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Cumann Matamaitice na hÉireann

Bulletin

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Irish Mathematical Society Bulletin

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

Correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

mailto:ims.bulletin@gmail.com

and only if not possible in electronic form to the address

The Editor
Irish Mathematical Society Bulletin
Department of Mathematics and Statistics
Maynooth University
Co. Kildare W23 HW31

Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

http://www.irishmathsoc.org/

EDITORIAL

In the last issue, I invited Irish schools to contribute news. There was a limited response. I imagine there were developments elsewhere. Please send reports to

mailto://ims.bulletin+news@gmail.com.

As before, to facilitate members who might wish to print the whole issue, the website will carry a pdf file of the whole Bulletin 84, in addition to the usual pdf files of the individual articles. As a further convenience (which may suit some Departments and Libraries), for a limited time a printed and bound copy of this Bulletin may be ordered online on a print-on-demand basis at a minimal price[1].

Next year's IMS Annual Scientific Meeting (also known as the "September meeting") will be held in DCU, on 27-28 August 2020. Please make a note in your diaries. The 2-day programme will consist of a number of invited talks by speakers from Ireland and abroad. The aim of the meeting is to reflect the diversity of the mathematical community in Ireland and the scientific interests of the members of the IMS. All are welcome to attend.

Michael Mackey's report on the AGA meeting, held last May to mark the achievements of the late Richard Timoney, appears in this issue. Just to hand is the special issue of the Mathematical Proceedings of the Royal Irish Academy (Volume 119A, Number 2, December 2019) that contains articles by eight of the speakers at that meeting, including an article by Seán Dineen on the mathematical legacy of Richard Timoney that complements his obituary in Bulletin 83. In the words of Martin Mathieu, editor of the Mathematical Proceedings, we all miss him a lot.

This issue of the Bulletin includes a short paper by C.T.C.Wall on solving cubic and quartic equations by radicals. For your editor, this provided his first encounter with the terms *transvectant* and *catalecticant*, and he counted the day well-spent on that account. Ireland has had a significant rôle in the theory of invariants, with important contributions from George Boole and George Salmon. The December 2019 number of the Notices of the AMS has an article by Moira Chas (*The Extraordinary Case of the Boole Family,* pp 1853–1866), in which she points out that Boole, in one of his earliest published papers, was first to point out the covariance of the discriminant of a binary quadratic form under linear transformations, and that this prompted Cayley to invent the theory of algebraic invariants for forms of higher degree in any number of variables. Dixmier, who was involved in a new wave of work on invariants in the nineteen eighties, kept a copy of Salmon's *Higher Plane Curves* in pride of place on his desk at IHES.

We are in the process of setting up Digital Object Identifiers (DOI's) for the Bulletin and its individual issues and articles. David Malone is looking after this for the Society, and Maynooth University Library staff are kindly assisting.

As we go to press comes the exciting news that zbMATH (the current incarnation of the venerable reviewing journal Zentrallblatt für Mathematik) is about to become an open access platform, thanks to an enlightened decision of the German authorities. See

https://www.zbmath.org/static/newsletter/zbNEWS-12.pdf

[1]Go to www.lulu.com and search for *Irish Mathematical Society Bulletin*.

Links for Postgraduate Study

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: mailto://maths@dcu.ie
DIT: mailto://chris.hills@dit.ie
NUIG: mailto://james.cruickshank@nuigalway.ie
MU: mailto://mathsstatspg@mu.ie
QUB: http://www.qub.ac.uk/puremaths/Funded_PG_2016.html
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: http://www.ucc.ie/en/matsci/postgraduate/
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://sarah.mitchell@ul.ie
UU: http://www.compeng.ulster.ac.uk/rgs/

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor. All links are live, and hence may be accessed by a click, when read in a suitable pdf reader.

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E-mail address: ims.bulletin@gmail.com

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Officers and Committee Members 2019

President	Dr Pauline Mellon	UCD
Vice-President	Dr Tom Carroll	UCC
Secretary	Dr D. Malone	Maynooth University
Treasurer	Prof G. Pfeiffer	NUI Galway

Dr P. Barry, Prof S. Buckley, Dr L. Creedon, Dr R. Levene, Dr D. Mackey, Dr M. Mathieu, Dr A. Mustata, Dr J. O'Shea .

Officers and Committee Members 2020

President	Dr Pauline Mellon	UCD
Vice-President	Dr Tom Carroll	UCC
Secretary	Dr D. Malone	Maynooth University
Treasurer	Dr C. Kelly	UCC

Prof S. Buckley, Dr L. Creedon, Dr R. Flatley, Dr D. Mackey, Dr M. Mathieu, Dr R. Ryan, Dr H. Smigoc, Dr N. Snigireva .

Local Representatives

Belfast	QUB	Dr M. Mathieu
Carlow	IT	Dr D. Ó Sé
Cork	IT	Dr J. P. McCarthy
	UCC	Dr S. Wills
Dublin	DIAS	Prof T. Dorlas
	TUD, City	Dr D. Mackey
	TUD, Tallaght	Dr C. Stack
	DCU	Dr B. Nolan
	TCD	Prof K. Soodhalter
	UCD	Dr R. Levene
Dundalk	IT	Mr Seamus Bellew
Galway	NUIG	Dr J. Cruickshank
Limerick	MIC	Dr B. Kreussler
	UL	Mr G. Lessells
Maynooth	MU	Prof S. Buckley
Tralee	IT	Dr B. Guilfoyle
Waterford	IT	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€200
Ordinary member	€30
Student member	€15
DMV, I.M.T.A., NZMS or RSME reciprocity member	€15
AMS reciprocity member	$20

The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 40.
If paid in sterling then the subscription is £30.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 40.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form with one year's subscription to:

Dr Cónall Kelly
School of Mathematical Sciences
Western Gateway Building, Western Road
University College Cork
Cork, T12 XF62
Ireland

Deceased Members

It is with regret that we report the deaths of members:

Dónal O'Donovan of TCD died on 9 November 2019.

E-mail address: subscriptions.ims@gmail.com

PRESIDENT'S REPORT 2019

Committee Changes: The IMS committee had a change of both President and Vice President in 2019, with S. Buckley (MU) being replaced as President by P. Mellon (UCD), and P. Mellon being replaced as Vice-President by T. Carroll (UCC). Many thanks to the outgoing President for his excellent and wide-ranging work in this role and, in particular, for bringing the EMS Meeting of Presidents to Ireland in April 2018. We are grateful that S. Buckley opted to remain on the committee. I would also like to welcome T. Carroll to the committee and to thank J. Gleeson (UL) who left the committee after six years of dedicated work to the Society.

IMS Bulletin: This was a watershed year as the Bulletin moved to electronic format. This decision was communicated to members and to those institutions with whom the Society has exchange or reciprocity agreements. The Edinburgh Mathematical Society responded that they were actively considering making the same decision. Individuals may order a printed copy of the Bulletin for a small fee online. For archiving purposes the Editor has agreed to send printed copies to the small number of copyright libraries. I extend the Society's continuing thanks to A. O'Farrell for his work as Editor of the Bulletin and I also thank the editorial board of the Bulletin. We are grateful also to Michael Mackey for his considerable work in streamlining the Society's webpages.

IMS-LMS Reciprocity Agreement: Many thanks to M. Mathieu for acting as liaison with the London Mathematical Society and securing a unique IMS-LMS Reciprocity Agreement, which as a legal agreement was signed by the Presidents of both societies in June 2019 and came into effect in July 2019.

IMS meetings: The Society's 2019 annual "September Meeting" was held on September 5-6 in Galway and we thank NUIG and the local organisers, namely, A. Carneval, M. Destrade, G. Pfeiffer and R. Quinlan for their work in making this meeting a success. The AGM of the Society was held during this time, with six committee members being elected or re-elected, of whom, D. Malone will remain on as Secretary and Cónall Kelly (UCC) will be the incoming Treasurer. I'm delighted to announce that Dublin City University have agreed to host the 2020 annual conference on August 27-28.

On December 6 Sligo IT hosted the Society's 4th Annual IMS Christmas Lecture given by Alastair Wood (Emeritus DCU). Alastair travelled from France to lecture on *George Gabriel Stokes Life, Science and Faith.* The lecture celebrated the 200th anniversary of the birth of the renowned Sligo mathematician. The Society's December committee meeting was held afterwards. Thanks to Leo Creedon, Kieran Hughes and Sligo IT for support.

IMS Conference Support: The Society supported the following conferences in 2019:

1. Analysis, Geometry and Algebra, TCD, May 8-11, 2019.
2. Groups in Galway, May 10-11, 2019.
3. Linear Algebra and Matrix Conference, May 23-24, 2019.
4. Irish Geometry Conference May 16-17, 2019.

Thanks to our treasurer, G. Pfeiffer, for overseeing our Conference Support Scheme and keeping our finances in a healthy condition.

Other: The EMS meeting of Presidents was held this year in TU Berlin on March 23-24, 2019 and, as I unfortunately could not attend, Stephen Buckley kindly represented the Society instead.

I thank J. Grannell (UCC) for making an excellent proposal to the NCCA on behalf of the Society, regarding updating the Leaving Certificate Applied Mathematics course. The recommendations in the proposal were largely adopted and the new course creates many opportunities for students in a wide range of subjects.

To finish I want to pay the Society's particular thanks to a late, great friend of the IMS, Richard Timoney, RIP, who tragically passed away on New Year's Day 2019. Apart from serving as Secretary, Vice President and President of the Society, Richard set up and maintained our Society's webpages until just before his death. He is sadly missed.

Pauline Mellon
December 2019

E-mail address: `pauline.mellon@ucd.ie`

IRISH DOCTORATES COMPLETED

The following are the names and thesis titles for PhD degrees in Mathematics completed at Irish universities in the period from April 2018 to March 2019, inclusive. Departments are requested to send the information for each year to the end of March to the editor at the address below.

DCU:
Luca Bernardinelli. Dynamic information aggregation in asset prices.
Diarmaid Hyland. Investigating students' learning of differential equations in physics.
Ben Quigley. Noncrossing partitions and subgroups of Artin groups of finite type.

MU:
Áine Dooley. Modelling techniques for biodiversity and ecosystem multifunctionality - theoretical development and application.
Jonathan Dunne. Endless data.
Jack McDonnell. Predicting grass growth at farm level to allow producers to adapt to changing and volatile weather conditions
Stephen McGuire. Extensions to a Lemma of Bernik with Applications in the area of Metric Diophantine Approximation.
Giulio Prevedello. A mathematical framework for clonal data analysis.

QUB:
Meabh McCurdy. Improving the Computational Efficiency for Calculating Matrix Exponentials using Krylov Subspace Methods.

TCD:
Francisco Jose Garcia Abad. Complexity of holographic flavors and causality in Gauss-Bonnet dual QFTs.
Lorenzo Gerotto. Form Factors, Integrability and the AdS/CFT Correspondence.
Philipp Hähnel. Higher Spin Theories in Twistor Space.
Vanessa Koch. String breaking from lattice QCD with $Nf = 2 + 1$ dynamical fermions.
Cian OHara. Towards excited radiative transitions in charmonium.

UCD:
Brendan Murray. Fourier Phase Dynamics in Turbulent Nonlinear Systems.
Lampros Bouranis. Advances in the Bayesian Analysis of statistical models with intractable nRMalising constants.
Gurpreet Singh. A Qualitative study of elliptic partial differential equations motivated by real life phenomena.
Tin Lok Ng. Network Analysis.
Emrah Sercan Yilmaz. On cosets of weight 4 of binary NHC codes with minimum distance.
Stiofáin Fordham. On a class of differential operators and artin-schreier extensions in arithmetic geometry.

UL:
Gary OKeeffe. Mathematical Modelling of Nanofluid-Based Direct Absorption Solar Collectors.
Daria Semochkina. Bayesian Approach to disease Model Calibration.

Kevin Brosnan Statistical Modelling of Lattice Data with Applications in Flow Cytometry.
Israel Ikoyi. The Impact of Phosphorus and Sulfur Fertilizer Application on Soil Microbiota, Nematodes and Grass Growth in Grassland Columns.

E-mail address: `ims.bulletin+phds@gmail.com`

NEWS

The following is the news reported from Irish universities for the year 2019. Departments are requested to send the information for each year to the end of November to the editor at the address below.

MU:

David Redmond retired December 2018.
Appointments:
Rafael de Andrade Moral, Lecturer 2018.
Niamh Cahill, Lecturer 2019.
Peter Mulligan, University Tutor 2019
Departure:
Caroline Brophy left for Trinity Nov 2019.

UCD:

Appointments made in 2019:
1 year temporary in Mathematics: Dr. Nina Snigereva (start September 2019)
Ad Astra Fellows in Mathematics (5 year appointment): Dr. John Skeekey (start September 2019), Dr. Myrto Manolaki (start January 2020)
Ad Astra Fellow in Applied Mathematics (5 year appointment): Dr. Sarp Akcay (start March 2020).
Ad Astra Fellow in Applied Mathematics and Statistics (3 year appointment): Dr. Aine Byrne (start September 2019).

E-mail address: ims.bulletin@gmail.com

Minutes of the Irish Mathematical Society General Annual General Meeting
held on September 6, 2019 at National University of Ireland, Galway

Present: P. Barry, S. Balagopalan, S. Buckley, A. Carnevale, T. Carroll, L. Creedon, E. Donlon, R. Egan, G. Ellis, M. Hanley, B. Kreußler, R. Levene, D. Mackey, N. Maddem, D. Malone, A. McCluskey, P. Mellon, H. Murray, M. Ní Chobhthaigh, A. O'Farrell, M. O'Reilly, G. Pfeiffer, R. Quinlan, R. Ryan, H. Šmigoc, N. Snigireva.

1 Minutes
Minutes of the last meeting, as in the Bulletin, were accepted.

2 Matters Arising
All matters will come up later.

3 Correspondence

- All members have been contacted about the Bulletin and other societies with reciprocity agreements. The Edinburgh Mathematical Society are considering a similar course of action and the Korean Mathematical society will no longer circulate hard copies, but electronic copies continue to be available to IMS members.
- The EMS have requested news items from members. Items can be sent to the president.

4 Membership Applications
Thirteen new members had been approved: E. Lingham, F. O'Reilly, F. Healy, Y. Mathieu, A. Carnevale, K. Soodhalter, S. Balagopalan, K. Hughes, M. Wieteska, S. Unnikrishnan, M. McAfee, K. Mulrennan, M. Manolaki. This including members taking advantage of the LMS reciprocity arrangement. This is a record, in recent years at least.

5 President's Report

- The IMS committee has a change of president (with P. Mellon taking over from S. Buckley) and change of vice-president (with T. Carroll taking over from P. Mellon). The president extended thanks to the previous president for his wide ranging activities, support and, in particular, for bringing the EMS Presidents Meeting to Ireland. The president also extended thanks to J. Gleeson for all his work on behalf of the society.
- The president extended thanks to A. O'Farrell for all his work on the Bulletin, moving it to electronic production and thanked Anthony Waldron of Maynooth for help in contacting other societies with which we have reciprocity agreements. The president also thanked the editorial board for their work.
- M. Mathieu had completed all the work on a membership reciprocity agreement with the LMS, with the document signed and came into force at the beginning of July. This was a special agreement, allowing members in Northern Ireland to choose their primary society. There was a lot of good feeling between the IMS and LMS following the agreement.
- Thanks were extended to the G. Pfeiffer, R. Quinlan, A. Carnevale, M. Destrade and the rest of the team in Galway for organising the annual meeting. The 2020 meeting will be in DCU, organised by B. Nolan, N. O'Sullivan and others.
- The society had supported four conferences:Analysis, Geometry and Algebra; Groups in Galway, Linear Algebra and Matrix Conference and the Irish Geometry Conference. There were a number of conferences in May that overlapped or nearly overlapped, and we should aim to avoid overlapping conferences where possible.

G. Pfeiffer was thanked for running the scheme. The society is always happy to provide support, where possible, and act as a kick-start funder.

- S. Buckley attended the EMS Presidents Meeting in TU Berlin on behalf of the society.
- The president also recorded thanks to the long-term support provided by Richard Timoney, who had been active in every aspect of the society over the years. Recently, he had been maintaining the society's website and had been helping M. Mackey with the handover right up to the end.

6 Treasurer's Report

A report on finances were circulated. The treasurer noted that this doesn't fully reflect the new steady state after changes in membership rates and the Bulletin becoming electronic. Our income is mainly from the membership and our outgoings include the Bulletin, conference support, the annual meeting and EMS subscription. We do have a small surplus. Thanks were extended to G. Pfeiffer.

7 Bulletin

The Bulletin, in its electronic form, depends heavily on M. Mackey both for putting up the electronic version and responding to queries. A. Waldron also manages the journal exchanges. Of course, the editorial board continues to provide guidance on issues. Thanks were extended to all for their work.

A small number of paper copies are still being produced for the copyright libraries. Paper copies are available for sale (US Letter size) for those who want them.

As always, the Bulletin welcomes good material, including surveys, book reviews and news.

8 Educational Subcommittee

The education subcommittee is currently focused on curriculum and textbooks. They hope to produce a document providing guidance on these.

9 Ethics

It had been suggested that the committee look at an ethics statement for the society. A short statement had been drawn up and it was proposed to add this to the society's web page. This was approved by the meeting.

10 Elections

D. Malone, L. Creedon and M. Mathieu have reached the end of their term, but are happy to run again. G. Pfeiffer, P. Barry, A. Mustata, J. O'Shea and R. Levene have all reached six years on the committee and thanks were extended to all for their work for the society. Nominations were received for R. Ryan, R. Flatley, N. Snigireva and H. Smigoc for general committee positions. C. Kelly was also nominated to act as treasurer. All were elected unanimously.

11 AOB

- The European Congress of Mathematics will be held in Slovenia on July 5–11. It will be preceded by the EMS Council meeting.
- B. Kreußler asked about how much money had been saved by moving the Bulletin to electronic format, and what new activities could be supported by the society. The full saving is not yet clear, but there may be the possibility of extending conference support or having more invited speakers.
- The Fergus Gains cup is usually presented by the Irish Mathematical Trust in May. This year, however, it was presented in the winner's school.
- M. O'Reilly asked what should be done with journal hard copies from societies with which we have reciprocity agreements. The old system was that various

Irish institutions housed copies, but in practice most hard copies were stored in Maynooth. It was felt that these should be retained.

- M. O'Reilly asked about checking the list of local IMS representatives. The meeting reviewed the matter, and resolved to update the list for the next Bulletin. K. Soodhalter was thanked for standing in as local representative in TCD.

David Malone
david.malone@mu.ie

IMS Annual Scientific Meeting 2019
National University of Ireland, Galway

September 5th-6th

The 29th annual scientific meeting of the society was hosted by the School of Mathematics, Statistics and Applied Mathematics at the National University of Ireland, Galway on the first Thursday and Friday in September. The organisers were Angela Carnevale, Michel Destrade and Götz Pfeiffer.

Opening welcome remarks were made by Rachel Quinlan, Head of the School of Mathematics, Statistics and Applied Mathematics at NUIG, and by Pauline Mellon, President of the IMS.

The scientific programme contained a mixture of lectures of varying lengths, given by speakers from Irish institutions and from abroad at various career stages, on topics ranging from algebra, geometry and combinatorics to applied mathematics and mathematics education. (The full list of talks follows below.)

Friday's schedule also included the Society's AGM which is reported on separately in these pages.

The web page for the meeting, which includes abstracts of presented talks, is archived at

http://september.nuigalway.ie/

The organisers are grateful to all who participated in the meeting but especially to our speakers whose lectures were given in the following sequence.

Thursday 5th September

Martin Kerin (WWU Münster/NUI Galway) - *Highly connected 7-manifolds with non-negative sectional curvature.*

Cian O'Brien (NUI Galway) - *Alternating Signed Bipartite Graph Colourings.*

Tobias Rossmann (NUI Galway) - *Groups, growth, and graphs.*

Aoife Hennessy (Waterford IT) - *Riordan arrays and weighted lattice paths.*

Francesco Brenti (Universita' di Roma "Tor Vergata") - *Permutations, tensor products, and Cuntz algebra automorphisms.*

Hazel Murray (Maynooth University) - *Guessing passwords.*

Friday 6th September

Dónal O'Regan (NUI Galway) - *A result of Andrzej Granas (1929-2019).*

David Henry (UCC) - *Exact, free-surface equatorial flows.*

Marianne Leitner (TCD/DIAS) - *Convolutions on the complex torus.*

Valentina Balbi (UL) - *The mechanics of a "twisted" brain.*

Helena Šmigoc (UCD) - *Some Results on Completely Positive Matrices.*

Mark Dukes (UCD) - *Chip-firing, toppling regimes, and combinatorial structures.*

Mike Welby (NUI Galway) - *Genus g Zhu Recursion for Vertex Operator Algebras.*

Maurice OReilly (DCU) - *Ireland and its place in European Research in Mathematics Education.*

Report by G. Pfeiffer (email: goetz.pfeiffer@nuigalway.ie)

Reports of Sponsored Meetings

AGA: Analysis, Geometry, Algebra
8-10 May 2019, Trinity College Dublin

In the early summer of 2018 it was decided that there were few people more deserving of a conference marking their 65th birthday and (notional) retirement than Richard Timoney of Trinity College. Quietly, a scientific committee was formed, a host volunteered (TCD, naturally), an initial few speakers invited, and a web page cobbled together. With Margaret Timoney's approval, this was revealed to Richard on his 65th birthday, July 17th 2018. Although Richard was undergoing treatment in hospital at the time, he looked forward to the conference in May 2019 with optimism.

The title of the meeting reflects some of the principle themes that can be seen in Richard's work: analysis, geometry and algebra, hence the moniker AGA.[1] Generous financial support was received from the principle sponsor of the meeting, the Hamilton Mathematics Institute, itself funded from the Simons Foundation, as well as from the cognate academic schools at UCD and TCD, and of course the Irish Mathematical Society.

Sadly, Richard passed away on New Year's Day 2019. His obituary appeared in Bulletin 83. His loss inevitably meant that the AGA meeting assumed a more reflective tone but it remained a celebration of Richard's life and work. For many, those outside Ireland especially, who knew or had worked with Richard, the meeting provided an opportunity to mark the passing of a fellow mathematician who was held in the highest regard.

The scientific programme consisted of three days of talks and, to cater for the number of people who wished to contribute, each lecture was quite short at 25 minutes duration. Many speakers further curtailed their time in order to share much appreciated reminiscences of Richard. The first to do so was Professor Patrick Prendergast, Provost of Trinity College, who opened the meeting on Wednesday 8th May and pointed out that Richard had lectured him in his earlier days as an engineering student in Trinity.

Throughout, a great depth and range of mathematical work was evident in the high quality presentations which, reflecting the title of the meeting, provided many reminders that no topic in mathematics can be fully understood without recourse to the others.

The last lecture on Thursday 9th was delivered by S. Dineen, entitled "A sample of Richard's contributions to infinite dimensional analysis". Afterwards, there were warm tributes from those assembled, including a short address by Vice-Provost of Trinity, Professor Chris Morash. Happily, Margaret and several members of Richard's family were able to join us for this session and dinner afterwards.

There were about sixty participants altogether. Twenty-nine lectures were presented with all but four of the speakers travelling from overseas. The full programme and the slides of most talks are available on the conference website, `https://maths.ucd.ie/aga`. Photos of the conference, and of Richard, are also available there. (If asked, the password is `richardt`.)

The organisers of AGA were Seán Dineen, Chris Boyd, Michael Mackey and Pauline Mellon of UCD, and Vladimir Dotsenko and Donal O'Donovan of TCD with administrative assistance from Karen O'Doherty and Emma Clancy at TCD School of Mathematics.

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[1]An alternate permutation was considered for a short time but we quickly realised that the meeting would not be very visible in a web search for "Dublin GAA".

Report by Michael Mackey, University College, Dublin
mackey@maths.ucd.ie

International Conference on Linear Algebra and Matrix Theory
23-24 May 2019, O'Brien Centre for Science – UCD

Organisers
Anthony Cronin and Helena Smigoc

On May 23rd and 24th 2019 an International Conference on Linear Algebra and Matrix Theory was held at UCD to honour Professor Thomas J. Laffey on the occasion of his 75th birthday. The event was supported by Science Foundation Ireland, the Irish Mathematical Society, Optum Technology and the UCD School of Mathematics and Statistics, all of whom we give great thanks to.

The international and national speakers reflected Tom's varied and long career from talks on matrix algebras, centralizers, spectra and idealizers to group representation theory and the nonnegative inverse eigenvalue problem. Day one of the event concluded with testimonials reflecting the many contributions Tom has made to mathematics, education and outreach both in Ireland and internationally. Professor Rod Gow regaled us with many entertaining stories of Tom's encyclopedic knowledge of politics, Irish weather and of course mathematics over his career at UCD spanning more than forty years. Rod also spoke eloquently on Tom's contribution to Linear Algebra and Matrix Theory research including his seminal work on providing a constructive proof of the celebrated Boyle-Handelman theorem on the existence of a nonnegative matrix with given non-zero spectrum, and the awarding to Tom of the Hans Schneider prize in 2013, which recognises research, contributions, and achievements at the highest level of linear algebra . Professor Pat Guiry gave thanks for Tom's 30 year involvement as a judge at the BT Young Scientist and Technology Exhibition. Gordon Lessells gave an entertaining pictorial account of Tom's considerable contribution to the Irish and International

Mathematical Olympiads, work which Tom is still involved with. Tom's first cousin Tom Doherty gave a beautiful account of three acts of kindness Tom Laffey bestowed on him as a young boy in addition to the beautiful recital of a section of the poem A Psalm of Life by H.W. Longfellow. Thankfully Tom's brother and sisters and their families joined us for these glowing tributes and attended the conference dinner afterwards. There were over 50 participants over the two days, with a total of 23 talks, 11 from international speakers including the president of the European Mathematical Society Professor Volker Mehrmann. Details of the programme and abstracts of all the talks are available at the conference website `http://www.maths.ucd.ie/lamt/` The meeting concluded with the announcement of the Thomas J. Laffey Wikipedia page going live. This page is available to view at `https://en.wikipedia.org/wiki/Thomas_J._Laffey/`

Report by Anthony Cronin, UCD, Dublin
anthony.cronin@ucd.ie

Irish Geometry Conference 2019
17-18 May 2019, Maynooth University

Geometry and the closely related field of topology were largely absent from Irish mathematics at a research level until relatively recently, despite being major areas of mathematical research internationally since (at least) the middle of the twentieth century. This situation has changed in recent years, with the annual Irish Geometry Conference being one of the fruits of this development.

The meeting has two main aims. One is to provide a platform where Irish geometers and topologists can present their current research among their peers in order to inform, but also to foster new collaborations within the community. The second is to invite one or more internationally prominent researchers to give invited addresses, allowing local mathematicians the chance to interact for a couple of days with a major figure, possibly leading to new research directions.

This year the main speaker was Professor Wilhelm Klingenberg from the University of Durham in the UK who spoke on joint work with Brendan Guilfoyle (Tralee) regarding a conjecture of Toponogov on complete convex surfaces. Professor Klingenberg is a renowned expert in the areas of complex and symplectic geometry, as well as geometric analysis and mathematical optics. He is also the Managing Editor of the journal

Proceedings of the Greek Mathematical Society. A lively presence throughout the conference, Professor Klingenberg's expertise and engaging style made for a wonderful talk.

Also speaking at this year's event was Dr Vladimir Dotsenko from TCD, Graham Ellis from NUI Galway, Madeeha Khalid from UCD, Eduardo Mota Sanchez from UCC, Arne Ruffer from UL as well as Stephen Buckley and Mark Walsh from Maynooth. A wide range of subjects from Geometry and Topology were covered. At the algebraic end we heard about certain derived categories and group cohomology while at the more geometric analytical end there were talks on constant mean curvature surfaces, quasihyperbolic geodesics and positive scalar curvature. Moduli spaces were a significant theme, with talks on both moduli spaces of vector bundles on K3 surfaces as well as those of stable rational curves.

A stimulating first round of talks on Friday 17th was followed by great conversation and dinner at local Thai restaurant, Kin Khao. The IGC this year was organised by Mark Walsh and David Wraith of Maynooth University. The conference traditionally has no registration fee and this year was no different. It relies mostly on department and university support. The organisers are grateful to Maynooth University and to the Irish Mathematical Society for their financial assistance in hosting this event.

The conference website is at

`https://www.maynoothuniversity.ie/mathematics-and-statistics/irish-geometry-conference-2019`

It gives the details of the programme and has abstracts of all the talks.

Report by Mark Walsh, Mathematics and Statistics, Maynooth University
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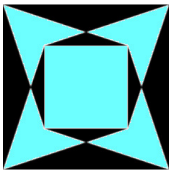
Groups in Galway 2019
10-11 May 2019, NUI Galway

The conference website is at

`http://www.maths.nuigalway.ie/conferences/gig19/`

It gives the details of the programme and has abstracts of all the talks, as well as a photograph of the participants.

Report by the editor,



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Proof-without-words: Markov's inequality

ROBIN E.HARTE, JANE HORGAN AND JAMES POWER

Abstract. We offer a "proof without words" of the inequalities of Markov and Chebyshev

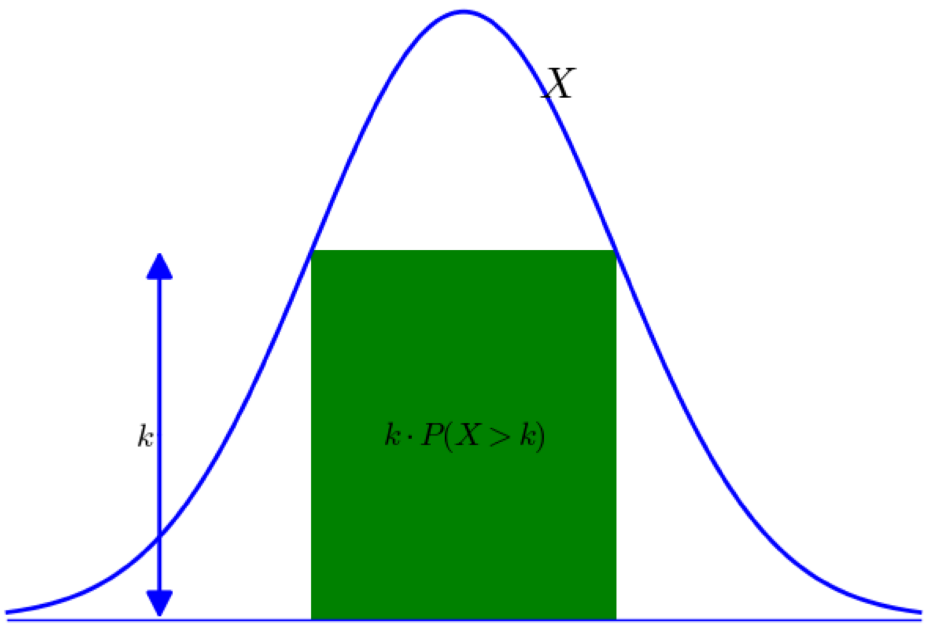

Figure 1. Markov's Inequality

Markov's inequality ([1] §20.1), for a random variable $X : \Omega \to [0, \infty) \subseteq \mathbb{R}$, living on a sample space Ω carrying a probability measure P, says that, if $0 \le k \in \mathbb{R}$,

$$k\ P(X \ge k) \le E(X)\ . \tag{1}$$

Here $E(X)$ is the *expectation* of X. The proof of (1) is rather simple: if

$$F_X : t \mapsto P(X \le t) \tag{2}$$

is the *cumulative distribution* of X, then

$$k\ P(X \ge k) = \int_{t=k}^{\infty} k dF_X(t) \le \int_{t=k}^{\infty} t dF_X(t) \le \int_{t=0}^{\infty} t dF_X(t) = E(X)\ . \tag{3}$$

In our picture, necessarily set in two dimensional real paper, the total "area" under the graph of X is the expectation $E(X)$, while the product $k(P(X \ge k)$ is the area of the

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shaded "rectangle". We do not assume that $\Omega \subseteq \mathbb{R}$, and certainly not that the base of the rectangle is a single real interval.

A proof of Markov's inequality is also a proof of Chebyshev's inequality: Chebyshev's inequality for X is neither more nor less than Markov's inequality for $\varphi(X)$, where, if $0 \le t \in \mathbb{R}$,

$$\varphi(t) = |t - E(X)|^2 \ . \tag{4}$$

In the middle of this writing the third author suddenly left us, far, far too soon: we offer it now as a tribute.

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[1] Jane M. Horgan, Probability with R, Wiley, 2009

Robin E. Harte taught for twenty years at University College, Cork, and retired a long time ago, but somehow never goes away; his ghost can sometimes be seen at the TCD-UCD Analysis Seminar.

Jane Horgan was in at the beginning of NIHE Dublin, now DCU, where she helped to set up the School of Computer Applications, and initiated the Ballymun Access Scheme for admissions to Dublin City University. In some sort of retirement, she has developed an obsession with the Turkish language, and the second edition of *Probability with R* is about to hit the shelves.

James Power taught Computer Science in DCU for a number of years before moving to Maynooth University in 2005. His research centred on parsers and front-end compiler technology for object-oriented languages, and also fuzzy systems. Computer Science in Ireland, and the academic world generally, are diminished by his sudden and untimely death, of a heart attack, in late August 2019.

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On the order of a smallest group with a given representation degree

ROBERT HEFFERNAN AND DESMOND MACHALE

ABSTRACT. We consider the problem of finding the minimal order of a finite group G that has an irreducible complex representation of degree n for small values of n.

It is well known that every finite group G with $k(G)$ conjugacy classes has $k(G)$ inequivalent irreducible complex matrix representations of degrees d_i, $1 \leq i \leq k(G)$, and that the degree equation

$$\sum_{i=1}^{k(G)} d_i^2 = |G|$$

holds [3, Cor. 2.7]. In this note we ask the question: For each n, what is the order $f(n)$ of a smallest group G with an irreducible complex representation of degree n?

For small n, the answer is provided by the following table:

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	6	12	20	55	42	56	72	144	110
n	11	12	13	14	15	16	17	18	19	20
$f(n)$	253	156	351	336	240	272	1751	342	3420	500
n	21	22	23	24	25	26	27	28	29	30
$f(n)$	672	506	1081	600	2525	702	1512	812	1711	930
n	31	32								
$f(n)$	992	1440								

The purpose of this note is to discuss and justify the entries in this table. For very small values of n we can proceed by hand but, as n increases, more theory is needed. As n becomes larger, we make extensive use of the Small Groups library, which we access using GAP [1].

It is well known that each d_i is a divisor of $|G|$ [3, Thm. 3.11], and the number of d_i equal to 1 is precisely $|G : G'|$, the index of the commutator subgroup of G [3, Cor. 2.23]. Moreover, if A is an abelian normal subgroup of G, then $d_i \leq |G : A|$ [3, Thm. 6.15].

If $n = 1$, then the answer is clearly the trivial cyclic group C_1. So $f(1) = 1$.

From now on all the groups we consider are nonabelian, since for all abelian groups, $d_i = 1$ for all i.

If $n = 2$, then since d_i divides $|G|$ and $\sum d_i^2 = |G|$, we have $|G| \geq 2^2 + 2 = 6$. Luckily, there is a nonabelian group of order 6, S_3, with degree equation

$$6 = 1^2 + 1^2 + 2^2,$$

and so S_3 has an irreducible representation of degree 2. S_3 is the unique nonabelian group of order 6 with this property. So $f(2) = 6$.

In general we can say that $f(n) \geq n^2 + n$ as $f(n)$ is a multiple of n, $n^2 < f(n)$ and G has a trivial degree $d_1 = 1$.

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Next, consider $n = 3$. We know that $f(3) \geq 3^2 + 3 = 12$ and among the groups of order 12 there is just one, A_4, with degree equation

$$12 = 1^2 + 1^2 + 1^2 + 3^2.$$

So, $f(3) = 12$.

If $n = 4$, then $f(4) \geq 4^2 + 4 = 20$, and there is a unique group of order 20, namely $\mathrm{Hol}(C_5)$, with degree equation

$$20 = 1^2 + 1^2 + 1^2 + 1^2 + 4^2$$

and so $f(4) = 20$.

If $n = 5$, then $f(5) \geq 5^2+5 = 30$. But every group of order 30 has a normal subgroup of order 15, which is abelian, so $d_i \leq \frac{30}{15} = 2$ for all i. However, 5 divides the minimal $|G|$, so

$$|G| = 35, 40, 45, 50, 55, 60, \cdots$$

Now, Sylow theory easily gives that groups of order 35 and 45 are abelian, so

$$|G| = 40, 50, 55, 60, \cdots.$$

Diophantine analysis can be used to rule out 40 and 50; For example, if $40 = \sum d_i^2 + 5^2$, we can have only one representation of degree 5, and none of degree 4, since $25 + 16 = 41 > 40$. Thus the other representations are of degrees 1 or 2, the only allowable divisors of 40. Thus the degree equation becomes

$$x + 4y + 25 = 40$$

or

$$x + 4y = 15,$$

which turns out to have no viable solutions, given that d_i must divide 40. Similarly, every group of order 50 has an abelian subgroup of order 25 and index 2, which forces d_i to be at most 2, for all i. So, 50 is ruled out as a possibility.

Now consider in general the case where $p < q$ are odd primes and p divides $q - 1$. It is well known that in this case there is a unique nonabelian group G of order pq which has $k(G) = p + \frac{q-1}{p}$ conjugacy classes, and $|G : G'| = p$. The degree equation of this group is easily seen to be

$$pq = p + \left[\frac{q-1}{p}\right] p^2,$$

and G has a representation of degree p. This gives in general an upper bound for $f(p)$ where p is an odd prime: find a prime q with p dividing $q - 1$. Then $f(p) \leq pq$. So we see finally that $f(5) = 55$. In like manner we find that $f(11) = 11 \cdot 23 = 253$.

In fact, according to James and Liebeck [4], we have the following: let q be a prime and let p divide $q - 1$, where p is not necessarily a prime, and let $r = (q - 1)/p$. Then there is a group G of order qp with $|G : G'| = p$ and $k(G) = p + r$ with r irreducible representations of degree p. So, $f(p) \leq qp$.

If $n = 6$, then $f(6) \geq 6^2 + 6 = 42$. Luckily, there is a unique group of order 42 with degree equation

$$42 = 6 \cdot 1^2 + 6^2,$$

which has a representation of degree 6. So $f(6) = 42$.

Notice that $f(5) = 55 > 42 = f(6)$, so that the function $f(n)$ is not in general increasing.

If $n = 7$, then $f(7) \geq 7^2 + 7 = 56$ and there is a group of order 56 with degree equation

$$56 = 7 \cdot 1^2 + 7^2$$

and so $f(7) = 56$.

If $n = 8$, then $f(8) \geq 8^2 + 8 = 72$ and there is a group of order 72 with an irreducible representation of degree 8. So, $f(8) = 72$.

Now we introduce some heavier machinery. See Sloane's integer sequence A220470 [2] for details.

(i) $f(n) = n^2 + n$ if and only if $n + 1$ is a prime or a power of a prime. This is consistent with the results above and means we can write down the values of $f(n)$ for $n = 10, 12, 15, 16, 18, 22, 24, 26, 28, 30$ and 31. See Harden [2].

(ii) An upper bound for $f(n)$ in general is given by nq^n where q^n is the smallest prime power which is congruent to 1 modulo n. This is because the group of affine transformations $x \mapsto ax + b$ from the finite field $\mathrm{GF}(q^n)$ to itself, where $a^n = 1$ and b is an arbitrary element of $\mathrm{GF}(q^n)$, has order nq^n and has a representation of degree n.

(iii) $f(n)$ is a sub-multiplicative function, i.e. $f(ab) \leq f(a)f(b)$ because if A has a representation of degree a and B has a representation of degree b, then $A \times B$ has a representation of degree ab.

Now, if $n = 9$, then since 10 is not a prime power, $f(9) > 9^2 + 9 = 90$. By the above, $f(9) \leq f(3)f(3) = 12 \cdot 12 = 144$. GAP can be used to rule out values of $|G|$ between 90 and 144, so $f(9) = 144$. We again note that $f(9) = 144 > 110 = f(10)$. Indeed, there are infinitely many instances of this phenomenon.

The remaining values in the table can be filled in using GAP, but the values for $n = 17$ and $n = 19$ have also been derived by Harden [2] using representation theory and extensive non-trivial calculations.

To find values of $f(n)$ using GAP we can simply search through nonabelian groups in the Small Groups library whose orders are multiples of n greater than or equal to n^2+n looking for a group with a character of degree n. For small n this works reasonably well but in some cases, such as $n = 32$, the large number of groups to be considered becomes an issue. For instance, there are 1,060,391 nonabelian groups of order 1280 and we must compute the character degrees of each of these in turn to rule out 1280 as a possible value for $f(32)$. This computation does not take long for an individual group, but when such a large number of groups must be checked this approach is impractical. However, an elementary result in character theory [3, Cor. 2.30] says that $d_i^2 \leq |G : Z(G)|$ and so, in particular, $n^2 \leq |G : Z(G)|$. Checking this condition for a given group G is generally much quicker than computing the character degrees, allowing us to find $f(32)$ in a reasonable amount of time. We know that $f(32) \geq 32^2 + 32 = 1056$ and, by (iii) above, we can also say $f(32) \leq f(4)f(8) = 20 \cdot 72 = 1440$. We can now inspect orders that are multiples of 32 between these two bounds to find that $f(32) = 1440$.

We note that there exist n for which two or more groups realise $f(n)$. For example, small groups 72/39 and 72/41 both have a representation of degree 8. Other examples occur for $n = 20, 21, 24$ and 32.

The result that if $n + 1$ is prime or a prime power then $f(n) = n(n + 1)$, has some interesting connections with several difficult and unsolved problems in number theory:

(a) Sophie Germain primes. If p is a prime such that $2p+1$ is also prime, then $f(2p) = 2p(2p + 1)$. Since $f(2p) \leq f(2)f(p) = 6f(p)$, we have $p(2p + 1)/3 \leq f(p)$.

(b) Mersenne primes. If p is a prime such that $2^p - 1$ is also prime, then $f(2^p - 1) = (2^p - 1)(2^p)$. In fact in general, $f(2^n - 1) = (2^n - 1)(2^n)$.

(c) Fermat primes. If $2^n + 1$ is prime, it is known that $n = 2^k$, for some natural number k. Then $f(2^n) = (2^n)(2^n + 1)$.

We conclude with a number of questions:

(1) Is it possible to have $f(a) = f(b)$ for different values of a and b?

(2) Can we have arbitrarily long sequences where $f(n)$ is decreasing?

(3) Are there infinitely many primes p for which $f(p) = pq$, where q is the smallest prime such that p divides $q-1$? We note that many of the values of $f(n)$ which we have found arise from Frobenius groups, such as these groups of order pq. However, we do not know of any conceptual reason why this should be the case

(4) Is it true that a smallest group with a representation of degree n, will always have trivial centre? This is true for all the cases we have presented.

Some of the results in this paper were presented at the Munster Groups conference held at UCC, Cork in September 2018.

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Student from Ireland wins Silver Medal at the 60th International Mathematical Olympiad

BERND KREUSSLER

The 60^{th} International Mathematical Olympiad (IMO) took place in Bath, United Kingdom, from 11–22 July 2019. A total of 621 students (65 of whom were girls) participated from 112 countries. These numbers make it the IMO with the largest participation so far.

The Irish delegation consisted of six students (see Table 1) accompanied by Gordon Lessells (Deputy Leader, UL), Bernd Kreussler (Team Leader, MIC Limerick), and Andrew Smith (Observer A, UCD).

Name	School	Year
Lucas Bachmann	Glenstal Abbey School, Murroe, Co. Limerick	6^{th}
Tianyiwa Xie	Alexandra College, Milltown Road, Dublin 6	5^{th}
Linhong Chen	The Institute of Education, Lower Leeson St, Dublin 2	5^{th}
Alex Hanley	Lucan Community College, Lucan, Co. Dublin	5^{th}
Laura Cosgrave	Midleton College, Midleton, Co. Cork	5^{th}
Yunjie Wang	Christian Brothers College, Wellington Road, Cork	5^{th}

Table 1. The Irish contestants at the 60^{th} IMO

1. Team selection and preparation

The team detailed in Table 1 consisted of those six students (in order) who scored highest in the Irish Mathematical Olympiad (IrMO), which was held for the 32^{nd} time on Saturday, 11^{th} May, 2019. The IrMO contest consists of two 3-hour papers on one day with five problems on each paper. The students who participated in the IrMO sat the exam simultaneously in one of five *Mathematics Enrichment Centres* (UCC, UCD, NUIG, UL and MU). This year, a total of 86 students took part in the IrMO, 32 of whom were girls. The top performer is awarded the *Fergus Gaines cup*; congratulations to Lucas Bachmann, who achieved this honour in IrMO 2019.

The students who participate in the IrMO usually attend extra-curricular Mathematics Enrichment classes, which are offered at the five Mathematics Enrichment Centres listed in the previous paragraph. These classes run each year from January until April and are offered by volunteer academic mathematicians from these universities or nearby third-level institutions. More information on the organisation of these classes, as well as links to the individual maths enrichment centres, can be found at the Irish Maths Enrichment/IrMO website `http://www.irmo.ie/`.

The selection and training for IMO 2019 followed procedures which are by now well-established. First, an Irish Maths Olympiad "Squad" was identified, consisting of the top performers in IrMO 2018 who were eligible to qualify for the Irish IMO team in 2019. For these students, a number of training camps was organised. Such training camps are very important, as during these mathematically intense 3–5 day events, students

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have the opportunity to socialise with their peers, exchange their mathematical ideas, and increase their motivation for their work throughout the year. A kick-start training camp, organised by Anca Mustata, was held in UCC from 22–25 August 2018. Classes were conducted by Eugene Gath, Conall Kelly, Claus Koestler, Bernd Kreussler, Declan Manning, Anca, Andrei and Anna Mustata, Cillian O'Doherty and Steve Wills.

Between the end of the UCC kick-start training camp and the beginning of the 2019 Mathematics Enrichment classes, the members of the Irish IMO Squad participated in the "remote training" programme, which operates as follows. At the beginning of each month from September to December inclusive, two sets of three problems are emailed to the participating students. They return their solutions or attempts by email to the proposer of the problems before the end of the month. The problem proposer then provides feedback on their work, as well as full solutions. This programme is very important for the successful engagement of "returning" students, and helps to develop the students' independence in mathematical problem solving. In 2018, 24 students comprised the Irish IMO Squad. The eight trainers involved in the remote training were Mark Flanagan, Eugene Gath, John Murray, Anca Mustata, Andrei Mustata, Prasanna Ramakrishnan, Harun Siljak and Andrew Smith.

Each year in November, the Irish Mathematical Olympiad starts with *IrMO Round 1*, a contest that is held in schools during a regular class period. In 2018, more than $14,000$ students, mostly in their senior cycle, participated in Round 1. Teachers were encouraged to hand out invitations to their best performing students to attend the mathematics enrichment classes in their nearest mathematics enrichment centre.

Having participated in other Mathematical Olympiads before is an advantage for students when they participate in the IMO. In the past six years, opportunities to do so have been created for members of the Irish Maths Olympiad Squad. In the current year, in addition to the possibility to compete in the European Girls' Mathematical Olympiad (EGMO), which is for female students only, the members of the Irish Maths Olympiad Squad were invited to participate in the Iranian Geometry Olympiad (6 September 2018), as well as the British Mathematical Olympiad Round 1 (30 November 2018) and Round 2 (24 January 2019). The exams in these three Olympiads are sat by the students at one of the five Enrichment Centres; no travel abroad was necessary. Thanks to the organisers of the IGO and to the UKMT, and in particular Geoff Smith, for giving our students these opportunities.

For all students who participate in enrichment classes, not only for the Squad members, each of the five maths enrichment centres hosts a local contest for the students, which takes place in February or March (each local contest is specific to its enrichment centre).

A number of training camps were organised in advance of IMO 2019. For the 2019/2020 Squad, to which five of the six members of the Irish team for IMO 2019 belong, a camp was held at Mary Immaculate College, Limerick, from 5–7 June 2019. At this camp, students gained exam experience in a $3\frac{1}{2}$ hours IMO-style exam in which they had to solve 3 problems. Two further training camps for the members of the Irish IMO team, featuring similar practice exams, were held in Limerick: at Mary Immaculate College from 2–4 July 2019 and at the University of Limerick from 9–12 July 2019. The second of these camps was a joint camp with the IMO team from Trinidad and Tobago. The camps were organised by Bernd Kreussler and Gordon Lessells. The sessions at these camps were conducted by Mark Burke, Mark Flanagan, Ronan Flatley, Eugene Gath, Bernd Kreussler, Jim Leahy, Gordon Lessells, Anca Mustata, Andrei Mustata, Anna Mustata, Jagdesh Ramnanan and Andrew Smith.

2. The days in Bath

The Jury of the 60$^{\text{th}}$ IMO gathered at the Celtic Manor Resort near Newport in South Wales. The Team Leader and the Observer A travelled to the Celtic Manor on Thursday, 11 July. The Jury, which is composed of the Team Leaders of the participating countries and a Chairperson who is appointed by the organisers, is the prime decision making body for all IMO matters. Its most important task is choosing the six contest problems out of a shortlist of 32 problems provided by the IMO Problem Selection Committee, also appointed by the host country.

This year's Chairperson of the IMO Jury was Prof. Adam McBride, who carried out the same role also in 2002, the last time the IMO took place in the United Kingdom. He chaired the Jury meetings in a gentle yet very efficient manner, without sitting down for a single minute.

During the first Jury meetings, Leaders articulated their first impressions about the merits and beauty of all the shortlisted problems. Four of the easier problems needed to be removed from the shortlist, because they were too similar to problems that were used in other competitions in the past. Like in recent years, the Leaders felt that there was a lack of sufficiently many suitable easy problems on the shortlist. An appeal was made to submit more problems that may fit in this category for future IMOs. After intense discussion and debate, in the early afternoon of Saturday, the six problems for this year's IMO paper were selected.

It was the seventh year in a row that a problem selection protocol was followed whereby one problem from each of the four areas (algebra, combinatorics, geometry and number theory) would be included in problems 1, 2, 4 and 5. This protocol has the principal advantage of ensuring a balance between the four areas among the less difficult problems in the contest. As is now standard procedure at the IMO, an electronic voting mechanism was used during the Jury meetings, ensuring both efficiency and anonymity in voting procedures.

The opening ceremony of IMO 2019 took place on the 15$^{\text{th}}$ of July in The Forum in Bath, a former cinema built in the 1930s, which is now used as a venue for concerts and other large events as it has the largest seating capacity in Bath. There were two very short speeches, one by Geoff Smith, the President of the IMO Advisory Board, and the other by the Deputy Director of the UK Mathematics Trust, Stephen O'Hagan. The main part of the opening ceremony consisted of the parade of the teams. The Master of Ceremony of this less-than-one-hour event was a DJ who tried to entertain the contestants.

The two IMO contest exams took place on the 16$^{\text{th}}$ and 17$^{\text{th}}$ of July, starting at 8:30 each morning. During the first 30 minutes of the exams each day, students can ask questions regarding the IMO paper. Such questions can resolve ambiguities and ensure that students understand clearly the formulation of any contest problem. The answers are composed to resolve these difficulties, without providing any hint as to how to solve the problem. The questions of the students were scanned and sent to the leaders' site, from where the answers are returned in the same way. This year's Q&A sessions were very efficient. On each day, 34 students had questions and these were answered by 9:15, this is only 45 minutes after the start of the contest.

The students' scripts from Day 1 became available at 8pm on the evening of the first day of the contest. After an initial brief study of the scripts it seemed that Lucas had solved all three problems on Day 1, an exciting surprise. Also, Tianyiwa's and Alex's solutions to Problem 1 looked promising. On Day 2, Andrew, Gordon and myself met with the team directly after the contest. Thereafter we began a detailed study of the scripts of the second day.

The final marks for each contestant are agreed in a process known as coordination. This important part of the IMO is well-established and ensures that the scripts of the students from so many different nations are marked fairly and consistently. The decisions in this process are based on very detailed and strict marking schemes which were prepared by the coordination teams, presented and defended to the Jury by the problem captains and agreed by the Jury – in some cases after significant changes to the first draft.

The marking of the scripts of each participating country is undertaken by two independent groups. One group consists of the Team Leader, the Deputy Leader and the Official Observer. The second group consists of the coordinators, who were appointed by the local organisers. This year's coordination schedule for our team was particularly tight: the half-hour meetings with the coordinators for problems 1, 2, 4 and 5 were scheduled to take place on the day immediately after the contest.

The help of Andrew as Observer A was essential to get us prepared in time for the coordination sessions. In total we had to study 227 pages of solutions and rough work of our contestants. In preparation for the coordination meetings, we needed to have a full understanding of the solution or attempts of each of our six students so that we could explain the merits of the students' work to the coordinators.

Each coordinator works on one problem only, but has to look at the solutions of the students from almost 20 teams, more than 100 students. Even though we had more time than the coordinators per problem and student, in most cases we came to the same conclusion regarding the points to be awarded. Due to the leadership of the chief coordinator, Imre Leader, and the professionalism of all coordinators, the coordination process went very smoothly at this year's IMO.

Problem 1, a functional equation, was solved in an efficient and straightforward manner by Lucas. Tianyiwa's solution was more complicated to understand. To agree on seven points with the coordinators, we needed to explain her writings in detail. We were able to do so successfully in the short time available to us thanks to the presence of an Irish Observer A. Alex's work on this problem fell short of a full solution because he did not make the connection between a usefully simplified form of the functional equation and the linearity of the solution function. This was a narrowly missed Honourable Mention.

Problem 4, a number theory problem, was more difficult than usual. Laura was the only member of the Irish team who solved this problem. Reading through the 22 pages she submitted for this problem was enjoyable, because she combined in a clever way estimates coming from the 2-adic and the 5-adic valuations of both sides of the equation. As a result, she only needed to check a few cases for small k by direct calculation. We needed to point the coordinators to a particular detail in Laura's solution to save her full marks for this question. Again, the availability of an Irish Observer A made it possible for us to enter the coordination meeting with such a detailed preparation.

During the two days of coordination, excursions and other activities were organised for the students. Our students enjoyed mostly the trip to Oxford on Friday, 19$^{\text{th}}$ July, where Andrew Wiles gave a lecture to them.

The final Jury meeting, at which the medal cut-offs were decided, took place on Friday evening. At this meeting, the chief invigilator, Jeremy King, reported an irregularity: at the end of the exam at 1pm on Day 1, two students continued to write their solutions even after repeatedly being told to stop doing so. The Jury decided to set the score for their best question on Day 1 to zero.

The closing ceremony was held on Sunday, 21$^{\text{st}}$ July, followed by a Fun Fair and a Farewell Banquet with live music that evening. The team returned to Ireland on 22$^{\text{nd}}$ July.

3. The problems

The two exams took place on the 16th and 17th of July, starting at 8:30 each morning. On each day, $4\frac{1}{2}$ hours were available to solve three problems.

First Day

Problem 1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

(South Africa)

Problem 2. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

(Ukraine)

Problem 3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

Second Day

Problem 4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4)\cdots(2^n - 2^{n-1}).$$

(El Salvador)

Problem 5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

(USA)

Problem 6. Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

(India)

4. The results

The Jury tries to choose the problems such that Problems 1 and 4 are the most accessible, while Problems 2 and 5 are more challenging. Problems 3 and 6 are usually the most difficult problems, whose existence on the paper is justified in posing a sizeable challenge even to the top students in the IMO competition. Table 2, which shows the scores achieved by all contestants on the 6 problems, illustrates that this gradient of difficulty was generally maintained this year also. However, comparing average scores it can be said that Problem 4 was slightly harder and Problem 5 much easier than problems 4 and 5 have been in the past decade.

	P1	P2	P3	P4	P5	P6
0	73	251	520	211	156	558
1	65	135	46	63	20	25
2	6	30	3	4	168	7
3	24	6	6	7	12	0
4	14	6	5	13	5	1
5	5	3	9	19	7	0
6	52	92	4	47	3	3
7	382	98	28	257	250	27
average	5.179	2.399	0.572	3.736	3.567	0.403

Table 2. The number of contestants achieving each possible number of points on Problems 1–6

The medal cut-offs were as follows: 31 points needed for a Gold medal (52 students), 24 for Silver (94 students) and 17 for Bronze (156 students). A further 144 students were awarded an Honourable Mention (an Honourable Mention is awarded to any student who did not win a medal, but achieved 7 points out of 7 on at least one problem). Overall, 37.8 % of the possible points were scored by the contestants, which is one percent more than last year. A higher percentage of the possible points was achieved only at two IMOs in the past 20 years, in 2004 and 2014.

Table 3 shows the results of the Irish contestants. The team scored a total of 61 points, the fifth best score of an Irish team at the IMO. Lucas Bachmann won a Silver medal, having completely solved four problems – this is a fantastic achievement. Also, Tianyiwa Xie and Laura Cosgrave won an Honourable Mention for their complete solutions to Problems 1 and 4, respectively.

The figures in Table 4 have the following meaning. The first figure after the topic indicates the percentage of all points scored out of the maximum possible. The second number is the same for the Irish team and the final column indicates the Irish average score as a percentage of the overall average. This year the relative performance of the Irish team on problems 1 and 4 was not as good as in the past three years.

Name	P1	P2	P3	P4	P5	P6	total	relative ranking	award
Lucas Bachmann	7	7	7	1	7	0	29	91.29 %	Silver Medal
Tianyiwa Xie	7	1	0	1	2	0	11	37.90 %	Hon. Mention
Laura Cosgrave	1	0	0	7	0	0	8	28.55 %	Hon. Mention
Alex Hanley	4	0	0	1	2	0	7	25.48 %	
Linhong Chen	1	2	0	0	2	0	5	22.90 %	
Yunjie Wang	1	0	0	0	0	0	1	13.06 %	

Table 3. The results of the Irish contestants

Problem	topic	all countries	Ireland	relative
1	algebra	74.0	50.0	67.6
2	geometry	34.3	23.8	69.5
3	combinatorics	8.2	16.7	204.1
4	number theory	53.4	23.8	44.6
5	combinatorics	51.0	31.0	60.7
6	geometry	5.8	0.0	0.0
all		37.8	24.2	64.1

Table 4. Relative results of the Irish team for each problem

It is also worth mentioning here that some young Irish mathematicians won awards this year in Mathematical Olympiads other than IMO. At the European Girls' Mathematical Olympiad (EGMO) 2019 in Kyiv, Ukraine, Tianyiwa Xie won a Bronze Medal and Laura Cosgrave and Yixin Huang won Honourable Mentions. In addition, Lucas Bachmann won a Bronze medal at the 5^{th} Iranian Geometry Olympiad (IGO) in September 2018.

This year, six students achieved a “perfect score” (42 points) at the IMO. Although the IMO is a competition for individuals only, it is interesting to compare the total scores of the participating countries. This year's top teams were China and the USA (both 227 points) closely followed by South Korea (226 points). Ireland, with 61 points in total, came in 71^{st} place, which corresponds to a relative ranking of 36.94%. This is the fourth best relative ranking an Irish team achieved since the start of its involvement with the IMO in 1988. Three of the four top relative rankings of the Irish team were achieved within the past 6 years.

The detailed results can be found on the official IMO website, which is located at `http://www.imo-official.org`.

5. Outlook

The next countries to host the IMO will be

2020	Russian Federation	8–18 July
2021	United States of America	7–16 July
2022	Norway	6–16 July
2023	Japan	2–13 July

6. Conclusions

The outstanding result of this year's IMO, from an Irish perspective, is Lucas Bachmann's Silver Medal. This is the second Silver Medal ever achieved by an Irish student – the first one was won by Fiachra Knox in 2005. Lucas' success was reported in newspapers such as the Irish Times, the Limerick Leader and the Limerick Post.

This is the third consecutive year in which the Irish team came home with at least one medal, this has not happened before in the history of Irish participation in the IMO. More than half of all Honourable Mentions achieved by Irish IMO contestants since Ireland's first IMO participation in 1988 were achieved within the last seven years. This is evidence that while there are fluctuations in performance year on year, a generally sustained team-level improvement can be detected within the last few years. The extra effort being invested in training activities in the last few years shows a clear correlation with this improvement.

It is of primary importance that sufficient funding becomes available for the activities detailed above, in particular for the training camps. An increased level of funding would also allow the scope of these initiatives to be widened further, so that the performance of Irish students in international mathematics contests can continue to improve year on year.

It is interesting to note that the four top relative rankings of the Irish team at the IMO were achieved either when all six team members got an award (2014, 2017), or one team member received a Silver Medal (2005, 2019). One obvious conclusion from this observation is that to improve the performance of the Irish team we need to increase efforts to enable a larger number of potential Irish contestants to perform at an internationally competitive level.

In recent years, initiatives have been started all over Ireland that aim at involving Junior Cycle students in problem-solving activities. The most noteworthy are Junior Maths Enrichment programmes, the PRISM (Problem Solving for Post-Primary Schools) competition and the Maths Circles initiative. Some of these activities are externally funded and their continuation depends on the availability of future funding. It seems essential for the long-term improvement of Irish teams at the IMO that the problem-solving activities offered for younger students are maintained or even extended, because students who become involved in problem-solving activities at an earlier age have a much enhanced probability to reach an internationally competitive level.

The sending of a full team of six students, together with Leader, Deputy Leader and Observer, to the IMO contest requires sustained funding. It would be very beneficial for the team leadership at future IMOs if the practice of sending an Irish Observer to the IMO could be continued in subsequent years.

7. Acknowledgements

Ireland could not participate in the International Mathematical Olympiad without the continued financial support of the Department of Education and Skills (DES), which is gratefully acknowledged. Thanks to its Minister, Joe McHugh TD, and the members of his department, especially Matthew O'Reilly-Kavanagh, for their continuing help and support. Also, thanks to the Royal Irish Academy, its officers, its Physical, Chemical and Mathematical Sciences Committee, and especially Marie Coffey, for support in obtaining funding.

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Financial support for the IMO-team training camps came from the Mary Vesey Fund at the Community Foundation for Ireland. This is gratefully acknowledged.

The principal foundation for the success of the contestants is the work done with the students in the Maths Enrichment Programmes at the five universities. This work is carried out for free by volunteers in their spare time. Thanks go to this year's trainers at the five Irish centres:

- At UCC:
  Tom Carroll, Michael Cronin, David Goulding, David Henry, Philipp Hoevel, Conall Kelly, Arundhathi Krishnan, Declan Manning, Anca Mustata, Andrei Mustata, Jacob Bennett Woolf.
- At UCD:
  Kazim Buyukboduk, Anthony Cronin, Fabio Deelan Cunden, Mark Flanagan, Marius Ghergu, Mary Hanley, Kevin Hutchinson, Samuel Johnston, Thomas Laffey, Myrto Manolaki, Gary McGuire, Harun Siljak, Helena Smigoc, Andrew Smith.
- At NUIG:
  John Burns, Angela Carnevale, Graham Ellis, Niall Madden, Goetz Pfeiffer, Kirsten Pfeiffer, Rachel Quinlan, Tobias Rossmann, Nina Snigireva.
- At UL:
  Mark Burke, Ronan Flatley, Mary Frawley, Sarah Frawley, Eugene Gath, Bernd Kreussler, Jim Leahy and Gordon Lessells.
- At MU:
  Stefan Bechtluft-Sachs, Jacquie Birkett, Stephen Buckley, Peter Clifford, Rafael de Andrade Moral, Liam Jordon, David Malone, Ollie Mason, Jack Mc Donnell, John Murray, Anthony O'Farrell, Adam Ralph, David Redmond, Mark Walsh.

Many thanks to all those involved in the training camps. Thanks also to the above named universities for permitting the use of their facilities in the delivery of the national Maths Enrichment Programme, and especially to the University of Limerick and to Mary Immaculate College, Limerick, for their continued support and hosting of the pre-olympiad training camps. Finally, thanks to the hosts for organising this year's IMO in the United Kingdom and especially to the team guide in Bath, Tasos Stylianou.

Bernd Kreussler is Lecturer in Mathematics at Mary Immaculate College Limerick. Since 2006 he is actively involved in the delivery of Mathematics Enrichment Classes, in organising the Irish Maths Olympiad and in the training of talented second level students for maths competitions at national and international level.

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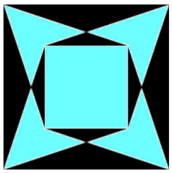

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Reflections of Retired UCC Mathematicians

GARY MCGUIRE AND COLM MULCAHY

Abstract. Six decades of UCC history as remembered by five mathematicians

1. Introduction

A good overview of mathematical life at UCD over a several decade period was provided by a series of three articles published in this Bulletin (Nos 63 and 64) in 2009. Those were based on interviews conducted by the first author with three colleagues of his who had just retired, namely Tom Laffey, Seán Dineen and Dave Lewis.

This article can be viewed as a Cork counterpart to those. We emailed questions to numerous retired University College Cork mathematicians and mathematical physicists, with invitations to address other things they wish they'd been asked about. Here we feature the five responses received, from Finbarr Holland (on the UCC staff 1965-2005), Michael Mortell (1973-2006), Donal Hurley (1973-2009), Des MacHale (1972-2011), and Patrick Fitzpatrick (1979-2013). These academics got their primary degrees in 1961, 1961, 1965, 1967 and 1973, respectively. The first three are Corkmen through and through, who also did their undergraduate and master's studies at UCC; the other two hail from Mayo and Belfast, respectively. All did their doctoral work outside Ireland. While they are officially retired, they remain active in various ways. (An extensive 2017 inteview conducted by the second author with Vincent Hart, who spent the early part of his career at UCC, appeared in issue 79 of this Bulletin).

Finbarr Holland, one of the founding fathers of the Irish Mathematical Society, sets the stage with a focus on the striking transformation of the UCC maths department over the course of the 1960s. The story is then taken up by his applied maths classmate Mick Mortell, who later served as UCC president, and younger men who became their colleagues, including Des MacHale who designed the distinctive IMS logo.

2. Finbarr Holland

What were your early interests, and who were your teachers of note?

FH: I was born at home, 10 Fernside Villas, Cork city, on May 12, 1939. My parents came from farming stock, and grew up in Barryroe and Grange, near Timoleague in West Cork, where the Hollands were known as "*Fir na leabhair*". One of the clan, Rev. W. Holland, PP, wrote "History of West Cork and the Diocese of Ross", a book that is well regarded by historians of the area. After their marriage, my parents settled in Cork city. My father became an insurance agent; he enjoyed quizzes, crosswords and Gaelic football. I received my early education first from Presentation nuns, and then Presentation brothers and a small sprinkling of lay teachers, at Scoil Críost Rí, Turners Cross. In my final years there, I attended what was loosely described as a secondary top, and was fortunate in have been taught mathematics at a level a little

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beyond Leaving Cert standard by Mr Con O'Keefe — Concubhuir (Con) Ó Chaoimh, who afterwards joined the Inspectorate of the Department of Education, where he left his mark.

After doing my Leaving Cert in 1957, I attended the Sharman and Crawford Technical Institute to study the first part of a Diploma Course in Industrial Science, which was designed to cater for future technicians, soon to be needed to operate the emerging oil refinery at Whitegate. Simultaneously, I studied on my own for the UCC Entrance Scholarship examination, and a special scholarship awarded very infrequently by the City of Cork Vocational Committee. In preparation for the former, I often met Con on his way home, hunched over a racing bike with a pipe dangling from his mouth — cycling off to infinity according to the class wit! We would chat at the side of the road to discuss scholarship questions, such as: what's the smallest integer bigger than $(2+\sqrt{3})^n$? Con was doing an MA by thesis under Paddy Kennedy at the time, and when I pointed out an error to Con in one of Kennedy's scholarship questions, and supplied a correction, he told Kennedy who apparently replied "*Nutet et Homer*". Needless to say, that pleased me no end!

As a recipient of both of the scholarships I applied for, I went to UCC where, for the next three years, I read mathematics and mathematical physics, and qualified for a BSc with first class honours. As well, during that time, I taught mathematics courses by day and night at the Crawford Tech.

How did your 1964 PhD at Cardiff with Lionel Cooper come about and how did you end up back at UCC?

FH: In my postgraduate year, I decided to sit the examination for the NUI Travelling Studentship, which was being offered that year. In those days, it was customary for this to be offered — if at all — only every two years, and prospective candidates at UCC, if they were lucky with the cycle, usually took two years to prepare for it. Hence, if I didn't sit the TS exam in my first postgrad year I would have had to wait for three years before my next opportunity, something I couldn't afford to do. So, I decided to prepare for the TS in one year, which I did with the help of Paddy Barry who laid on extra courses. (The same year a similar decision to attempt the TS in one year was made by Martin Newell in UCG and John Galvin in UCD.)

I revealed my desire to do a PhD to the then NUI mathematics extern, Lionel Cooper, during his summer visit. I also revealed my intention to put the Dirac delta function, which I had learned about from Paddy Quinlan, on a firm foundation. Cooper told me that this had already been achieved by Laurent Schwartz! However, he offered to supervise me, and once I had secured a Travelling Studentship, I set off for Cardiff to work on a thesis under him. This largely involved a generalization of Bochner's representation theorem on continuous positive-definite functions to functions that were only locally square-summable. I was awarded my PhD from the National University of Wales in 1964, and the same year accompanied Cooper to Caltech where I served for one year as a research fellow. I returned to take up a position at UCC in 1965.

The emphasis was largely on teaching in those days, correct?

FH: As a temporary lecturer, I joined a staff of two permanent members, Paddy Barry, as professor — by then Paddy Kennedy was the first professor of mathematics at the new University of York — and Siobhán O'Shea, the sole statutory lecturer. We were assisted by Freddie Holland (no relation) who taught one course to engineers, and a couple of postgraduates who were appointed as student demonstrators to teach two courses throughout the year in exchange for free tuition. At that time, although the total student population was small, a full range of courses was taught which placed a heavy teaching and examining burden on all the staff. First year students doing either pre-med, pre-dent, commerce or ag science received basic level courses in mathematics. Civil and electrical engineering students received mathematics courses in their first three years, as did arts and science students at pass and honours level. Pass courses were examined in June with repeats in September, when the honours courses were also examined. In addition to examining for the NUI, we were expected to set and mark one or two entrance scholarship papers for the UCC entrance scholarship examination, and assist with the marking of the matriculation examination both of which were generally held in late July after the Leaving Certificate examination. Naturally, this encroached on our research time which was very limited during term time.

At the time I joined UCC, before they could be recommended for appointment to the Senate of the University, candidates for a statutory lectureship in mathematics in UCC were expected to display proficiency in both written and oral Irish in a test conducted by professors in the subject. I took the requisite test, but failed it. However, within a couple of years of this unhappy event, the relevant statute was amended, and when I applied a second time, I was successful and appointed to the position of Statutory Lecturer in Mathematics Number 2 by the Senate of the NUI. The offending statute was completely abolished in 1974.

Yet, growth was just around the corner, both in terms of curriculum and personnel?

FH: In my first year back I taught a postgraduate course on functional analysis to a small group of students that included Donal Hurley and David Walsh, who went on to have successful careers in mathematics. With Paddy Barry's assent, I revamped the undergraduate programme and began by introducing metric spaces and the Lebesgue integral in third year. l was given free rein over the taught postgraduate courses until staff numbers increased and four-year degree courses for honours mathematics became the norm, which led largely to the demise of taught postgrad courses.

The year 1966 was significant for a few reasons: Lennart Carleson proved Luzin's 1915 conjecture that the Fourier series of a continuous function converged to the function not only in the sense of Cesàro summability in a uniform manner, but almost everywhere in a pointwise manner. This result had an immediate consequence for me because it signalled the end of a line of research connected with Luzin's problem that had been driving the subject of harmonic analysis, my main research interest at the time. It caused me to take up the study of Toeplitz and Hankel operators. Later on, I was able to advise David Walsh when he decided to study the latter objects for his PhD.

Sadly, that same year, Paddy Kennedy took his own life (in England) shortly before he was about to take on the role of external examiner for the NUI. His untimely and unexpected death affected us greatly.

The years following my appointment were momentous. It was the post-Sputnik era, and suddenly everybody wanted to do science; also free education was introduced in Ireland by education minister Donogh O'Malley. To cater for the increased intake of students, more staff had to be employed and lecture space built to accommodate them. Fortunately, UCC had a newly appointed and dynamic president in Donal McCarthy, who was well versed in the ways of government, and he winkled enough money to hire new staff and build the necessary lecture rooms. This marked an era of immense growth for the college. During his time, the science building was completed and the building of the Boole library had begun. In recognition of its importance in the development of college, the mathematics department received its rightful share; it responded by expanding its staff and student numbers. In the early 1970s, I took a year's sabbatical leave at Chelsea College, London, and on my return took over the headship of the department to allow Paddy Barry to take on the role of vice-president for a couple of years. Things rested so for a decade!

The 1970s also saw the opening up of communication between various Irish universities and the setting up of the Irish Maths Soc.

FH: Shortly after taking up my position at UCC, I renewed my acquaintance with Trevor West and we started a fruitful collaboration about building up the strength of mathematics in Ireland. We were particularly anxious to improve communication levels between mathematicians in the different colleges. Our first effort led to the establishment of a visiting-lecturer scheme operated by TCD and UCC, whereby foreign visitors were passed from one college to the other in turn, and expenses were shared. Also, to assess the needs of our colleagues, we devised and circulated a questionnaire to them seeking advice on what should be done. As a result of the feedback a series of Summer Schools were instituted under the aegis of the Royal Irish Academy. The first of these was organised by Trevor in TCD, and was hugely successful, both academically and financially, generating enough funds to run future ones. Later, one on analysis was held in Cork, and several more were held at various centres throughout the 1970s. Ultimately, these led to the establishment of the Irish Mathematical Society. A further outgrowth was the setting up a national mathematical contest for secondary students, which laid the foundation for Ireland's participation in the International Mathematics Olympiad, first in 1988 and thenceforth. More detail about this has been published in a recent issue of the BIMS (Number 82 (2018) 69-78).

What do you see as the role of problems in maths?

FH: Problems are the lifeblood of mathematics. At the International Congress of Mathematicians in 1900, David Hilbert broke with tradition, and instead of focusing on his own work, startled the assembled gathering by announcing twenty three problems, several dealing with the very foundations of the subject, that influenced its directions for the twentieth century. This sparked a trend, and was followed a little later, for instance, by the formulation of Luzin's conjecture about the convergence of Fourier series, and Bieberbach's about the growth of the coefficients of univalent power series. These latter questions revitalised and invigorated different areas of analysis for many years to come before they were settled. While a few of Hilbert's problems were solved soon after they were published, the Riemann Hypothesis remains open.

My research interests spanned the areas of harmonic analysis, univalent functions, functions with positive real parts, Hankel operators on function spaces and inequalities. I've collaborated with and/or published joint work on a variety of research topics, problems or solutions to problems, with about eighteen mathematicians of various stripes.

I started composing problems in my late teens when I began teaching at the Crawford Tech, a practice I continued during my UCC days — even to this day, although those I devise nowadays are unlikely to ever see the light of day! I particularly enjoyed setting challenging problems for competitive exams for University Scholarship and local and international Mathematical Olympiads. Over the years, I've also availed of the invitation issued to participants at conferences to submit problems. One in particular that I submitted to a conference proceedings pleased me immensely: it sought a description of the generators of Hankel operators of trace class. This was taken up by the Russian mathematician Vladimir V. Peller, who gave a complete characterization of the class of such operators that belong to the more general Schattenvon-Neumann classes. Since I retired officially from UCC, I've been a regular contributor of original problems — and solutions of published problems — to several journals such as BIMS and the *American Mathematical Monthly.* One such problem appeared last year in the latter, a joint proposal with Tom Laffey and Roger Smyth about the set of eigenvalues of a particular tri-diagonal matrix. It spawned a research article that is due to appear in *Linear Algebra* about a more general class of similar matrices. *Lean leis an obair!*

3. Michael Mortell

What were your early interests, and who were your teachers of note?

MM: I was born in Cork city but lived in Charleville from age five after the death of my father. I attended the Charleville Christian Brothers schools for primary and secondary education, and did my Leaving Certificate in 1958.

My interests were general reading, hurling and school and I was generally viewed as being good at mathematics. Charleville CBS was a small school, there being seventeen students in my Leaving Certificate class. The teacher who had the most influence on me was Brother D.F. Williams. For the Leaving Certificate he taught honours courses in maths, physics, and chemistry, and the pass course in applied maths. Ironically, the latter was the area in which I did my PhD at Caltech. It was the time of Sputnik and the above range of courses pointed me in the direction of science when I went to UCC. The other influential teacher was James O'Sullivan who taught me English. I got a lifelong love of reading and literature from him.

I was not someone who plotted out a life's trajectory but had the philosophy of just taking the next best step and then doing as well as I could until the next decision-time came. At the time of my Leaving Cert the probability of my going to UCC was slight due to the family financial circumstances. However, due to the great generosity of my family and friends and in the light of my Leaving Cert results I was able to go to UCC. If I had not been able to go to UCC I suppose I would have ended up in England, which is where my mother was from.

When I walked through the gates of UCC for the first time in October 1958, there was a queue of students for each faculty and I chose the queue for science and my fate was determined. For first science I chose maths, maths physics, physics, and chemistry, and did the honours course in each of them. I was living in the Honan hostel under the watchful eye of Prof Cormac Ó Cuilleanáin. I worked hard and played a lot of hurling, getting a Fitzgibbon medal before I was eighteen. I added two more, plus the County championship before I was finished. On the basis of the results of the first science exam I was awarded a college scholarship, which greatly eased my financial position.

The next question to be dealt with was what subjects would I choose for the BSc degree. I think I had decided that physics or chemistry were not for me as I didn't particularly like doing experiments and was not very good at them, and I also enjoyed being able to go down to the Mardyke in the afternoon rather than being stuck in labs. So it was to be maths and maths physics for my degree. Again I didn't spend too much time pondering the pros and cons of the question.

The maths physics department consisted of Prof P.M. Quinlan and Dr V.G. Hart plus some MSc students like R.A. Scott and J.N. Flavin both of who later became professors, Scott at University of Michigan and Flavin at UCG. The mathematics department consisted of Prof P.B. Kennedy, Dr Siobhán O'Shea and Mr Paddy O'Donohoe who later was a faculty member at QUB. Dr Paddy Barry arrived in the final year of my BSc.

It is clear from the small number of staff in the departments that a broad syllabus was not possible. In both departments we followed the engineering syllabus and this was augmented by special honours courses. In maths we were exposed to modern mathematics involving proofs and this was focussed on real and complex variables. We did a tiny bit of group theory and a full course on matrices. V.G. Hart taught us dynamics, while P.M. Quinlan taught us complex variable fluid dynamics as well as his own work on the λ-method. The BSc course was clearly quite limited, but given the resources available the teaching staff did a very good job. The person who made the deepest impression on us students was Prof P.B. Kennedy. He seemed to have a somewhat stern exterior, but personally was a warm individual. He brought mathematics in UCC into the 20th century, and was a brilliant, concise lecturer.

How did your 1968 PhD at Caltech with Jim Knowles come about and was that area what you ended up pursuing?

MM: I chose to do an MSc in mathematical physics mainly as I was more interested in applied problems. Prof P.M. Quinlan had continuing contacts at Caltech and a number of UCC students had gone there to do PhDs. Among them were Prof M.E.J. O'Kelly at UCG, Prof P.G. O'Regan at UCC, and Dr Bernard Reardon at UCD. So it was no surprise that I should choose to go there. My MSc course consisted of real variables, complex variables, and a major in fluid dynamics taught by V.G. Hart. I found these UCC courses to be invaluable to me at Caltech.

There was a new applied math department at Caltech headed up by G.B. Whitham and I decided to do my PhD in applied math under the direction of J.K. Knowles in the area of elasticity. But before beginning research there were one and a half years of coursework to be done. I came in contact with the likes of A. Erdélyi, J.D. Cole, P.A. Lagerstorm, J.K. Knowles, Marshall Hall Jnr, and G.B. Whitham. This was among the best applied maths departments in the world. So I now had to operate and compete in a different league. The standards were very high and the work very hard as you had to turn in homework every week. However, you learn a lot and have to rise to the occasion. This was the best training I ever got! I now understood hard work, and what standards are. My PhD research was in a branch of elasticity called shell theory. A shell is a thin elastic body. I worked on the propagation and focusing of linear waves on a spherical shell – think of a bullet fired into the shell – and on low frequency linear waves on a cylindrical shell, e.g., what speed do they travel with. Since then almost all my research publications are involved with nonlinear waves and consequent shocks. After my PhD, I got a tenure track position in the department of the applications of mathematics at Lehigh University, headed by R.S. Rivlin, a founder of continuum mechanics. I worked on nonlinear waves in bounded materials, where the effect of reflections must be taken into account, and I joined forces with B.R. Seymour. I was promoted to associate professor with tenure.

Describe how you ended up at UCC and how mathematics developed during your time there.

MM: Despite my tenured position I returned to UCC in 1973 to a lectureship in mathematical physics, and have remained there ever since – except for a sabbatical or two. The department I came into was changed little from what I had left. V.G. Hart was gone to Australia and P.D. McCormack had arrived from Trinity. I did not think the department was in good health even though we still had very good students. I had a relatively very high teaching load – as much as 11 hours per week at one stage – but continued with a good research output by joining B.R. Seymour at UBC in Vancouver over the summer. My training at Caltech stood to me! I did not think the department had moved with the times. My time at Caltech and at Lehigh has impressed on me the necessity for research if the dept and UCC were to have a standing in the wider academic world. During this time I had my one PhD student, Ted Cox, now an associate professor at UCD. Ted and I continued to work together through my time as registrar and president and into my "retirement". I became Registrar of UCC in 1979 and President in 1989. A previous president, Donal McCarthy, had introduced a promotion scheme that was changing the culture of the College. It was becoming clear that staff were expected to do and publish research if they were to progress in their careers. There was a new generation of younger mathematicians. They did well in this regime, and under Prof P.D. Barry the department broadened and strengthened. The students, still very good, now had a much broader and deeper degree.

What papers, books, lectures or mathematicians influenced you?

MM: In my student days at UCC, I depended mainly on notes given by the lecturers. The lectures by P.B. Kennedy and P.D. Barry in mathematics and by V.G. Hart in maths physics were very important to me. Each knew their subject in depth and explained it clearly to the student. P.B. Kennedy was particularly precise and concise and showed us how a proof should go.

How do you feel about the role of teaching and what is your approach to teaching?

MM: Teaching, and good teaching, is very important particularly if you want to engage the student. Mastery of the content and clear delivery, with an eye on standards, are essential. I am not a great believer in teaching at 3rd level as distinct from lecturing. At university, the good honours student should normally be able to work things out for himself from the lecture notes. Pass level students should be given plenty of help, but must learn to stand on their own feet also. My approach to teaching/lecturing was very simple: be well prepared, arrive on time, do not skip lectures, give good ordered notes. In recent years I assigned homework and worked through it in tutorials.

What is the future of mathematics in general and at UCC?

MM: So mathematics generally is flourishing in UCC and some of the best students are attracted to these departments. In the area of applied maths the number of staff is small, they are all active with a new professor, Sebastian Wieczorek, and the future looks bright. However, resources are a problem, as always.

During my time as registrar and president of UCC I emphasised the role of research and reported on it annually to the Governing Body. I saw my job as facilitating the academic work of the college. To that end significant amounts of money were raised to extend the area of the College and improve the infrastructure. Many academic positions were unfilled due to the austerity of the 1980s and before and all these were filled during the 1990s. Unfortunately this ground has now been lost under recent government policies. The universities are now significantly underfunded.

How do you view the Irish contribution to mathematics?

MM: If you think of Hamilton, Boole and Stokes we are in the major league immediately. Overall, given we are a small nation, I think we have punched above our weight.

What have you been doing since your official retirement in 2006?

MM: I continued to teach until about 2014. I have published about fifteen papers (jointly with other authors) in various international journals, and I continue to do research. I have also published two books (jointly), *Singular Perturbations: Introduction to System Order Reduction Methods with Applications* (Springer, 2014) and *Nonlinear Waves in Bounded Media* (World Scientific, 2017).

4. Donal Hurley

What were your early interests, and who were your teachers of note?

DH: From an early age, I was interested in solving mathematical problems and I was very fortunate to have had inspiring teachers right through schooling. At primary level, attending Clonakilty Boys N.S., my teacher in my final two years was Mr C. O'Rourke who constantly challenged us with problems. There were some really bright guys in the class (including Seán Dineen) so trying to be the first with the correct solution was very competitive. I remained on for a year after the Primary Certificate Examination (a state examination at end of primary school education) and during that year, Mr O'Rourke covered quite an amount of Euclidean geometry. When I came to doing the Intermadiate Certificate Examination, I realised that during that particular year, I had covered most of the prescribed course in Euclidian geometry. I had also been taught most of the material on the Leaving Certificate arithmetic paper (in those days there were 3 papers in the Honours mathematics course).

At secondary school in Farranferris College, Cork, I had a very enthusiastic teacher, Fr Tom Clancy, in my final year. I remember, in particular, the calculus book we had. Each chapter had a brief presentation of some theory and then there were about twenty problems of an applied nature which ensured that we appreciated the ways calculus could be used. As a result of this, a few of us sat the applied mathematics paper in the Leaving Certificate Examination even though we did not have any clsses in the subject. Fr Clancy also did quite an amount of Euclidean geometry with us, much more than the syllabus required.

At UCC, we had fantastic teachers; Paddy Kennedy, Paddy Barry, Finbarr Holland, Siobhán O'Shea as well as Vincent Harte and George Kelly in mathematical physics. All of them had superb styles of lecturing and made the material accessible.

Paddy Kennedy was memorable because of the performance of his lectures. He required that we all wear undergraduate gowns at lectures. I think he was the only academic in UCC who insisted on that. We put on the gowns in the lecture hall before he arrived and took them off again as soon as he departed as we were embarrassed to wear them on the campus. In fact, honours mathematics students were recognised as the guys (at the time there were no women in the class) who had gowns rolled up under their arms. Kennedy was a chain smoker and lit one cigarrette from the butt of the one that was just smoked. Early on in the term, he passed around a pack of cigarettes and invited the smokers to take one. When the pack was returned, he saw that none was taken and remarked that each year he hoped that some brave guy would take up his offer of a "fag". However, the really memorable aspect of his lecturing was his presentation. I still have the notes I took in his course and they are a clear, organised presentation of the introduction to the abstract algebra course. As he lectured, he gave the impression that he was just thinking up the material as he went along; he would stand back, look at the blackboard and reflect for a minute or two and then go and fill the board with his beautiful writing and crystal clear material. He was by far the best lecturer I ever had. Classmates included Michael Brennan (WIT) and David Walsh (MU).

How did your 1970 PhD at Yale under Gustav Hedlund came about and was that the area you ended up pursuing?

DH: I decided to do graduate studies in the USA as funding was a little easier to obtain there as opposed to going to the UK (no funding available in Ireland at that time). The programme there was four years, the first two of which were courses across the whole span of mathematics. That appealed to me as I thought that I should do some more algebra as well as geometry and topology. Vincent Harte explained to me how to apply to USA universities and where I should apply. I applied to several, Yale being my top choice, and I was very pleased to be accepted there.

UCC had a very strong reputation in analysis, especially complex analysis, when I studied there. Going to Yale, my plan was to specialise in functional analysis. Interestingly, during my first few weeks there, I discovered that most of the 20 students in the class were planning to do algebra. I asked a fellow student why this was the case and he told me that it was because of Walter Feit, Nathan Jacobson and some other very strong algebraists who were at Yale. I had not heard of these people! Charles Rickart, who had a reputation in functional analysis, was there and I had heard of him so he was the person I hoped to study with. In our second year of study, we took our PhD qualifying examination which was a three-hour oral. The custom was that one asked the person one wished to study with to chair the examination and to select the other three or four members of the board. Rickart chaired my board and I took the examination at the beginning of second semester of my second year. I took some more advanced analysis courses and graduate seminars during the second semester of that year.

At the beginning of my third year, I decided to take a course on topological dynamics with Hedlund and he suggested that I also take the differential dynamics course being given by Ziggy Nitecki who had just arrived as a postdoc from the University of California at Berkeley. At that time there was geat excitement in the mathematical community about the work being done at Berkeley by Stephen Smale and his coworkers. The combination of Hedleud's course with Nitecki's was very attractive and I got engaged in the area of differential dynamical systems. I spoke to Hedlund and he said he had a problem for which Niteck's course should give me useful background. Since I had also been discussing topics with Rickart, I informed him of my new found interest, and he encouraged me to work with Hedlund.

Hedlund told me of his problem in geodesic flows about a certain class of geodesics on manifolds of hyperbolic type. It was related to a conjecture of Ya Pesin about the entropy of flows on these manifolds.

What papers, books, lectures or mathematicians influenced you?

DH: When I began research, the main reference for work in geodesic flows was the book by D.V. Anosov, *Geodesic flows on Closed Riemannian Manifolds with Negative Curvature*, as well as papers writted by Hedlund and M. Morse. For differential dynamical systems, the several volumes of the Proceedings of AMS Summer Research Institutes on *Differential Dynamical Systems* (1968) and *Smooth Ergodic Theory* (1969) were the references. People whose work I followed in later years while working on geodesic flows included P. Eberlein (University of North Carolina), K. Burns (Northwestern) and Ya Pesin (Pennsylvania State University).

I began collaborating with Michel Vandyck (Physics, UCC) around 1990, and we developed a differential operator which we called “D-differentiation”. This collobaration has proved to be very successful and our work has been applying this operator to several areas in mathematical physics.

Describe how you ended up at UCC and how mathematics developed during your time there.

DH: I spent 3 years at UCG (now NUIG) before being appointed to a lectureship in UCC in 1973. At the same time, 3 others were appointed; Des MacHale, Tim Porter, and Tony Seda. Paddy Barry, who was head of department, was keen to spread the expertise in the department. He was also interested in developing team spirit so we had regular weekly colloquia and regular departmental meetings. The department had the reputation as being one of the most democratic ones in UCC.

At that time, we had no internet or email so doing research was difficult as there were few opportunities to keep in contact with researchers at other institutions or keep abreast of advances being made. One depended on attending conferences but this was of limited value. When email, and later the WWW, became available that changed opportunities dramatically.

The major development in UCC was the formation of the school of mathematical sciences when the departments of mathematics, applied mathematics and statistics merged. We had many meetings discussing the merger as opinions varied widely about the possibly structure. Des Clarke (philosophy department) guided us towards the structure which eventually emerged from discussions.

How do you feel about the role of teaching and what is your approach to teaching?

DH: I tried to follow the example of the better lectures I had while a student and so prepared my lectures to be well organised and accessible to the students. In lecturing to honours mathematics students, it was easy to find the appropriate level as students were fairly uniform. The situation was more difficult with classes of non-mathematics majors as the level of students was occasionally very varied and determining the correct level could be difficult. But I enjoyed giving the lectures and always got great satisfaction in seeing how well the students had mastered the material while marking the examinations.

I got involved with Finbarr Holland, in the mid 1980s, in giving the Enrichment Classes to students being prepared for the International Mathematical Olympiad. Exposing bright second level students to topics not covered in schools curriculum was very rewarding and a number of these students were motivated to study mathematics when they went on to university.

What is the future of mathematics in general and at UCC?

DH: One of the very exciting aspects of mathematics is the way in which research topics evolve as seminal results are obtained. Because of the developments in computing, I think that data analysis and computer graphics are changing the directions of research. UCC has made some recent staff appointments of people who are very skilled in these areas. Not only are they exploiting these tools in developing their own research but students are being exposed to these new developments.

How do you view the Irish contribution to mathematics

DH: Mathematics is providing the foundation for STEM subjects which the government sees as the basis for future economic development. However, I do have some concern about the Project Maths programme and would like to see a thorough independent analysis of it in the near future.

Many mathematics graduates of Irish universities are making valuable contributions to the sciences at top universities and instiutions abroad. I would like to see more government funding for mathematical research to broaden the opportunities for our graduates to work in Ireland and to attract researchers from abroad. In Ireland, we are still a long way short of the critical mass of professional mathematicians required to make a significant impact on mathematics internationally.

5. Des MacHale

What were your early interests, and who were your teachers of note?

DMH: As a kid, I loved counting and geometric figures. I would ask everyone in the house how many potatoes they wanted for dinner and make out a little chart and present it to my mother. My party piece was to recite very quickly $1+1=2$, $2+2=4$, $4+4=8$, $8+8=16$, $16+16=32$, $32+32=64$, $64+64=128$, etc. When I helped my father in the garden to plant cabbages and potatoes, all the drills had to be perfectly parallel to each other and all the plants the same uniform distance apart before he was allowed to cover them with earth.

In secondary school I was lucky enough to have an excellent mathematics teacher, a De La Salle brother called James Sheridan (Brother George). He really stimulated my interest in mathematics and challenged me. He encouraged me to solve a given problem in as many different ways (often up to ten!) as possible, which is a wonderful teaching technique. I used to live for his *agusíní* or "cuts" which he gave me as a reward for finishing routine problems. He had a handwritten journal in which he stored his own special solutions, specially on geometry and the parabola. These were often original of his own making, and I am sorry I did not inherit it. He loved too harmonic ranges, pole and polar, inversion – wonderful classical geometry, sadly unknown to today's generation of students. I really think I owe him my career as a mathematician and my love of the subject. I met him once in UCC after he retired where he told me that some person with the same name as me had written a shameful book of jokes and we agreed this was very embarrassing! He never found out it was me.

For the Leaving Certificate, I did applied mathematics entirely on my own without a teacher, which was a difficult thing to do, and one of my proudest achievements was to obtain honours in it, despite the fact that I was not doing physics.

How did your 1972 PhD at Keele with Hans Liebeck come about and was that area what you ended up pursuing?

DMH: As an undergraduate at University College Galway, taught by Seán Tobin and Tom Laffey, I fell in love with algebra, especially group theory and ring theory. I'm afraid I tolerated analysis and mathematical physics for which I had little talent, but I enjoyed a course on projective geometry given by Tom McDonough, and the MSc course on complex analysis delivered by Sean McDonagh. I lectured for a year (1968-1969) at UCG and got a taste of research on commutativity in finite groups and decided to pursue a career of research in groups by doing a PhD in the UK. Seán Tobin suggested I try Hans Liebeck at the University of Keele and I got a studentship there. I also had offers from Leeds, Aberdeen, London, and Belfast, but I had been to Keele for interview and the country estate campus where all staff and students lived proved very attractive, especially for a keen tennis player! Hans Liebeck was an excellent supervisor and we got on very well together – if the mathematics was not progressing, we often played squash together, until things seemed clearer. At first I was working on a very difficult problem – the breath and class of a finite p-group – still unresolved for groups of odd order. Then one day in the library I came across *The Collected Works of G.A. Miller*, an eccentric American group theorist. This was a treasure trove of about five hundred papers, concerning automorphisms, abelian subgroups, conjugacy classes, etc. This led to my thesis on "Finite Groups with an Automorphism inverting Many Elements" for which I was awarded a PhD in 1972 (conferred by Princess Margaret). I have since had many publications based on the work of G.A. Miller.

What papers, books, lectures or mathematicians influenced you?

DMH: I was very influenced by my supervisor Hans Liebeck and my external examiner Peter Neumann; also by Miller and Tom Laffey. I owe a lot to the great Philip Hall and his great concept of isoclinism – I still have a twenty-page letter he wrote me, full of ideas and encouragement. The text that most influenced me was Herstein's *Topics in Algebra*, a beautiful book that would make an algebraist of anyone.

Describe how you ended up at UCC and how mathematics developed during your time there.

DMH: I ended up at UCC because there were jobs there, one of which I obtained on my second attempt. I had a temporary position there 1972-73, and worked very hard to persuade them that they could not do without me — Martin Stynes won a Studentship that year taking my MSc course in algebra. Under Professor Paddy Barry mathematics thrived at UCC for the next forty years; an excellent mathematician himself, he was a super head of department, and I do not remember a single incident of conflict in that time. He chose staff very wisely, both for their mathematical ability and emotional maturity and I was blessed to be surrounded by colleagues too numerous to mention. As a perk, I got to give all my favourite algebra courses, but I enjoyed teaching calculus to engineers also. We had some excellent undergraduate students, as good as anywhere in the world – and many went on to fill mathematical positions worldwide. For example, we had Stephen Buckley, Peter Hegarty, Diarmuid Early, Martin Stynes and many others. I initiated the Superbrain Competition and the Irish Intervarsity Examination which gave student mathematics at UCC a cutting edge and made us very successful.

We wound up with a very balanced undergraduate course with strengths in analysis, real and complex, algebra, combinatorics, and geometry, and good options at postgrad level including cryptology, introduced by Pat Fitzpatrick.

How do you feel about the role of teaching and what is your approach to teaching?

DMH: I loved teaching mathematics, which I regard as inseparable from research. My favourite class was the first year honours class – very bright young boys and girls in from school, just bursting with talent. I loved to challenge them and widen their mathematical horizons. Every couple of weeks we had a ten minute spot on a special topic, not necessarily examinable – paradoxes, countability, intransitive dice, orthogonal Latin squares, unsolved problems in number theory etc.. These sessions really made them sit up. I enjoyed too giving the undergraduate modules in groups, rings and fields, geometric constructibility etc.. As a result, I became interested in commutativity in rings in which I have worked with Stephen Buckley a lot. But I enjoyed teaching practical calculus to engineers and scientists too, and one of the highlights of my career was an MA course we gave to secondary teachers on Curriculum Studies.

With Tom Carroll and Donal Hurley, I was awarded a UCC prize for a new course on Problem Solving and Mathematical Creativity we put together. This proved very popular with students and included a project.

What is the future of mathematics in general and at UCC?

DMH: The future of mathematics is promising but the future of pure mathematics is more shaky. You cannot be a rich pure mathematician – that is the price you pay for the exquisite pleasure the subject gives you. You can be comfortable and have a good living being paid to do what you enjoy most (remember the old quip – a pure mathematician is someone who has found something more enjoyable than sex!) but you will not become rich. Computer science, statistics, engineering, applied mathematics, financial mathematics, and other areas have seduced many students who would have contributed greatly to pure mathematics. Ironically, all those other lucrative areas depend vitally on progress in pure mathematics, *a la* Boole. The future of mathematics at UCC seems secure – we have good enthusiastic staff, and a geographical catchment area that every year produces world class mathematical students.

How did you get interested in George Boole?

DMH: In 1974, the late Dr Sean Pettit of UCC was doing his doctoral thesis on the history of third level education in Cork and told me he had discovered some letters of Boole's in the archives. I started to write a short article on them, which grew into a long article, which grew into a short book. Then when I realised there was no full-length biography of Boole, I decided to write one. It took me over ten years. Boole's widow had destroyed many of his letters and papers before 1900 as she felt they reflected badly on the religious movements he had been involved in. The Life and Work of George Boole was published in 1985 by Boole Press in Dublin and reprinted by Cork University Press in 2014 for the bicentenary Of Boole's birth. Then in 2018, with my former student Yvonne Cohen, we published *New Light On George Boole* (Cork University Press) a 500-page book of new material on Boole's social, educational, and family life, which had two highlights – the meeting of Boole and Babbage in London in 1862, which if they both had lived, could have led to the first computer a hundred years earlier, and the sensational theory that Boole was the inspiration for Professor James Moriarty, the arch-villain of the Sherlock Holmes stories.

What are the connections between mathematics and humour?

DMH: This is a largely unexplored topic, but very fruitful. Contrary to popular belief, most mathematicians have a very well-developed sense of humour and love jokes. Logic seems to be the link – mathematics is based on logic, but humour turns logic on its head. Paradox is another strong link. The riddle is close to the claim and the conjecture, and the joke has very much the same structure as the theorem, with the assertion and the punch line in reverse order. I have even written a book on the topic – *Comic Sections* (Boole Press 1983). The Mathematical Association in the UK are soon to reprint an expanded version – so watch this space. I believe that humour and mathematical ability go hand in hand – they both require insight, ingenuity, creativity and the ability to see connections. Look at Tom Lehrer, Lewis Carroll and Stephen Leacock, and maybe myself.

How do you view the Irish contribution to mathematics?

DMH: I think that Ireland, North and South, has definitely punched above its weight in mathematics. Poorly financed, but needing little resources, Irish people, historically and currently, have contributed greatly to mathematics, pure and applied. Hamilton, Stokes, Casey, Murphy, Kelvin, Graves, Newell, Laffey, Kennedy, and many others have given us an international reputation in the subject. It would be an interesting topic to investigate – does the Irish mentality lend itself to mathematics as it does to humour?

Final Remark

DMH: Mathematics has theological implications for me, and is one of the reasons I believe in God. I can't believe that the Universe is just random when I see the beauty, ingenuity, and consistency of mathematics. As some philosopher has said, God exists because mathematics is consistent, but the Devil exists because we cannot prove it!

6. Patrick Fitzpatrick

What were your early interests, and who were your teachers of note?

PF: I discovered mathematics at the age of ten, in first year at St Mary's CBS in Belfast. Our school mathematics programme was divided into Arithmetic, Algebra and Geometry (of the Euclidean variety) and I was introduced to proofs which I loved. One particular incident stands out: we were doing ruler and compass constructions and having learned how to bisect an angle, I recall the teacher telling us "no-one has ever discovered a way to trisect an angle". I was undaunted and the next day when the teacher asked "did anyone try it?" of course I had. Finding out years later why no-one had ever managed to do it triggered that memory, and I identify it in retrospect as when I became a mathematician. I loved mathematics throughout school and had several brilliant teachers especially that one, a young man just out of college, and the other a venerable Christian Brother from whom I learned mathematics and applied mathematics for A-Level. I did my undergraduate degree at the University of Surrey, where I was very influenced by Donald Keedwell who taught me algebra and combinatorics.

How did your 1980 PhD at Australian National University with László Kovács come about and was that area what you ended up pursuing?

PF: After graduation from Surrey in 1973, I took the PGCE and spend the best part of 3 years in second level teaching. That proved ultimately unsatisfying – mathematics chased me in my spare time. I did not have the financial means to do a PhD, but in 1976 I was very fortunate to win a scholarship to ANU. My wife Johanna and I travelled together for what was a great “adventure”, both personally and professionally. At the Research School of Physical Sciences at ANU, I was immersed in the mathematical environment created by Bernhard and Hanna Neumann which focussed on the theory of varieties of groups, those defined by laws. Laci Kovács was my supervisor and Mike Newman was also very much involved. I studied the variety of nilpotent groups of class four, that is, those defined by the commutator law $[x_1, x_2, x_3, x_4] = 1$, and achieved a full parametric classification. I learned a great deal, both in my thesis work and in the weekly seminar, and perhaps the most important lesson was to understand how little I actually knew.

Describe how you ended up at UCC and how mathematics developed during your time there.

PF: In 1979, I returned to England for the last three months of my scholarship, partly because Laci was going to be visiting Queen Mary College for the summer. I had an interview for a 1-year position at UCC which was successful, so Johanna and I arrived in Cork, with our 10-month old baby, in September 1979. I wrote up my thesis during that year and was awarded the PhD in June 1980. Fortunately, I was selected for appointment to a permanent position at UCC from September 1980.

In UCC the main structural change in mathematics was the amalgamation of the departments of mathematics, mathematical physics and statistics into a school in the mid-90’s, and somewhat later the introduction of the degree in financial mathematics. One of the main changes in the development of mathematics from my perspective was the introduction of the computer as a tool. This made doing calculations easier, and allowed an “experimental” element into conjecture. In a completely different direction it made doing mathematics so much easier because it enabled communication with colleagues by email, with TeX/LaTeX to write things down in a common editable language, and obviated the necessity of multiple sequences of typesetting and editing drafts of papers. It allowed us much more time to focus on proof, as opposed to proof-reading.

What papers, books, lectures or mathematicians influenced you?

PF: To begin with at UCC, I worked on some problems in group theory with Des MacHale, who was a huge support in my early years as an academic. But sometime in the mid-1980s I had a conversation with my father who, as a BT planning engineer, was heavily involved in the roll-out of the fibre-optic telephone network in Northern Ireland, and he told me there was a very interesting connection with algebra. One thing led to another and I started to look at error-correcting codes, very interesting mathematical objects that sit between algebra and combinatorics, and are essential for digital communications. I also relished the experience of working with engineers: when I was an undergraduate anyone who was interested in algebra was destined to be a “pure” mathematician, and it was a revelation to discover that applications in the discrete domain, as opposed to the continuous, needed classical algebraic objects like Galois fields, and more recent inventions such as Gröbner bases. So I was able to tap into my “inner applied mathematician” and I enjoyed the experience!

I spent a sabbatical year 1986-87 in Alain Poli’s group in Toulouse, and made the changeover from research in group theory to algebraic coding theory. I spent another sabbatical in 1994-95 with Don O’Shea and taught at Mount Holyoke and Amherst Colleges. Together with David Cox and John Little, he had just written the well-known book *Ideals, Varieties and Algorithms* (their later book *Using Algebraic Geometry* included some of my work). I very much enjoyed the interaction with the circle of mathematicians in the Five Colleges, and benefitted greatly from the weekly seminar at

UMass Amherst. It's also interesting to recall that there was an early introduction to the internet at Mount Holyoke, led by the staff in the mathematics department. I still recall the meaning of terms like URL, HTTP, HTML, etc.

How do you feel about the role of teaching and what is your approach to teaching?

PF: To me teaching is an essential part of the work of "doing mathematics". I have always loved interaction with students and the challenges it brings to clarity of thought. Developing new courses is a fruitful way to learn new areas, while guiding PhD students is possibly the most rewarding. I have been very fortunate in that respect. The relationship between teaching and research should never be underestimated.

What is the future of mathematics in general and at UCC?

PF: I think there will always be a (relatively small) number of students who are motivated to study mathematics at an advanced level. The key is to identify and nurture those at an early age, and that means taking specific actions. At UCC, we have held the Mathematical Enrichment classes for at least 40 years, and engaged the students in preparation for a wide range of challenging competitions, not least the national and International Mathematical Olympiads. An essential component of this activity is the nation-wide network that supports it. Our university schools of mathematics have always produced champions and I'm sure that will continue.

However, there is an obvious danger that targeted funding by government agencies, and indirectly by industry, will become so dominant, that mathematics in general, and pure mathematics in particular, will find it increasingly difficult to survive. I think every Irish university is threatened by this, and it is not obvious that university administrations are alive to the danger. I think it's more important than ever for mathematicians to tell the story of mathematics, and include further examples of its "unreasonable effectiveness", not just in physics, but, now more pertinently, in biology, computer science, and communications, *inter alia*. There needs to be a concerted effort to communicate this message. Every generation needs a Eugene Wigner and in the incessant noise of the modern day, more than one.

What have you been doing since your official retirement in 2013?

PF: I retired formally in December 2013 after almost nine years as dean of science and subsequently, after UCC restructuring, as head of college of science, engineering and food science. I have continued since then working at UCC Academy three days per week and I do some mathematics in my spare time. I'm looking forward to doing more of that when I eventually fully retire!

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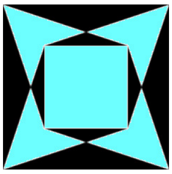

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Solving cubic and quartic equations by radicals

C. T. C. WALL

Abstract. The rule for the solution of a cubic or quartic equation by radicals is obtained from elementary considerations of the geometry of the projective line.

1. Introduction

The formula $\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$ for the roots of the quadratic equation $ax^2+bx+c=0$ is well known and easy to establish. The search for a corresponding formula for a cubic equation met with success in the 16th century, as can be found in one of the more colourful chapters of the history of mathematics. A method for solving quartic equations by radicals was also discovered in the 16th century, relatively soon after the solution of cubics, and other methods were then found.

Our object is neither to present the history nor the early versions of these arguments, but to give an account in the language of (projective) geometry to clarify the reasons for the formulae. A related version was given in [1].

2. Cubic equations

A cubic equation may be written as $ax^3+bx^2+cx+d=0$. We work over a field K containing the coefficients a,b,c,d; to find a general formula, we may pick a field k (my personal preference is the field $\mathbb{C}$ of complex numbers) and take K as the pure transcendental extension $K=k(a,b,c,d)$, or just take $K=\mathbb{C}$; we will also discuss the case $K=\mathbb{R}$. Even the above rule for solving quadratic equations fails if k has characteristic 2; for cubic and quartic equations we must further assume that k does not have characteristic 2 or 3.

To know what to expect, we refer to Galois Theory. As the group of the equation is the symmetric group $\mathfrak{S}_3$, to pass from K to the root field of the cubic we need a quadratic extension (taking a square root), and a cubic extension which, provided K contains a cube root of unity, involves taking a cube root.

From now on, we view the situation geometrically, so write the equation in homogeneous form as $h(x,y):=ax^3+3bx^2y+3cxy^2+dy^3$, and regard the root α as defining the point $P_\alpha=(\alpha:1)$ on the projective line P^1, and correspondingly for the roots β and γ. We assume that the roots are distinct.

A homography of P^1 is a map of the form $(x:y)\to(px+qy:rx+sy)$; it is determined by the images of any 3 distinct points. Thus there is a unique $\phi:P^1\to P^1$ such that $\phi(P_\alpha)=P_\beta$, $\phi(P_\beta)=P_\gamma$ and $\phi(P_\gamma)=P_\alpha$; it follows that ϕ^3 is the identity. This ϕ has just 2 fixed points, which we denote by Q_0 and Q_1.

The cubic $h(x,y)$ has Hessian $H(h):=h_{xx}h_{yy}-h_{xy}^2$, which is a quadratic covariant of h. Explicitly, if we change coordinates by $X=px+qy$, $Y=rx+sy$ and $h(x,y)=k(X,Y)$, we find that $h_{xx}h_{yy}-h_{xy}^2=(ps-qr)^2(k_{XX}k_{YY}-k_{XY}^2)$.

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The key observation is that the roots of $H(h)$ determine the points $Q_0, Q_1 \in P^1$. To see this, we change coordinates in P^1 so that in the new coordinates $(X : Y)$, $Q_0 = (0 : 1)$ and $Q_1 = (1 : 0)$. Since $H(h)$ is a covariant, $H(k)$ is a multiple of XY. Thus ϕ has the form $(X : Y) \to (\omega X : Y)$, where ω is a cube root of unity, the roots of k may be written as $(\xi : 1)$, $(\omega\xi : 1)$, $(\omega^2\xi : 1)$, so

$$k(X, Y) = a(X - \xi Y)(X - \omega\xi Y)(X - \omega^2\xi Y) = a(X^3 - \xi^3 Y^3),$$

and $H(k) = -36a^2\xi^3 XY$, which indeed has zeros at Q_0 and Q_1.

To solve the cubic h, we thus first factorise $H(h)$ in the form $(px+qy)(rx+sy)$ (which involves solving a quadratic equation); then make the coordinate change $(X, Y) = (px + qy, rx + sy)$, which puts h in the form $k(X, Y) = AX^3 + DY^3$. Then extracting the cube root of $-D/A$ allows us to factorise k, and changing coordinates back gives the desired result.

Returning to our original coordinates, we calculate:

$$H(h) = 36(ax + by)(cx + dy) - (bx + cy)^2 = 36[(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2].$$

This quadratic (removing the factor 36) has discriminant

$$\Delta(h) := (ad - bc)^2 - 4(ac - b^2)(bd - c^2) = a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd,$$

which coincides with the usual formula for the discriminant of h. We remark that in the case $a = 1, b = 0$ this formula reduces to $\Delta(h) = d^2 + 4c^3$.

If the roots of h are α, β, γ, the quadratic extension is the one containing $\sigma := (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$, since σ^2 is a symmetric function of α, β and γ. Since the vanishing of Δ is also the condition for h to have equal roots, σ^2 agrees with Δ up to a scalar factor. To check the scalar, suppose $b = c = 0$. Then $\sigma = (1-\omega)(\omega-\omega^2)(\omega^2-1)\alpha^3 = 3\sqrt{-3}\alpha^3$, $\Delta = a^2d^2$ and $d = a\alpha^3$. Thus $a^4\sigma^2 = -27a^4\alpha^6 = -27a^2d^2 = -27\Delta$.

3. Quartic equations I

In this case, it is convenient to think of numbers as points on the conic S_0, with parametrisation $(t^2, t, 1)$ and equation $g_0 = 0$ in the plane P^2, where $g_0(x, y, z) = y^2 - xz$. Write the quartic equation as $f(t) = 0$, where

$$f(t) \equiv at^4 + 4bt^3 + 6ct^2 + 4dt + e. \tag{1}$$

The basic invariants of f are the transvectant

$$T(f) = ae - 4bd + 3c^2$$

and the catalecticant

$$K(f) = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

The roots of f correspond to the points of intersection of S_0 with the conic given by $g = 0$, where

$$g(x, y, z) = ax^2 + 4bxy + c(4y^2 + 2xz) + 4dyz + ez^2.$$

Here we can replace $g = 0$ by any of the conics S_λ given by $g - 2\lambda g_0 = 0$. The equation of S_λ has matrix

$$\begin{pmatrix} a & 2b & c+\lambda \\ 2b+c & 4c-2\lambda & 2d \\ c+\lambda & 2d & e \end{pmatrix},$$

and its determinant evaluates to

$$R(f) := 2(\lambda^3 - \lambda T(f) + 2K(f)).$$

$R(f)$ is called the *cubic resolvent* of f. If λ takes a value making $R(f)$ vanish, the conic S_λ is singular, and so breaks up into a pair of lines. Each of these lines meets S_0 in two points, and these points are the 4 points on S_0 giving the roots of f. Thus we have an arrangement of this set of 4 points in two pairs. Conversely, for each of the 3 such arrangements, the two lines each joining one of the pairs form a line-pair giving one of the singular conics S_λ.

The procedure for solving the quartic $f = 0$ is now
(i) Choose a value of λ such that $R(f) = 0$.
(ii) Factorise $g - 2\lambda g_0$ as $\ell\ell'$, where ℓ and ℓ' are linear expressions.
(iii) Substitute $x = t^2, y = t, z = 1$ in ℓ, and solve the resulting quadratic equation; then do the same for ℓ'.

Step (i) involves the solution of a cubic, which we can achieve by taking a square root, then a cube root.

A short calculation shows that step (ii) also reduces to solving a quadratic, hence involves taking a square root; and step (iii) again involves solving a quadratic.

The first step in the solution involves taking the square root of the discriminant of the resolvent cubic, which is $\Delta(f) := \Delta(R(f)) = T(f)^3 - 27K(f)^2$. The quadruple of roots in P^1 is determined up to equivalence by the single invariant

$$j(f) := \frac{T(f)^3}{T(f)^3 - 27K(f)^2}.$$

4. Quartic equations II

In the same style as our treatment of cubic equations, we consider involutions, i.e. homographies of P^1 of order 2. Here it is convenient to use the inhomogeneous coordinate t on P^1. An involution I may be written as $att' + b(t + t') + c = 0$, or equivalently as $I(t) = -\frac{bt+c}{at+b}$. I has 2 fixed points on P^1 and is determined by them. Two points are paired by I if and only if they harmonically separate the fixed points. An involution I' commuting with I must either preserve the fixed points of I (in which case it coincides with I) or interchange them.

The four roots of a quartic equation $f = 0$ determine, as above, four points of P^1. For each arrangement of these four points in two pairs, say $(\alpha, \beta)(\gamma, \delta)$ there is a unique involution of P^1 interchanging the pairs (α, β) and (γ, δ). These involutions correspond to the singular conics S_λ just described: the above involution corresponds to the conic which is the union of the lines $P_\alpha P_\beta$ and $P_\gamma P_\delta$. The 3 such arrangements yield 3 involutions, which form a group isomorphic to the four group. Since any two of the involutions commute, the corresponding pairs of fixed points separate harmonically, so the 6 fixed points (which can be found as the zeros of the jacobian of f with its Hessian $H(f)$) form the vertices of a regular octahedron under a suitable identification of $P^1(\mathbb{C})$ with the 2-sphere. The symmetry group of the octahedron is $\mathfrak{S}_4$, isomorphic to the group of f.

The calculations simplify if I takes the form $I(t) = -t$. To achieve this, choose a root λ of the resolvent cubic: then the quadratic S_λ is a line pair. The point P of intersection of these lines can be found by solving the linear equations

$$\partial(g - 2\lambda g_0)/\partial x = \partial(g - 2\lambda g_0)/\partial y = \partial(g - 2\lambda g_0)/\partial z = 0$$

(that these are consistent follows since λ is a root of R). The corresponding involution is cut on S_0 by lines through P. If P has coordinates (x_0, y_0, z_0), then the points with parameters t and t' lie on a line through P if the determinant

$$\begin{vmatrix} x_0 & y_0 & z_0 \\ t^2 & t & 1 \\ t'^2 & t' & 1 \end{vmatrix}$$

vanishes, or equivalently, removing the factor $(t-t')$, if

$$x_0 - y_0(t+t') + z_0tt' = 0.$$

Let ξ be a solution of $\xi^2x_0 + 2\xi y_0 + z_0 = 0$, and set $w = \frac{\xi t+1}{z_0t/\xi+x_0}$; then indeed the involution takes the form $I(w) = -w$, and now taking w as coordinate reduces f to the form $aw^4 + 6cw^2 + e$. In this situation, the procedure for solving $f = 0$ reduces to first solving the quadratic equation for w^2, and then taking square roots of the solutions.

When $b = d = 0$, the invariants reduce to $T(f) = ae + 3c^2$ and $K(f) = c(ae - c^2)$, and $R(f)$ factorises as $R(g) = 2(\lambda - 2c)(\lambda^2 + 2c\lambda + c^2 - ae)$; the root $\lambda = 2c$ corresponds to the chosen involution.

5. Over the real numbers

For a cubic equation $h = 0$, if $\Delta(h) < 0$, the quadratic equation $H(h)$ has real roots, we can reduce the Hessian $H(h)$ to xy, and then require the 3 cube roots of a real number, so only one of the roots of h is real.

If however $\Delta(h) > 0$, the quadratic has conjugate complex roots, and we can reduce $H(h)$ to $x^2 + y^2$. Geometrically, the Hessian points are now $(\pm i : 1)$ and ϕ is a real rotation through $2\pi/3$. In this case all 3 roots of h are real.

For a quartic equation, at each stage of the above procedure where the square root of an expression E is taken, there are two cases according to the sign of E; this seems to lead to huge numbers of cases. However there are just 3 cases for the quartic, according as it has 0, 2 or 4 real roots.

If f has 4 real roots p, q, r, s , the resolvent cubic has 3 real roots, corresponding to the arrangements of the roots in pairs as $(p, q)(r, s)$, $(p, r)(q, s)$ and $(p, s)(q, r)$. In each of these cases, the conic S_λ consists of 2 real lines, and each of these lines meets S_0 in 2 real points.

If f has 2 real roots p, q and a conjugate complex pair $z, \overline{z}$, then the conics corresponding to the arrangements $(p, z)(q, \overline{z})$ and $(p, \overline{z})(q, z)$ are conjugate to each other, so R has just 1 real root. For the arrangement $(p, q)(z, \overline{z})$ we have 2 real lines, with one line meeting S_0 in 2 real points, the other in none.

If f has 0 real roots, the roots form 2 conjugate complex pairs $(w, \overline{w})$ and $(z, \overline{z})$, and the conic corresponding to the arrangement $(w, \overline{w})(z, \overline{z})$ consists of 2 real lines, neither having a real point of intersection with S_0, while each of the conics corresponding to $(w, z)(\overline{w}, \overline{z})$ and $(w, \overline{z})(z, \overline{w})$ is real, but consists of a pair of conjugate complex lines. Here again R has 3 real roots.

We have seen that the sign of the discriminant determines whether R has 1 or 3 real roots. If it has 3, deciding whether f has 0 or 4 real roots is less simple, but it can be shown that f has 4 real roots if and only if both $ac - b^2$ and $a^3e - 4a^2bd - 9a^2c^2 + 24ab^2c - 12b^4$ are negative.

We can rewrite the above in terms of involutions, following [2]. Write J for complex conjugation on P^1: then an involution I is real, i.e. has real coefficients, if and only if $JI = IJ$. If I is a real involution, then either

type (r): its fixed points are both real (an example is $I_0(t) = -t$); or

type (c): its fixed points are complex conjugates (an example is $I_i(t) = -t^{-1}$).

If f is a real quartic, the map J must preserve the octahedron O formed by the fixed points of the 3 involutions. There are two cases:

(a) J interchanges a pair of opposite vertices of O and fixes the other vertices;

(b) J fixes one pair of opposite vertices and interchanges the other two pairs.

In case (a), we can take the vertices as $(0,\infty)(\pm1)(\pm i)$ (with J the usual complex conjugation); the involutions are then $t \to -t, t \to 1/t, t \to -1/t$; all are real, the first two of type (r), the other of type (c).

In case (b), we can take the involutions as $t \to -t, t \to i/t, t \to -i/t$; the first is real of type (r), the other two are complex conjugates; and the vertices are $(0,\infty)(\pm e^{i\pi/4})(\pm e^{-i\pi/4})$.

In each case there is at least one real involution of type (r) preserving f.

Conversely, given a real involution I preserving f, we seek to follow the above procedure for reducing I to the form $t \to -t$. First we solve linear equations (so can work over $\mathbb{R}$), to find a point P_0 with coordinates (x_0, y_0, z_0). We then require the square root of $y_0^2 - x_0z_0$. The sign of $y_0^2 - x_0z_0$ depends whether P_0 is inside or outside the conic S_0, hence on whether the involution I has 0 or 2 real fixed points. We can thus reduce I to $t \to -t$ provided I has type (r).

In case (a), the roots of f have the form $\pm\alpha, \pm\alpha^{-1}$, so $f(z) = (z^2-\alpha^2)(z^2-\alpha^{-2}) = z^4 + 6cz^2 + 1$, with $-6c = \alpha^2 + \alpha^{-2}$. There are 3 cases:

(i) ($c < -\frac{1}{3}$) all roots are on the real axis,

(ii) ($c > \frac{1}{3}$) all roots on the imaginary axis,

(iii) ($|c| < \frac{1}{3}$) all roots on the unit circle.

However we could also have begun with the other real involution of type (r). The involutions $t \to -t$ and $t \to 1/t$ are interchanged by the substitution $u = \frac{t+1}{t-1}$; making this change replaces c by $\frac{1-c}{1+3c}$, and interchanges cases (ii) and (iii).

In case (a), we have $\Delta = T^3 - 27K^2 = (1-9c^2)^2 > 0$, and $j = \frac{T^3}{T^3-27K^2} = \frac{(1+3c^2)^3}{(1-9c^2)^2}$, which is > 1. If $c = \pm\frac{1}{3}$, $j = \infty$, if $c = \pm1$, $j = 1$; if $c = 0$, $j = 1$. In each of the cases (i)-(iii), j can take any value > 1; so j is of no use to distinguish these cases.

In case (b), the roots have the form $\pm\alpha, \pm i/\alpha$; two are real and two are complex conjugate, and f takes the form $t^4 + 6ct^2 - 1$. Here $T = 3c^2 - 1$, $K = -c(1+c^2)$, and $\Delta = -(1+9c^2)^2 < 0$.

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[2] C. T. C. Wall, *Pencils of real binary cubics*, Math. Proc. Camb. Phil. Soc. **93** (1983), 477–484.

C.T.C.Wall took his Ph.D. in Cambridge in 1960, and held a teaching post at the University of Liverpool from 1965-1999, except from 1983-88 when he held an EPSRC Senior Fellowship. His research has mainly been in differential topology and singularity theory

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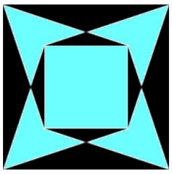

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PROBLEMS

IAN SHORT

Problems

The first problem is a corrected version of Problem 82.1, which was missing some hypotheses. The problem uses the usual notation $x_1, x_2, \dots, x_n$ for the components of a vector x in $\mathbb{R}^n$.

Problem 84.1. Suppose that u and v are linearly independent vectors in $\mathbb{R}^n$ with

$$0 < u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \quad \text{and} \quad v_1 > v_2 > \cdots > v_n > 0.$$

Given $x \in \mathbb{R}^n$, let y be the orthogonal projection of x onto the subspace spanned by u and v; thus $y = \lambda u + \mu v$, for uniquely determined real numbers λ and μ. Prove that if

$$x_1 > x_2 > \cdots > x_n > 0,$$

then μ is positive.

The second problem was contributed by Finbarr Holland, of University College Cork.

Problem 84.2. Given any finite collection $L_1, L_2, \dots, L_n$ of infinite straight lines in the complex plane, find a formula in terms of data specifying $L_1, L_2, \dots, L_n$ for a differentiable function $f \colon \mathbb{R} \longrightarrow \mathbb{C}$ with the property that each line L_i is tangent to the curve $f(\mathbb{R})$.

For the third problem, we use the definition of a directed graph that allows loops and multiple directed edges with the same source and target vertex.

Problem 84.3. Suppose that each edge of a finite directed graph G is coloured in one of some finite collection of different colours, with the property that for each colour c and vertex v, there is precisely one directed edge with colour c and target vertex v. Prove that for any infinite sequence of colours $c_1, c_2, \dots$ there is an infinite walk $e_1, e_2, \dots$ of directed edges of G such that, for each index i, e_i has colour c_i and the target vertex of e_i equals the source vertex of e_{i+1}.

Solutions

Here are solutions to the problems from *Bulletin* Number 82.

Problem 82.1 was false. It is replaced by Problem 84.1. Thanks to Omran Kouba of the Higher Institute for Applied Sciences and Technology, Damascus, Syria and the North Kildare Mathematics Problem Club for providing examples to demonstrate the falsehood of Problem 82.1.

The next problem was solved by Omran Kouba, the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland. We present the solution of the Problem Club.

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Problem 82.2. Prove that

$$\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x}\, dx = \log 2.$$

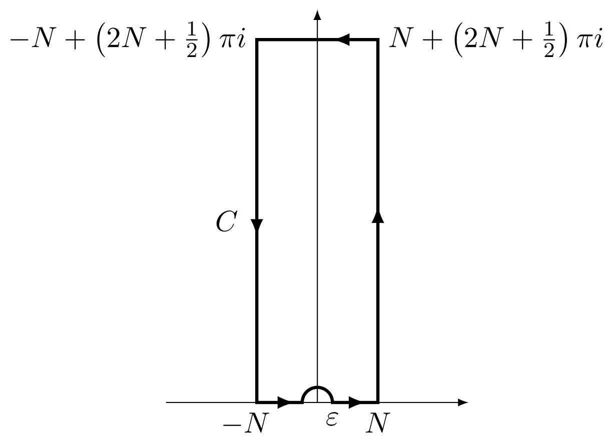
Solution 82.2. Let

$$f(z) = \frac{\sinh z - z}{z^2 \sinh z},$$

and observe that f is an even function, so

$$\int_0^\infty f(x)\, dx = \frac{1}{2}\int_{-\infty}^\infty f(x)\, dx.$$

Let N be a positive integer and let ε be a positive constant, less than 1. Let C be the contour shown in the figure, traversed once anticlockwise.



One can check that the integral of f along the semicircle of radius ε tends to 0 as $\varepsilon \to 0$. Next, we wish to show that the integral of f along the vertical edges and top edge of C tends to 0 as $N \to \infty$. By writing

$$f(z) = \frac{1}{z^2} - \frac{1}{z \sinh z}$$

we see that the main task is to check that the integral of $1/(z \sinh z)$ along these contours tends to 0 as $N \to \infty$. This is easily done for the two vertical contours of C by using the inequality $|\sinh z| \geq \sinh N$ for any point z on one of the vertical contours.

Now consider a point $z = x + \left(2N + \frac{1}{2}\right)\pi i$ on the top contour Γ of C. Observe that $\sinh z = i \cosh x$. Hence

$$\left|\int_\Gamma \frac{1}{z \sinh z}\right| \leq \frac{1}{\left(2N + \frac{1}{2}\right)\pi}\int_{-N}^N \frac{1}{\cosh x}\, dx \leq \frac{1}{2N + \frac{1}{2}},$$

so the integral of $1/(z \sinh z)$ along this contour tends to 0 as $N \to \infty$ also.

Hence, by applying the residue theorem and then taking limits, we see that

$$\int_{-\infty}^\infty f(x)\, dx$$

is equal to $2\pi i$ times the sum of the residues of f in the upper half-plane. The poles of f in the upper half-plane occur at $\pi n i$, for each positive integer n, and the residue of f at $\pi n i$ is $(-1)^{n+1}/(\pi n i)$. Hence

$$\int_0^\infty f(x)\, dx = \frac{1}{2} \times 2\pi i \sum_{n=1}^\infty \frac{(-1)^{n+1}}{\pi n i} = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} = \log 2. \qquad \square$$

No solutions were received for the extended version of Problem 82.2, which asks for a proof of the integral formula

$$\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x}\, e^{-x}\, dx = \log \pi - 1.$$

The third problem was solved by Omran Kouba, the North Kildare Mathematics Problem Club, and Henry Ricardo of the Westchester Area Math Circle, New York, USA. Solutions also appeared in Issue 255 of the M500 Society of the Open University, from which the problem was taken. The solution we present is an amalgamation of these solutions.

*Problem 82.3***.** Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(5n-3)(5n-2)} = \frac{\pi}{5}\sqrt{1-\frac{2}{\sqrt{5}}}.$$

Solution 82.3. Recall the well-known result that

$$\sum_{n=1}^{\infty} \frac{1}{n^2-a^2} = \frac{1}{2a}\left(\frac{1}{a} - \pi \cot \pi a\right), \qquad (*)$$

for $0 < a < 1$. This can be proved by methods of contour integration, or by taking the logarithm and differentiating each side of the equation

$$\frac{\sin \pi a}{\pi a} = \prod_{n=1}^{\infty}\left(1 - \frac{a^2}{n^2}\right)$$

with respect to a. Now observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2-a^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2-a^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2-a^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2-a^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2-(a/2)^2}.\end{aligned}$$

By applying $(*)$ and simplifying we can check that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2-a^2} = \frac{\pi}{4a}\tan(\pi a/2).$$

Next, we have

$$\frac{1}{(5n-3)(5n-2)} = \frac{1}{25(n-1/2)^2-(1/2)^2} = \frac{4}{25((2n-1)^2-(1/5)^2)}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{(5n-3)(5n-2)} = \frac{4}{25} \times \frac{5\pi}{4}\tan(\pi/10) = \frac{\pi}{5}\sqrt{1-\frac{2}{\sqrt{5}}}. \qquad \square$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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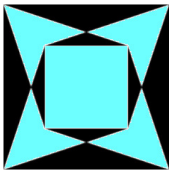

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