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Values of $f(G)$ for groups G of odd order with Pr$(G) \geq 11/75$

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Abstract. We augment the 2011 table of Das and Nath by finding all possible values of the commutativity ratio $f(G)$ for a finite group G of odd order, where another commutativity ratio $\Pr(G)$ satisfies $\Pr(G) \geq 11/75$.

1. Introduction

Throughout, let G be a finite group and let $\Pr(G)$ be the probability that two elements of G, chosen at random with replacement, commute with each other. Since $\Pr(G) = 1$ if and only if G is abelian, $\Pr(G)$ may be regarded as a commutativity ratio for groups. It is well known that $\Pr(G) = \frac{k(G)}{|G|}$, where G has $k(G)$ conjugacy classes. In 2011, Das and Nath [3] found all possible values of $\Pr(G)$ where $|G|$ is odd and $\Pr(G) \geq \frac{11}{75}$. They also found the structures for G', $G' \cap Z(G)$ and $G/Z(G)$ corresponding to each of these values of $\Pr(G)$.

We define $f(G)$ to be

$$\frac{1}{|G|}\sum_{i=1}^{k(G)} d_i$$

where d_i, $1 \leq i \leq k(G)$, are the degrees of the irreducible complex representations of G. Since $f(G) = 1$ if and only if G is abelian, $f(G)$ may also be regarded as a commutativity ratio for finite groups.

The commuting probability $\Pr(G)$ has been extensively studied [5, 9, 12, 10, 13, 11, 4] and the ratio $f(G)$ has also been considered by several authors [8, 7, 1, 13].

One's intuitive feeling is that if the values of one commutativity ratio $\Pr(G)$ for a given set of groups are 'large', then the values of another commutativity ratio $f(G)$ should be 'large' also. For the groups G of odd order with $\Pr(G) \geq \frac{11}{75}$, we find the corresponding values of $f(G)$ and show that if $\Pr(G) \geq \frac{11}{75}$, then $f(G) > \frac{15}{75}$.

In general

$$(f(G))^2 \leq \Pr(G) \leq f(G)$$

with equality if and only G is abelian [2].

We note that, for non-abelian G, saying $\Pr(G)$ and $f(G)$ are 'large' is another way of saying that G is close to being abelian.

Finally, it is clear that $\Pr(G) = 1 = f(G) = |G'| = |G/Z(G)|$ if and only if G is abelian and this corresponds to row 1 of the table in [3]. So, from now on we may assume that G is non-abelian of odd order.

We employ Philip Hall's very useful concept of isoclinism [6], which is not specifically mentioned in [3]. Two groups H and K are said to be *isoclinic* if there exist isomorphisms $\theta : H/Z(H) \to K/Z(K)$ and $\phi : H' \to K'$ such that the isomorphism ϕ is induced by the isomorphism θ. Isoclinism is an equivalence relation on finite groups and

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the isoclinism classes are called families. Each family contains a stem group G, with the property that $G' \supseteq Z(G)$. Thus, for a stem group G we have $G' \cap Z(G) = Z(G)$ and $|G| = |Z(G)||G/Z(G)| = |G' \cap Z(G)||G/Z(G)|$ and these values of the orders of stem groups can be read off from the following table taken from [3]:

Row	$\Pr(G)$	G'	$G' \cap Z(G)$	$G/Z(G)$	$f(G)$
1	1	$\{1\}$	$\{1\}$	$\{1\}$	1
2	$\frac{1}{3}\left(1+\frac{2}{3^{2s}}\right)$	C_3	C_3	$(C_3 \times C_3)^s$	$\frac{3^{2s}+2}{3^{2s+1}}$
3	$\frac{1}{5}\left(1+\frac{4}{5^{2s}}\right)$	C_5	C_5	$(C_5 \times C_5)^s$	$\frac{5^{2s}+4}{5^{2s+1}}$
4	$\frac{5}{21}$	C_7	$\{1\}$	$C_7 \rtimes C_3$	$\frac{3}{7}$
5	$\frac{55}{343}$	C_7	C_7	$C_7 \times C_7$	$\frac{13}{49}$
6	$\frac{17}{81}$	C_9 or $C_3 \times C_3$	C_3	$(C_3 \times C_3) \rtimes C_3$	$\frac{11}{27}$
6A	$\frac{17}{81}$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3 \times C_3$	$\frac{11}{27}$
7	$\frac{121}{729}$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3 \times C_3 \times C_3$	$\frac{25}{81}$
8	$\frac{7}{39}$	C_{13}	$\{1\}$	$C_{13} \rtimes C_3$	$\frac{5}{13}$
9	$\frac{3}{19}$	C_{19}	$\{1\}$	$C_{19} \rtimes C_3$	$\frac{7}{19}$
10	$\frac{29}{189}$	C_{21}	C_3	$C_3 \times (C_7 \rtimes C_3)$	$\frac{23}{63}$
11	$\frac{11}{75}$	$C_5 \times C_5$	$\{1\}$	$(C_5 \times C_5) \rtimes C_3$	$\frac{9}{25}$

We aim to justify the values of $f(G)$ appearing in the final column of this augmented table. Both $\Pr(G)$ and $f(G)$ are isoclinic invariants [10, 2], so we may confine our attention in general to the case where G is a stem group.

2. Values of $f(G)$

Consider the unique non-abelian group G_{pq} of order pq, where $p < q$ are odd primes and p divides $q-1$.

It is easy to see that $Z(G_{pq})$ is trivial and that $|G_{pq} : G'_{pq}| = p$, since the Sylow q-subgroup is normal with abelian factor group. Furthermore, each representation of G_{pq} has degree 1 or p, since the Sylow q-subgroup is normal and abelian.

Routine calculations show that G_{pq} has $p + (q-1)/p$ conjugacy classes so that

$$\Pr(G_{pq}) = \frac{p^2+q-1}{p^2 q}.$$

The degree equation

$$|G| = \sum_{i=1}^{k(G)} d_i^2$$

of G_{pq} is now given by

$$|G_{pq}| = p + \left[\frac{q-1}{p}\right] p^2$$

so

$$f(G_{pq}) = \frac{p + [(q-1)/p]\,p}{pq} = \frac{p+q-1}{pq}.$$

We are now in a position to fill in the values of $f(G)$ for several rows of the table.

Row 4. $\Pr(G) = \frac{5}{21}$; a stem group G has order $21 = 3 \cdot 7$, so $f(G) = \frac{7+3-1}{7 \cdot 3} = \frac{9}{21} = \frac{3}{7}$.

Row 8. $\Pr(G) = \frac{7}{39}$; a stem group G has order $39 = 3 \cdot 13$, so $f(G) = \frac{3+13-1}{3 \cdot 13} = \frac{15}{39} = \frac{5}{13}$.

Row 9. $\Pr(G) = \frac{3}{19}$; a stem group G has order $57 = 3 \cdot 19$, so $f(G) = \frac{3+19-1}{3 \cdot 19} = \frac{7}{19}$.

Row 11. $\Pr(G) = \frac{11}{75}$; a stem group has order $|Z(G)||G/Z(G)| = 75$ and is the unique non-abelian group of this order. Since the Sylow 5-subgroup is abelian, normal and of index 3, each $d_i = 1$ or 3 for all i. Thus G has eleven conjugacy classes, so the degree equation can only be

$$75 = 1 + 1 + 1 + 8 \cdot 3^2.$$

Thus, $f(G) = \frac{3+8\cdot 3}{75} = \frac{27}{75} = \frac{9}{25}$.

Row 10. $\Pr(G) = \frac{29}{189}$; a stem group G has order 189, has 29 conjugacy classes and $|G : G'| = \frac{189}{21} = 9$. The degree equation can only be

$$189 = 9 \cdot 1 + 20 \cdot 3^2.$$

So, $f(G) = \frac{9+20\cdot 3}{189} = \frac{23}{63}$.

Row 6. $\Pr(G) = \frac{17}{81}$; a stem group G has order 81 and 17 conjugacy classes. We have $|G : G'| = 9$, so the only possible degree equation is

$$81 = 9 \cdot 1^2 + 8 \cdot 3^2.$$

Thus $f(G) = \frac{9+8\cdot 3}{81} = \frac{11}{27}$.

Row 6A. $\Pr(G) = \frac{17}{81} = \frac{51}{243}$; a stem group has order $27 \cdot 9 = 243$ and 51 conjugacy classes. $|G'| = 9$, so $|G : G'| = 27$ and there are 24 other conjugacy classes. The only possible degree equation is

$$243 = 27 \cdot 1^2 + 24 \cdot 3^2,$$

so $f(G) = \frac{27+24\cdot 3}{243} = \frac{11}{27}$.

Note that rows 6 and 6A are an example of different families which have the same $\Pr(G)$ and $f(G)$ values.

Row 5. $\Pr(G) = \frac{55}{343}$; a stem group G has order $7^3 = 343$ and 55 conjugacy classes. $|G'| = 7$, so $|G : G'| = 49$ and there are 6 other classes. Thus the only possible degree equation is

$$343 = 49 \cdot 1^2 + 6 \cdot 7^2$$

and $f(G) = \frac{49+6\cdot 7}{343} = \frac{13}{49}$.

Row 7. $\Pr(G) = \frac{121}{729}$; a stem group G has order $3^2 \cdot 3^4 = 729$. $|G'| = 9$ and $|G : G'| = 81$. G has 40 other classes.

Now, $81 + 40 \cdot 9 < 729$, so we must consider the possibility that G has representations of degrees 3 and 9. Thus the degree equation is

$$729 = 81 + a \cdot 3^2 + b \cdot 3^4$$

for some non-negative integers a and b. We get $9a + 81b = 648$ and $a + b = 40$. This gives $a = 36$ and $b = 4$. So, the degree equation is

$$729 = 81 + 36 \cdot 3^2 + 4 \cdot 3^4.$$

Thus

$$f(G) = \frac{81 + 36 \cdot 3 + 4 \cdot 9}{729} = \frac{25}{81} = \left(\frac{5}{9}\right)^2.$$

Now all that remains is to examine the extra-special 3-group and 5-group cases.

Row 2. $\Pr(G) = \left(\frac{1}{3}\right)\left(1+\frac{2}{3^s}\right)$, $s \geq 1$. Here $|G'| = 3$, $|G' \cap Z(G)| = |Z(G)| = 3$ and $|G/Z(G)| = 3^{2s}$. So, a stem group has order 3^{2s+1}. Now, $|G : G'| = \frac{3^{2s+1}}{3} = 3^{2s}$ and

$$\Pr(G) = 3^{2s}\left(\frac{1+2/3^{2s}}{3^{2s+1}}\right) = \frac{3^{2s}+2}{3^{2s+1}}.$$

So G has $3^{2s}+2$ classes, so we have two extra classes to consider. The degree equation can only be

$$3^{2s+1} = 3^{2s} + (3^s)^2 + (3^s)^2.$$

So

$$f(G) = \frac{3^s+2}{3^{s+1}}$$

after simplification.

Row 3. $\Pr(G) = \frac{1}{5} + \frac{4}{5^{2s+1}}$. In like manner to the above, we find

$$f(G) = \frac{5^s+4}{5^{s+1}}.$$

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