}}$ to the category of $\mathbf{Z}$-modules, i.e. to the category of abelian groups. In particular, using the bisets M_1 and M_2 from §1.2, we obtain the maps

$$\begin{array}{rcl} \mathrm{Rep}_{\mathbf{F}_p}(\mathrm{S}_3) & \xrightarrow{\mathrm{Rep}_{\mathbf{F}_p}([M_1])} & \mathrm{Rep}_{\mathbf{F}_p}(\mathrm{S}_2) \\ {[V]} & \mapsto & [\text{restriction of } V \text{ to } \mathrm{S}_2] \end{array}$$

and

$$\begin{array}{rcl} \mathrm{Rep}_{\mathbf{F}_p}(\mathrm{C}_3) & \xrightarrow{\mathrm{Rep}_{\mathbf{F}_p}([M_2])} & \mathrm{Rep}_{\mathbf{F}_p}(\mathrm{C}_3) \\ {[V]} & \mapsto & [\text{twist of } V \text{ with } \alpha] \ . \end{array}$$

Note that, if $p \leq n$, even the simple $\mathbf{F}_p$-representations of S_n are not entirely known: One knows a construction, due to James [9], but one does not know their $\mathbf{F}_p$-dimensions. Biset functors do not directly aim to solve this problem, but at any rate they are a tool to work with these representations.

1.4. **Globally-defined Mackey functors.** There is an equivalence of categories between the category of biset functors over R and the category of *globally-defined Mackey functors* $\mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}$ [6, §8]. Here, a globally-defined Mackey functor, with respect to $\mathcal{X}$ and $\mathcal{Y}$, maps groups to R-modules and each group morphism α covariantly to an R-module morphism $\alpha_\star$, provided $\mathrm{kern}(\alpha) \in \mathcal{Y}$, and contravariantly to $\alpha^\star$, provided $\mathrm{kern}(\alpha) \in \mathcal{X}$. It is required that these morphisms satisfy a list of compatibilities, amongst which a Mackey formula, see e.g. [6, §8]. By that equivalence, these requirements on a Mackey functor can now be viewed as properties that result from being a contravariant functor from $\mathrm{Biset}_R^{\mathcal{X},\mathcal{Y}}$ to R-Mod.

1.5. **Further examples.** We list two examples of biset functors, [6, §8].

- Let $\mathcal{X} = \{1\}$ and let $\mathcal{Y}$ consist of all finite groups. Let $n \geq 0$. Consider the biset functor $\mathrm{Biset}_{\mathbf{Z}}^{\mathcal{X},\mathcal{Y}} \to \mathbf{Z}$-Mod that maps a finite group G to the algebraic K-theory $\mathrm{K}_n(\mathbf{Z}G)$ of $\mathbf{Z}G$.
- Let $\mathcal{X}$ consist of all finite groups and let $\mathcal{Y} = \{1\}$. Let $n \geq 0$. Consider the biset functor $\mathrm{Biset}_R^{\mathcal{X},\mathcal{Y}} \to R$-Mod that maps a finite group G to the cohomology $\mathrm{H}^n(G,R)$ of G with trivial coefficients.

For some more examples, see [6, §8]. The example of the classical Burnside ring, depending on a group G, is also explained in [4, §6.1].

1.6. **The double Burnside algebra.** Suppose given a finite group G, i.e an object of Biset_R. Its endomorphism ring $\mathrm{B}_R(G,G)$ in the category Biset_R is called *double Burnside algebra* of G.

The isomorphism classes of finite transitive (G,G)-bisets form an R-linear basis of $\mathrm{B}_R(G,G)$. In particular, if we choose a system $\mathcal{L}_{G\times G}$ of representatives for the conjugacy classes of subgroups of $G \times G$, we have the R-linear basis $([(G\times G)/U] : U \in \mathcal{L}_{G\times G})$.

If G is cyclic and if R is a field in which $|G|$ and $\varphi(|G|)$ are invertible, where φ denotes Euler's totient function, then the double Burnside algebra $\mathrm{B}_R(G,G)$ is semisimple. This is shown in [7, Theorem 8.11, Remark 8.12(a)].

In case of $G = \mathrm{S}_3$, we have 22 conjugacy classes of subgroups of $\mathrm{S}_3 \times \mathrm{S}_3$ and thus $\mathrm{rk}_R(\mathrm{B}_R(\mathrm{S}_3,\mathrm{S}_3)) = 22$. The double Burnside $\mathbf{Q}$-algebra $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ has been described by B. Masterson [1, §8] and then by B. Masterson and G. Pfeiffer [2, §7]. We describe $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ independently, using a direct Magma-supported calculation [10], with the aim of being able to pass from $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ to $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$ in the sequel.

In order to do that, we first restate some preliminaries on bisets and the double Burnside ring in §2 and construct a $\mathbf{Z}$-linear basis of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$ in §3.

In §4 we tackle the problem that the double Burnside $\mathbf{Q}$-algebra $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ is not semisimple [5, Proposition 6.1.5], thus not isomorphic to a direct product of matrix rings. As a substitute, we use a suitable isomorphic copy A of $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$. We obtain this copy using a Peirce decomposition of $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$. In addition, we give a description of $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ as path algebra modulo relations.

The next step, in §5, is to pass from $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ to $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$. We find a $\mathbf{Z}$-order $A_{\mathbf{Z}}$ inside A such that $A_{\mathbf{Z}}$ contains an isomorphic copy of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$, which we describe via congruences, cf. Proposition 5, Theorem 8.

$$\begin{tikzcd} \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3) \arrow[r, "\sim"] & A \\ \mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) \arrow[u, hook] \arrow[r, "\text{injective}"] & A_{\mathbf{Z}} \arrow[u, hook] \end{tikzcd}$$

We calculate a path algebra for $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3,\mathrm{S}_3)$, cf. Proposition 11. We deduce that $\mathrm{B}_{\mathbf{F}_2}(\mathrm{S}_3,\mathrm{S}_3)$ is Morita equivalent to the path algebra

$$\mathbf{F}_2\left[\begin{tikzcd} \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde\tau_2"] & \tilde{\mathrm{e}}_5 \arrow[l, bend left, "\tilde\tau_1"] \arrow[loop right, "\tilde\tau_7"] \arrow[r, bend right, "\tilde\tau_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde\tau_4"'] \end{tikzcd}\right] \Big/ \begin{pmatrix} \tilde\tau_2\tilde\tau_1 & , & \tilde\tau_2\tilde\tau_3 & , & \tilde\tau_2\tilde\tau_7, \\ \tilde\tau_4\tilde\tau_1 & , & \tilde\tau_4\tilde\tau_3 & , & \tilde\tau_4\tilde\tau_7, \\ \tilde\tau_7\tilde\tau_1 & , & \tilde\tau_7\tilde\tau_3 & , & \tilde\tau_7^2 - \tilde\tau_1\tilde\tau_2 \end{pmatrix},$$

cf. Corollary 12.

We calculate a path algebra for $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3,\mathrm{S}_3)$, cf. Proposition 15. We deduce that $\mathrm{B}_{\mathbf{F}_3}(\mathrm{S}_3,\mathrm{S}_3)$ is Morita equivalent to the path algebra

$$\mathbf{F}_3\left[\begin{tikzcd} \tilde{\mathrm{e}}_5 & \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde\tau_2"] & \tilde{\mathrm{e}}_6 \arrow[l, bend left, "\tilde\tau_1"] \arrow[r, bend right, "\tilde\tau_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde\tau_4"'] \end{tikzcd}\right] \Big/ (\tilde\tau_4\tilde\tau_3,\ \tilde\tau_4\tilde\tau_1,\ \tilde\tau_2\tilde\tau_1,\ \tilde\tau_2\tilde\tau_3)\ ,$$

cf. Corollary 16.

2. Preliminaries on bisets and the double Burnside algebra

Bisets. Recall that an (G,G)-*biset* X is a finite set X together with a left G and a right G-action that commute with each other, i.e. $(h\cdot x)\cdot g = h\cdot(x\cdot g) =: h\cdot x\cdot g$ for $h,g\in G$ and $x\in X$.

Every (G,G)-biset X can be regarded as a left $(G\times G)$-set by setting $(h,g)x := hxg^{-1}$ for $(h,g)\in G\times G$ and $x\in X$. Likewise, every left $(G\times G)$-set Y can be regarded as an (G,G)-biset by setting $h\cdot y\cdot g := (h,g^{-1})y$ for $h,g\in G$ and $y\in Y$. We freely use this identification.

Tensor product. Let M be an (G,G)-biset and let N be a (G,G)-biset. The cartesian product $M\times N$ is a (G,G)-biset via $h(m,n)p=(hm,np)$ for $h,p\in G$ and $(m,n)\in M\times N$. It becomes a left G-set via $g(m,n)=(mg^{-1},gn)$ for $g\in G$ and $(m,n)\in M\times N$. We call the set of G-orbits on $M\times N$ the *tensor product* $M\underset{G}{\times}N$ of M and N. This also is an (G,G)-biset. The G-orbit of the element $(m,n)\in M\times N$ is denoted by $m\underset{G}{\times}n\in M\underset{G}{\times}N$. Moreover, let L be a (G,G)-biset. Then $M\underset{G}{\times}(N\underset{G}{\times}L)\xrightarrow{\sim}(M\underset{G}{\times}N)\underset{G}{\times}L$, $m\underset{G}{\times}(n\underset{G}{\times}\ell)\mapsto(m\underset{G}{\times}n)\underset{G}{\times}\ell$ as (G,G)-bisets.

Double Burnside R-algebra. We denote by $\mathrm{B}_R(G,G)$ the double Burnside R-algebra of G. Recall that $\mathrm{B}_R(G,G)$ is the R-module freely generated by the isomorphism classes of finite (G,G)-bisets, modulo the relations $[M\sqcup N]=[M]+[N]$ for (G,G)-bisets M,N. Multiplication is defined by $[M]\underset{G}{\cdot}[N]=[M\underset{G}{\times}N]$ for (G,G)-bisets M,N. An R-linear basis of $\mathrm{B}_R(G,G)$ is given by $([(G\times G)/U]:U\in\mathcal{L}_{G\times G})$, where we choose a system $\mathcal{L}_{G\times G}$ of representatives for the conjugacy classes of subgroups of $G\times G$. Moreover, $1_{\mathrm{B}_{\mathbf{Z}}(G,G)}=[G]$.

Abbreviation. In case of $G=\mathrm{S}_3$, we often abbreviate $\mathrm{B}_R:=\mathrm{B}_R(\mathrm{S}_3,\mathrm{S}_3)$.

3. $\mathbf{Z}$-linear basis of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$

The following calculations were done using the computer algebra system Magma [10].

The group S_3 has the subgroups

$$V_0:=\{\mathrm{id}\},\ V_1:=\langle(1,2)\rangle,\ V_2:=\langle(1,3)\rangle,\ V_3:=\langle(2,3)\rangle,\ V_4:=\langle(1,2,3)\rangle,\ V_5:=\mathrm{S}_3\ .$$

The set $\{V_0,\ V_1,\ V_4,\ V_5\}$ is a system of representatives for the conjugacy classes of subgroups of S_3. In S_3, we write $a:=(1,2)$, $b:=(1,2,3)$ and $1:=\mathrm{id}$. So $V_1=\langle a\rangle$, $V_4=\langle b\rangle$ and $V_5=\langle a,b\rangle$.

A system of representatives for the conjugacy classes of subgroups of $\mathrm{S}_3\times\mathrm{S}_3$ is given by

$$\begin{array}{llllllll}
U_{0,0} & := V_0\times V_0 & = & \{(1,1)\}, & U_{4,1} & := V_4\times V_1 & = & \langle(b,1),(1,a)\rangle,\\
U_{1,0} & := V_1\times V_0 & = & \langle(a,1)\rangle, & U_{1,4} & := V_1\times V_4 & = & \langle(a,1),(1,b)\rangle,\\
U_{0,1} & := V_0\times V_1 & = & \langle(1,a)\rangle, & U_7 & := & & \langle(a,a),(b,1)\rangle,\\
\Delta(V_1) & = & & \langle(a,a)\rangle, & \Delta(V_5) & = & & \langle(a,a),(b,b)\rangle,\\
U_{4,0} & := V_4\times V_0 & = & \langle(b,1)\rangle, & U_{4,4} & := V_4\times V_4 & = & \langle(b,1),(1,b)\rangle,\\
U_{0,4} & := V_0\times V_4 & = & \langle(1,b)\rangle, & U_{1,5} & := V_1\times V_5 & = & \langle(a,1),(1,a),(1,b)\rangle,\\
\Delta(V_4) & = & & \langle(b,b)\rangle, & U_{5,1} & := V_5\times V_1 & = & \langle(a,1),(b,1),(1,a)\rangle,\\
U_{1,1} & := V_1\times V_1 & = & \langle(a,1),(1,a)\rangle, & U_{4,5} & := V_4\times V_5 & = & \langle(b,1),(1,a),(1,b)\rangle,\\
U_{5,0} & := V_5\times V_0 & = & \langle(a,1),(b,1)\rangle, & U_{5,4} & := V_5\times V_4 & = & \langle(a,1),(b,1),(1,b)\rangle,\\
U_{0,5} & := V_0\times V_5 & = & \langle(1,a),(1,b)\rangle, & U_8 & := & & \langle(a,a),(b,1),(1,b)\rangle,\\
U_6 & := & & \langle(a,a),(1,b)\rangle, & U_{5,5} & := V_5\times V_5 & = & \langle(a,1),(1,a),(b,1),(1,b)\rangle.
\end{array}$$

Let $H_{i,j}:=[(\mathrm{S}_3\times\mathrm{S}_3)/U_{i,j}]$ for $i,j\in\{0,1,4,5\}$, $H_s:=[(\mathrm{S}_3\times\mathrm{S}_3)/U_s]$ for $s\in[6,8]$ and $H_t^{\Delta}:=[(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(V_t)]$ for $t\in\{1,4,5\}$.

So we obtain the $\mathbf{Z}$-linear basis

$$\mathcal{H}:=\begin{array}{l}(H_{0,0},H_{1,0},H_{0,1},H_1^{\Delta},H_{4,0},H_{0,4},H_4^{\Delta},H_{1,1},H_{5,0},H_{0,5},H_6,\\ H_{4,1},H_{1,4},H_7,H_5^{\Delta},H_{4,4},H_{1,5},H_{5,1},H_{4,5},H_{5,4},H_8,H_{5,5})\end{array}$$

of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$. Of course, $\mathcal{H}$ is also a $\mathbf{Q}$-linear basis of $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$.

4. $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$

4.1. **Peirce decomposition of** $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$**.** Using Magma [10] we obtain an orthogonal decomposition of $1_{\mathrm{B}_{\mathbf{Q}}}$ into the following idempotents of $\mathrm{B}_{\mathbf{Q}} = \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$.

$$\begin{aligned}
e &:= -\tfrac{1}{2}H_{0,0} + H_{1,0} + \tfrac{1}{2}H_{4,0}\\
g &:= \tfrac{4}{3}H_{0,0} - 2H_{1,0} - \tfrac{4}{3}H_{0,1} - H_{4,0} + 2H_{1,1} + H_{4,1}\\
h &:= -\tfrac{1}{12}H_{0,0} + \tfrac{1}{3}H_{0,1} + \tfrac{1}{4}H_{4,0} - \tfrac{1}{4}H_{0,4} + \tfrac{3}{4}H_{4,4} - H_{4,1}\\
\varepsilon_2 &:= -H_{0,0} + H_{1,0} + H_{0,1} + H_1^{\Delta} - 2H_{1,1}\\
\varepsilon_3 &:= -\tfrac{1}{4}H_{0,0} + \tfrac{1}{4}H_{4,0} + \tfrac{1}{4}H_{0,4} + \tfrac{1}{2}H_4^{\Delta} - \tfrac{3}{4}H_{4,4}\\
\varepsilon_4 &:= \tfrac{1}{2}H_{0,0} - H_1^{\Delta} - \tfrac{1}{2}H_4^{\Delta} + H_5^{\Delta}
\end{aligned}$$

Write $\varepsilon_1 := e + g + h$. In Remark 1 and Remark 3, we shall see that these idempotents are primitive. In a next step, we fix $\mathbf{Q}$-linear bases of the Peirce components.

Peirce component	$\mathbf{Q}$-linear basis
$e\,\mathrm{B}_{\mathbf{Q}}\,e$	$e = -\frac{1}{2}H_{0,0} + H_{1,0} + \frac{1}{2}H_{4,0}$
$e\,\mathrm{B}_{\mathbf{Q}}\,g$	$b_{e,g} := \frac{1}{2}H_{0,0} - H_{1,0} - \frac{1}{2}H_{0,1} - \frac{1}{2}H_{4,0} + H_{1,1} + \frac{1}{2}H_{4,1}$
$e\,\mathrm{B}_{\mathbf{Q}}\,h$	$b_{e,h} := -\frac{1}{8}H_{0,0} + \frac{1}{4}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{8}H_{4,0} - \frac{3}{8}H_{0,4} - H_{1,1} - \frac{1}{2}H_{4,1} + \frac{3}{4}H_{1,4} + \frac{3}{8}H_{4,4}$
$g\,\mathrm{B}_{\mathbf{Q}}\,e$	$b_{g,e} := -\frac{4}{3}H_{0,0} + 2H_{1,0} + H_{4,0}$
$g\,\mathrm{B}_{\mathbf{Q}}\,g$	$g = \frac{4}{3}H_{0,0} - 2H_{1,0} - \frac{4}{3}H_{0,1} - H_{4,0} + 2H_{1,1} + H_{4,1}$
$g\,\mathrm{B}_{\mathbf{Q}}\,h$	$b_{g,h} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{4}{3}H_{0,1} + \frac{1}{4}H_{4,0} - H_{0,4} - 2H_{1,1} - H_{4,1} + \frac{3}{2}H_{1,4} + \frac{3}{4}H_{4,4}$
$h\,\mathrm{B}_{\mathbf{Q}}\,e$	$b_{h,e} := -\frac{1}{3}H_{0,0} + H_{4,0}$
$h\,\mathrm{B}_{\mathbf{Q}}\,g$	$b_{h,g} := \frac{1}{3}H_{0,0} - \frac{1}{3}H_{0,1} - H_{4,0} + H_{4,1}$
$h\,\mathrm{B}_{\mathbf{Q}}\,h$	$h = -\frac{1}{12}H_{0,0} + \frac{1}{3}H_{0,1} + \frac{1}{4}H_{4,0} - \frac{1}{4}H_{0,4} + \frac{3}{4}H_{4,4} - H_{4,1}$
$e\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$b_{e,\varepsilon_4} := -\frac{1}{8}H_{0,0} + \frac{1}{4}H_{1,0} + \frac{1}{4}H_{0,1} + \frac{1}{8}H_{4,0} + \frac{1}{8}H_{0,4} - \frac{1}{2}H_{1,1} - \frac{1}{4}H_{0,5} - \frac{1}{4}H_{4,1}$ $- \frac{1}{4}H_{1,4} - \frac{1}{8}H_{4,4} + \frac{1}{2}H_{1,5} + \frac{1}{4}H_{4,5}$
$g\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$b_{g,\varepsilon_4} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{2}{3}H_{0,1} + \frac{1}{4}H_{4,0} + \frac{1}{3}H_{0,4} - H_{1,1} - \frac{2}{3}H_{0,5} - \frac{1}{2}H_{4,1}$ $- \frac{1}{2}H_{1,4} - \frac{1}{4}H_{4,4} + H_{1,5} + \frac{1}{2}H_{4,5}$
$h\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$b_{h,\varepsilon_4} := -\frac{1}{12}H_{0,0} + \frac{1}{6}H_{0,1} + \frac{1}{4}H_{4,0} + \frac{1}{12}H_{0,4} - \frac{1}{6}H_{0,5} - \frac{1}{2}H_{4,1} - \frac{1}{4}H_{4,4} + \frac{1}{2}H_{4,5}$
$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$\varepsilon_2 = -H_{0,0} + H_{1,0} + H_{0,1} + H_1^{\Delta} - 2H_{1,1}$
$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$b_{\varepsilon_2,\varepsilon_4} := -\frac{1}{2}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_1^{\Delta} + \frac{1}{2}H_{0,4} - H_{1,1} - \frac{1}{2}H_{0,5} - \frac{1}{2}H_6 - \frac{1}{2}H_{1,4}$ $+ H_{1,5}$
$\varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3$	$\varepsilon_3 = -\frac{1}{4}H_{0,0} + \frac{1}{4}H_{4,0} + \frac{1}{4}H_{0,4} + \frac{1}{2}H_4^{\Delta} - \frac{3}{4}H_{4,4}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,e$	$b_{\varepsilon_4,e} := \frac{1}{6}H_{0,0} - \frac{1}{3}H_{1,0} - \frac{1}{6}H_{4,0} + \frac{1}{3}H_{5,0}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,g$	$b_{\varepsilon_4,g} := -\frac{1}{6}H_{0,0} + \frac{1}{3}H_{1,0} + \frac{1}{6}H_{0,1} + \frac{1}{6}H_{4,0} - \frac{1}{3}H_{1,1} - \frac{1}{3}H_{5,0} - \frac{1}{6}H_{4,1} + \frac{1}{3}H_{5,1}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,h$	$b_{\varepsilon_4,h} := \frac{1}{24}H_{0,0} - \frac{1}{12}H_{1,0} - \frac{1}{6}H_{0,1} - \frac{1}{24}H_{4,0} + \frac{1}{8}H_{0,4} + \frac{1}{3}H_{1,1} + \frac{1}{12}H_{5,0}$ $+ \frac{1}{6}H_{4,1} - \frac{1}{4}H_{1,4} - \frac{1}{8}H_{4,4} - \frac{1}{3}H_{5,1} + \frac{1}{4}H_{5,4}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$b_{\varepsilon_4,\varepsilon_2} := -\frac{1}{2}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_1^{\Delta} + \frac{1}{2}H_{4,0} - H_{1,1} - \frac{1}{2}H_{5,0} - \frac{1}{2}H_{4,1}$ $- \frac{1}{2}H_7 + H_{5,1}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$\varepsilon_4 = \frac{1}{2}H_{0,0} - H_1^{\Delta} - \frac{1}{2}H_4^{\Delta} + H_5^{\Delta}$, $b'_{\varepsilon_4,\varepsilon_4} := \frac{1}{24}H_{0,0} - \frac{1}{12}H_{1,0} - \frac{1}{12}H_{0,1} - \frac{1}{24}H_{4,0} - \frac{1}{24}H_{0,4} + \frac{1}{6}H_{1,1} + \frac{1}{12}H_{5,0} + \frac{1}{12}H_{0,5}$ $+ \frac{1}{12}H_{4,1} + \frac{1}{12}H_{1,4} + \frac{1}{24}H_{4,4} - \frac{1}{6}H_{1,5} - \frac{1}{6}H_{5,1} - \frac{1}{12}H_{4,5} - \frac{1}{12}H_{5,4} + \frac{1}{6}H_{5,5}$, $b''_{\varepsilon_4,\varepsilon_4} := \frac{1}{4}H_{0,0} - \frac{3}{4}H_{1,0} - \frac{3}{4}H_{0,1} + \frac{1}{4}H_1^{\Delta} - \frac{1}{4}H_{4,0} - \frac{1}{4}H_{0,4} + \frac{3}{2}H_{1,1} + \frac{3}{4}H_{5,0}$ $+ \frac{3}{4}H_{0,5} - \frac{1}{4}H_6 + \frac{3}{4}H_{4,1} + \frac{3}{4}H_{1,4} - \frac{1}{4}H_7 + \frac{1}{4}H_{4,4} - \frac{3}{2}H_{1,5} - \frac{3}{2}H_{5,1}$ $- \frac{3}{4}H_{4,5} - \frac{3}{4}H_{5,4} + \frac{1}{4}H_8 + \frac{3}{2}H_{5,5}$

Remark 1. The idempotents $e, g, h, \varepsilon_2, \varepsilon_3$ are primitive, as $e\,\mathrm{B}_{\mathbf{Q}}\,e \cong \mathbf{Q}$, $g\,\mathrm{B}_{\mathbf{Q}}\,g \cong \mathbf{Q}$, $h\,\mathrm{B}_{\mathbf{Q}}\,h \cong \mathbf{Q}$, $\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 \cong \mathbf{Q}$ and $\varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3 \cong \mathbf{Q}$.

We have the following multiplication table for the basis elements of $\mathrm{B}_{\mathbf{Q}} = \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$.

$(\cdot)$	e	$b_{e,g}$	$b_{e,h}$	$b_{g,e}$	g	$b_{g,h}$	$b_{h,e}$	$b_{h,g}$	h	b_{e,ε_4}	b_{g,ε_4}	b_{h,ε_4}	ε_2	$b_{\varepsilon_2,\varepsilon_4}$	ε_3	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	$b_{\varepsilon_4,\varepsilon_2}$	ε_4	$b'_{\varepsilon_4,\varepsilon_4}$	$b''_{\varepsilon_4,\varepsilon_4}$
e	e	$b_{e,g}$	$b_{e,h}$	0	0	0	0	0	0	b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0
$b_{e,g}$	0	0	0	e	$b_{e,g}$	$b_{e,h}$	0	0	0	0	b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0
$b_{e,h}$	0	0	0	0	0	0	e	$b_{e,g}$	$b_{e,h}$	0	0	b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0
$b_{g,e}$	$b_{g,e}$	g	$b_{g,h}$	0	0	0	0	0	0	b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0
g	0	0	0	$b_{g,e}$	g	$b_{g,h}$	0	0	0	0	b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0
$b_{g,h}$	0	0	0	0	0	0	$b_{g,e}$	g	$b_{g,h}$	0	0	b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0
$b_{h,e}$	$b_{h,e}$	$b_{h,g}$	h	0	0	0	0	0	0	b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0
$b_{h,g}$	0	0	0	$b_{h,e}$	$b_{h,g}$	h	0	0	0	0	b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0
h	0	0	0	0	0	0	$b_{h,e}$	$b_{h,g}$	h	0	0	b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0
b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_{e,ε_4}	0	0
b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_{g,ε_4}	0	0
b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_{h,ε_4}	0	0
ε_2	0	0	0	0	0	0	0	0	0	0	0	0	ε_2	$b_{\varepsilon_2,\varepsilon_4}$	0	0	0	0	0	0	0	0
$b_{\varepsilon_2,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b_{\varepsilon_2,\varepsilon_4}$	0	0
ε_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ε_3	0	0	0	0	0	0	0
$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	0	0	0	0	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0
$b_{\varepsilon_4,g}$	0	0	0	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	0	0	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0
$b_{\varepsilon_4,h}$	0	0	0	0	0	0	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0
$b_{\varepsilon_4,\varepsilon_2}$	0	0	0	0	0	0	0	0	0	0	0	0	$b_{\varepsilon_4,\varepsilon_2}$	$b''_{\varepsilon_4,\varepsilon_4} - 12b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0
ε_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	$b_{\varepsilon_4,\varepsilon_2}$	ε_4	$b'_{\varepsilon_4,\varepsilon_4}$	$b''_{\varepsilon_4,\varepsilon_4}$
$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0
$b''_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b''_{\varepsilon_4,\varepsilon_4}$	0	0

We see that ε_3 is even a central element.

Lemma 2. *Consider* $\mathbf{Q}[\eta,\xi]/(\eta^2,\eta\xi,\xi^2) = \mathbf{Q}[\overline{\eta},\overline{\xi}]$, *where* $\overline{\xi} := \xi + (\eta^2,\eta\xi,\xi^2)$ *and* $\overline{\eta} := \eta + (\eta^2,\eta\xi,\xi^2)$.

We have the $\mathbf{Q}$*-algebra isomorphism*

$$\begin{array}{rccc} \mu: & \mathbf{Q}[\overline{\eta},\overline{\xi}] & \to & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ & \overline{\eta} & \mapsto & b'_{\varepsilon_4,\varepsilon_4} \\ & \overline{\xi} & \mapsto & b''_{\varepsilon_4,\varepsilon_4}\ . \end{array}$$

Proof. Since $\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 = {}_{\mathbf{Q}}\langle \varepsilon_4, b'_{\varepsilon_4,\varepsilon_4}, b''_{\varepsilon_4,\varepsilon_4}\rangle$ is commutative and $(b'_{\varepsilon_4,\varepsilon_4})^2 = 0$, $(b''_{\varepsilon_4,\varepsilon_4})^2 = 0$ and $b'_{\varepsilon_4,\varepsilon_4} b''_{\varepsilon_4,\varepsilon_4} = 0$, the map μ is a well-defined $\mathbf{Q}$-algebra morphism.

As the $\mathbf{Q}$-linear basis $(1,\overline{\eta},\overline{\xi})$ is mapped to the $\mathbf{Q}$-linear basis $(\varepsilon_4, b'_{\varepsilon_4,\varepsilon_4}, b''_{\varepsilon_4,\varepsilon_4})$, it is bijective. □

Remark 3. *The ring* $\mathbf{Q}[\overline{\eta},\overline{\xi}]$ *is local. In particular,* ε_4 *is a primitive idempotent of* $\mathrm{B}_{\mathbf{Q}}$.

Proof. We have $\mathrm{U}(\mathbf{Q}[\overline{\eta},\overline{\xi}]) = \mathbf{Q}[\overline{\eta},\overline{\xi}] \setminus (\overline{\eta},\overline{\xi})$, as for $u := a + b\overline{\eta} + c\overline{\xi}$ the inverse is given by $u^{-1} = a^{-1} - a^{-2}b\overline{\eta} - a^{-2}c\overline{\xi}$ for $a,b,c \in \mathbf{Q}$, with $a \neq 0$. Thus the nonunits of $\mathbf{Q}[\overline{\eta},\overline{\xi}]$ form an ideal and so $\mathbf{Q}[\overline{\eta},\overline{\xi}]$ is a local ring. □

To standardize notation, we aim to construct a $\mathbf{Q}$-algebra $A := \bigoplus_{i,j} A_{i,j}$ such that $A \cong \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$.

In a first step to do so, we choose $\mathbf{Q}$-vector spaces $A_{i,j}$ and $\mathbf{Q}$-linear isomorphisms $\gamma_{i,j} : A_{i,j} \xrightarrow{\sim} \varepsilon_i \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_j$ for $i, j \in [1,4]$. We define the tuple of $\mathbf{Q}$-vector spaces

$$\begin{array}{llll} (A_{1,1}\,, & A_{1,2}\,, & A_{1,3}\,, & A_{1,4}\,, \\ \;A_{2,1}\,, & A_{2,2}\,, & A_{2,3}\,, & A_{2,4}\,, \\ \;A_{3,1}\,, & A_{3,2}\,, & A_{3,3}\,, & A_{3,4}\,, \\ \;A_{4,1}\,, & A_{4,2}\,, & A_{4,3}\,, & A_{4,4}) \end{array} := \begin{array}{llll} (\mathbf{Q}^{3\times 3}\,, & 0\,, & 0\,, & \mathbf{Q}^{3\times 1}\,, \\ \;0\,, & \mathbf{Q}\,, & 0\,, & \mathbf{Q}\,, \\ \;0\,, & 0\,, & \mathbf{Q}\,, & 0\,, \\ \;\mathbf{Q}^{1\times 3}\,, & \mathbf{Q}\,, & 0\,, & \mathbf{Q}[\overline{\eta},\overline{\xi}])\,, \end{array}$$

cf. Lemma 2.

We have $\gamma_{s,t} = 0$ for $(s,t) \in \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(3,4),(4,3)\}$.

Let

$$\begin{array}{rcl} \gamma_{1,1} : \; A_{1,1} & \xrightarrow{\sim} & \varepsilon_1 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_1 \\ \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix} & \mapsto & \begin{array}{rrr} r_{1,1}e & + \; r_{1,2}b_{e,g} & + \; r_{1,3}b_{e,h} \\ + \; r_{2,1}b_{g,e} & + \; r_{2,2}g & + \; r_{2,3}b_{g,h} \\ + \; r_{3,1}b_{h,e} & + \; r_{3,2}b_{h,g} & + \; r_{3,3}h \end{array} \end{array},$$

$$\begin{array}{rcl} \gamma_{1,4} \quad : \; A_{1,4} & \xrightarrow{\sim} & \varepsilon_1 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} & \mapsto & \begin{array}{r} u_1 b_{e,\varepsilon_4} \\ + \, u_2 b_{g,\varepsilon_4} \\ + \, u_3 b_{h,\varepsilon_4} \end{array} \end{array},$$

$$\begin{array}{rcl} \gamma_{2,2} \quad : \; A_{2,2} & \xrightarrow{\sim} & \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2 \\ u & \mapsto & u\varepsilon_2 \end{array}, \qquad \begin{array}{rcl} \gamma_{2,4} \quad : \; A_{2,4} & \xrightarrow{\sim} & \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ u & \mapsto & u b_{\varepsilon_2,\varepsilon_4} \end{array},$$

$$\begin{array}{rcl} \gamma_{3,3} \quad : \; A_{3,3} & \xrightarrow{\sim} & \varepsilon_3 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_3 \\ u & \mapsto & u\varepsilon_3 \end{array},$$

$$\begin{array}{rcl} \gamma_{4,1} \quad : \; A_{4,1} & \xrightarrow{\sim} & \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_1 \\ \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} & \mapsto & v_1 b_{\varepsilon_4,e} + v_2 b_{\varepsilon_4,g} + v_3 b_{\varepsilon_4,h} \end{array},$$

$$\begin{array}{rcl} \gamma_{4,2} \quad : \; A_{4,2} & \xrightarrow{\sim} & \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2 \\ u & \mapsto & u b_{\varepsilon_4,\varepsilon_2} \end{array},$$

$$\begin{array}{rcl} \gamma_{4,4} \overset{\text{L.2}}{:=} \mu : \; A_{4,4} & \xrightarrow{\sim} & \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ a + b\overline{\eta} + c\overline{\xi} & \mapsto & a\varepsilon_4 + b b'_{\varepsilon_4,\varepsilon_4} + c b''_{\varepsilon_4,\varepsilon_4}. \end{array}$$

Let $\beta : \mathrm{B}_{\mathbf{Q}} \times \mathrm{B}_{\mathbf{Q}} \to \mathrm{B}_{\mathbf{Q}}$ be the multiplication map on $\mathrm{B}_{\mathbf{Q}}$. Write

$$\beta_{i,j,k} := \beta|^{\varepsilon_i \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_k}_{\varepsilon_i \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_j \times \varepsilon_j \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_k} : \varepsilon_i \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_j \times \varepsilon_j \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_k \to \varepsilon_i \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_k \; .$$

Now, we construct $\mathbf{Q}$-bilinear multiplication maps $\alpha_{i,j,k}$ for $i, j, k \in [1,4]$ such that the following quadrangle of maps commutes.

$$\begin{tikzcd} A_{i,j} \times A_{j,k} \arrow[r, "\alpha_{i,j,k}"] \arrow[d, "\gamma_{i,j} \times \gamma_{j,k}"'] & A_{i,k} \arrow[d, "\gamma_{i,k}"] \\ \varepsilon_i \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_j \times \varepsilon_j \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_k \arrow[r, "\beta_{i,j,k}"] & \varepsilon_i \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_k \end{tikzcd}$$

I.e. we set $\alpha_{i,j,k} := \gamma_{i,k}^{-1} \circ \beta_{i,j,k} \circ (\gamma_{i,j} \times \gamma_{j,k})$. This leads to

- $\alpha_{i,j,k} = 0$ if (i,j), (j,k) or (i,k) is contained in
  $\{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(3,4),(4,3)\}$
- $\alpha_{1,1,1} : A_{1,1} \times A_{1,1} \to A_{1,1},\ (X,Y) \mapsto XY$
- $\alpha_{1,1,4} : A_{1,1} \times A_{1,4} \to A_{1,4},\ (X,u) \mapsto Xu$
- $\alpha_{1,4,1} = 0$
- $\alpha_{1,4,4} : A_{1,4} \times A_{4,4} \to A_{1,4},\ (u, a + b\overline{\eta} + c\overline{\xi}) \mapsto ua$
- $\alpha_{2,2,2} : A_{2,2} \times A_{2,2} \to A_{2,2},\ (u,v) \mapsto uv$
- $\alpha_{2,2,4} : A_{2,2} \times A_{2,4} \to A_{2,4},\ (u,v) \mapsto uv$
- $\alpha_{2,4,2} = 0$
- $\alpha_{2,4,4} : A_{2,4} \times A_{4,4} \to A_{2,4},\ (u, a + b\overline{\eta} + c\overline{\xi}) \mapsto ua$
- $\alpha_{3,3,3} : A_{3,3} \times A_{3,3} \to A_{3,3},\ (u,v) \mapsto uv$
- $\alpha_{4,1,1} : A_{4,1} \times A_{1,1} \to A_{4,1},\ (v,X) \mapsto vX$
- $\alpha_{4,1,4} : A_{4,1} \times A_{1,4} \to A_{4,4},\ (v,u) \mapsto vu\overline{\eta}$
- $\alpha_{4,2,2} : A_{4,2} \times A_{2,2} \to A_{4,2},\ (u,v) \mapsto uv$
- $\alpha_{4,2,4} : A_{4,2} \times A_{2,4} \to A_{4,4},\ (u,v) \mapsto uv(\overline{\xi} - 12\overline{\eta})$
- $\alpha_{4,4,1} : A_{4,4} \times A_{4,1} \to A_{4,1},\ (a + b\overline{\eta} + c\overline{\xi}, v) \mapsto av$
- $\alpha_{4,4,2} : A_{4,4} \times A_{4,2} \to A_{4,2},\ (a + b\overline{\eta} + c\overline{\xi}, v) \mapsto av$
- $\alpha_{4,4,4} : A_{4,4} \times A_{4,4} \to A_{4,4},\ (a+b\overline{\eta}+c\overline{\xi}, \tilde{a}+\tilde{b}\overline{\eta}+\tilde{c}\overline{\xi}) \mapsto (a+b\overline{\eta}+c\overline{\xi}) \cdot (\tilde{a}+\tilde{b}\overline{\eta}+\tilde{c}\overline{\xi})$
  where $a, b, c, \tilde{a}, \tilde{b}, \tilde{c} \in \mathbf{Q}$

For convenience, we fix a notation similar to matrices and matrix multiplication.

Notation 4. *Suppose given $r \in \mathbf{Z}_{\geq 0}$. Suppose given R-modules $M_{i,j}$ for $i,j \in [1,r]$. We write*

$$\bigoplus_{i,j\in[1,r]} M_{i,j} =: \begin{bmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,r} \\ M_{2,1} & M_{2,2} & \dots & M_{2,r} \\ \vdots & \vdots & \dots & \vdots \\ M_{r,1} & M_{r,2} & \dots & M_{r,r} \end{bmatrix} .$$

Accordingly, elements of this direct sum are written as matrices with entries in the respective summands, i.e. in the form $[m_{i,j}]_{i,j}$ with $m_{i,j} \in M_{i,j}$ for $i,j \in [1,r]$.

Proposition 5. *Let*

$$A := \bigoplus_{i,j\in[1,4]} A_{i,j} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^{3\times 3} & 0 & 0 & \mathbf{Q}^{3\times 1} \\ 0 & \mathbf{Q} & 0 & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} & 0 \\ \mathbf{Q}^{1\times 3} & \mathbf{Q} & 0 & \mathbf{Q}[\overline{\eta},\overline{\xi}] \end{bmatrix} .$$

Define the multiplication

$$\begin{array}{ccccc} A & \times & A & \to & A \\ ([a_{i,j}]_{i,j} & , & [a'_{s,t}]_{s,t}) & \mapsto & [\sum\limits_{r\in[1,4]} \alpha_{i,r,j}(a_{i,r}, a'_{r,j})]_{i,j} \end{array} .$$

We obtain a $\mathbf{Q}$-algebra isomorphism

$$\begin{array}{ccc} A & \xrightarrow{\sim\ \gamma} & \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3) \\ [a_{i,j}]_{i,j\in[1,4]} & \mapsto & \sum\limits_{i,j\in[1,4]} \gamma_{i,j}(a_{i,j}) \end{array} .$$

4.2. $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ **as path algebra modulo relations.** We aim to write

$$\mathrm{B}_{\mathbf{Q}} = \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3) \cong\ A\ ,$$

up to Morita equivalence, as a path algebra modulo relations.

We denote by $\mathrm{e}_{i,j} \in A_{1,1} = \mathbf{Q}^{3\times 3}$ the elements that have a single non-zero entry 1 at position (i,j). We have $a_{1,1} := \gamma^{-1}(e) = \mathrm{e}_{1,1} \in \mathbf{Q}^{3\times 3} \subseteq A$, $\gamma^{-1}(g) = \mathrm{e}_{2,2} \in \mathbf{Q}^{3\times 3} \subseteq A$, $\gamma^{-1}(e) = \mathrm{e}_{3,3} \in \mathbf{Q}^{3\times 3} \subseteq A$ and $a_{k,k} := \gamma^{-1}(\varepsilon_k)$ for $k \in [2,4]$, cf. Proposition 5.

We have $Aa_{1,1} \cong A\,\mathrm{e}_{2,2}$ as A-modules, using multiplication with $\mathrm{e}_{1,2}$ from the right from $Aa_{1,1}$ to $A\,\mathrm{e}_{2,2}$ and multiplication with $\mathrm{e}_{2,1}$ from the right from $A\,\mathrm{e}_{2,2}$ to $Aa_{1,1}$. Note that $\mathrm{e}_{1,2}\,\mathrm{e}_{2,1} = a_{1,1}$ and $\mathrm{e}_{2,1}\,\mathrm{e}_{1,2} = \mathrm{e}_{2,2}$. Similarly $Aa_{1,1} \cong A\,\mathrm{e}_{3,3}$.

Therefore, A is Morita equivalent to

$$A' := \Big(\sum_{i\in[1,4]} a_{i,i}\Big)A\Big(\sum_{i\in[1,4]} a_{i,i}\Big) = \bigoplus_{i,j\in[1,4]} a_{i,i}Aa_{j,j} = \bigoplus_{i,j\in[1,4]} a_{i,i}A_{i,j}a_{j,j}\ .$$

Write $A'_{i,j} := a_{i,i}A_{i,j}a_{j,j} = A_{i,j}$ for $i,j \in [2,4]$.

Identify $A'_{1,1} := \mathbf{Q} = \begin{pmatrix} \mathbf{Q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = a_{1,1}A_{1,1}a_{1,1} \subseteq A_{1,1} = \mathbf{Q}^{3\times 3}$.

Identify $A'_{1,4} := \mathbf{Q} = \begin{pmatrix} \mathbf{Q} \\ 0 \\ 0 \end{pmatrix} = a_{1,1}A_{1,4}a_{4,4} \subseteq A_{1,4} = \mathbf{Q}^{3\times 1}$.

Identify $A'_{4,1} := \mathbf{Q} = \begin{pmatrix} \mathbf{Q} & 0 & 0 \end{pmatrix} = a_{4,4}A_{4,1}a_{1,1} \subseteq A_{4,1} = \mathbf{Q}^{1\times 3}$. Let $A'_{1,j} := 0$ and $A'_{j,1} := 0$ for $j \in [2,3]$.

We have the $\mathbf{Q}$-linear basis of A'

$$a_{1,1} = \begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \quad a_{2,2} = \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \quad a_{3,3} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}$$

$$a_{4,4} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix} \quad a_{1,4} := \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \quad a_{4,1} := \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix}$$

$$a_{2,4} := \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} \quad a_{4,2} := \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{bmatrix} \quad a'_{4,4} := \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&\overline{\eta} \end{bmatrix}$$

$$a''_{4,4} := \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&\overline{\xi} \end{bmatrix}$$

We have the following multiplication table for the basis elements.

$(\cdot)$	$a_{1,1}$	$a_{1,4}$	$a_{2,2}$	$a_{2,4}$	$a_{3,3}$	$a_{4,1}$	$a_{4,2}$	$a_{4,4}$	$a'_{4,4}$	$a''_{4,4}$
$a_{1,1}$	$a_{1,1}$	$a_{1,4}$	0	0	0	0	0	0	0	0
$a_{1,4}$	0	0	0	0	0	0	0	$a_{1,4}$	0	0
$a_{2,2}$	0	0	$a_{2,2}$	$a_{2,4}$	0	0	0	0	0	0
$a_{2,4}$	0	0	0	0	0	0	0	$a_{2,4}$	0	0
$a_{3,3}$	0	0	0	0	$a_{3,3}$	0	0	0	0	0
$a_{4,1}$	$a_{4,1}$	$a'_{4,4}$	0	0	0	0	0	0	0	0
$a_{4,2}$	0	0	$a_{4,2}$	$a''_{4,4} - 12a'_{4,4}$	0	0	0	0	0	0
$a_{4,4}$	0	0	0	0	0	$a_{4,1}$	$a_{4,2}$	$a_{4,4}$	$a'_{4,4}$	$a''_{4,4}$
$a'_{4,4}$	0	0	0	0	0	0	0	$a'_{4,4}$	0	0
$a''_{4,4}$	0	0	0	0	0	0	0	$a''_{4,4}$	0	0

We have $a'_{4,4} = a_{4,1} \cdot a_{1,4}$ and $a''_{4,4} = a_{4,2} \cdot a_{2,4} + 12a_{4,1} \cdot a_{1,4}$. Hence, as a $\mathbf{Q}$-algebra A' is generated by $a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, a_{1,4}, a_{4,1}, a_{2,4}, a_{4,2}$.

Consider the quiver $\Psi := \left[\begin{tikzcd} \tilde{a}_{3,3} & \tilde{a}_{2,2} \arrow[r, bend left, "\sigma"] & \tilde{a}_{4,4} \arrow[l, bend left, "\vartheta"] \arrow[r, bend right, "\rho"'] & \tilde{a}_{1,1} \arrow[l, bend right, "\pi"'] \end{tikzcd}\right]$.

We have a surjective $\mathbf{Q}$-algebra morphism $\varphi : \mathbf{Q}\Psi \to A'$ by sending

$$\begin{array}{ccccccccccccccc} \tilde{a}_{1,1} & \mapsto & a_{1,1} & , & \tilde{a}_{2,2} & \mapsto & a_{2,2} & , & \tilde{a}_{3,3} & \mapsto & a_{3,3} & , & \tilde{a}_{4,4} & \mapsto & a_{4,4} & , \\ \rho & \mapsto & a_{4,1} & , & \pi & \mapsto & a_{1,4} & , & \vartheta & \mapsto & a_{4,2} & , & \sigma & \mapsto & a_{2,4} & . \end{array}$$

We establish the following multiplication trees, where we underline the elements that are not in a $\mathbf{Q}$-linear relation with previously underlined elements.

$$\begin{tikzcd} \underline{a_{1,1}} \arrow[r, "a_{1,4}"'] & \underline{a_{1,4}} \arrow[r, "a_{4,1}"'] \arrow[d, "a_{4,2}"'] & a_{1,4}a_{4,1} = 0 \\ & a_{1,4}a_{4,2} = 0 & \end{tikzcd} \qquad \begin{tikzcd} \underline{a_{2,2}} \arrow[r, "a_{2,4}"'] & \underline{a_{2,4}} \arrow[r, "a_{4,2}"'] \arrow[d, "a_{4,1}"'] & a_{2,4}a_{4,2} = 0 \\ & a_{2,4}a_{4,1} = 0 & \end{tikzcd}$$

$$\begin{tikzcd} & & \underline{a_{4,4}} \arrow[dl, "a_{4,2}"'] \arrow[dr, "a_{4,1}"] & & & \\ (a''_{4,4} - 12a'_{4,4})a_{4,1} = 0 & \underline{a_{4,2}} \arrow[dl, "a_{2,4}"] & & \underline{a_{4,1}} \arrow[r, bend left, "a_{1,4}"] & \underline{a_{4,1}a_{1,4}} \arrow[r, "a_{4,1}"'] \arrow[d, "a_{4,2}"'] & a'_{4,4}a_{4,1} = 0 \\ \underline{a_{4,2}a_{2,4}} \arrow[u, "a_{4,1}"'] \arrow[d, "a_{4,2}"'] & & & & a'_{4,4}a_{4,2} = 0 & \\ (a''_{4,4} - 12a'_{4,4})a_{4,2} = 0 & & & & & \end{tikzcd}$$

The multiplication tree of the idempotent $a_{3,3}$ consists only of the element $a_{3,3}$.
So the kernel of φ contains the elements:

$$\begin{array}{ccccccc} \pi\rho & , & \sigma\vartheta & , & \rho\pi\rho & , & \vartheta\sigma\rho \ , \\ \pi\vartheta & , & \sigma\rho & , & \rho\pi\vartheta & , & \vartheta\sigma\vartheta \ . \end{array}$$

Let I be the ideal in $\mathbf{Q}\Psi$ generated by those elements. So $I \subseteq \mathrm{kern}(\varphi)$. Therefore, φ induces a surjective $\mathbf{Q}$-algebra morphism from $\mathbf{Q}\Psi/I$ to A' .

Note that $\mathbf{Q}\Psi/I$ is $\mathbf{Q}$-linearly generated by

$$\mathcal{N} := \{\tilde{a}_{3,3}+I, \tilde{a}_{2,2}+I, \tilde{a}_{4,4}+I, \tilde{a}_{1,1}+I, \sigma+I, \pi+I, \vartheta+I, \rho+I, \vartheta\sigma+I, \rho\pi+I\},$$

cf. the underlined elements above. To see that, note that a product ξ of k generators may be written as a product in $\mathcal{N}$ of k' generators and a product of k'' generators, where $k = k'+k''$ and where k' is chosen maximal. We call k'' the excess of ξ. If $k'' \geq 1$ then, using the trees above, we may write ξ as an $\mathbf{Q}$-linear combination of products of generators that have excess $\leq k''-1$. In the present case, we even have $\xi = 0$.

Moreover, note that $|\mathcal{N}| = 10 = \dim_{\mathbf{Q}}(A')$.

Since we have a surjective $\mathbf{Q}$-algebra morphism from $\mathbf{Q}\Psi/I$ to A', this dimension argument shows this morphism to be bijective. In particular, $I = \mathrm{kern}(\varphi)$.

We may reduce this list to obtain $\mathrm{kern}(\varphi) = (\pi\rho, \sigma\vartheta, \pi\vartheta, \sigma\rho)$.

So we obtain the

Proposition 6. *Recall that* $I = (\pi\rho, \sigma\vartheta, \pi\vartheta, \sigma\rho)$. *We have the isomorphism of* $\mathbf{Q}$*-algebras*

$$A' \xrightarrow{\sim} \mathbf{Q}\left[\begin{tikzcd} \tilde{a}_{3,3} & \tilde{a}_{2,2} \arrow[r, bend left, "\sigma"] & \tilde{a}_{4,4} \arrow[l, bend left, "\vartheta"] \arrow[r, bend right, "\rho"'] & \tilde{a}_{1,1} \arrow[l, bend right, "\pi"'] \end{tikzcd}\right]/I = \mathbf{Q}\Psi/I$$

$$\begin{array}{rcl}
a_{1,1} & \mapsto & \tilde{a}_{1,1}+I \\
a_{2,2} & \mapsto & \tilde{a}_{2,2}+I \\
a_{3,3} & \mapsto & \tilde{a}_{3,3}+I \\
a_{4,4} & \mapsto & \tilde{a}_{4,4}+I \\
a_{4,1} & \mapsto & \rho+I \\
a_{1,4} & \mapsto & \pi+I \\
a_{4,2} & \mapsto & \vartheta+I \\
a_{2,4} & \mapsto & \sigma+I\ .
\end{array}$$

In particular, $\mathbf{Q}\Psi/I$ *is Morita equivalent to* $A \cong \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$.

5. The double Burnside R-algebra $\mathrm{B}_R(\mathrm{S}_3, \mathrm{S}_3)$ for $R \in \{\mathbf{Z}, \mathbf{Z}_{(2)}, \mathbf{F}_2, \mathbf{Z}_{(3)}, \mathbf{F}_3\}$

5.1. $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$ via congruences.

Recall that

$$A = \bigoplus_{i,j\in[1,4]} A_{i,j} \xrightarrow[\gamma]{\sim} \mathrm{B}_{\mathbf{Q}}\ ,$$

cf. Proposition 5. In the $\mathbf{Q}$-algebra A, we define the $\mathbf{Z}$-order

$$A_{\mathbf{Z}} := \begin{bmatrix} A_{\mathbf{Z},1,1} & A_{\mathbf{Z},1,2} & A_{\mathbf{Z},1,3} & A_{\mathbf{Z},1,4} \\ A_{\mathbf{Z},2,1} & A_{\mathbf{Z},2,2} & A_{\mathbf{Z},2,3} & A_{\mathbf{Z},2,4} \\ A_{\mathbf{Z},3,1} & A_{\mathbf{Z},3,2} & A_{\mathbf{Z},3,3} & A_{\mathbf{Z},3,4} \\ A_{\mathbf{Z},4,1} & A_{\mathbf{Z},4,2} & A_{\mathbf{Z},4,3} & A_{\mathbf{Z},4,4} \end{bmatrix} := \begin{bmatrix} \mathbf{Z}^{3\times 3} & 0 & 0 & \mathbf{Z}^{3\times 1} \\ 0 & \mathbf{Z} & 0 & \mathbf{Z} \\ 0 & 0 & \mathbf{Z} & 0 \\ \mathbf{Z}^{1\times 3} & \mathbf{Z} & 0 & \mathbf{Z}[\overline{\eta}, \overline{\xi}] \end{bmatrix} \subseteq A.$$

In fact, $A_{\mathbf{Z}}$ is a subring of A, as $\alpha_{i,j,k}(A_{\mathbf{Z},i,j} \times A_{\mathbf{Z},j,k}) \subseteq A_{\mathbf{Z},i,k}$ for $i, j, k \in [1,4]$.

Remark 7. As $A \cong \mathrm{B}_{\mathbf{Q}}$ is not semisimple, there are no maximal $\mathbf{Z}$-orders in A, [8, §10]. So $A_{\mathbf{Z}}$ is not a canonical choice of a $\mathbf{Z}$-order in A, but it nonetheless enables us to describe Λ inside $A_{\mathbf{Z}}$ via congruences.

Consider the following elements of $\mathrm{U}(A)$.

$$x_1 := \begin{bmatrix} 0 & -2 & 0 & 0 & 0 & 0 \\ 6 & 6 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; x_2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; x_3 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 1 \end{bmatrix}.$$

We define the injective ring morphism $\delta : \mathrm{B}_{\mathbf{Z}} \to A,\ y \mapsto x_3^{-1} \cdot x_2^{-1} \cdot x_1^{-1} \cdot \gamma^{-1}(y) \cdot x_1 \cdot x_2 \cdot x_3$. The conjugating element x_1 was constructed such that the its image lies in $A_{\mathbf{Z}}$. The elements $x_2,\ x_3$ serve the purpose of simplifying the congruences of $\delta(\mathrm{B}_{\mathbf{Z}})$.

Theorem 8. *The image $\delta(\mathrm{B}_{\mathbf{Z}})$ in $A_{\mathbf{Z}}$ is given by*

$$\Lambda := \delta(\mathrm{B}_{\mathbf{Z}}) = \left\{ \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{bmatrix} \in A_{\mathbf{Z}} : \begin{array}{ll} 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 & \\ x_1 & \equiv_4 0 \\ x_2 & \equiv_4 0 \\ x_3 & \equiv_4 0 \\ y & \equiv_2 0 \\ t_1 & \equiv_2 0 \\ t_2 & \equiv_2 0 \\ t_3 & \equiv_2 0 \\ v & \equiv_2 0 \\ & \\ x_1 & \equiv_3 0 \\ x_2 & \equiv_3 0 \\ x_3 & \equiv_3 0 \\ z_2 & \equiv_3 0 \end{array} \right\}.$$

In particular, we have $\mathrm{B}_{\mathbf{Z}} = \mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3) \cong \Lambda$ as rings.

More symbolically written, we have

$$\Lambda = \begin{bmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) & & \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) & & \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) & & \\ 0 & 0 & 0 & \mathbf{Z} & 0 & (2) & & \\ 0 & 0 & 0 & 0 & \mathbf{Z} & 0 & & 8 \\ (12) & (12) & (12) & (2) & 0 & \mathbf{Z} & +(12)\overline{\eta} & +(4)\overline{\xi} \end{bmatrix}.$$

Proof. We identify $\mathbf{Z}^{22\times 1}$ and $A_{\mathbf{Z}}$ along the isomorphism

$$\begin{pmatrix} s_{1,1}, s_{2,1}, s_{3,1}, s_{1,2}, s_{2,2}, s_{3,2}, s_{1,3}, s_{2,3}, s_{3,3}, \\ x_1, x_2, x_3, u, y, w, t_1, t_2, t_3, v, z_1, z_2, z_3 \end{pmatrix}^{\mathrm{t}} \mapsto \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{bmatrix}.$$

Let M be the representation matrix of δ, with respect to the bases $\tilde{\mathcal{H}} = (H_{0,0},H_{0,1},H_{1,0},H_1^{\Delta},H_{0,4},H_{4,0},H_4^{\Delta},H_{1,1},H_{0,5},H_{5,0},H_7,H_{1,4},H_{4,1},H_6,H_5^{\Delta},H_{4,4},$ $H_{5,1},H_{1,5},H_{5,4},H_{4,5},H_8,H_{5,5})$ of $\mathrm{B}_{\mathbf{Z}}$ and the standard basis of $A_{\mathbf{Z}}$.

We obtain that $M =$

$$\begin{pmatrix}
0 & 0 & 15 & -3 & 0 & 20 & 8 & 6 & 0 & 25 & 7 & 9 & 8 & -3 & 1 & 12 & 10 & 3 & 15 & 4 & 3 & 5 \\
0 & 0 & -18 & 0 & 0 & -24 & 0 & -9 & 0 & -30 & -12 & -6 & -12 & 0 & 0 & -8 & -15 & -3 & -10 & -4 & -4 & -5 \\
0 & 0 & 126 & -6 & 0 & 168 & 12 & 60 & 0 & 210 & 78 & 48 & 80 & -6 & 0 & 64 & 100 & 21 & 80 & 28 & 26 & 35 \\
-5 & -2 & -60 & 9 & -3 & -55 & -23 & -24 & -1 & -85 & -16 & -36 & -22 & 10 & 0 & -33 & -34 & -12 & -51 & -11 & -5 & -17 \\
6 & 3 & 72 & 3 & 2 & 66 & 2 & 36 & 1 & 102 & 33 & 24 & 33 & 1 & 1 & 22 & 51 & 12 & 34 & 11 & 11 & 17 \\
-42 & -20 & -504 & 2 & -16 & -462 & -46 & -240 & -7 & -714 & -208 & -192 & -220 & 15 & 0 & -176 & -340 & -84 & -272 & -77 & -65 & -119 \\
0 & 0 & -10 & 2 & 0 & -10 & -4 & -4 & 0 & -15 & -3 & -6 & -4 & 2 & 0 & -6 & -6 & -2 & -9 & -2 & -1 & -3 \\
0 & 0 & 12 & 0 & 0 & 12 & 0 & 6 & 0 & 18 & 6 & 4 & 6 & 0 & 0 & 4 & 9 & 2 & 6 & 2 & 2 & 3 \\
0 & 0 & -84 & 4 & 0 & -84 & -6 & -40 & 0 & -126 & -38 & -32 & -40 & 4 & 1 & -32 & -60 & -14 & -48 & -14 & -12 & -21 \\
0 & 0 & -756 & 36 & 0 & -1008 & -72 & -360 & 0 & -1260 & -468 & -288 & -480 & 72 & 0 & -384 & -600 & -108 & -480 & -144 & -120 & -180 \\
252 & 120 & 3024 & -12 & 96 & 2772 & 276 & 1440 & 36 & 4284 & 1248 & 1152 & 1320 & -228 & 0 & 1056 & 2040 & 432 & 1632 & 396 & 252 & 612 \\
0 & 0 & 504 & -24 & 0 & 504 & 36 & 240 & 0 & 756 & 228 & 192 & 240 & -48 & 0 & 192 & 360 & 72 & 288 & 72 & 48 & 108 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -10 & 2 & 0 & -10 & -4 & -4 & 0 & -10 & -4 & -6 & -4 & 2 & 0 & -6 & -4 & -2 & -6 & -2 & -2 & -2 \\
0 & 0 & 12 & 0 & 0 & 12 & 0 & 6 & 0 & 12 & 6 & 4 & 6 & 0 & 0 & 4 & 6 & 2 & 4 & 2 & 2 & 2 \\
0 & 0 & -84 & 4 & 0 & -84 & -6 & -40 & 0 & -84 & -40 & -32 & -40 & 4 & 0 & -32 & -40 & -14 & -32 & -14 & -14 & -14 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 504 & -24 & 0 & 504 & 36 & 240 & 0 & 504 & 240 & 192 & 240 & -48 & 0 & 192 & 240 & 72 & 192 & 72 & 24 & 72 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0
\end{pmatrix}.$$

Let

$$\lambda := \begin{bmatrix}
s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\
s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\
s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\
0 & 0 & 0 & u & 0 & v \\
0 & 0 & 0 & 0 & w & 0 \\
x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi}
\end{bmatrix} \in A_{\mathbf{Z}},$$

identified with $\lambda \in \mathbf{Z}^{22\times 1}$.

We have $\lambda \in \Lambda$
$\Leftrightarrow \exists\, q \in \mathbf{Z}^{22\times 1}$ such that $\lambda = Mq$
$\Leftrightarrow \exists\, q \in \mathbf{Z}^{22\times 1}$ such that $M^{-1} \cdot \lambda = q$
$\Leftrightarrow 24M^{-1} \cdot \lambda \in 24\mathbf{Z}^{22\times 1}$
and this is equivalent to

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 18 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}
\begin{pmatrix}
s_{1,1} \\ s_{2,1} \\ s_{3,1} \\ \vdots \\ s_{3,3} \\ x_1 \\ x_2 \\ x_3 \\ u \\ y \\ w \\ t_1 \\ t_2 \\ t_3 \\ v \\ z_1 \\ z_2 \\ z_3
\end{pmatrix} \in 24\mathbf{Z}^{11\times 1}$$

and hence equivalent to

$$\left\{\begin{array}{lll} 2w-2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 & & \\ \quad x_1 & \equiv_4 & 0 \\ \quad x_2 & \equiv_4 & 0 \\ \quad x_3 & \equiv_4 & 0 \\ \quad y & \equiv_2 & 0 \\ \quad t_1 & \equiv_2 & 0 \\ \quad t_2 & \equiv_2 & 0 \\ \quad t_3 & \equiv_2 & 0 \\ \quad v & \equiv_2 & 0 \\ & & \\ \quad x_1 & \equiv_3 & 0 \\ \quad x_2 & \equiv_3 & 0 \\ \quad x_3 & \equiv_3 & 0 \\ \quad z_2 & \equiv_3 & 0 \end{array}\right\}.$$

$\square$

5.2. **Localisation at 2:** $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3,\mathrm{S}_3)$ **via congruences.** Write $R := \mathbf{Z}_{(2)}$. In the $\mathbf{Q}$-algebra A, cf. Proposition 5, we have the R-order

$$A_R := \begin{bmatrix} A_{R,1,1} & A_{R,1,2} & A_{R,1,3} & A_{R,1,4} \\ A_{R,2,1} & A_{R,2,2} & A_{R,2,3} & A_{R,2,4} \\ A_{R,3,1} & A_{R,3,2} & A_{R,3,3} & A_{R,3,4} \\ A_{R,4,1} & A_{R,4,2} & A_{R,4,3} & A_{R,4,4} \end{bmatrix} := \begin{bmatrix} R^{3\times 3} & 0 & 0 & R^{3\times 1} \\ 0 & R & 0 & R \\ 0 & 0 & R & 0 \\ R^{1\times 3} & R & 0 & R[\overline{\eta},\overline{\xi}] \end{bmatrix} \subseteq A\ .$$

Corollary 9. *We have*

$$\Lambda_{(2)} = \left\{ \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1+z_2\overline{\eta}+z_3\overline{\xi} \end{bmatrix} \in A_R : \begin{array}{lll} 2w-2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 & & \\ \quad x_1 & \equiv_4 & 0 \\ \quad x_2 & \equiv_4 & 0 \\ \quad x_3 & \equiv_4 & 0 \\ \quad y & \equiv_2 & 0 \\ \quad t_1 & \equiv_2 & 0 \\ \quad t_2 & \equiv_2 & 0 \\ \quad t_3 & \equiv_2 & 0 \\ \quad v & \equiv_2 & 0 \end{array} \right\} \subseteq A_R\,.$$

In particular, we have $\mathrm{B}_R = \mathrm{B}_R(\mathrm{S}_3,\mathrm{S}_3) \cong \Lambda_{(2)}$ *as* R*-algebras.*

More symbolically written, we have

$$\Lambda_{(2)} = \begin{bmatrix} R & R & R & 0 & 0 & (2) & & \\ R & R & R & 0 & 0 & (2) & & \\ R & R & R & 0 & 0 & (2) & & \\ 0 & 0 & 0 & R & 0 & (2) & & \\ 0 & 0 & 0 & 0 & R & 0 & & 8 \\ (4) & (4) & (4) & (2) & 0 & R & +(4)\overline{\eta} & +(4)\overline{\xi} \end{bmatrix}.$$

Remark 10. *We claim that* $1_{\Lambda_{(2)}} = e_1 + e_2 + e_3 + e_4 + e_5$ *is an orthogonal decomposition into primitive idempotents, where*

$$e_1 := \begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \; e_2 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \; e_3 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},$$

$$e_4 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \; e_5 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}.$$

Proof. We have $\mathrm{e}_1\,\Lambda_{(2)}\,\mathrm{e}_1 \cong R$, $\mathrm{e}_2\,\Lambda_{(2)}\,\mathrm{e}_2 \cong R$, $\mathrm{e}_3\,\Lambda_{(2)}\,\mathrm{e}_3 \cong R$ and $\mathrm{e}_4\,\Lambda_{(2)}\,\mathrm{e}_4 \cong R$. So, it follows that $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4$ are primitive.

As R-algebras, we have

$$\begin{aligned} \mathrm{e}_5\,\Lambda_{(2)}\,\mathrm{e}_5 \;&\cong\; \left\{ \left(\, w,\; z_1 + z_2\overline{\eta} + z_3\overline{\xi}\,\right) \in R \times R[\overline{\eta}, \overline{\xi}] : 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \right\} =: \Gamma \\ &\subseteq\; R \times R[\overline{\eta}, \overline{\xi}]\;. \end{aligned}$$

To show that e_5 is primitive, we show that Γ is local.

We have the R-linear basis (b_1, b_2, b_3, b_4) of Γ, where

$$\begin{aligned} b_1 &= (\,1,\; 1\,), & b_2 &= (\,0,\; 2 + 4\overline{\eta}\,), \\ b_3 &= (\,0,\; 8\overline{\eta}\,), & b_4 &= (\,0,\; 4\overline{\xi}\,)\;. \end{aligned}$$

We *claim* that the Jacobson radical of Γ is given by $J := {}_R\langle 2b_1, b_2, b_3, b_4\rangle$, that $\Gamma/J \cong \mathbf{F}_2$ and that Γ is local.

In fact, the multiplication table for the basis elements is given by

$(\cdot)$	b_1	b_2	b_3	b_4
b_1	b_1	b_2	b_3	b_4
b_2	b_2	$2b_2 + b_3$	$2b_3$	$2b_4$
b_3	b_3	$2b_3$	0	0
b_4	b_4	$2b_4$	0	0

.

This shows that J is an ideal. Moreover, J is topologically nilpotent as

$$J^3 = {}_R\langle 8b_1, 4b_2, 2b_3, 4b_4\rangle \subseteq 2\,\mathrm{e}_5\,\Lambda_{(2)}\,\mathrm{e}_5\;.$$

Since $\Gamma/J \cong \mathbf{F}_2$, the *claim* follows. ∎

5.3. $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3,\mathrm{S}_3)$ and $\mathrm{B}_{\mathbf{F}_2}(\mathrm{S}_3,\mathrm{S}_3)$ as path algebras modulo relations. Write $R := \mathbf{Z}_{(2)}$. We aim to write $\Lambda_{(2)}$, up to Morita equivalence, as path algebra modulo relations. The R-algebra $\Lambda_{(2)}$ is Morita equivalent to $\Lambda'_{(2)} := (\mathrm{e}_3+\mathrm{e}_4+\mathrm{e}_5)\Lambda_{(2)}(\mathrm{e}_3+\mathrm{e}_4+\mathrm{e}_5)$ since $\Lambda_{(2)}\,\mathrm{e}_1 \cong \Lambda_{(2)}\,\mathrm{e}_2 \cong \Lambda_{(2)}\,\mathrm{e}_3$ using multiplication with elements of $\Lambda_{(2)}$ with a single nonzero entry 1 in the upper (3×3)-corner.

We have the R-linear basis of $\Lambda'_{(2)}$ consisting of

$$\mathrm{e}_3 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\quad \mathrm{e}_4 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\quad \mathrm{e}_5 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},$$

$$\tau_1 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&4&0&0&0 \end{bmatrix},\quad \tau_2 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&2\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\quad \tau_3 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&2&0&0 \end{bmatrix},$$

$$\tau_4 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&2\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\quad \tau_5 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&8\overline{\eta} \end{bmatrix},$$

$$\tau_6 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&4\overline{\xi} \end{bmatrix},\quad \tau_7 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&2+4\overline{\eta} \end{bmatrix}$$

We have $\tau_5 = \tau_1\tau_2$ and $\tau_6 = \tau_3\tau_4 + 6\tau_1\tau_2$. Hence, as an R-algebra $\Lambda'_{(2)}$ is generated by $\mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5, \tau_1, \tau_2, \tau_3, \tau_4, \tau_7$.

Consider the quiver $\Psi := \left[\begin{tikzcd} \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_5 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[loop right, "\tilde{\tau}_7"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right]$.

We have a surjective R-algebra morphism $\varphi : R\Psi \to \Lambda'_{(2)}$ by sending

$$\begin{array}{llllllllllll} \tilde{\mathrm{e}}_3 & \mapsto & \mathrm{e}_3 & , & \tilde{\mathrm{e}}_4 & \mapsto & \mathrm{e}_4 & , & \tilde{\mathrm{e}}_5 & \mapsto & \mathrm{e}_5 & , \quad \tilde{\tau}_1 \mapsto \tau_1 \;, \\ \tilde{\tau}_2 & \mapsto & \tau_2 & , & \tilde{\tau}_3 & \mapsto & \tau_3 & , & \tilde{\tau}_4 & \mapsto & \tau_4 & , \quad \tilde{\tau}_7 \mapsto \tau_7 \;. \end{array}$$

We establish the following multiplication trees, where we underline the elements that are not in an R-linear relation with previous elements.

$$
\begin{tikzcd}
\underline{\mathrm{e}_3} \arrow[r, "\tau_2"'] & \underline{\tau_2} \arrow[r, "\tau_1"'] \arrow[dl, "\tau_3"'] \arrow[d, "\tau_7"] & \tau_2\tau_1 = 0 \\
\tau_2\tau_3 = 0 & \tau_2\tau_7 = 2\tau_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathrm{e}_4} \arrow[r, "\tau_4"'] & \underline{\tau_4} \arrow[r, "\tau_1"'] \arrow[dl, "\tau_3"'] \arrow[d, "\tau_7"] & \tau_4\tau_1 = 0 \\
\tau_4\tau_3 = 0 & \tau_4\tau_7 = 2\tau_4 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \tau_7^2 = 2\tau_7 + \tau_1\tau_2 & & & \\
\tau_7\tau_1 = 2\tau_1 & \underline{\tau_7} \arrow[l, "\tau_1"'] \arrow[r, "\tau_3"'] \arrow[u, "\tau_7"'] & \tau_7\tau_3 = 2\tau_3 & & \\
 & \underline{\mathrm{e}_5} \arrow[u, "\tau_7"'] \arrow[dl, "\tau_3"'] \arrow[dr, "\tau_1"] & & & \\
\underline{\tau_3} \arrow[d, "\tau_4"'] & & \underline{\tau_1} \arrow[r, "\tau_2"'] & \underline{\tau_1\tau_2} \arrow[r, "\tau_3"'] \arrow[d, "\tau_7"'] \arrow[dr, "\tau_1"] & \tau_1\tau_2\tau_3 = 0 \\
\underline{\tau_3\tau_4} \arrow[r, "\tau_3"'] \arrow[dr, "\tau_1"] \arrow[dd, "\tau_7"'] & \tau_3\tau_4\tau_3 = 0 & & \tau_1\tau_2\tau_7 = 2\tau_1\tau_2 & \tau_1\tau_2\tau_1 = 0 \\
 & \tau_3\tau_4\tau_1 = 0 & & & \\
\tau_3\tau_4\tau_7 = 2\tau_3\tau_4 & & & &
\end{tikzcd}
$$

So, the kernel of φ contains the elements:

$$
\begin{array}{ccccccccc}
\tilde{\tau}_2\tilde{\tau}_1 & , & \tilde{\tau}_4\tilde{\tau}_1 & , & \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_1 & , & \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_1 & , & \tilde{\tau}_7\tilde{\tau}_1 - 2\tilde{\tau}_1, \\
\tilde{\tau}_2\tilde{\tau}_3 & , & \tilde{\tau}_4\tilde{\tau}_3 & , & \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3 & , & \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_3 & , & \tilde{\tau}_7\tilde{\tau}_3 - 2\tilde{\tau}_3, \\
\tilde{\tau}_2\tilde{\tau}_7 - 2\tilde{\tau}_2 & , & \tilde{\tau}_4\tilde{\tau}_7 - 2\tilde{\tau}_4 & , & \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_7 - 2\tilde{\tau}_1\tilde{\tau}_2 & , & \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_7 - 2\tilde{\tau}_3\tilde{\tau}_4 & , & \tilde{\tau}_7^2 - 2\tilde{\tau}_7 - \tilde{\tau}_1\tilde{\tau}_2\ .
\end{array}
$$

Let I be the ideal generated by these elements. So $I \subseteq \mathrm{kern}(\varphi)$. Therefore, φ induces a surjective R-algebra morphism from $R\Psi/I$ to $\Lambda'_{(2)}$. We may reduce the list of generators to obtain

$$
I = (\tilde{\tau}_2\tilde{\tau}_1, \tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_7\tilde{\tau}_1 - 2\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_3, \tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_7\tilde{\tau}_3 - 2\tilde{\tau}_3, \tilde{\tau}_2\tilde{\tau}_7 - 2\tilde{\tau}_2, \tilde{\tau}_4\tilde{\tau}_7 - 2\tilde{\tau}_4, \tilde{\tau}_7^2 - 2\tilde{\tau}_7 - \tilde{\tau}_1\tilde{\tau}_2)\ .
$$

Note that $R\Psi/I$ is R-linearly generated by

$$
\mathcal{N} := \{\tilde{\mathrm{e}}_3 + I, \tilde{\mathrm{e}}_4 + I, \tilde{\mathrm{e}}_5 + I, \tilde{\tau}_1 + I, \tilde{\tau}_2 + I, \tilde{\tau}_3 + I, \tilde{\tau}_4 + I, \tilde{\tau}_7 + I, \tilde{\tau}_3\tilde{\tau}_4 + I, \tilde{\tau}_1\tilde{\tau}_2 + I\},
$$

cf. the underlined elements above. To see that, note that a product ξ of k generators may be written as a product in $\mathcal{N}$ of k' generators and a product of k'' generators, where $k = k' + k''$ and where k' is choosen maximal. We call k'' the excess of ξ. If $k'' \geq 1$ then, using the trees above, we may write ξ as an R-linear combination of products of generators that have excess $\leq k'' - 1$. Moreover, note that $|\mathcal{N}| = 10 = \mathrm{rk}_R(\Lambda'_{(2)})$.

Since we have a surjective R-algebra morphism from $R\Psi/I$ to $\Lambda'_{(2)}$, this rank argument shows this morphism to be bijective. In particular, $I = \mathrm{kern}(\varphi)$.

So, we obtain the

Proposition 11. *Recall that* $I = \left(\begin{array}{ccccc} \tilde{\tau}_2\tilde{\tau}_1 & , & \tilde{\tau}_2\tilde{\tau}_3 & , & \tilde{\tau}_2\tilde{\tau}_7 - 2\tilde{\tau}_2, \\ \tilde{\tau}_4\tilde{\tau}_1 & , & \tilde{\tau}_4\tilde{\tau}_3 & , & \tilde{\tau}_4\tilde{\tau}_7 - 2\tilde{\tau}_4, \\ \tilde{\tau}_7\tilde{\tau}_1 - 2\tilde{\tau}_1 & , & \tilde{\tau}_7\tilde{\tau}_3 - 2\tilde{\tau}_3 & , & \tilde{\tau}_7^2 - 2\tilde{\tau}_7 - \tilde{\tau}_1\tilde{\tau}_2 \end{array}\right)$.

We have the isomorphism of $\mathbf{Z}_{(2)}$*-algebras*

$$\Lambda'_{(2)} \quad \xrightarrow{\sim} \quad R\left[\begin{tikzcd} \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_5 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[loop right, "\tilde{\tau}_7"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right]/I$$

$$\begin{array}{rcl} \mathrm{e}_i & \mapsto & \tilde{\mathrm{e}}_i + I \text{ for } i \in [3,5] \\ \tau_j & \mapsto & \tilde{\tau}_j + I \text{ for } j \in [1,7] \setminus \{5,6\} \ . \end{array}$$

Recall that $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3, \mathrm{S}_3)$ *is Morita equivalent to* $\Lambda'_{(2)}$.

Corollary 12. *As* $\mathbf{F}_2$*-algebras, we have*

$$\Lambda'_{(2)}/2\Lambda'_{(2)} \;\cong\; \mathbf{F}_2\left[\begin{tikzcd} \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_5 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[loop right, "\tilde{\tau}_7"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right] \Big/ \left(\begin{array}{ccccc} \tilde{\tau}_2\tilde{\tau}_1 & , & \tilde{\tau}_2\tilde{\tau}_3 & , & \tilde{\tau}_2\tilde{\tau}_7, \\ \tilde{\tau}_4\tilde{\tau}_1 & , & \tilde{\tau}_4\tilde{\tau}_3 & , & \tilde{\tau}_4\tilde{\tau}_7, \\ \tilde{\tau}_7\tilde{\tau}_1 & , & \tilde{\tau}_7\tilde{\tau}_3 & , & \tilde{\tau}_7^2 - \tilde{\tau}_1\tilde{\tau}_2 \end{array}\right) \ .$$

Recall that $\mathrm{B}_{\mathbf{F}_2}(\mathrm{S}_3, \mathrm{S}_3)$ *is Morita equivalent to* $\Lambda'_{(2)}/2\Lambda'_{(2)}$.

5.4. **Localisation at 3:** $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3, \mathrm{S}_3)$ **via congruences.** Write $R = \mathbf{Z}_{(3)}$. In the $\mathbf{Q}$-algebra A, cf. Proposition 5, we have the R-order

$$A_R := \begin{bmatrix} A_{R,1,1} & A_{R,1,2} & A_{R,1,3} & A_{R,1,4} \\ A_{R,2,1} & A_{R,2,2} & A_{R,2,3} & A_{R,2,4} \\ A_{R,3,1} & A_{R,3,2} & A_{R,3,3} & A_{R,3,4} \\ A_{R,4,1} & A_{R,4,2} & A_{R,4,3} & A_{R,4,4} \end{bmatrix} := \begin{bmatrix} R^{3\times 3} & 0 & 0 & R^{3\times 1} \\ 0 & R & 0 & R \\ 0 & 0 & R & 0 \\ R^{1\times 3} & R & 0 & R[\overline{\eta}, \overline{\xi}] \end{bmatrix} \subseteq A \ .$$

Corollary 13. *We have*

$$\Lambda_{(3)} = \left\{ \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{bmatrix} \in A_R : \begin{array}{r} x_1 \equiv_3 0 \\ x_2 \equiv_3 0 \\ x_3 \equiv_3 0 \\ z_2 \equiv_3 0 \end{array} \right\} \subseteq A_R .$$

In particular, we have $\mathrm{B}_R = \mathrm{B}_R(\mathrm{S}_3, \mathrm{S}_3) \cong \Lambda_{(3)}$ *as* R*-algebras.*

More symbolically written, we have

$$\Lambda_{(3)} = \begin{bmatrix} R & R & R & 0 & 0 & R & & \\ R & R & R & 0 & 0 & R & & \\ R & R & R & 0 & 0 & R & & \\ 0 & 0 & 0 & R & 0 & R & & \\ 0 & 0 & 0 & 0 & R & 0 & & \\ (3) & (3) & (3) & R & 0 & R & +(3)\overline{\eta} & +R\overline{\xi} \end{bmatrix} .$$

Remark 14. *We claim that* $1_{\Lambda_{(3)}} = \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5 + \mathrm{e}_6$ *is an orthogonal decomposition into primitive idempotents, where*

$$\mathrm{e}_1 := \begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_2 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_3 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},$$

$$\mathrm{e}_4 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_5 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_6 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix}.$$

Proof. We have $\mathrm{e}_s \, \Lambda_{(3)} \, \mathrm{e}_s \cong R$ for $s \in [1, 5]$. Therefore it follows that $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5$ are primitive.

To show that that e_6 is primitive, we *claim* that the ring $\mathrm{e}_6 \, \Lambda_{(3)} \, \mathrm{e}_6 \cong R[\overline{\eta}, \overline{\xi}]$ is local.

We have $\mathrm{U}(R[\overline{\eta}, \overline{\xi}]) = R[\overline{\eta}, \overline{\xi}] \setminus (3, \overline{\eta}, \overline{\xi})$. In fact, for $u := a + b\overline{\eta} + c\overline{\xi}$ with $a \in R \setminus (3)$ and $b, c \in R$, the inverse is given by $u^{-1} = a^{-1} - a^{-2}b\overline{\eta} - a^{-2}c\overline{\xi}$ as

$$uu^{-1} = aa^{-1} + (-a^{-1}b + a^{-1}b)\overline{\eta} + (-a^{-1}c + a^{-1}c)\overline{\xi} = 1 \ .$$

Thus the nonunits of $R[\overline{\eta}, \overline{\xi}]$ form an ideal and so $R[\overline{\eta}, \overline{\xi}]$ is a local ring. This proves the *claim.* □

5.5. $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3, \mathrm{S}_3)$ **and** $\mathrm{B}_{\mathbf{F}_3}(\mathrm{S}_3, \mathrm{S}_3)$ **as path algebras modulo relations.** Write $R := \mathbf{Z}_{(3)}$. We aim to write $\Lambda_{(3)}$, up to Morita equivalence, as path algebra modulo relations. The R-algebra $\Lambda_{(3)}$ is Morita equivalent to $\Lambda'_{(3)} := (\mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5 + \mathrm{e}_6)\Lambda_{(3)}(\mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5 + \mathrm{e}_6)$ since $\Lambda_{(3)} \, \mathrm{e}_1 \cong \Lambda_{(3)} \, \mathrm{e}_2 \cong \Lambda_{(3)} \, \mathrm{e}_3$ using multiplication with elements of $\Lambda_{(3)}$ with a single nonzero entry 1 in the upper (3×3)-corner. We have the R-linear basis of $\Lambda'_{(3)}$ consisting of

$$
\mathrm{e}_3 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} , \ \mathrm{e}_4 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} , \ \mathrm{e}_5 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0 \end{bmatrix} ,
$$

$$
\mathrm{e}_6 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix} , \ \tau_1 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&3&0&0&0 \end{bmatrix} , \ \tau_2 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} ,
$$

$$
\tau_3 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0 \end{bmatrix} , \ \tau_4 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} , \ \tau_5 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&3\overline{\eta} \end{bmatrix} ,
$$

$$
\tau_6 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\overline{\xi} \end{bmatrix} .
$$

We have $\tau_5 = \tau_1\tau_2$ and $\tau_6 = \tau_3\tau_4 + 4\tau_1\tau_2$. Hence, as an R-algebra $\Lambda'_{(3)}$ is generated by $\mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5, \mathrm{e}_6, \tau_1, \tau_2, \tau_3, \tau_4$.

Consider the quiver $\Psi := \left[\begin{tikzcd} \tilde{\mathrm{e}}_5 & \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_6 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd} \right]$. We have a surjective R-algebra morphism $\varphi : R\Psi \to \Lambda'_{(3)}$ by sending

$$
\begin{array}{llllllllllll}
\tilde{\mathrm{e}}_3 & \mapsto & \mathrm{e}_3 & , & \tilde{\mathrm{e}}_4 & \mapsto & \mathrm{e}_4 & , & \tilde{\mathrm{e}}_5 & \mapsto & \mathrm{e}_5 & , \quad \tilde{\mathrm{e}}_6 \mapsto \mathrm{e}_6 \ , \\
\tilde{\tau}_1 & \mapsto & \tau_1 & , & \tilde{\tau}_2 & \mapsto & \tau_2 & , & \tilde{\tau}_3 & \mapsto & \tau_3 & , \quad \tilde{\tau}_4 \mapsto \tau_4 \ .
\end{array}
$$

We establish the following multiplication trees, where we underline the elements that are not in an R-linear relation with previous elements.

The multiplication tree of the idempotent e_5 consists only of the element e_5.

$$
\begin{tikzcd} \underline{\mathrm{e}_4} \arrow[r, "\tau_4"'] & \underline{\tau_4} \arrow[r, "\tau_3"'] \arrow[d, "\tau_1"'] & \tau_4\tau_3 = 0 \\ & \tau_4\tau_1 = 0 & \end{tikzcd}
\qquad
\begin{tikzcd} \underline{\mathrm{e}_3} \arrow[r, "\tau_2"'] & \underline{\tau_2} \arrow[r, "\tau_1"'] \arrow[d, "\tau_3"'] & \tau_2\tau_1 = 0 \\ & \tau_2\tau_3 = 0 & \end{tikzcd}
$$

$$\begin{tikzcd}
 & & \underline{\mathrm{e}_6} \arrow[dl, "\tau_3"'] \arrow[dr, "\tau_1"] & & & \\
\tau_3\tau_4\tau_3 = 0 & \underline{\tau_3} \arrow[dl, "\tau_4"'] & & \underline{\tau_1} \arrow[r, "\tau_2"] & \underline{\tau_1\tau_2} \arrow[r, "\tau_1"'] \arrow[d, "\tau_3"'] & \tau_1\tau_2\tau_1 = 0 \\
\underline{\tau_3\tau_4} \arrow[u, "\tau_3"] \arrow[d, "\tau_1"'] & & & & \tau_1\tau_2\tau_3 = 0 & \\
\tau_3\tau_4\tau_1 = 0 & & & & &
\end{tikzcd}$$

So the kernel of φ contains the elements:

$$\tilde{\tau}_4\tilde{\tau}_1,\ \tilde{\tau}_4\tilde{\tau}_3,\ \tilde{\tau}_2\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_3,\ \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_3,\ \tilde{\tau}_2\tilde{\tau}_3,\ \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_1,\ \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3,\ \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_1\ .$$

Let I be the ideal generated by these elements. So, $I \subseteq \mathrm{kern}(\varphi)$. Therefore, φ induces a surjective R-algebra morphism from $R\Psi/I$ to $\Lambda'_{(3)}$. We may reduce the list of generators to obtain $I = (\tilde{\tau}_4\tilde{\tau}_3,\ \tilde{\tau}_4\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_3)$.

Note that $R\Psi/I$ is R-linearly generated by

$$\mathcal{N} := \{\tilde{\mathrm{e}}_3 + I, \tilde{\mathrm{e}}_4 + I, \tilde{\mathrm{e}}_5 + I, \tilde{\mathrm{e}}_6 + I, \tilde{\tau}_1 + I, \tilde{\tau}_2 + I, \tilde{\tau}_3 + I, \tilde{\tau}_4 + I, \tilde{\tau}_3\tilde{\tau}_4 + I, \tilde{\tau}_1\tilde{\tau}_2 + I\},$$

cf. the underlined elements above. To see that, note that a product ξ of k generators may be written as a product in $\mathcal{N}$ of k' generators and a product of k'' generators, where $k = k' + k''$ and where k' is choosen maximal. If $k'' \geq 1$ then, using the trees above, we have $\xi = 0$. Moreover, note that $|\mathcal{N}| = 10 = \mathrm{rk}_R(\Lambda'_{(3)})$.

Since we have an surjective algebra morphism from $R\Psi/I$ to $\Lambda'_{(3)}$, this rank argument shows this morphism to be bijective. In particular, $I = \mathrm{kern}(\varphi)$.

So, we obtain the

Proposition 15. *Recall that* $I = (\tilde{\tau}_4\tilde{\tau}_3,\ \tilde{\tau}_2\tilde{\tau}_1,\ \tilde{\tau}_4\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_3)$.

We have the isomorphisms of R-algebras

$$\begin{array}{rcl}
\Lambda'_{(3)} & \xrightarrow{\sim} & R\left[\begin{tikzcd} \tilde{\mathrm{e}}_5 & \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_6 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right]/I \\
\mathrm{e}_i & \mapsto & \tilde{\mathrm{e}}_i + I \textit{ for } i \in [3,6] \\
\tau_i & \mapsto & \tilde{\tau}_i + I \textit{ for } i \in [1,4]
\end{array}$$

Recall that $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3, \mathrm{S}_3)$ *is Morita equivalent to* $\Lambda'_{(3)}$.

Corollary 16. *As* $\mathbf{F}_3$*-algebras, we have*

$$\Lambda'_{(3)}/3\Lambda'_{(3)} \ \cong\ \mathbf{F}_3\left[\begin{tikzcd} \tilde{\mathrm{e}}_5 & \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_6 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right]/(\tilde{\tau}_4\tilde{\tau}_3,\ \tilde{\tau}_4\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_3)\ .$$

Recall that $\mathrm{B}_{\mathbf{F}_3}(\mathrm{S}_3, \mathrm{S}_3)$ *is Morita equivalent to* $\Lambda'_{(3)}/3\Lambda'_{(3)}$.

References

[1] B. Masterson, *On the table of marks of a direct product of finite groups,* `http://hdl.handle.net/10379/5550`, PhD thesis, NUI Galway, 2016.

[2] B. Masterson, G. Pfeiffer *On the table of marks of a direct product of finite groups,* arXiv:1704.03433v3, 2018.
[3] S. Bouc, *Foncteurs d'ensembles munis d'une double action*, J. Algebra 183, 664-736, 1996.
[4] S. Bouc, *Burnside rings*, Handbook of Algebra, volume 2, section 6E, 739-804, 2000.
[5] S. Bouc, *Biset Functors for Finite Groups,* Springer, 2010.
[6] P. Webb, *A guide to Mackey functors*, Handbook of Algebra, volume 2, section 6E, 805-836, 2000.
[7] R. Boltje, S. Danz, *A ghost algebra of the double Burnside algebra in characteristic zero,* arXiv:1203.1346, 2013.
[8] I. Reiner, *Maximal Orders*, Oxford University Press, 2003.
[9] G. D. James, *The irreducible representations of symmetric groups*, Bull. London Math. Soc. 8, no. 3, 229-232, 1976.
[10] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24, 235-265, 1997.

Nora Krauss has written a Master's Thesis at the University of Stuttgart on double Burnside rings, where she in particular studied the case S_3 in detail. At present, she is a PhD student at the University of Stuttgart.

(author) University of Stuttgart
E-mail address: `kraussna@mathematik.uni-stuttgart.de`