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PROBLEMS

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Problems

The first problem is a corrected version of Problem 82.1, which was missing some hypotheses. The problem uses the usual notation $x_1, x_2, \dots, x_n$ for the components of a vector x in $\mathbb{R}^n$.

Problem 84.1. Suppose that u and v are linearly independent vectors in $\mathbb{R}^n$ with

$$0 < u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \quad \text{and} \quad v_1 > v_2 > \cdots > v_n > 0.$$

Given $x \in \mathbb{R}^n$, let y be the orthogonal projection of x onto the subspace spanned by u and v; thus $y = \lambda u + \mu v$, for uniquely determined real numbers λ and μ. Prove that if

$$x_1 > x_2 > \cdots > x_n > 0,$$

then μ is positive.

The second problem was contributed by Finbarr Holland, of University College Cork.

Problem 84.2. Given any finite collection $L_1, L_2, \dots, L_n$ of infinite straight lines in the complex plane, find a formula in terms of data specifying $L_1, L_2, \dots, L_n$ for a differentiable function $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ with the property that each line L_i is tangent to the curve $f(\mathbb{R})$.

For the third problem, we use the definition of a directed graph that allows loops and multiple directed edges with the same source and target vertex.

Problem 84.3. Suppose that each edge of a finite directed graph G is coloured in one of some finite collection of different colours, with the property that for each colour c and vertex v, there is precisely one directed edge with colour c and target vertex v. Prove that for any infinite sequence of colours $c_1, c_2, \dots$ there is an infinite walk $e_1, e_2, \dots$ of directed edges of G such that, for each index i, e_i has colour c_i and the target vertex of e_i equals the source vertex of e_{i+1}.

Solutions

Here are solutions to the problems from *Bulletin* Number 82.

Problem 82.1 was false. It is replaced by Problem 84.1. Thanks to Omran Kouba of the Higher Institute for Applied Sciences and Technology, Damascus, Syria and the North Kildare Mathematics Problem Club for providing examples to demonstrate the falsehood of Problem 82.1.

The next problem was solved by Omran Kouba, the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland. We present the solution of the Problem Club.

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Problem 82.2. Prove that

$$\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x}\, dx = \log 2.$$

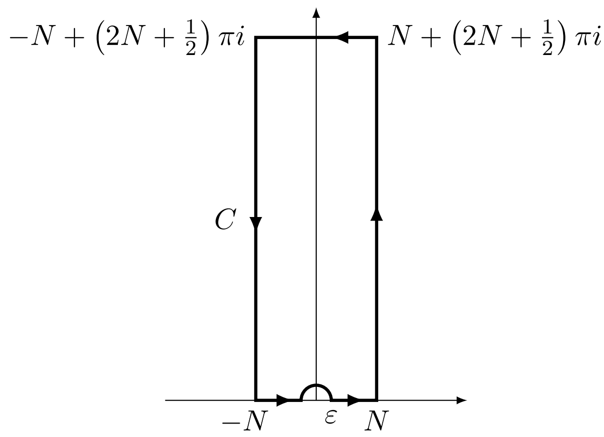
Solution 82.2. Let

$$f(z) = \frac{\sinh z - z}{z^2 \sinh z},$$

and observe that f is an even function, so

$$\int_0^\infty f(x)\, dx = \frac{1}{2}\int_{-\infty}^\infty f(x)\, dx.$$

Let N be a positive integer and let ε be a positive constant, less than 1. Let C be the contour shown in the figure, traversed once anticlockwise.



One can check that the integral of f along the semicircle of radius ε tends to 0 as $\varepsilon \to 0$. Next, we wish to show that the integral of f along the vertical edges and top edge of C tends to 0 as $N \to \infty$. By writing

$$f(z) = \frac{1}{z^2} - \frac{1}{z \sinh z}$$

we see that the main task is to check that the integral of $1/(z \sinh z)$ along these contours tends to 0 as $N \to \infty$. This is easily done for the two vertical contours of C by using the inequality $|\sinh z| \geq \sinh N$ for any point z on one of the vertical contours.

Now consider a point $z = x + \left(2N + \frac{1}{2}\right)\pi i$ on the top contour Γ of C. Observe that $\sinh z = i \cosh x$. Hence

$$\left|\int_\Gamma \frac{1}{z \sinh z}\right| \leq \frac{1}{\left(2N + \frac{1}{2}\right)\pi}\int_{-N}^N \frac{1}{\cosh x}\, dx \leq \frac{1}{2N + \frac{1}{2}},$$

so the integral of $1/(z \sinh z)$ along this contour tends to 0 as $N \to \infty$ also.

Hence, by applying the residue theorem and then taking limits, we see that

$$\int_{-\infty}^\infty f(x)\, dx$$

is equal to $2\pi i$ times the sum of the residues of f in the upper half-plane. The poles of f in the upper half-plane occur at $\pi n i$, for each positive integer n, and the residue of f at $\pi n i$ is $(-1)^{n+1}/(\pi n i)$. Hence

$$\int_0^\infty f(x)\, dx = \frac{1}{2} \times 2\pi i \sum_{n=1}^\infty \frac{(-1)^{n+1}}{\pi n i} = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} = \log 2. \qquad \square$$

No solutions were received for the extended version of Problem 82.2, which asks for a proof of the integral formula

$$\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x}\, e^{-x}\, dx = \log \pi - 1.$$

The third problem was solved by Omran Kouba, the North Kildare Mathematics Problem Club, and Henry Ricardo of the Westchester Area Math Circle, New York, USA. Solutions also appeared in Issue 255 of the M500 Society of the Open University, from which the problem was taken. The solution we present is an amalgamation of these solutions.

*Problem 82.3***.** Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(5n-3)(5n-2)} = \frac{\pi}{5}\sqrt{1 - \frac{2}{\sqrt{5}}}.$$

Solution 82.3. Recall the well-known result that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a}\left(\frac{1}{a} - \pi \cot \pi a\right), \tag{$*$}$$

for $0 < a < 1$. This can be proved by methods of contour integration, or by taking the logarithm and differentiating each side of the equation

$$\frac{\sin \pi a}{\pi a} = \prod_{n=1}^{\infty}\left(1 - \frac{a^2}{n^2}\right)$$

with respect to a. Now observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 - a^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2 - a^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2 - (a/2)^2}.\end{aligned}$$

By applying $(*)$ and simplifying we can check that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 - a^2} = \frac{\pi}{4a}\tan(\pi a/2).$$

Next, we have

$$\frac{1}{(5n-3)(5n-2)} = \frac{1}{25(n-1/2)^2 - (1/2)^2} = \frac{4}{25((2n-1)^2 - (1/5)^2)}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{(5n-3)(5n-2)} = \frac{4}{25} \times \frac{5\pi}{4}\tan(\pi/10) = \frac{\pi}{5}\sqrt{1 - \frac{2}{\sqrt{5}}}. \qquad \square$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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