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The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

Correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

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and only if not possible in electronic form to the address

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Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

EDITORIAL

In the past, when the Bulletin was called the Newsletter, it carried a page of news, mainly concerning the comings and goings of mathematical personnel. Colm Mulcahy told us that these reports were very useful to him in his search for facts for his Archive of Irish Mathematics. He suggested that we reinstate the item on a regular basis, and this seems a good idea. So I now invite the Irish schools to contribute news. Typically, we expect each item to consist of one or two sentences, and suitable material would be new appointments and sabbatical visitors. It would be best if reports were channeled through the local representatives, but to begin with we do not want to be unduly restrictive. Please send reports to `mailto://ims.bulletin+news@gmail.com`.

The European Women in Mathematics association (EWM) was founded in 1986, and has several hundred members and coordinators in 33 European countries. Every other year, the EWM holds a general meeting and a summer school. A newsletter is published twice a year, the EWM has an e-mail network EWM-ALL, a facebook group and a website: `https://www.europeanwomeninmaths.org/`. The EWM aims to: encourage women to study mathematics, support women in their careers, and give prominence and visibility to women mathematicians. Moreover, the EWM awards early-career grants, offers mentoring for members in need of guidance regarding their mathematical career, and balancing career and family, and provides partial financial support for national meetings.

As promised in the last issue, the Bulletin's page layout has been changed, reducing the number of pages needed for a given amount of content, and hence reducing the cost to members of printing material. As before, to facilitate members who might wish to print the whole issue, the website will carry a pdf file of the whole Bulletin 83, in addition to the usual pdf files of the individual articles. As a further convenience (which may suit some Departments and Libraries), a printed and bound copy of this Bulletin may be ordered online on a print-on-demand basis[1].

This year's IMS September meeting will be held in NUI Galway, from Thursday September 5 to Friday September 6, 2019. Please make a note in your diaries. The 2-day programme will consist of a number of invited talks by speakers from Ireland and abroad. The aim of the meeting is to reflect the diversity of the mathematical community in Ireland and the scientific interests of the members of the IMS. All are welcome to attend. Further details will be announced soon. The organising committee comprises Angela Carnevale, Michel Destrade, Götz Pfeiffer and Rachel Quinlan.

The directed graph implicit in the database of the Mathematics Genealogy project `https://www.genealogy.math.ndsu.nodak.edu` has about 200,000 nodes, and one component includes about nine-tenths of the whole. It is probable that all doctorate-holding IMS members belong to this component. The earliest-dated root of this main component is, at present, one Sharaf al-Din al-Tusi, who has over 159,000 descendents in the graph. His scientific grandson, Nasir al-Din al-Tusi assembled a famous set of astronomical tables for the Il-Khan, a grandson of Genghiz Khan. These were passed on by his student Shams ad-Din al-Bukhari to the greek bishop Gregory Chioniades, and thus found their way into Byzantium, and thence through a chain of students to

[1]Go to `www.lulu.com` and search for *Irish Mathematical Society Bulletin*.

Verona, then to Padua and Nicholas Copernicus. Copernicus has over 156,000 descendants in the genealogy graph, so one may also say that most living mathematicians are his descendants. In this issue, we carry an article with a "tale of two cities" (Padua and Uppsala), that tells the interesting story of how the burial-place of Copernicus was identified.

We also publish, *inter alia*, a short, provocative article by Bernard Beauzamy, and an obituary article on Richard Timoney, who is greatly missed.

Links for Postgraduate Study

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: `mailto://maths@dcu.ie`
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MU: `mailto://mathsstatspg@mu.ie`
QUB: `http://www.qub.ac.uk/puremaths/Funded_PG_2016.html`
TCD: `http://www.maths.tcd.ie/postgraduate/`
UCC: `http://www.ucc.ie/en/matsci/postgraduate/`
UCD: `mailto://nuria.garcia@ucd.ie`
UL: `mailto://sarah.mitchell@ul.ie`
UU: `http://www.compeng.ulster.ac.uk/rgs/`

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor. All links are live, and hence may be accessed by a click, when read in a suitable pdf reader.

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CONTENTS

NOTICES FROM THE SOCIETY

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(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€200
Ordinary member	€30
Student member	€15
DMV, I.M.T.A., NZMS or RSME reciprocity member	€15
AMS reciprocity member	$20

The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 40.
If paid in sterling then the subscription is £30.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 40.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form with one year's subscription to:

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Real Life Decisions

BERNARD BEAUZAMY

Abstract. We show that everyday decisions, which should be taken on rational grounds, are most of the time taken on irrational basis, with very little room for mathematics. The reason is that data are not sufficient in order to ensure a proper scientific treatment of the problem, which is, therefore, treated in a completely non-scientific way. The mathematical community should develop tools for decision help in such situations.

1. All of us have to take decisions, on an everyday basis, and we all wish these decisions to be taken on rational grounds, perhaps even on scientific grounds: that would be great! Of course, we all know that this is not possible in general, since most decisions are taken on moral, religious, or emotional basis. Still, we have the impression that, in a few cases, Reason might enter. For instance, the width of a road (one lane, two lanes or more) might be decided from traffic data.

The academic community, in mathematics, has a strong feeling, even a conviction: they are completely certain that such problems should be addressed with the use of mathematics. However, this is not the case in practice, as we will see: even rational problems, such as the width of the road, are solved by irrational means, containing no mathematics. The reason is that, for the mathematical tools to apply, a lot of data would be needed: precise information about traffic at various times, the nature of the environment (characteristics of the landscape), pluviometry, and so on. Since these data are incomplete or missing, the decision will be taken based on rough information. Of course, if the land nearby belongs to a guy the Mayor does not like, the road will be promptly enlarged. Conversely, if a protected animal was seen nearby, the road will never be enlarged. Otherwise, it depends on budgets, public demand, severity of past accidents and many other factors, as usual in a modern democracy, with very little room for mathematics.

The result of such an attitude, for the academic community in mathematics, is quite negative: they have lost a lot of teaching duties, which are now taken by Engineering departments, Environment departments, and so on: since they could not provide a satisfactory answer, the mathematicians left the decision to people who come with "rough" tools, more appropriate in such situations.

Let us look more in detail at a specific example, which seems completely rational, namely predictive maintenance. An industry has a set of equipment (say for instance electricity distribution, and the equipment would be transformers), and wonders about the replacement of this equipment. All readers with good faith will admit that this problem has no moral contents. Some, with ecological minds, might consider that it should be addressed on a religious side: since the French transformers carry nuclear electricity, they should be destroyed, not replaced. Let us skip this debate.

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2. The collapse of Operation Research

Any mathematician from the academic world will consider that the above problem should be addressed by means of Operation Research Theory, which will provide a complete answer: not only which transformers and when, but also location of parts, size of teams, timetables for their actions, minimal budget and so on. But this is not so in practice: all necessary data are missing. One does not know much about the transformers (date of installation, dates of previous repairs), even less about the parts which might be necessary (are the old ones still produced?) and almost nothing about the people who are in charge. And about the budget, the question is: this year, or next decade?

Therefore, no Electricity Company solves this problem by means of Operation Research. What they do is this: they wait until something happens (usually after a storm), and then they fix as quickly as they can.

3. What the problem is

If we look more deeply at the problem, we see indeed that it is not simple at all. Roughly speaking, we have the choice between three possible attitudes:

(1) We double each item of equipment, so that there is a constant supply of electricity. This is very costly, and done only in critical situations, where no interruption is allowed (hospitals, for instance).
(2) We keep a lot of parts in some central place, and if some failure occurs, we take the part where it is and bring it quickly to the transformer to be repaired. This solution is rather cheap, but still some interruption will occur in case of a failure.
(3) We repair and replace the transformers in advance, according to some predefined scheme.

4. Approaching a solution

From a mathematical point of view, the third approach should have our preference. Each year, according to the existing budget, we can define how many transformers should be addressed. Then, a simple statistical survey will allow us to understand the structure of the availability. For instance, the transformers should last at least 50 years, but if they are close to the sea, they live much less. So, we will draw up very rough probability laws, saying: for this type, at this place, life expectancy is more than 40 years, and for this type at that place, less than 30 years.

Still, this is not a complete, solid, basis for decision, since some transformers are more critical than others. So one has to find, among the most critical, the ones which are oldest or more exposed.

One sees, from this example, that the decision will be taken on grounds which are not mathematical, but still contain a significant amount of mathematics, which is good for us. In any case, one should always remember that, as we say in French: *Satan conduit le bal.*

Bernard Beauzamy, born 1949, was University Professor 1979-1995. In 1995, he created the Company "Société de Calcul Mathmatique SA"and has been Chairman and Chief Executive Officer of this Company since then.

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Generalization of an Identity about Euler's totient function

MEI BAI, WENCHANG CHU AND NADIA N. LI

Abstract. A general formula about arithmetic functions is established that contains some interesting identities as special cases, including one about Euler's totient function proposed by Max A. Alekseyev (2011).

In number theory, Euler's totient function $\varphi(n)$ is well–known, which counts the natural numbers $\leq n$ that are relatively prime to n and satisfies the property

$$\sum_{d|n} \varphi(d) = n.$$

Max A. Alekseyev [1] proposed the following monthly problem. For a positive integer m, prove that

$$\sum_{k=0}^{m-1} \varphi(2k+1) \left\lfloor \frac{m+k}{2k+1} \right\rfloor = m^2, \tag{1}$$

where $\lfloor x \rfloor$ denotes the integer part for a real number x.

Inspired by this problem, we shall establish the following general formula.

Theorem 1 (Main theorem)**.** *Let* $F(n) = \sum_{d|n} f(d)$*, then*

$$\sum_{n=1}^{m} F(2n-1) = \sum_{k=1}^{m} f(2k-1) \left\lfloor \frac{m+k-1}{2k-1} \right\rfloor.$$

Proof. By replacing d by $2k-1$ and then interchanging the summation order, we have

$$\sum_{n=1}^{m} F(2n-1) = \sum_{n=1}^{m} \sum_{d|2n-1} f(d)$$

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$$=\sum_{n=1}^{m}\sum_{(2k-1)|(2n-1)} f(2k-1)$$
$$=\sum_{k=1}^{2m-1} f(2k-1)\sum_{\substack{n=1\\(2k-1)|(2n-1)}}^{m} 1.$$

The last sum counts the number of odd multiples of $2k-1$ from 1 to $2m-1$, which is equal to the number of all the multiples of $2k-1$ minus that of the even multiples of $2k-1$. Hence we have

$$\sum_{\substack{n=1\\(2k-1)|(2n-1)}}^{m} 1=\left\lfloor\frac{2m-1}{2k-1}\right\rfloor-\left\lfloor\frac{2m-1}{4k-2}\right\rfloor=\left\lfloor\frac{m+k-1}{2k-1}\right\rfloor.$$

This proves the formula displayed in Theorem 1. □

Replacing f by φ in Theorem 1, we confirm immediately Alekseyev's identity (1).

Instead, by assigning, in Theorem 1, F to be σ_ℓ, the ℓth power sum of the divisors of n

$$\sigma_\ell(n)=\sum_{d|n} d^\ell,$$

we derive another interesting formula:

Corollary 2.

$$\sum_{k=1}^{m}\sigma_\ell(2n-1)=\sum_{k=1}^{m}(2k-1)^\ell\left\lfloor\frac{m+k-1}{2k-1}\right\rfloor.$$

Furthermore, we can work out an identity for the Liouville function, which is defined by

$$\lambda(n)=(-1)^{k_1+k_2+\cdots+k_m}$$

if n is factorized into $n=\prod_{i=1}^m p_i^{k_i}$ with $\{p_i\}_{i=1}^m$ being distinct primes. The λ function admits the property

$$\sum_{d|n}\lambda(d)=\begin{cases}1, & \text{if } n \text{ is a square;}\\ 0, & \text{otherwise.}\end{cases}$$

Now take $f(n)=\lambda(n)$ in Theorem 1. We can see that $\sum_{n=1}^m F(2n-1)$ is, in fact, the number $\lfloor\sqrt{2m}\rfloor$ of odd squares from 1 to $2m$ minus the number $\lfloor\sqrt{\frac{m}{2}}\rfloor$ of even squares from 1 to $2m$:

$$\sum_{n=1}^{m}F(2n-1)=\left\lfloor\sqrt{2m}\right\rfloor-\left\lfloor\sqrt{\frac{m}{2}}\right\rfloor=\left\lfloor\sqrt{\frac{m}{2}}+\frac{1}{2}\right\rfloor.$$

We have consequently established the following identity.

Corollary 3.

$$\sum_{k=1}^{m}\lambda(2k-1)\left\lfloor\frac{m+k-1}{2k-1}\right\rfloor=\left\lfloor\sqrt{\frac{m}{2}}+\frac{1}{2}\right\rfloor.$$

There exist similar results in the literature. In fact, one can find in Burton [2, Theorem 6.11] the following formula. Suppose that

$$F(n)=\sum_{d|n} f(d)$$

then there holds

$$\sum_{n=1}^{m} F(n)=\sum_{k=1}^{m} f(k)\left\lfloor\frac{m}{k}\right\rfloor. \tag{2}$$

Analogously, we can deduce from (2) the following identities:

$$\sum_{k=1}^{m} \varphi(k)\left\lfloor\frac{m}{k}\right\rfloor=\binom{m+1}{2}, \tag{3}$$

$$\sum_{n=1}^{m} \sigma_{\ell}(n)=\sum_{k=1}^{m} k^{\ell}\left\lfloor\frac{m}{k}\right\rfloor, \tag{4}$$

$$\sum_{n=1}^{m} \lambda(n)\left\lfloor\frac{m}{n}\right\rfloor=\lfloor\sqrt{m}\rfloor. \tag{5}$$

Among these formulae, (3) and the two particular cases $\ell=0,1$ of (4) appeared previously in Burton's book (Page 120, Corollaries 1 & 2).

References

[1] M. A. Alekseyev, *Squares from totients: Problem 11544,* Amer. Math. Monthly 118 (2011), Page 84; Solution ibid 119 (2012), Page 807.

[2] David M. Burton, *Elementary Number Theory* (6th edition), McGraw Hill Higher Education, 2007.

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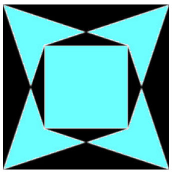

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The Hornich space and a subspace of BMOA

JOSEPH A. CIMA

Abstract. In this article we make a linear association between the Hornich space and a subspace of the BMOA space on the unit disc. We obtain a few new results that arise from the association and identify some classes of functions in BMOA that were hitherto unknown (or utilized).

1. Introduction

In this article we consider two normed spaces of analytic functions on the unit disc D and produce a linear isomorphism between them. The purpose is not just to make the identification: it suggests a large class of functions that were hitherto unknown in one of the spaces. The first space, which we denote by $\mathscr{H}$, is a real Banach space and is known to those specialists who work in geometric function theory. The second is a certain subspace $\mathscr{K}$ of the space BMO of functions of bounded mean oscillation. Indeed, $\mathscr{K}$ is a subspace of the space BMOA of analytic functions of bounded mean oscillation. The space BMO has broader applications and is found in research in the areas of partial differential equations, harmonic analysis and other areas. There are classes of functions in $\mathscr{H}$ described by their geometry and it is of interest to see what the implications of the connection to the subspace of BMOA will imply about these subsets, and on the obverse side of the relation it will be of interest to see what the geometry will imply about the behavior of the associated BMOA functions.

2. The Hornich space and some of its properties

Let $\mathcal{A}$ denote the set of functions f which are holomorphic on the unit disc D, and which are locally univalent there (i.e. $f'(z) \neq 0, |z| < 1$), and normalized by $f(0) = 0, f'(0) = 1$. For each $r \in (0,1)$ and $f \in \mathcal{A}$ define the quantity

$$I(r;f) \equiv \frac{1}{2\pi}\int_0^{2\pi} |\log|f'(re^{\imath t})||dt.$$

We define a set $\mathscr{L}$ as follows.

$$\mathscr{L} = \{f \in \mathcal{A} \mid \sup_r(I(r;f)) < \infty).$$

We define a few other sets which we wish to refer to later in this paper. Namely, set

$$\mathscr{M} = \{\log f'(z) \mid f \in \mathscr{L}\},$$

and

$$\mathscr{N} = \{f' \mid f \in \mathscr{L}\}.$$

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The set $\mathscr{M}$ is a subset of the space of Cauchy Transforms of finite Borel measures on the unit circle (see Cima and Pfaltzgraff [4] and Cima, Matheson, and Ross [3]). In particular there is a function $M(\zeta)$ of bounded variation on the unit circle $\mathbb{T}$ such that

$$\log f'(z) = \int_{\mathbb{T}} \frac{(\zeta + z)}{(\zeta - z)} dM(\zeta), \quad z \in D.$$

Note that it is easy to see that each functions f in $\mathscr{N}$ has derivative f' in the Nevanlinna class, and hence has radial boundary values $f'(\zeta)$ at almost all $\zeta \in \mathbb{T}$.

For f and g in $\mathscr{L}$ define

$$f \oplus g(z) = \int_o^z f'(w)g'(w)dw,$$

and for $r \in \mathbb{R}$ set

$$r \odot f(z) = \int_o^z (f'(w))^r dw.$$

With these operations, $\mathscr{L}$ becomes a real vector space, and the expression

$$||f||_{\mathscr{L}} \equiv \sup_{0<r<1} (I(r; f))$$

is a norm on $\mathscr{L}$, and in this norm the space is complete, that is, $\mathscr{L}$ is a real Banach space. Although not much is known about a generic vector in $\mathscr{L}$ or the relationship of the entire space $\mathscr{L}$ to other classical Banach spaces of analytic functions (e.g. weighted Bergman spaces or weighted Bloch spaces) there are interesting subspaces and subsets of $\mathscr{L}$ for which many interesting things are known. Further the interested reader will find most of what is necessary for univalent functions in the books of Pommerenke [10], and Duren [5], and facts about the classical Hardy spaces in books by Garnett [6] or Cima, Matheson,and Ross [3]. Further, set

$$\mathscr{L}_1 \equiv \left\{ f \in \mathscr{L} \;\mid\; \|f\|_{\mathscr{L}} = \int_{\mathbb{T}} |\log |f'(e^{\imath t})||dt \right\}.$$

It is shown in Cima and Pfaltzgraff [4], that $\mathscr{L}_1$ is a real, separable Banach space (a subset of the harmonic Hardy space, h^1) . In his paper [8], Hornich defined a subspace of $\mathscr{L}_1$ by requiring the following

$$\mathscr{H} \equiv \left\{ f \in \mathscr{L}_1 \;\mid\; \sup_{(|z|<1)} |\arg f'(z)| < \infty \right\}.$$

Hornich equipped this space with a norm

$$||f||_{\mathscr{H}} = \sup \left\{ \, (\, |\arg f'(z) - \arg f'(w)| \quad | \;\; z \in D \;\; ; \;\; w \in D \,) \right\}.$$

Equipped with this norm $\mathscr{H}$ becomes a real, separable Banach space. Note that the set S of all normalized univalent functions on D is not a subset of $\mathscr{H}$. For example univalent functions mapping onto spiral-like domains are not in $\mathscr{H}$. But there are many well known and well researched subsets of univalent functions in $\mathscr{H}$, see Cima and Pfaltzgraff [4], Ali and Vasudevarao [1]. Without attempting to supply a comprehensive list of references we give an abbreviated set of references with the understanding that the interested reader will find most of the necessary background in the references in the papers and books that are listed.

As noted, $S \cap \mathscr{H}$ contains several sets of functions which are distinguished by their additional geometric properties. Among these are the class $\mathcal{C}$ of normalized univalent functions of S mapping onto convex domains, the class $\mathcal{S}$ of functions in S mapping onto starlike domains and the class $\mathcal{K}$ of close to convex functions in S (that is functions $f \in S$ for which there exists a starlike function $g \in S$ for which

$$\Re\frac{zf'(z)}{g(z)} > 0, \ (|z| < 1).$$

However, $\mathscr{H} \setminus S$ contains several other sets of functions which are locally univalent but not necessarily univalent. As examples we mention the following which come from the interesting paper, Ali and Vasudervarao [1]. Let $\alpha > 0$ and set

$$G(\alpha) = \left\{ f \in \mathcal{A} | \ \Re\left(1 + \frac{zf"(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2} \right\}$$

Functions in $G(\alpha)$ for $\alpha \leq 1$ are univalent $(G(\alpha) \subset (S))$ but for $\alpha > 1$ the functions in $G(\alpha)$ need not be univalent. In particular in [1], it is shown that if $f \in \mathscr{H}$ with $||f||_{\mathscr{H}} \leq \alpha\pi$ then $f \in G(\alpha)$. Other examples of non-univalent classes containing functions in $\mathscr{H}$ are given there and we refer the interested reader to that source for these other examples and to many useful background references.

Most of the work on this topic has been done by researchers interested in geometric function theory, and much of it involves such topics as topological properties (separability and non-separability), extreme points of certain classical subsets of $\mathscr{H}$, the inclusion of certain well known subsets of analytic functions in $\mathscr{H}$, and inclusion of $\mathscr{H}$ in some classical Banach spaces of analytic functions on the unit disc. The following can be found in Cima and Pfaltzgraff [4] and Ali and Vasudevarao [1]. The notation $H^p(D)$ denotes the classical Hardy space of the unit disc.

Theorem 2.1. *The space $\mathscr{H}$ is a subset of $\bigcup_{0<p<\infty} H^p(D)$.*

In Section 5 we will strengthen this result.

3. The space BMOA

Now we wish to point out several pertinent properties of the space BMO and BMOA on the unit circle. An excellent source for material on these sets is the set of notes by Girela [7]. An $L^2(\mathbb{T}, dm)$ function f on the unit circle $\mathbb{T}$ is said to have *bounded mean oscillation* (BMO) if there is a positive number M such that for each interval $I \subset \mathbb{T}$ set

$$U(f, I) = \frac{1}{|I|}\int_I |f(\zeta) - f_I| dm(\zeta) \leq M,$$

where $\zeta = \exp^{\imath\theta}$, $dm(\zeta) = d\theta/2\pi$ and $f_I = \frac{\int_I f(\zeta)dm(\zeta)}{|I|}$. The set of such functions f for which $U = \sup_I U(f, I) < \infty$ with pointwise sum and scalar products is a complex Banach space with

$$||f||_b = |f(0)| + U.$$

The set can be made into a Banach space in several ways. The space BMOA is the subspace of BMO consisting of functions from the Hardy spaces $H^2 \bigcap \mathrm{BMO}(mathbbT)$. In addition we will be considering a subspace of BMOA for which $f(0) = 0$. The real variable version of this space BMO($\mathbb{T}$) (or more generally BMO($\mathbb{R}^n$)) suitably defined) has an illustrious history, but as we focus on BMOA($\mathbb{T}$) we will not try to outline this history but again mention the classic notes of Girela where the case BMO($\mathbb{T}$) is treated in detail. Some of the other useful equivalent definitions for the space BMOA are as follows. Assume $a \in D$ and ϕ_a is an automorphism of the unit disc mapping a to zero. For $p \in (0, \infty)$ and f analytic on D define

$$||f||_{\mathrm{BMOA}_p} \equiv |f(0)| + \sup_{a \in D} ||f_a||_{H^p},$$

where $f_a \equiv f \circ \phi_a - f(a)$.

Theorem 3.1. *With the above notation the following are equivalent:*
(a)$||f||_{\mathrm{BMOA}_p} < \infty$.
(b) *The family* $\{f \circ S - f(S(0)) \mid S \in Aut(D)\}$ *is a bounded set in* H^p.
(c) *The family* $\{f_a \mid a \in D\}$ *is a bounded set in* H^p.

All of these give equivalent definitions for BMOA. The space BMOA is not separable in this norm and the closure of the analytic polynomials in this norm is a separable closed subspace, labeled VMOA, Both VMOA and BMOA play roles in duality situations for the Hardy space theory.

In the case of analytic functions, for some time there were open problems related to some linear functional analysis questions for the classical Hardy spaces. In particular one such question was the following. What are the topological duals of the classical Hardy spaces $H^p(\mathbb{T})$? It was possible to satisfactorily identify the duals for $1 < p < \infty$, but the question for $p = 1$ had a functional analysis solution to the question stated in terms of a coset space. Namely to characterize the dual of $H^1(\mathbb{T})$ one uses duality between $L^1(T)$ and L^∞ and an annihilator. Fefferman's initial work on BMOA showed that the space of analytic functions on the unit disc with bounded mean oscillation was the correct function space dual.

There are several important ways to recognize whether or not an analytic function on the unit disc is in this space. It is known that if an analytic function has a bounded real part then it is in BMOA. Further,if $f(z) = U(z) + \imath\widetilde{U}(z) \in$ BMOA and $U \in L_\infty(D)$ then since $||\widetilde{U}||_{L^2(\mathbb{T})} \le C||U||_{L^\infty(\mathbb{T})}$ we have $||f||_b \le C||U||_{L^\infty}$. Further, if P denotes the Riesz transform on $L^2(\mathbb{T})$ then it is well known that $f \in$ BMOA if and only if there is $\psi \in L^\infty(T)$ for which $f(z) = P(\psi)(z)$.

For our purposes we need one important representation of BMOA functions. For U a bounded, harmonic function of the unit disc D, let $\widetilde{U}$ be a suitably normalized harmonic conjugate.

Theorem 3.2. *For F in BMOA there exist two bounded harmonic functions U and V so that*

$$F(z) = (U(z) + \imath\widetilde{U}(z)) + \imath(V(z) + \imath\widetilde{V}(z)) = F_1(z) + \imath F_2(z).$$

(Again see {Girela [7], Theorem 7.2} .)

Note although BMOA has many equivalent norms, the following is known.

Theorem 3.3. *If $F = F_1 + \imath F_2$ with $F_j, j = 1, 2$ as above then for some absolute constant C,*

$$||F||_b \quad \le C \quad \inf\{||U||_\infty + ||V||_\infty\}$$

as F_1 and F_2 vary over all such decompositions.

4. The subspace $\mathscr{K}$

So let us define the real subspace of BMOA that we are interested in. Let

$$\mathscr{K} = \{F \in \mathrm{BMOA} \mid F(0) = 0 \ , \ \Im(F(z)) \in L^\infty(\mathbb{T}).$$

Proposition 4.1. *The set $\mathscr{K}$ is a real subspace of BMOA. $\mathscr{K}$ is not a closed subset of BMOA.*

Before supplying the proof of Proposition 4.1 we want to define the linear mapping in question between $\mathscr{H}$ and $\mathscr{K}$. For $f \in \mathscr{H}$ define

$$L(f)(z) = F(z) = \ln(f'(z)) = \ln|f'(z)| + \imath \arg f'(z) = U(z) + \imath\widetilde{U}(z), \quad z \in D.$$

Note that by our assumptions L is well defined by the above comments and $F \in$ BMOA. (Since $\arg f'(z) \in L^\infty(D)$ one has $F \in$ BMOA by the above.) Further, $L(f)(0) = F(0) = 0$, implying that $L(f) \in \mathscr{K}$.

Note that

$$L(f \oplus g)(z) = L(f)(z) + L(g)(z),$$

and

$$L(r \odot g)(z) = rL(g)(z),$$

so that L is a linear mapping . It is now clear that L is bounded since

$$\|L(f)\|_b \le C \cdot \sup\{|\arg f'(z)| \mid z \in D\} \le 2C \cdot \|f\|_{\mathscr{H}}.$$

By the normalization it is obvious that L is one to one and further setting

$$f(z) = \int_o^z \exp(F(w))dw$$

that L has a linear inverse. Note that $\ln(1-z) \in \mathscr{K}$ but it is not in VMOA so that $\mathscr{K} \setminus VMOA$ is not empty.

We can now proceed to the proof of Proposition 4.1.

Proof. It is obvious that $\mathscr{K}$ is a real normed subspace of BMOA.

We give an example to show that $\mathscr{K}$ is not closed in BMOA. Let Ω be the unbounded domain, symmetric in the x and y axis, containing 0 and 1, bounded on the right by a smooth curve Γ. Assume that $2 \in \Gamma$ and as $z \in \Gamma, \Im(z) > 0$ tends to ∞ the curve tends smoothly to the positive real y axis and reflect to get the unbounded domain Ω in $\mathbb{C}$. Then take the Riemann mapping ϕ from the unit disc onto $\Omega, \phi(0) = 0, \phi'(0) = 1$. It is well known that ϕ is in VMOA. Hence, there is a sequence of polynomials, say p_n, tending to ϕ in VMOA and hence, uniformly on compacta of D. We can normalize the p_n to get a new family of polynomials, say q_n, with $q_n(0) = 0, q_n'(0) = 1$, and with q_n tending to ϕ in BMOA. Thus, ϕ is in $\overline{\mathscr{K}}$ but is not in $\mathscr{K}$. ☐

Let us state at this point another important fact about BMOA, known as the John-Nirenberg Theorem and for which we shall have a use. First, recall the equivalent norms for BMOA, and use the notation BMOA_p for them, $\infty > p > 0$. Fix p as above and $F \in \text{BMOA}$, and write

$$||F||_b^p = \sup_{z \in D} \int_{\mathbb{T}} |F - F(z)|^p P_z ds = \sup_{z \in D} (|F|^p(z) - |F(z)|^p)$$

where P_z is the Poisson kernel evaluated at z . The John-Nirenberg inequality (see Knese [9]) can be stated as follows.

Theorem 4.2. *For $F \in \text{BMOA}$ and any $\epsilon < \frac{2}{\sqrt{2}||F||_{\text{BMOA}_2}}$, we have*

$$\int_{\mathbb{T}} \exp^{\epsilon|F-F(z)|} P_z ds < \frac{3}{(1 - \frac{\epsilon\sqrt{e}||F||_{\text{BMOA}_2}}{2})^{3/2}}$$

5. Implications

Recall that if $f \in \mathscr{H}$ then f is in some Hardy space, and if $f \in \mathscr{L}_1$ then $\int_{\mathbb{T}} |\log|f'(\exp^{\imath\theta})||d\theta < \infty$. In general functions in Hardy spaces need not have derivatives which are well behaved. For example there are functions in the disc algebra (hence in all the Hardy spaces) for which the derivatives are not in the Nevanlinna class. So a question to consider is the following. If f is in $\mathscr{L}_1$ is f' in some Hardy space? We have not been able to answer this in this form but using information gleaned from the relationship established by the map L we can give an answer when f is in the Hornich space.

Theorem 5.1. *If $f \in \mathscr{H}$ then there is a $p > 0$ for which $f' \in H^p$.*

Proof. Assume $f \in \mathscr{H}$, and $L(f) = F \in$ BMOA. Apply the John-Nirenberg Theorem, with $z = 0$, to obtain

$$\int_{\mathbb{T}} \exp^{\epsilon|F(\exp^{\imath\theta})|} d\theta/2\pi = \int_{\mathbb{T}} \exp^{\epsilon|\log f'(\exp^{\imath\theta})|} d\theta/2\pi \geq \int_{\mathbb{T}} \exp^{\epsilon|\log|f'(\exp^{\imath\theta})} |d\theta/2\pi$$

$$\geq \int_{\mathbb{T}\bigcap(|f'|\geq 1)} |f'(\exp^{\imath\theta})|^{\epsilon} d\theta/2\pi$$

Hence, $f' \in H^{\epsilon}$. It is now well known that if $\epsilon < 1$ then $f \in H^q$, where $q = \frac{\epsilon}{(1-\epsilon)}$ and if $\epsilon > 1$, then $f \in \Lambda_{\alpha}, \alpha = 1 - 1/\epsilon$. The space Λ_{α} is the Lipschitz space of order α.

□

We wish to introduce another mathematical object that plays a major role in the BMOA theory and for which we think the pullback to the $\mathscr{H}$ space is of interest. Namely, given a positive measure $d\mu(z)$ on the unit disc, μ is called a *Carleson measure* if for $h > 0$, $|\zeta| = 1$ and $\Omega(\zeta, h)$ the Carleson box in D,

$$\Omega(\zeta, h) = (z \in D \mid 1 - h < |z| < 1, |\arg(z - \zeta)| < h),$$

one has

$$\mu(\Omega(\zeta, h)) \leq Ch.$$

It is known that for $F \in$ BMOA the measure $d\mu(z) = |F'|^2(1 - |z|)dxdy$ is a Carleson measure.

For our interests this supplies a straightforward proof of the following.

Theorem 5.2. *For every function $f \in \mathscr{H}$ the measure*

$$d\nu(z) = \left|\frac{f"}{f'}\right|^2 (1 - |z|^2)\, dxdy$$

is a Carleson measure. Consequently, for any p the map from H^p into $L^p(d\nu)$ is bounded. That is for $g \in H^p$ there is a C(p) for which

$$\left(\int |g(z)|^p d\nu(z)\right)^{1/p} \leq C(p)||g||_{H^p}.$$

In order to discuss a last point we need to make a few more definitions.

Definition 5.3. For $\lambda > 0$ let $\mathscr{G}(\lambda)$ denote the set of functions g, analytic and non-zero in the unit disc, satisfying

$$|\arg g(z)| \leq \lambda,$$

and

$$|g(0)| = 1.$$

The second definition is for a space known as weak H^1.

Definition 5.4. Denote by H^1_w the weak Hardy-type space of order one. It consists of those analytic functions f on the unit disc which are in the Smirnov (Nevanlinna) class and for which

$$|\{\exp\imath\theta \in \mathbb{T} \mid |f(\exp\imath\theta| > t\}| < \frac{b}{t}$$

The notation $| * |$ denotes Lebesgue measure of $*$ on the unit circle. That is, the distribution function of f on the unit circle is controlled. The letter b is a constant depending on f. The infimum over all such b is known as the weak H^1 norm and is denoted by $b(f)$.

With this notation Baernstein and Brown [2] proved the following.

Theorem 5.5. *Assume F is a function satisfying*

$$F(z) = \lambda \cdot \left(\frac{\mu - z}{\delta - z} \right) \cdot f(z),$$

where λ, μ, and δ are in the unit circle and $f \in \mathscr{G}(\nu)$, where $\nu < \pi/2$. Further, assume F satisfies

$$|F(z)| \leq 2/(1 - |z|).$$

Then F is in H^1_w.

We wish to add to this result for functions from the Hornich space.

Theorem 5.6. *Assume f is in the Hornich space, $\mathscr{H}$, and satisfies $|\arg f'(z)| \leq \nu \leq \pi/2$ and the function F satisfies*

$$F(z) = \lambda \cdot \left(\frac{\mu - z}{\delta - z} \right) \cdot f'(z),$$

where as above μ and δ are in the unit circle. Then $\ln |F'(z)|$ is in $L^p(\mathbb{T})$ for all $p < \infty$.

This follows immediately since $\ln F(z) \in$ BMOA.

6. Questions related to this material

The following are questions that we have considered in writing this and have been unable to answer.

Q 1. What is the closure of $\mathscr{H}$ in the space of BMOA functions that vanish at the origin?

Q 2. Assuming the operator L^{-1} has been extended to $\overline{\mathscr{H}}$ what is its range, relative to $\mathscr{L}$?

Q 3. Can the function F occurring in the Baernstein and Brown theorem be shown to be in the space CT of Cauchy-Transforms of finite Borel measures on the unit circle? Recall that CT $\subseteq H^1_w$.

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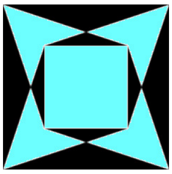

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In the footsteps of Copernicus: Padua and Uppsala

STEFANO DE MARCHI AND MACIEJ KLIMEK

Abstract. For over half a millennium the life and work of the Polish astronomer and mathematician Nicolaus Copernicus has been described and discussed in countless publications. And yet there are some aspects of his life, as well as the whereabouts of his remains that are relatively unknown to the general public. One of the less explored episodes is the period Copernicus spent at Padua University in Italy. Another that deserves more recognition is the recent detective story, involving work at the University of Uppsala in Sweden to identify his genetic material, permitting the definitive location of his grave. In this article we intend to shed light on these less familiar facts about Copernicus, hoping that the reader will be enticed to visit those two amazing historical cities: Padua and Uppsala.

1. Introduction

Nicolaus Copernicus, or to use his Polish name Mikołaj Kopernik, was one of the very few scientists throughout history whose discoveries have had a profound impact not just on research but on human civilization in general. From the point of view of mathematicians, there are some rather specific aspects of his scientific contributions that make them timeless.

Most of all, his work is the first example how a better mathematical model of reality can dramatically improve comprehension of empirical data and significantly simplify forecasting. His work also exemplifies brilliantly how vital for science is liberation from blurring of the boundaries between extrinsic beliefs, mathematical models and reality.

Modern scientists function in an increasingly interdisciplinary research environment, very often held together by various mathematical constructions. Even within mathematics, gone are the days when different branches of the Queen of Science could happily exist in separation from each other. In this respect Copernicus, alongside other great figures of the Renaissance period, can also serve as a role model for the intellectuals of the 21st century.

In this short article we would like to celebrate the varied educational background of Copernicus by conveying somewhat less known facts concerning his academic studies in Italy. In particular, while he studied mathematics and astronomy in Poland, there is some circumstantial evidence that the first draft of the heliocentric model was created during his medical studies in Padua. We will also describe the fascinating chain of events, which was a result of a cooperation of a Polish-Swedish team of archaeologists and experts in forensic medicine, and which lead to the identification of his remains just a few years ago. We are convinced that Copernicus, being himself a medical practitioner, would have loved the story.

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2. Tangible evidence of Copernicus' stay in Padua

The several years that Copernicus spent in Italy were divided between four towns: Bologna (1496-1500), Rome (1500), Padua (1501-1504) and Ferrara (1503). In Ferrara he received a doctorate in canon law, which in fact was the main and official reason why he was studying in Italy. Here we want to focus on the time he spent at the University of Padua, which was not only the second oldest university in Italy (1222) and one of the oldest in the world, but also the most liberal one. That period of Copernicus' life is considered by his biographers to be the most obscure part of his stay in Italy (see for example in [2, p. 67]).

Copernicus studied mathematics and astronomy during the period 1491-1495 at the Jagiellonian University of Kraków in Poland, prior to his trip to Italy. It is important to mention that since the 13th century the cities of Padua and Kraków have had close links, and since the late 14th century this has been especially true of their universities. A sort of mutual esteem between their citizens, has made it natural for the students to move between the two universities to complete their studies. By the time Copernicus came to Padua to study medicine, the Polish presence in the city had been well established, and dates back at least to 1271 when *Nicolaus de Polonia* (Nicholas of Poland), a healer and later a dominican friar in Kraków, was appointed the rector of the University of Padua. Other prominent Poles were studying in Padua in the fifteenth century. Among them we should mention *Maciej z Miechowa* (Matthew of Miechów), called Miechowita, the renowned author of the treatise *De duabus Sarmatiis*. Moreover in 1487 Jan Ursinus, the author of *Modus epistolandi*, studied medicine there. This indicates that when Copernicus arrived in Padua, he was following a well-established tradition of bilateral contacts and most likely it was easy for him to integrate with the academic community.

As another manifestation of this tradition, in 1873, on the occasion of the fourth centenary of Copernicus' birth, the University of Padua dedicated to him the inscription:

```
NIC. COPERNICO
QUO DIE XI K.AL MART. AN. MDCCCLXXIII
E1 US NATALITIA IV POST SAECULO CELEBRANTUR
UNIVERSITAS PATAVINA
TANTO LAETA VIRO IN SUUM OLIM SINUM RECEPTO
TIT. POS.
```

This is positioned on the right-hand side wall in the great-hall of the University, located in the *Palazzo Bo*, the main building that has hosted the rectorate since the fifteenth century. The plate, in the form of golden characters on a black background (see Fig. 1), is surrounded by other plates donated to the University by students after they completed their studies Fig. 2.

Figure 1. Inscription on the right wall of the great-hall of the University of Padua (*Photo courtesy of the University of Padua*)

Unfortunately there are no written sources directly concerning Copernicus' stay in Padua. In fact, for centuries the biographers of Copernicus could not provide any tangible evidence. Eventually, in 1876 a document certifying his diploma in canon law was found. The document was issued by the University of Ferrara in May 1503, confirming that prior to his studies in Ferrara Copernicus had studied for three years in Padua. This is detailed in the informative book by Bierkenmajer [1], published on the occasion of the seventh centenary of the foundation of the Polish Art and Science Academy of the University of Padua.

Figure 2. The Copernicus plate surrounded by plates donated to the University by students (*Photo courtesy of the University of Padua*)

During Copernicus' stay in Padua, officially he was studying medicine, but it is safe to assume that he was constantly in contact with the astronomers, mathematicians and philosophers around him. According to [1], the great Hellenistic philosopher Niccolò Leonico Tomeo had a particularly strong influence on Copernicus, to the extent that when Tomeo moved to Venice, Copernicus decided that it was time to return to Poland. It is very likely that the stimulating academic environment that he found in Padua, allowed him to formulate in writing a short outline his heliocentric theory of the world, either during his stay in Italy or shortly afterwards. The earliest evidence of the existence of the manuscript known as *Commentariolus* (A Little Commentary) comes from an inventory of the book collection belonging to Matthew of Miechów (see e.g. [4]). The inventory was compiled in 1514, when Matthew of Miechów was a professor and a rector of the Jagiellonian University of Kraków. Interestingly, while handwritten copies of *Commentariolus* were circulating during the best part of the sixteenth century, later the treatise fell into complete obscurity, to be eventually published in the nineteenth century.

No signs of the actual house in Padua where Copernicus stayed have been found. However, there exists a document from that time found in the book of deeds of the notary Stefano Venturato, *TOMO 1 degli istrumenti* depicted in Fig. 3, which determines Copernicus' whereabouts. The deed is a proxy issued to the canons Apicio Colo and Michele Iode of Wrocław's church, to take possession in his name of the *scolasteria* at the Collagiate Church of the Holy Cross in the same city. The date of this deed is 10 January 1503 and was written following instructions of Copernicus. Since the deed was initiated by a member of the clergy, it was written at the Padua bishop's chancery in front of two priests – father Leonardo Redinger of the diocesis of Padua and father Niccolò Monsterberg of the diocesis of Wrocław (in the deed written as Vradislavia). The document, Fig. 3, can be seen and consulted at the State Archive in Padua (the address is Via dei Colli, 24 - Padova), in the volume 2245, f. 175 (f. stands for "folio", that is page). In [5] the page is 173, but as you see from Fig. 3 is indeed 175.

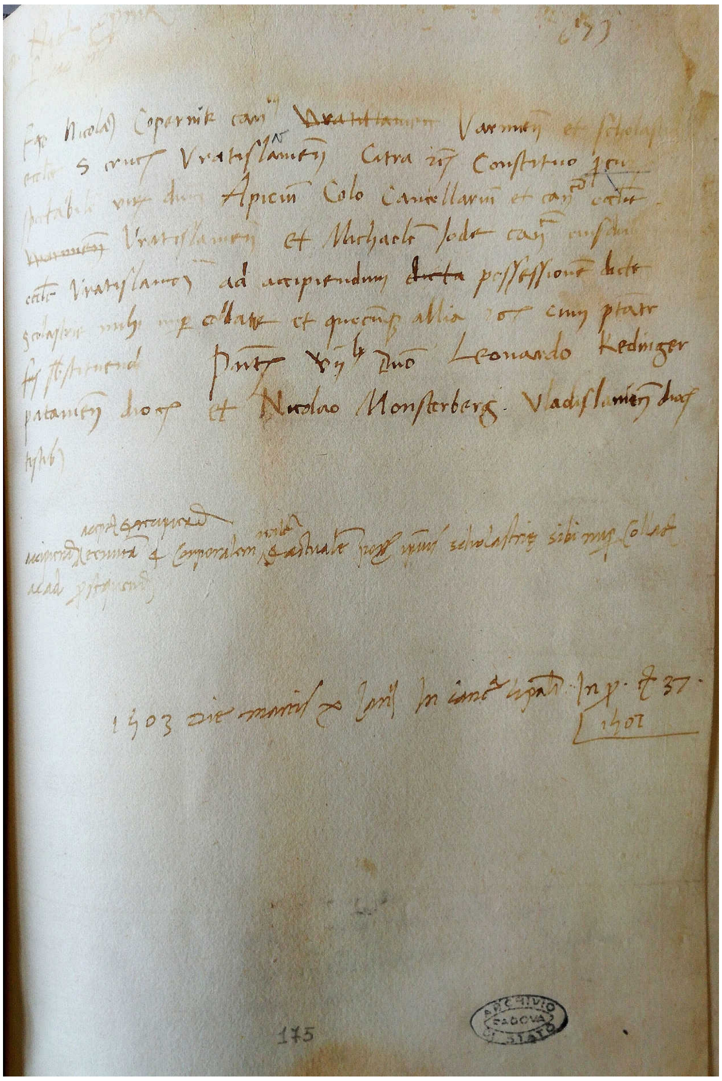

FIGURE 3. The deed signed by Copernicus. The paper is at the State Archive of Padua, Notary Archive Vol. 2245, f. 175 (*Photo courtesy of the Archivio Notarile - Padova*)

The deed in Latin says the following:

> ***Procura reverendi domini Nicolai Copernik Per hoc presens******
> ***Ego Nicolaus Copernik, canonicus Varmiensis et scholasticus ecclesie Sancte Crucis Vratislaviensis citra et cetera, constituo procuratores spectabilem virum dominum Apicium Colo, cancellarium et canonicum ecclesie Vratislaviensis,***
> ***et Michaelem Iode, canonicum eiusdem ecclesie Vratislaviensis, ad accipiendum possessionem diete scolastrie mihi nuper***

> ***coliate et quecumque allia et cetera cum potestate substituendi. Presentibus venerabili domino Leonardo Redinger Pataviensis diocesis et Nicolao Monsterberg Vladislaviensis diocesis***
>
> ***testibus accipiendum, aceptandum et recipiendum tenutam et corporalem, realem et actualem possessionem ipsius scholastrie sibi nuper collatae ac ad prosequendum.***
>
> ***1503, die martis x ianuarii, in cancellaria episcopali. In prothocollo, charta 37, 1502.***

This document is very important: before it was uncovered there had been only hypotheses, but not a real proof of the whereabouts of the astronomer during those years.

3. The Polish-Swedish detective story

Someone familiar with any of the published biographies of Copernicus might be justifiably wondering what could be his possible connection to Sweden, a country never visited by the great astronomer. In order to explore that link, we need to move back in time to a small medieval fishing town situated on the Vistula Lagoon in Northern Poland called Frombork (known also by its German name as Frauenburg). It was there that Copernicus had lived for over three decades prior to his death in 1543. It was also there that he wrote his groundbreaking work *De revolutionibus orbium coelestium*. For a long time it was assumed that Copernicus, who had served as a lay canon at the Frombork cathedral – was buried inside the church itself, under the floor, in an unmarked grave. Some historians suspected that the grave was located next to the altar of the Holy Cross for the maintenance of which Copernicus himself was responsible (see [7] and [8]). However, before explaining how we know that this is indeed the case we have to take a short historical detour to explain how through the trials and tribulations of history a Swedish thread is interwoven into the Copernican legacy.

To this end, we have to move forward in time to the year 1587. After a coronation ceremony at the end of that year, Sigismund III Vasa became the King of Poland and the Grand Duke of Lithuania. This was the beginning his 45 year long rule of the Polish-Lithuanian Commonwealth, which lasted right until his death. Sigismund III Vasa was the son of the king of Sweden Johan III and a Polish-Lithuanian princess Catherine Jagiellon. Being the only male offspring of that union, he was also the crown prince of Sweden and inevitably inherited the Swedish throne upon the death of his father in 1592. However, his reign in Sweden was cut short in 1599 when he was deposed by his own uncle acting in collusion with the early incarnation of the Swedish parliament. This marks the beginning of an especially dark period in the history of Polish-Swedish relations. Over the next six decades Poland and Sweden endured five wars with cumulative duration of approximately 25 years. During one of these wars, in 1626 the Swedish troops attacked and subsequently occupied Frombork, causing significant loss of human life and substantial material destruction. In particular, the cathedral treasury was robbed. The loot taken back to Sweden included a priceless collection of books which Copernicus left behind when he died. The fate of his astronomical instruments is uncertain, but 22 of his books ended up at Uppsala University in Sweden.

FIGURE 4. Inscription at Uppsala University (*Photo by M. Klimek*)

The traces of these Polish-Swedish conflicts can be still found today. For example, the Mathematics Department of Uppsala University was housed until recently in a group of buildings inherited from the Uppland Regiment, a Swedish army infantry unit established in the middle of 16th century and disbanded only in 1957. In one of those buildings, to this day one can find a large wall inscription commemorating major battles fought by the regiment. As shown in Figure 4, fair share of the locations included on this list were in Poland. The place where the former barracks are situated is known as Polacksbacken. According to the prominent 17th century Uppsala University professor Johannes Schefferus, the name originates form the fact that Polish soldiers serving under Sigismund III Vasa set their camp there. More pertinent to our story is the fact that the Uppsala University library Carolina Rediviva is still in possession of some old books which made their way to Sweden in the 17th century, courtesy of Swedish troops returning home with their spoils of war. For centuries, the 22 books that used to belong to Copernicus remained in total obscurity, buried among other old books and documents stored at Uppsala University. A change of their status came about just over a decade ago.

In 2004, the Polish bishop Jacek Jezierski from the Warmian Cathedral Chapter, who was also at the time the provost of the Frombork's Archcathedral, asked a prominent archaeologist professor Jerzy Gąssowski if he would be interested in trying to locate the remains of the great astronomer. Knowing that the vast cathedral church hides over a hundred mostly anonymous graves, the archaeologist initially declined the undertaking [8]. It should be mentioned that there had been earlier unsuccessful attempts — the first one was ordered by Napoleon himself in 1807 [3]. The bishop however did not give up and told Gąssowski about a theory posited by the historian Jerzy Sikorski, according to which canons were buried close to the altars for the upkeep of which they had been responsible. Working on this assumption would narrow down the search to a manageable fragment of the church. In the end, the professor relented. As a consequence, the archaeological quest commenced in 2004 and continued on and off until 2006 [8]. The initial results seemed very promising. The researchers found the remains of a man of the right age, with well preserved cranium, but no mandible, and a handful of other bones. On the basis of the cranium a scientist from the Central Forensic Laboratory at the Polish Police Headquarters, inspector Dariusz Zajdel, reconstructed the face of the deceased, which according to anthropologists showed significant likeness to existing portraits of the astronomer [7, 8, 10]. Even so, all this was at best only circumstantial evidence for the claim that the grave of Copernicus was identified.

In October 2006, a serendipitous cultural event in Uppsala marked the beginning of the process of finding an empirical justification of this claim. As part of celebrations of the so called Polish Days in Sweden, Museum Gustavianum in Uppsala organised an exhibition showing astronomical instruments from the time of Copernicus on loan from the Jagiellonian University in Kraków. It also displayed some of the old books mentioned earlier. The opening address was given by professor Gąssowski and generated a considerable interest among the audience. In particular, two professors from Uppsala University, the astronomer Göran Henricksson and the archaeologist Władysław Duczko, suggested getting in touch with their university colleague who was a renowned expert in forensic and historical DNA analysis. The person they had in mind was a professor in forensic medicine named Marie Allen, who as it turned out was very much interested in the challenge.

Among the remains recovered, a few molars as well as the femur bones contained viable genetic material, and the mitochondrial DNA analysis confirmed they all came from the same individual [3]. The tests were done independently by professor Allen, professor Wiesław Bogdanowicz from the Polish Academy of Science in Warsaw and two forensic geneticists from Kraków Dr. Andrzej Beanicki and Dr. Tomasz Kupiec. This did not however answer the question about the identity of the deceased, as an extensive search for DNA from contemporary relatives of Copernicus or their descendants turned out to be a dead end. Luckily, one of Copernicus' books in the Uppsala University collection, namely *Calendarium Romanum Magnum* (see Figure 5) contained 2 hair strands with enough workable DNA to perform comparative analysis. The results finally confirmed that the remains belong to Copernicus and his grave has been located.

4. Final remarks

While admiring the truly Renaissance aspects of the Copernicus legacy and the wonderful diversity of the modern science that helped to unravel the mystery of the astronomer's grave, it is only fitting to close our story with a quotation from *De revolutionibus orbium coelestium* [6] expressing the great astronomer's motivation for undertaking research:

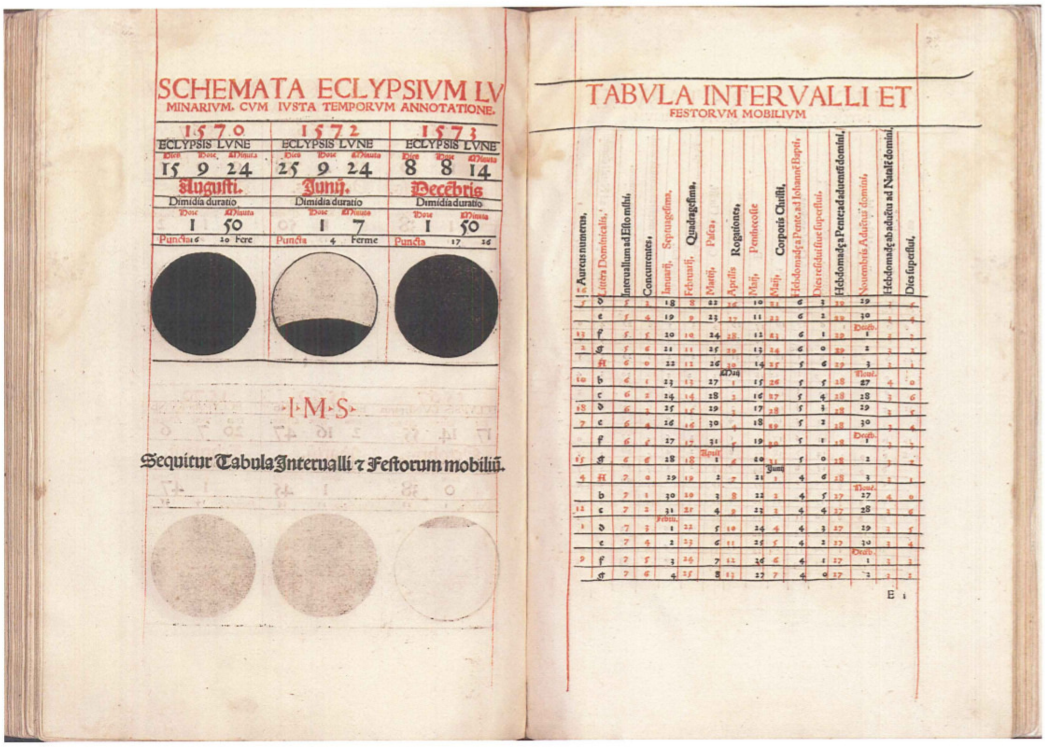

FIGURE 5. Pages from Kopernik's own copy of Calendarium Romanum Magnum (*Scan courtesy of the Uppsala University Library*)

Among the many various literary and artistic pursuits which invigorate men's minds, the strongest affection and utmost zeal should, I think, promote the studies concerned with the most beautiful objects, most deserving to be known. […] What indeed is more beautiful that heaven, which of course contains all things of beauty?

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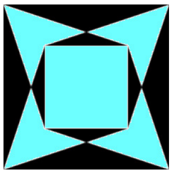

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Richard Martin Timoney, 1953-2019

SEÁN DINEEN

I first met Richard in 1972 when, at his father's suggestion, I drove my struggling Ford Anglia to their house in Stillorgan Grove for some badly needed repairs. Richard and his brother David did the repairs while I chatted in the house with his father. Richard's father Dick was, at that time, Professor of Mathematical Analysis and head of the Department of Mathematics at University Colllege Dublin (UCD) and I was a first year lecturer in the same department. A few years later, Richard attended my masters degree course on analysis at UCD. In 1980 Richard joined the staff at Trinity College Dublin (TCD) as a lecturer and we jointly organised the TCD-UCD weekly Analysis Seminar for the next thirty plus years. In 1982 we began a research collaboration, which continued for almost fifteen years, on different aspects of infinite dimensional holomorphy. Over the years our paths, both personal and professional, criss-crossed: we met at conferences in Ireland and abroad, in our various roles with the Irish Mathematical Society, as members of the National Committee for Mathematics, on assessment boards, as external examiners, etc., and in 1987 I became god-father to Richard's youngest son Kevin. These interactions continued right up until Richard's death on January 1, 2019. Still, until relatively recently I never expected to be writing Richard's obituary.

Richard was a quiet person with a wry sense of humour and he was very self-effacing. He was not verbose nor given to self-advertisement but a good listener who rarely bothered to fill a silence with small talk. I have heard it said that Richard was shy, but this is not my opinion. Whenever our seminar had an unforseen gap, Richard would

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volunteer. His carefully structured seminars, his style of lecturing, and his reaction to questions showed a personality with a confident style who enjoyed what he was doing and not those of someone who was shy. It was easy, however, especially for the impatient, to misinterpret Richard's sparse conversations as aloofness. Generally, appreciations of Richard's personality developed over time and could not be rushed. However, such generalisations are never universally true and I was rather surprised recently to receive an email from John P. D'Angelo, who had a very different experience on first meeting Richard. I include here D'Angelo's self-explanatory letter.

I am saddened to hear of Richards passing. I cannot come to Dublin for the fest[1], *but I have a story I wish to share.*

In Spring 1978 I was a Postdoc at MIT and I gave a job talk at Illinois. There was a party for me, at which a senior faculty member was pontificating about investments.

I wasnt interested so instead I chatted at length with the only graduate student there, Richard Timoney. He told me a bit about his work on Bloch functions in several complex variables.

He asked me a few questions about strongly pseudoconvex domains; I was impressed with his questions and his interests.. I took the job, and I have been at Illinois ever since.

He gave me a very positive impression of the quality of the graduate program, which was a decisive reason for me to accept the offer. (In fact, the program was nowhere near as good as he was!)

Over the years we had several interactions; I invited him back here for something in honor of his advisor which he attended and spoke, and I refereed papers for him a few times.

I owe him a huge debt for being such a great graduate student; he was a crucial reason for my choosing where I spent my mathematical career. I once told him so, and his response was quite modest. I think the world of him.

In talking to people and examining documents as preparation for this obituary I realised that I, like many others, had underestimated the extent of Richard's contributions. He hid himself very well and just did things. We could all see that certain things had been done and obviously someone was responsible but too often we didn't look closely at the situation. If we had, we would have seen that Richard was behind much more than we realised. Just two years after his return to Ireland he joined the committee of the Irish Mathematical Society (IMS). He served as Secretary for five years, 1982-1987, as Vice-President during 1988-1989, as President during 1990-1991 and from 1981 until 2018 he was the TCD local representative of the IMS. Richard was never a passive committee member but was always looking ahead and initiating projects that would have long-term consequences. In 1984, Richard compiled a directory of Irish mathematicians and revised it annually until it was transformed into the web-site of the Irish Mathematical Society (IMS). For many years Richard maintained this web-site and through it made available all newsletters and bulletins of the IMS – a priceless resource for the mathematical community.

Richard Martin Timoney was born in Dublin on July 17, 1953. His parents were James Richard ('Dick') Timoney[2] from Belleek in County Fermanagh, later Professor of Mathematical Analysis at UCD, and Nora (nee Fallon) Timoney. The family homestead in Belleek has remained in the Timoney family having being inherited by Dick, by Richard and now by Richard's eldest son, Pádraig. Richard's paternal grandfather

[1] A reference to AGA, the memorial conference for Richard, held in TCD, May 8-10, 2019.

[2] Richard's and his father's careers had a number of similarities, see for instance p.126-127 of *Report of the President, 1978-79, University College Dublin* and T. J. Laffey and S. O'Brien, *Obituary – Professor J. R. Timoney,* Irish Mathematical Society Bulletin, 16, 10-13, 1986.

was an associate of Count Horace Plunkett, founder and father figure of the Irish co-operative movement, and was part of a delegation to California in the 1920's led by the Count. Richard's maternal grandfather was a secondary school inspector from Galway and he had a keen interest in the Irish language. This he passed onto his daughter Nora and through her to the succeeding generations of Timoneys. Under her influence, Richard and his siblings, David, Nicola and Norma attended the Irish speaking primary school, Scoil Lorcáin, in Monkstown, as did Richard's three children Nuala, Pádraig and Kevin. In primary school he used the Irish form of his name, Risteárd and this is still used within the family. From his early teens Richard was a voracious reader and readily absorbed and retained news and details about many different topics. His analysis was never superficial and this often surprised those who naively assumed he had no interest in certain topics. This broad culture often allowed Richard to economically summarise a situation in a particularly striking fashion: for example in describing the teaching style of his friend and colleague, Trevor West, as being unique he said it was the *type of interaction that might be more common between a sports coach and his team.* A remark that says as much about Richard as Trevor.

Richard attended Blackrock College for his second level education. Significant later, but of course not apparent at the time, was the fact that Richard Hendron and Richard Timoney were classmates in Blackrock and that Richard Timoney married Margaret Hendron, a sister of the other Richard, in 1981. Blackrock is noted for both its high academic standards and as a rugby school and nursery. Richard was never an active rugby player but attended games particularly when family members were involved. In recent years he enjoyed watching his nephew Nick Timoney playing professional rugby with Ulster. The Timoney brothers, accompanied by Margaret, Nicola, Norma, Kevin, Nuala, her husband Matthew and son Connall, and by David's wife Darina and daughter Keelin would travel in convoy to Kingspan stadium in Belfast and celebrate the game and the day as a family reunion.

Richard's father studied, as a first year university student, both mathematics and engineering but continued with mathematics. Both disciplines, however, were very much present in the Timoney household during Richard's childhood and adolescence. The driveway of the family home in Stillorgan always contained a car, more often than not a Citroen, that was being worked on or even being assembled and Richard readily absorbed the Timoney passion for cars. Richard and his brother David became skilled mechanics in their teenage years and David went on to become a professor of mechanical engineering at UCD while Richard's eldest son, Pádraig, graduated from UCD with a PhD in mechanical engineering[3]. Richard's daughter Nuala followed more closely Richard's calling by studying theoretical physics at TCD and afterwards completing a PhD in quantum computing at Siegen in Germany[4]. The bias towards science and engineering was broken by Richard's youngest son Kevin who studied economics at TCD and completed a masters in economics at the Barcelona Graduate School of Economics. Kevin now works as an economist with the Irish Fiscal Advisory Council.

While Richard was a Teaching Assistant at the University of Illinois he was also in receipt of a Travelling Studentship stipend from the National University of Ireland. With this discretionary income he bought a blue Plymouth Fury 3 car and indulged himself by spending as much as the car was worth on a set of Michelin tyres. During the summer of 1976, Richard and David drove over 8000 miles across and around the states visiting all the well known Formula 1 racing tracks: the Indianapolis 500 track, the Watkin's Glen track in upstate New York and the famed Utah Salt Flats at Bonneville where many world records were recorded. They set their own speed records of approximately

[3]Pàdraig is currently the principal engineer in metrology at GlobalFoundries in upstate New York.

[4]Today, Nuala works for Intel in Leixlip and lives in Maynooth.

180 kmph in Bonneville. In Ireland, Richard and his growing family, usually fortified by a roast dinner prepared by Margaret, maintained an interest in Formula 1 racing by attending Grand Prix races in Mondello and the Phoenix Park. In 1998, while on a holiday in the south of France, the whole family visited Monte Carlo and all drove a lap of the famous circuit. Richard never smoked nor did he ever drink alcoholic drinks, a suitable trait for someone with his interest in cars.

This interest in cars may have been responsible for Richard's lifelong interest in solving complicated mechanical problems and in dismantling and reassembling cars, computers and his grandchildren's toys. His academic career spanned the period 1978-2018 and this more or less coincided with the birth, the development and finally the dominant presence of the personal computer as an indispensable piece of furniture in every academic's office. Like many other mathematics departments TCD School of Mathematics had staff members who had foresight about what might digitally lie ahead and who were adept, theoretically and practically, at implementing the changes that kept their department ahead of others along the digital curve. All doubts for mathematicians were put aside once Knuth's *expert system* of TEXwas unveiled. In the School of Mathematics at TCD, the two experts were Richard and Tim Murphy. Of course, all their colleagues, even those who could not change a light bulb, knew who these experts were and their advice and expertise were frequently sought. Right up until ten days before he died Richard was remotely solving computer and internet related problems for his colleagues. It is indeed ironic that Richard's obituary of Tim Murphy appeared in the December 2018 Bulletin of the IMS.

Richard attended University College Dublin (UCD) during the years 1970-1974. At UCD he obtained a B.Sc.(Hons) and an M.Sc.(Hons) in Mathematical Science. Political and educational changes in Irish society during the mid-1960's resulted in a large increase in the student population and a corresponding increase in the academic staff during the period 1968-1972. The new staff included Tom Laffey, David Lewis, Brendan Quigley, David Tipple, Phil Boland and David Williams and, a little earlier, Fergus Gaines. All were under thirty and as a group they brought new energy and enthusiasm to the department. This doubling of the staff resulted in a reduced teaching load and facilitated research. These were the teachers that nurtured Richard's interest and mathematical talent during his undergraduate days. In my experience freshly minted PhDs usually set very demanding standards for their students as they learn their teaching trade and if the students are not overwhelmed they benefit greatly. Equally influential was the fact that Richard was a member of an honours mathematics class that many remember as one of the most talented ever seen in UCD. Five of that group of 14, Richard, David Redmond, Leslie Daly, Denis P. O'Brien and Joe Hogan, completed PhD's in the mathematical sciences and pursued academic careers while the others were equally successful outside academia. An idea of the even distribution of talent within the group may be gauged by noting that Joe Hogan got first place in first year, Richard came out top in the masters examinations with David Redmond in second place while Leslie Daly was reputed to have asked good questions in *every* class. Having completed their masters at UCD, Richard and his classmate, David Redmond, continued their studies at Champaign-Urbana. David completed his PhD in algebra in 1977 under the direction of Michio Suzuki. Afterwards he accepted a lecturing position in Maynooth and recently retired as Registrar of Maynooth University. Richard's thesis, *Bloch Functions in Several Complex Variables* was supervised by Professor Lee Rubel.

On completing his doctorate, Richard spent two years as Vaclav Hlavaty Research Assistant Professor at Indiana University in Bloomington. In 1980, Richard returned to Ireland as a Lecturer in Mathematics at Trinity College Dublin (TCD). Apart from a one year visiting position during 1984-1985 at the University of North Carolina in

Chapel Hill and shorter academic trips elsewhere, Richard spent the rest of his career at TCD. Richard was a traditional, almost conservative, type of person. Trinity College, which has remained an educational institution and retained academics in prominent decision making positions[5], suited his personality. Over his almost forty years at TCD he came to appreciate the traditions and congeniality of TCD. Richard got on well with most people, but there were some with whom he had differences and others who had differences with him. I have been told that even with those with whom he had problems he was willing to help if help was needed. On occasion, and these were rare, if he felt he was being overburdened with duties or that the spirit of certain rules and regulations were being too loosely interpreted he was capable of writing a logical but devastating critique of the situation and, if necessary, of taking the matter further.

Richard had a wide mathematical culture and published quality research articles in a variety of mathematical areas: functions of one complex variable; several complex variables; operator theory; Jordan structures in analysis; functional analysis; harmonic analysis; infinite dimensional holomorphy and differential equations. Richard's mathematical research was highly regarded internationally but, since I am currently writing an article[6] on Richard's mathematical legacy, I will not discuss his research here.

Richard took a holistic approach to life as a mathematician and teacher and contributed to the administrative, institutional, social, societal and personal aspects of his profession. He wrote survey articles, book reviews, letters to the national papers, guides to projects for second level students, obituaries, technical explanatory[7] papers on TEXand Mathematica, policy papers on the role and importance of mathematics in society, he organised conferences and he was an editor for three international research journals. All his writings are well researched and clearly written. Here and there within them we find unexpected insights into his character and beliefs. His well written and thoughtful obituaries and tributes contained concrete facts that will help future historians but also playful insightful images that resonated with those of us who knew his subjects. In 2013 he wrote a spirited defence on the contributions of mathematics to society and mentions what we may regard as part of his own philosophy: *the essential ingredient of inspiration provided by highly motivated individuals and the long-established record of surprise.* He wrote survey articles on specialised topics, both to help himself get an overall view of a subject and as an aid to any locals who might be interested in researching the topic. He was constantly expanding his own expertise by unselfishly accepting requests to referee articles tangential to his main interests and encouraging colloquium speakers to give in his words *intelligent talks in a wide variety of subjects so that we can all have an idea what's going on in a variety of fields.* Along-side his work for the Irish Mathematical Society he was for many years a member, and for some years Chairman, of the National Committee for Mathematics of the Royal Irish Academy. In this capacity he represented Ireland at two international congresses.

During the 1990s Richard was very involved in the Euromath project and in various computer-related topics, such as interactive mathematical editors, databases and computer algebra and in the OpenMath project which aims to provide a means to exchange mathematical information between computer programs. This involved a lot of travelling, especially to Eastern Europe, and considerable administrative skill. Although the overall aims of EuroMath were achieved, the main products are not widely used

[5]In contrast to other Irish universities which have been transformed in recent years into businesses where students and standards are subservient to the requirement to attract research grants and high-fee paying students from outside the EU.

[6]*The Mathematical Legacy of Richard M. Timoney*, to appear in Math. Proc. of the Royal Irish Academy, December 2019.

[7]Long before we all had our own computers and at a time when it took seven minutes to print three pages.

today. As a result of his involvement in the TEMPUS project, Richard was awarded the *Medal of Merit for the Development of the University* by Nicolaus Copernicus University, Toruń, Poland.

Within the School of Mathematics, Richard was appointed a Senior Lecturer in 1990 and an Associate Professor in 2008 and he was head of department for three years. Richard volunteered, or was volunteered, for numerous administrative tasks both within the department and the university: timetabling, examinations coordinator, college parking, the departmental library (with David Simms), liaison for the refurbishment of 19 Westland Row, member of the TCD Hamilton Mathematics Institute executive, responsibility for the School's web-site, Director of Graduate Studies in the School of Mathematics, etc. etc. Many of these were time consuming and he was always looking for ways in which computers could be used to simplify and make these tasks less onerous, e.g. in processing examination results and the automatic generation of transcripts. All these he undertook while also being the Principal Investigator on a number of research grants for Science Foundation Ireland. Richard was elected a Fellow of TCD in 1989 and in time became a Senior Fellow. In 2016, Richard was elected a member of the College Board by the fellows. The college board sits frequently and considers everything to do with the running of the college. Richard was a traditional voice on the board. He subjected proposals to a thorough analysis and he was always willing to be convinced by a clear logical argument. During the period 2016-2018 he was Junior Proctor at TCD.

Richard taught large and small groups of engineering, general science, physics and mathematics students. With the larger service classes he gradually introduced, as the computer facilities improved within TCD, the use of computer algebra and pioneered the classroom use of Maple and Mathematica. He taught complex analysis, functional analysis, measure theory, harmonic analysis and algebra to mathematics students and also introduced new subjects such as wavelets to the curriculum. Richard's courses were considered demanding but fair and his lecturing style was methodical and organised. He had a talent for pacing lectures so that students had time to understand and ask questions. One student told me: *I used to write bullet points in his lectures rather than paragraphs since these provided the background structure for the material covered and you could piece together the details later. I owe him a great debt of gratitude for showing me how enjoyable analysis could be when broken down into small ideas in this way.* Richard provided detailed printed notes and, in the smaller classes, written feedback to students. His accessibility and reputation for taking care of his students meant that he was highly sought after as a PhD adviser and he directed seven doctoral students and a number of masters students in a variety of different areas within analysis. He was patient with his research students, he gave them time to find their own level and he would meet each individually in his office for a scheduled two or three hours per week. His office had piles of books, etc, here, there and everywhere. Richard knew where everything was but students frequently had to move things about in order to make space to sit down.

Richard was very proud of his three children and two grandchildren Connall and Cían while Margaret was the anchor that enabled him to cope with a full and productive life. She brought music into his life and in his later years they enjoyed going to concerts in the National Concert Hall. Richard and Margaret were regular and active in their local parish of Newtownpark Avenue in Blackrock. They both enjoyed travelling to many parts of the world, sometimes for the purposes of Richard's work and sometimes for family visits. Just five days before he died Richard attended the marriage of Kevin and Aisling. His brave speech at the wedding reception was delivered with his typical humour and honesty.

Ar dheis Dé go raibh a anam dilís. Ní bheidh a leithéid ann arís.

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We omit Richard's mathematical research publications, which will be listed in the forthcoming article on his mathematical legacy.

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17. *Why we need Mathematics in the RPE era*, Irish Math. Soc. Bull., 71 2013, 5-12.

Editorial Work.

2006-18: Associate Editor, Journal of Mathematical Analysis and Applications.
2008-18: Subject Editor, Proceedings of the Edinburgh Mathematical Society.
2009-18: Associate Editor, Journal of Geometric Analysis.

PhD Theses Supervised.

1996 Colum Watt, *Complex Sprays, Finsler Metrics and Horizontal Curves.*
2000 David Malone, *Solutions to dilation equations.*
2004 Bernard Keville, *Multidimensional second order generalised stochastic processes.*
2008 Derek Kitson, *Methods of ascent and descent in multivariable spectral theory.*
2012 Robert Pluta, *Ranges of bimodule projections and conditional expectations.*
2014 David McConnell, $C_0(X)$*-structure in* C^**-algebras, multiplier algebras and tensor products.*
2016 James Boland, *The Herrero conditions on norm limits of hypercyclic operators.*

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Donald E. Marshall: Complex Analysis, Cambridge University Press, 2019. ISBN:978-1-107-13482-9, GBP 49.99, 286pp.

REVIEWED BY JAMES E. BRENNAN

Prior to introducing a new subject certain questions always arise, such as what should be emphasized, and where should the presentation begin? In the preface to the book under review the author identifies four points of view each central to the subject, and associated with Cauchy, Weierstrass, Riemann, and Runge, respectively. Each individual point of view offers a possible starting point where the emphasis would be on:

1. Cauchy: functions having a complex derivative, and integral formulas;
2. Weierstrass: functions locally expressible as a power series;
3. Riemann: functions or mappings which preserve angles, a more geometric viewpoint;
4. Runge: functions that can be expressed as the limits of rational functions.

As indicated by the author, the seminal text in this area was written by Ahlfors [1] and stresses Cauchy's viewpoint, while most subsequent texts have followed that lead. Marshall, on the other hand has chosen Weierstrass' point of view and to begin with functions locally expressible as a power series. That approach leads almost immediately to a feature of complex analytic functions not found anywhere in real function theory; namely, to the distinctive property of unique continuation.

Prior to engaging in such such details there is a nice introduction to complex numbers, and the historical events that led to these seemingly imaginary quantities being taken seriously. The initial impetus was the publication of the *Ars Magna* by Cardano in 1545 in which a complete algebraic solution of the depressed cubic

$$x^3 + px + q = 0$$

was presented for the first time, the roots being given by the formula

$$x = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3}.$$

The first anomaly appeared about 30 years later when Bombelli drew attention to the fact that the equation $x^3 - 15x - 4 = 0$ has threes distinct real roots ($x = 4$ is one), but in terms of Cardano's formula they are expressed as

$$x = (2 + 11i)^{1/3} + (2 - 11i)^{1/3},$$

which clearly involves complex quantities. Here one might have thought that there should be another formula that avoids this difficulty, but almost 270 years elapsed until in 1891 Hölder, making use of Galois theory and the concept of a normal field extension, proved that there can be no formula expressing the roots of the general cubic that does not pass through the complex domain (cf. [2, pp. 450-453]). I have always felt due to its impact on the subsequent development of complex function theory that this is at least as

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important and interesting as the fact that there can be no similar formula expressing the roots of an equation of degree 5 or higher. Other texts mention Bombelli's example as providing evidence that complex numbers must be taken seriously, but to my knowledge Marshall's is the first to acknowledge Hölder's contribution to closing the door on any possibility of avoiding complex numbers. There is another oddity in connection with Cardano's formula that needs to be considered. In particular if the cube roots appearing in the formula are allowed to be specified in all possible ways, then the formula predicts more roots than a cubic can have. This is dealt with in exercise 1.9 on p. 11 in which the student is carefully led in a series of steps to rederive Cardano's formula, where in that process of doing so it becomes clear how the appropriate branch is to be selected. There are a number of such exercises where the student is encouraged with guidance to experience a bit of the joy of discovery.

Before turning to the study of analytic functions proper, Chapter 2 opens with an elegant proof of the fundamental theorem of algebra, a subject usually taken up much later in most texts, and is then based on Lioville's theorem to the effect that a bounded entire function is constant. The proof presented here, however, was first suggested by d'Alembert in 1746 and depends on the fact that a continuous positive function on a compact set attains a minimum, a fact unproven at the time. Since then, of course, the gap in d'Alembert's proof has been filled (cf. [6, p. 266]). At this point an analytic function is formally defined as a function locally expressible as a convergent power series, and the principle of unique continuation is established, along with certain basic properties such as the sum, product and composition of analytic functions are again analytic. Evidentally, certain functions such as e^x, $\sin x$, and $\cos x$ can be extended analytically to the entire complex plain, but the question remains as to how (or whether) other elementary functions such as

$$\log x,\ \sqrt{x},\ x^{4/3},\ldots$$

can be extended analytically from the real line into the plane. That question is first addressed on p. 29 (Ex. 2.10) along with some hints in connection with extending the function $x^{1/n}$ provided that n is a positive integer. Although it is not mentioned in the text, there is at this point sufficient information available to proceed directly to extending $\log x$ from the interval $\{x : |x-1| < 1\}$ to the disc $\{z : |z-1| < 1\}$ by setting

$$\log z = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(z-1)^n.$$

Since it has already been established that the composition of analytic functions is also analytic one can simply set $z^{1/n} = e^{(1/n)\log z}$, and this is the desired unique analytic extension since equality is clearly satisfied on an interval in the real line.

Having begun from the point of view of power series, within 30 pages there is already a rich collection of analytic functions which can serve as examples from which to infer what might be true in more general situations. From here the discussion moves quickly into the heart of complex function theory to such topics as: the maximum principle, the local behavior of analytic functions, contour integration, Cauchy's theorem, Runge's theorem on rational approximation, the argument principle, and so forth. Along the way it is shown that if a function is continuously differentiable in an open set Ω, then it is in fact analytic. Goursat's theorem to the effect that continuity can be dropped and that the mere existence of a derivative throughout an open set is sufficient for analyticity is another of those instances where the student is encouraged in an exercise, along with hints, to fill in an important gap (cf. p. 62, Ex. 4.12). As a kind of sequel on p. 121 a function f is defined to be *weakly-analytic* in a region Ω if

$$\int_\Omega f \frac{\partial \varphi}{\partial \bar{z}}\, dA = 0$$

for all continuously differentiable functions φ defined on Ω, where dA denotes two-dimensional Lebesgue (or area) measure. The problem for the student is to verify *Weyl's lemma*, which states that a function is weakly-analytic if, and only if, it is analytic, and to state a similar result for harmonic functions replacing $\partial/\partial\bar{z}$ by the Laplacian.

Another example of this particular teaching device occurs earlier on p. 61, where the student is asked to show that the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges and is analytic in the half-plane $\{s : \operatorname{Re} s > 1\}$, to prove that whenever $\operatorname{Re} s > 1$

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \int_n^{n+1} \left(n^{-s} - t^{-s}\right) dt$$

and to conclude from this that $\zeta(s)$ can be continued analytically to $\{\operatorname{Re} s > 0\} \setminus \{1\}$. Although this is not as strong as Riemann's proof that the zeta function can be continued analytically to $\mathbb{C} \setminus \{1\}$, it is sufficient, however, to establish the prime number theorem which is presented much later on p. 191 as an exercise with hints.

These are marvelous problems, but I do not want to leave the impression that all exercises are of the same difficulty as those I have chosen to highlight. Throughout the text exercises are arranged in groups designated A, B and C. Those in group A are meant to be routine and intended to be solved as the student is reading the text. Problems in the other two groups are more challenging, the groups being listed in the order of increasing difficulty. In some cases, as in the preceding two paragraphs, a problem assigned at one stage will reappear later in a more challenging context.

Finally, there are certain features of this book that distinguish it from other texts currently available. Here are two examples:

First, normal families are treated in the context of the chordal metric, a concept introduced on p. 10. Based on Marty's theorem from 1931 characterizing normal families of meromorphic functions on plane domains (cf. [1, p. 226]) together with a lemma of Zalcman (cf. [7, p. 216]), both the great and little theorems of Picard are obtained in a short efficient manner (cf. Marshall pp.162-166). Marty's work is mentioned in Ahlfors, but not in Stein and Shakarchi, while Zalcman's lemma appears in neither.

Second, there is a strong emphasis on conformal mapping beyond what one usually encounters in an introductory text. There is an extensive discussion concerning the actual construction conformal maps. Moreover, Marshall presents two different proofs of the Riemann mapping theorem. One is somewhat constructive and based on what the author refers to as the *zipper algorithm*. The other is based on normal families and is usually associated with Koebe. And, in the end there is a beautiful exposition of the *uniformization theorem* for simply connected Riemann surfaces, not found in either of the texts mentioned above.

When I entered Brown University in 1961 as a graduate in mathematics I had the great pleasure of being introduced to complex analysis through a masterful series of lectures delivered by John Wermer, and starting from the point of view of power series. Whenever I have had the opportunity to teach the subject I have always taken that point of view, and begun in the same way. Although Cartan [3] also begins from the point of view of power series, I never felt that there was a suitable text to assign that students could follow in connection with the course lectures. THERE IS NOW!

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PROBLEMS

IAN SHORT

Problems

The first problem uses arithmetic involving ∞ (such as $1/\infty = 0$).

Problem 83.1. Find positive integers a, b, c, d, e such that

$$\cfrac{1}{a - \cfrac{1}{b - \cfrac{1}{c - \cfrac{1}{d - \cfrac{1}{e}}}}} \;=\; 0,$$

and such that this equation remains true if a, b, c, d, e is replaced by any cyclic permutation of those five letters in that order.

The second problem was sent to the Open University by a student, and solved by Phil Rippon, once editor of these problem pages.

Problem 83.2. Prove that for each positive integer m,

$$\tan^{-1} m = \sum_{n=0}^{m-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right).$$

The third problem is a classic.

Problem 83.3. Find all positive integers x and y such that $x^y = y^x$.

A more difficult problem is to find all positive rational solutions of the same equation.

Solutions

Here are solutions to the problems from *Bulletin* Number 81.

Problem 81.1. Find a homogenous linear ordinary differential equation of order two that is satisfied by the function

$$y(x) = \int_0^{\pi} \sin(x\cos t)\, dt. \qquad \square$$

We are grateful to Omran Kouba of the Higher Institute for Applied Sciences and Technology, Damascus, Syria for pointing out that y is the zero function, making this question somewhat absurd. The problem pages editor accepts the blame for this one. Apologies to all!

We move swiftly on to a problem suggested by Finbarr Holland of University College Cork. This uses the standard notation

$$f(x) \sim g(x) \quad \text{as} \quad x \to \infty,$$

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where f and g are positive functions, to mean that

$$\frac{f(x)}{g(x)} \to 1 \quad \text{as} \quad x \to \infty.$$

It was solved by Omran Kouba, the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland. We present Omran's solution.

Problem 81.2. Let

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2, \quad n = 0, 1, 2, \dots.$$

Prove that

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{(n!)^2} \sim \frac{e^{4\sqrt{x}}}{4\pi\sqrt{x}} \quad \text{as} \quad x \to \infty.$$

Solution 81.2. Observe that a_n is the coefficient of x^n in the polynomial expansion of

$$(1+x)^n(1+x)^n = (1+x)^{2n}.$$

Hence $a_n = \binom{2n}{n}$. We define

$$b_n = \frac{a_n}{(n!)^2} \quad \text{and} \quad c_n = \frac{1}{2\pi} \cdot \frac{4^{2n+1}}{(2n+1)!}.$$

The c_n's are chosen so that for all t we have

$$\sum_{n=0}^{\infty} c_n t^{2n+1} = \frac{1}{2\pi} \sinh(4t).$$

In particular, we may define $g(x)$ for $x > 0$ by

$$g(x) = \frac{1}{2\pi} \frac{\sinh(4\sqrt{x})}{\sqrt{x}} = \sum_{n=0}^{\infty} c_n x^n.$$

Using Stirling's formula we see that

$$b_n \cdot \frac{(2n+1)!}{4^{2n}} = (2n+1)\frac{((2n)!)^2}{2^{4n}(n!)^4} \sim 2n\frac{(\sqrt{4\pi n}(2n/e)^{2n})^2}{(\sqrt{2\pi n}(n/e)^n)^4 2^{4n}} = \frac{2}{\pi}.$$

Thus

$$\lim_{n\to\infty} \frac{b_n}{c_n} = 1.$$

This proves that the series

$$\sum_{n=0}^{\infty} b_n x^n$$

defines an entire analytic function f.

Now, given $\varepsilon \in (0,1)$ there exists $n_\varepsilon > 0$ such that if $n \geqslant n_\varepsilon$, then

$$\left(1 - \frac{\varepsilon}{2}\right) c_n \leqslant b_n \leqslant \left(1 + \frac{\varepsilon}{2}\right) c_n.$$

It follows that, for $x > 0$, we have

$$\left(1 - \frac{\varepsilon}{2}\right) \sum_{n=n_\varepsilon}^{\infty} c_n x^n \leqslant \sum_{n=n_\varepsilon}^{\infty} b_n x^n \leqslant \left(1 + \frac{\varepsilon}{2}\right) \sum_{n=n_\varepsilon}^{\infty} c_n x^n.$$

Thus

$$\left(1 - \frac{\varepsilon}{2}\right) g(x) - \left(1 - \frac{\varepsilon}{2}\right) g_{n_\varepsilon}(x) \leqslant f(x) - f_{n_\varepsilon}(x) \leqslant \left(1 + \frac{\varepsilon}{2}\right) g(x),$$

where

$$f_m(x) = \sum_{k=0}^{m-1} b_k x^k \quad \text{and} \quad g_m(x) = \sum_{k=0}^{m-1} c_k x^k.$$

Hence

$$1 - \frac{\varepsilon}{2} - \frac{g_{n_\varepsilon}(x)}{g(x)} \leqslant \frac{f(x)}{g(x)} \leqslant 1 + \frac{\varepsilon}{2} + \frac{f_{n_\varepsilon}(x)}{g(x)}.$$

Now clearly

$$\lim_{x\to\infty} \frac{f_{n_\varepsilon}(x)}{g(x)} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{g_{n_\varepsilon}(x)}{g(x)} = 0.$$

Consequently, there exists $x_\varepsilon > 0$ such that for all $x > x_\varepsilon$ we have

$$\frac{f_{n_\varepsilon}(x)}{g(x)} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{g_{n_\varepsilon}(x)}{g(x)} < \frac{\varepsilon}{2}.$$

Thus, for $x > x_\varepsilon$, we have

$$1 - \varepsilon \leqslant \frac{f(x)}{g(x)} \leqslant 1 + \varepsilon.$$

This proves that $f(x) \sim g(x)$ as $x \to +\infty$. But obviously $g(x) \sim e^{4\sqrt{x}}/(4\pi\sqrt{x})$ as $x \to +\infty$, so

$$f(x) \sim \frac{e^{4\sqrt{x}}}{4\pi\sqrt{x}} \quad \text{as} \quad x \to +\infty,$$

as desired. □

The third problem was solved by Omran Kouba, the North Kildare Mathematics Problem Club, and Neil Dobbs of University College Dublin, and the three solutions were all different. Neil points out that there are detailed discussions of this question on-line with numerous solutions, including his one, which we present here.

Problem 81.3. Find a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that the restriction of f to any open interval I is a surjective function from I to $\mathbb{R}$.

Solution 81.3. We express each real number x in binary form as $a.x_1x_2x_3\dots$, where a is an integer and $x_i \in \{0, 1\}$. Let

$$f_N(x) = \sum_{n=1}^{N} \frac{(-1)^{x_n}}{n}.$$

If $f_N(x)$ converges to a finite value as $N \to \infty$, then we denote the limit by $f(x)$; otherwise, we set $f(x) = \pi$. □

The verification that f has the required properties is left to the reader! A non-constructive solution (from MathOverflow, due to Jim Belk) goes as follows. Let $\pi\colon \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Q}$ be the projection homomorphism, and let $\rho\colon \mathbb{R}/\mathbb{Q} \longrightarrow \mathbb{R}$ be a bijection. Then $f = \rho \circ \pi$ has the required property.

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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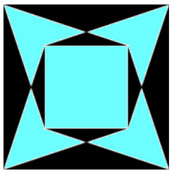

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