Irish Mathematical Society
Cumann Matamaitice na hÉireann

Bulletin

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Irish Mathematical Society Bulletin

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the *Bulletin* for 30 euro per annum.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new. See the inside back cover for submission instructions.

Correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

`ims.bulletin@gmail.com`

and only if not possible in electronic form to the address

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Further information about the Irish Mathematical Society and its *Bulletin* can be obtained from the IMS webpage

`http://www.irishmathsoc.org/`

Printed in the University of Limerick

EDITORIAL

Mr Éamonn O'Connor retired in 2017, after many years' service in the printing works of the University of Limerick. Members will be aware that the production and distribution of the Bulletin is organised by Gordon Lessells at UL, and he relied on Éamonn for the printing and binding. The Society thought it appropriate to express its appreciation, and a plaque was presented to Éamonn at a ceremony to mark his retirement. In the photograph are Gordon, Éamonn and James Gleason.

Colm Mulcahy has drawn our attention to the sad story of Francis Dolan. See `https://tinyurl.com/yckuby2r`. The article raises serious issues about the conditions of service of junior academic researchers.

We received no thesis abstracts at all this year, despite an effort to enlist the IMS local representatives to beat the bushes. We take this to mean that the publication of their abstracts is not a priority for new doctoral graduates in Mathematics, although it was reported from one institution that there were 'no theses in real mathematics'. We've concluded that times have changed, and that resources such as institutional and personal web-pages, social media, and ArXiv.org have absorbed the function formerly performed by the publication of the abstracts. Accordingly, we shall no longer publish thesis abstracts in the Bulletin. Some members feel that the abstracts were of interest in giving us some idea of the range of mathematical activity tround the country, and to serve this interest we shall explore the possibility of publishing a brief annual list, giving thesis authors and titles.

In this issue we have, *inter alia*, part I of a substantial survey on curvature and topology by Mark Walsh. Part II is scheduled to appear in the next issue.

Links for Postgraduate Study

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: (Olaf Menkens)
http://www.dcu.ie/info/staff_member.php?id_no=2659
DIT: mailto://chris.hills@dit.ie
NUIG: mailto://james.cruickshank@nuigalway.ie
NUIM: http://www.maths.nuim.ie/pghowtoapply
QUB: http://www.qub.ac.uk/puremaths/Funded_PG_2016.html
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: http://www.ucc.ie/en/matsci/postgraduate/
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://sarah.mitchell@ul.ie
UU: http://www.compeng.ulster.ac.uk/rgs/

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor, a url that works. All links are live, and hence may be accessed by a click, in the electronic edition of this Bulletin[1].

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[1] http://www.irishmathsoc.org/bulletin/

Irish Mathematical Society Bulletin 80, Winter 2017

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President	Prof S. Buckley	Maynooth University
Vice-President	Dr Pauline Mellon	University College Dublin
Secretary	Dr D. Malone	Maynooth University
Treasurer	Prof G. Pfeiffer	NUI Galway

Dr P. Barry, Prof J. Gleeson, Dr B. Kreussler, Dr R. Levene, Dr M. Mac an Airchinnigh, Dr D. Mackey, Dr A. Mustata, Dr J. O'Shea .

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Local Representatives

Belfast	QUB	Dr M. Mathieu
Carlow	IT	Dr D. Ó Sé
Cork	IT	Dr D. Flannery
	UCC	Dr S. Wills
Dublin	DIAS	Prof T. Dorlas
	DIT	Dr D. Mackey
	DCU	Dr B. Nolan
	SPD	Dr S. Breen
	TCD	Prof R. Timoney
	UCD	Dr R. Higgs
Dundalk	IT	Mr Seamus Bellew
Galway	UCG	Dr J. Cruickshank
Limerick	MIC	Dr B. Kreussler
	UL	Mr G. Lessells
Maynooth	NUI	Prof S. Buckley

Tallaght	IT	Dr C. Stack
Tralee	IT	Dr B. Guilfoyle
Waterford	IT	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€160
Ordinary member	€25
Student member	€12.50
DMV, I.M.T.A., NZMS or RSME reciprocity member	€12.50
AMS reciprocity member	$15

The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 30.00.
If paid in sterling then the subscription is £20.00.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 30.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form with one year's subscription to:

The Treasurer, IMS
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway
Ireland

Deceased Members

It is with regret that we report the deaths of members:
Timothy Murphy, of TCD, died on 15 February 2017.
The Society learned only this year of the death of Professor James D. Reid (1930-2013) of Wesleyan University.

E-mail address: `subscriptions.ims@gmail.com`

President's Report 2017

Introduction and thanks

My term as President begin in January 2017. Pauline Mellon's term as Vice-President began at the same time. I would like to thank my predecessor, Michael Mackey, for his leadership and hard work during his term of office. He did an excellent job of representing the Society, and he helped to make the transition very smooth.

The new members of the Committee for 2017 are Pauline Mellon and Dana Mackey, replacing outgoing members Michael Mackey and Martin Mathieu, both of whom I would like to thank for their dedicated work on the Committee during the previous six years.

I wish to record the Society's thanks to Anthony O'Farrell for his ongoing excellent work as editor of the IMS Bulletin. Thanks also go to the other members of the Bulletin team: Ian Short (problem page maintainer), Gordon Lessells (who oversees printing and distribution), and the Editorial Board (Anthony O'Farrell, Tom Carroll, Jim Cruickshank, Ian Short, and Thomas Unger).

Finally, I would also like to thank Richard Timoney for maintaining the Society's webpages.

IMS meetings and IMS-supported meetings

The Society's 2017 "September meeting" was held at IT Sligo (31 August–1 September). As usual, this functioned as both the main scientific meeting of the Society and its AGM. Special thanks go to the main organiser, Leo Creedon, for organising this event. The 2018 "September meeting" will be held in UCD.

The Committee also met on 19 December in Queen's University Belfast. That meeting was followed immediately by the IMS Invited Lecture *Temporal Profiles of Avalanches on Networks*, given by James Gleeson. It is my hope that future end-of-year Committee meetings can also be followed by invited lectures.

The Society supported the following meetings held during 2017:

- Minimal surfaces: Integrable systems and visualisation, UCC: March 27–29.
- CERME 10, DCU: February 1–5.
- The 14th Annual Workshop on Numerical Methods for Problems with Layer Phenomena, NUIG: April 6–7.

- Groups in Galway, NUIG: May 18–20.
- Irish Geometry Conference 2017, Maynooth University: May 25–26.
- 6th National Student Chapter Conference of the UK and Ireland Section of SIAM, NUIG: May 26.
- 3rd Irish Linear Algebra and Matrix Theory Meeting, Maynooth University, September 8.
- 11th Annual Irish Workshop on Mathematics Learning and Support, North West Regional College, Derry: December 1.

It was a difficult task to choose which meetings to support. Thanks go to the Treasurer, Goetz Pfeiffer, for efficiently overseeing the application process, and for making the task of deciding on funding less onerous.

EMS Meetings of Presidents

I represented the Society at the annual Meeting of Presidents held by the European Mathematical Society. This year's meeting was on 1–2 April in Lisbon. It was a useful opportunity to discuss matters of mutual interest with other national mathematical society presidents and representatives. The main discussion on 2 April concerned open access publication. Various documents, including a report on the meeting, can be found at `http://ems2017.spm.pt/`

I accepted an invitation to hold the next Meeting of Presidents in Ireland. That meeting will be held at Maynooth University, 14–15 April 2018.

Education Subcommittee

The Education Subcommittee has been working very actively on a number of matters throughout the year, and I would like to thank it for its hard work. As a result of a recommendation from the Subcommittee, I wrote to the National Council for Curriculum and Assessment (NCCA), seeking to add at least one more academic mathematician to the membership of the Mathematics Development Group (MDG). (The MDG is a committee that was formed to advise the NCCA on matters relating to Mathematics, and it is currently working on a revised Applied Mathematics curriculum. David Wraith was the only third level mathematician on the MDG as of early 2017.) Thanks to pressure from ourselves and other sources, the NCCA approved James Gleeson as an additional member of the MDG.

Another initiative of the Subcommittee resulted in the formulation of a *Short Guide to Second-Level Maths*. This webpage on the IMS website contains a first introduction to the structure of mathematics education in the Irish second-level education system, together with various useful links relating to curriculum and exams. As a source of detailed information on what mathematics is taught at second level in Ireland—bearing in mind that the curriculum has undergone significant changes in recent years because of the roll-out of the new "Project Maths" syllabus—this page should be of interest to all IMS members working in Irish third level institutions, as well as colleagues in other science departments.

S. Buckley
December, 2017

E-mail address: sbuckley@nuim.ie

Minutes of the IMS Annual General Meeting
September 1, 2017 at the Institute of Technology Sligo

Present: P. Barry, S. Buckley, S. Dineen, T. Dorlas, F. Gallagher, M. Hanley, T. Hurley, B. Kreussler, G. Lessells, P. Lynch, D. Mackey, D. Malone, M. Mathieu, P. Mellon, A. O'Farrell, M. OReilly, G. Pfiffer, R. Quinlan, R.M. Timoney.

Apologies: J. O'Shea, J. Gleeson, R. Levene.

1 Minutes

Minutes of the last meeting were accepted.

2 Matters Arising

There were no matters arising that would not be covered in other items.

3 Correspondence

The Charities Regulator had contacted the society, as charity status for the society needs to be updated to come in line with new charity regulations.

4 Membership Applications

Applications from M. Golitsyna (Student), A.J. Kavanagh, E.J. Clerkin, S.L. Kitson and F. Gallagher (Student) had been approved by the committee.

5 President's Report

- The president extended his thanks to M. Mackey, the previous president, for handing over the society in good shape and officers of the society for their continued activity. L. Creedon was thanked for hosting the 2017 Conference and AGM. A. O'Farrell, G. Lessells and R. Timoney were thanked for producing, distributing and maintaining the Bulletin and the web pages.
- M. Mackey and M. Mathieu have stepped down from the committee. P. Mellon has joined as vice-president and D. Mackey as a general committee member.
- The educational subcommittee has been active and details to be presented later in the meeting.

- M. Mackey had organised for the society to be represented on the RIA's Physical, Chemical and Mathematical Sciences Committee. This representation is active and the committee is working on a number of issues that will be of interest to the mathematical community.
- The society is part of the European Mathematical Society. The president attended the Presidents' Meeting of the IMS and discussed a number of upcoming scientific meetings. The 2018 EMS Presidents' Meeting is to be hosted in Maynooth, on 14–15 April 2018.
- The Fergus Gaines cup is usually presented at the IMS AGM. However, this year it will be coordinated with the Irish Mathematical Trust's teachers' award.
- A number of different meetings have been supported by the society this year (including the *Minimal surfaces: Integrable systems and visualisation*, *Groups in Galway 2017*, *6th National Student Chapter Conference of the UK and Ireland Section of SIAM*, *The 14th Annual Workshop on Numerical Methods for Problems with Layer Phenomena*, *CERME 10*, *Irish Geometry Conference 2017*, *3rd Irish Linear Algebra and Matrix Theory Meeting*, *11th Annual Irish Workshop on Mathematics Learning and Support*).
- The 2018 meeting will be hosted in UCD in September. Arrangements for the 2019 meeting are underway.
- M. OReilly proposed thanks for all the hard work.

6 Treasurer's Report

A report on finances was circulated. Membership income is as expected. Outgoings are also as expected, with a slight reduction in costs and a slightly larger surplus as a consequence. The report was accepted and thanks were proposed by A. O'Farrell.

7 Bulletin

A. O'Farrell encouraged the submission of any articles that may be of interest to members, and also Ph.D. reports, etc. Thanks were extended to R. Timoney and G. Lessells and to the editorial board (T. Carroll, J. Cruickshank, I. Short, T. Unger). M. Mathieu suggested that IMS local reps could encourage people to submit articles and thesis reports.

8 Education Subcommittee

- The Educational Subcommittee has been working hard. As its job is to bring things to the attention of the society, it doesn't communicate with other bodies directly.
- One issue that has been considered is the revision of Applied Mathematics at the leaving certificate. In this respect, extra representation for professional mathematicians has been secured on the NCCA's MDG (Mathematics Development Group). D. Wraith was already on this body but J. Gleeson is also now in attendance. To support the work of these members of the MDG a document has been produced by the educational subcommittee. A discussion of the document followed covering issues such as the IAMTA, availability of applied mathematics teachers, availability of timetable slots to teach applied mathematics, software appropriate for use in applied mathematics and so on. The document was accepted with a minor amendment, to not specifically advocate for any software package. More representation on the MDG was identified as worth pursuing.

9 Elections

- B. Kreussler has reached the end of his term and M. Mac an Airchinnigh has asked to step down.
- This year the secretary and treasurer positions are to be filled. G. Pfiffer and D. Malone are happy to be nominated again. M. Mathieu and L. Creedon were proposed to fill the two arising positions on the committee.
- M. OReilly noted that geographic and institutional representation on the committee was, at least informally, important. It was noted that the current spread of representation was good.

10 AOB No items were raised.

Report by David Malone, Secretary.
david.malone@nuim.ie

IMS Annual Meeting
Institute of Technology Sligo

August 31 and September 1, 2017

The meeting was hosted by the Mathematical Modelling Research Group of IT Sligo and was organised by Dr Leo Creedon, Kieran Hughes and Fergal Gallagher.

On the first day, Registration (and refreshments) ran from 11:00-12:00 and there were breaks for lunch and coffee. There was a poster session at 17:30, and the conference dinner was held at the Glasshouse Hotel in the city centre, from 19:30. On the second day, there was a meeting of the IMS Executive Committee from 08:30–10:00, and the Society's Annual General Meeting was held from 13:00–14:30. The meeting closed at 17:30.

The meeting was opened by Dr Dr Brendan McCormack, President, IT Sligo. About 50 members attended.

The schedule of invited talks was as follows:

Thursday, 31 August

13:00-14:00: Ted Hurley (NUI Galway): *Algebraic structures for communications*

14:00-15:30: Eoin Gill (WIT): *Maths Week*

Peter Lynch (UCD): *Pedro Nunes and the Retrogression of the Sun*

Fergal Gallagher (IT Sligo): *Optimal Codes and Group Algebras*

16:00 - 16:30: Kieran Hughes (IT Sligo): *Group Algebras, Codes and Derivations*

16:30 - 17:30: Des MacHale (UCC): *Sherlock Holmes, James Moriarty, and George Boole*

Friday, September 1

10:00 - 11:30: Olga O'Mahony (NUI Galway): *Edge-minimal graphs of exponent 2*

Malgorzata Wieteska (IT Sligo): *Mathematical Modelling of the Bovine Estrus Cycle*

Konrad Mulrennan (IT Sligo): *A comparative study of machine learning algorithms applied as soft sensors in a polymer extrusion process*

12:00 - 12:40: Dana Mackey (DIT): *Interval Filling and Immunodiagnostics*

14:30 - 16:00: Richard Karsten (Acadia University, Nova Scotia, Canada): *Applying mathematics to Tidal Energy*
Rogrio Villafranca (UFABC, Sao Paulo, Brazil): *Abelian codes: some results from group rings*
Stephen McGuire (Maynooth University): *On a Lemma of Bernik*

16:30 - 17:20: Rod Gow (UCD): *George Gabriel Stokes: the work of a versatile and influential mathematician*

Reports of Sponsored Meetings

Minimal Surfaces: Integrable Systems and Visualization.
Spring 2017 Workshop
27–29 March 2017, University College Cork

m:iv is an international network funded by The Leverhulme Trust. The network partners consist of Tim Hoffman (Munich), Martin Kilian (Cork), Katrin Leschke (Leicester), Francisco Martín (Granada) and Katsuhiro Moriya (Tsukuba). The network will run a series of workshops to bring together experts from the fields of minimal surfaces, integrable systems and computer visualization. More details can be found at `http://www2.le.ac.uk/projects/miv`. The first workshop, held in Cork, ran over three days and consisted of twelve talks by speakers from around the globe. The speakers, titles and abstract were as follows:

Benoît Daniel (Université de Lorraine)
Minimal isometric immersions into $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$*.*
For a given simply connected Riemannian surface Σ, we relate the problem of finding minimal isometric immersions of Σ into $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ to a system of two partial differential equations on Σ. We prove that a constant intrinsic curvature minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ is either totally geodesic or part of an associate surface of a certain limit of catenoids in $\mathbb{H}^2 \times \mathbb{R}$. We also prove that if a non constant curvature Riemannian surface admits a continuous one-parameter family of minimal isometric immersions into $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, then all these immersions are associate.

Isabel Fernández (University of Seville)
Surfaces of critical constant mean curvature and harmonic maps.
Minimal surfaces (H=0) in euclidean 3-space and Bryant surfaces (H=1) in hyperbolic 3-space are a special family among all the CMC surfaces in spaces forms. Similarly, surfaces of critical CMC in homogenous 3-spaces present a special behavior among all the CMC surfaces. For example, Fernández and Mira found a hyperbolic Gauss map for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that turns out to be harmonic for surfaces of critical CMC. Later on, Daniel discovered a Gauss map for surfaces

in Heisenberg space that is also harmonic when the mean curvature is critical. However, both definitions are quite different and it was unclear how to extend them to the general setting. In this talk we will review some properties of critical CMC surfaces in homogeneous 3-spaces and present a unified definition of a Gauss map for surfaces in these ambient spaces that is harmonic when the mean curvature of the surface is critical.

Leonor Ferrer (University of Granada)

Properly embedded minimal annuli in $\mathbb{H}^2 \times \mathbb{R}$.

In this talk we ask for properly embedded minimal annuli in $\mathbb{H}^2 \times \mathbb{R}$ which bound a pair of vertical graphs over $\partial_\infty \mathbb{H}^2 \equiv \mathbb{S}^1$. We present some compactness results for these surfaces. We also give some existence results for proper, Alexandrov-embedded, minimal annuli. Contrary to what might be expected, we show that, in general, one can not prescribe the two components of the boundary at infinity. However, we can prescribe one of the boundary data, the position of the neck and the vertical flux of the annulus. This is a joint work with F. Martín, R. Mazzeo and M. Rodríguez.

Sebastian Klein (University College Cork)

Asymptotic methods for finite gap curves in the 3-dimensional space forms.

I will discuss how asymptotic methods can be used to study closed curves, and in particular finite gap curves, in the 3-dimensional space forms.

Laurent Hauswirth (Université Paris-Est)

Asymptoticity of Minimal surfaces in Heisenberg space.

I will describe the asymptotic behavior of minimal ends using the Dirac operator. Joint work with Taimanov.

Felix Knöppel (Technical University Berlin)

Complex line bundles over simplicial complexes.

Over the last years we could apply bundle theory to several problems in Computer Graphics. This led to a quite good understanding of the underlying discrete theory: We present a discrete analogue of a classification theorem due to Kobayashi and then focus on hermitian line bundles with curvature. For these a discrete analogue of Weil's theorem and a discrete Poincaré-Hopf theorem hold. Furthermore, we generalize the well-known cotan-Laplace operator.

Ben McKay (University College Cork)
Isometric immersions and integrability
Some examples of surfaces for which isometric immersion to a space form is an integrable system.

Barbara Nelli (University of L'Aquila)
Minimal Surfaces in the Heisenberg Space.
We discuss the behaviour of some minimal surfaces in the Heisenberg space. In particular, we deal with existence and growth of non compact graphs and stability properties.

Pablo Mira (Technical University of Cartagena)
Constant mean curvature spheres in homogeneous three-manifolds, I
The aim of these two talks (which are based on joint work with Bill Meeks and Antonio Ros) is to prove the following theorem: any two spheres of the same constant mean curvature immersed in a homogeneous three-manifold only differ by an ambient isometry. Our study will also determine the exact values of the mean curvature for which such CMC spheres exist, together with some of their most important geometric properties. For instance, we will show that CMC spheres in simply connected metric Lie groups have index one, are Alexandrov embedded and maximally symmetric, their left invariant Gauss maps are diffeos, and the corresponding moduli space of CMC spheres is a connected one-dimensional manifold.

Franz Pedit (University of Massachusetts)
Energy quantization for harmonic 2-spheres in non-compact symmetric spaces.
It is well known from results by Uhlenbeck, Chern-Wolfson and Burstall-Rawnsley that harmonic 2-spheres in compact symmetric spaces have quantized energies. Using the reformulation of the harmonic map equation as a family of flat connections, we construct an energy preserving duality between harmonic maps into a non-compact symmetric space (typically pseudo-Riemannian) and harmonic maps into compact real forms of its complexification. Applying this construction to the conformal Gauss maps of Willmore 2-spheres in the n-sphere provides a generalization and unifying approach to existing

quantization results in special cases: Bryant for n=3; Montiel for n=4; and Ejiri for Willmore 2-spheres admitting a dual Willmore surface.

Joaquín Pérez (University of Granada)
Constant mean curvature spheres in homogeneous three-manifolds, II
See Pablo Mira.

Ulrich Pinkall (Technical University Berlin)
From Smoke Ring Flow to Real Fluids.
The so-called "smoke ring flow" for space curves was introduced in 1910 by Da Rios (who was a PhD student of Levi-Civita) for describing the time evolution of vortex filaments in an ideal fluid. Starting from the 1970's it became clear that the smoke ring flow actually is the most basic integrable system that originates in Differential Geometry. It is closely related to the one-dimensional Landau-Lifshitz equation and to the one-dimensional nonlinear Schrödinger equation. As an asymptotic limit it is also crucial for understanding the geometry of CMC surfaces in space forms. Vortex filaments are the solitons of fluid dynamics, so in this sense fluid flow can be viewed as a perturbed integrable system. In this talk we will show a method for fluid simulation that reflects this fact. This method is closely related to the three-dimensional Landau-Lifshitz equation and to the three-dimensional nonlinear Schrödinger equation.

Martin Schmidt (University of Mannheim)
Willmore energy of conformal maps $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$

Report by Martin Kilian, UCC
m.kilian@ucc.ie

Fourteenth Irish Geometry Conference
25–26 May 2017, Maynooth University

The Irish Geometry Conference was held at Maynooth University on 25-26 May 2017, and was organized by David Wraith. This annual meeting in now in its fourteenth year, and this was the fourth time it had been held in Maynooth. As well as participants based in Ireland, this year's event included two speakers from Germany and one from Argentina, with talks covering a diverse range of geometric and topological research topics. The speakers, titles and abstracts were as follows:

Stefan Bechtluft-Sachs, N.U.I.M.

Green's function of Dirac Operators on rank-1-symmetric spaces.

In harmonic spaces the introduction of radial functions reduces the scalar Laplace equation to an ordinary differential equation. The Laplace operator has a right inverse whose integral kernel, Green's function, is radial in the sense that it is a function of the distance only. Our main aim is to extend this radial calculus to Dirac operators and operators that can be derived from these, like the Cartan differential and the Laplacian on differential forms.

On compact manifolds a solution of the Dirac equation on differential forms yields a solution of the Cartan equation via Hodge Theory. This in turn provides a density for Gauss-type formulas for the linking number (in 3 dimensions this is essentially the Biot-Savart law for the magnetic flux induced by a stationary current). On non-compact rank one symmetric spaces, in the absence of Hodge Theory, we get the same results from our radial equation. (joint with E.Samiou)

Nikos Georgiou, W.I.T.

Minimal surfaces in the product of 2-manifolds

In this talk we first introduce a Kaehler structure of neutral signature on the product of two pseudo-Riemannian manifolds. Then we discuss surface theory in such a product, and in particular we describe a classification result about Lagrangian surfaces with parallel mean curvature. Finally, we present some recent results on minimal surfaces in the product of a two-dimensional real space form with itself.

Thomas Huettemann, Q.U.B.

From topology to strongly graded algebra

Finiteness properties of topological spaces are relevant in many areas of topology (e.g. manifold theory) and algebra (e.g. geometric group theory). Starting from a specific homological result on "finite domination" going back to Ranicki and others, I will explain how seemingly simple generalisations lead to unexpected complications, and how strongly graded algebra provides a natural setting to formulate statements and proofs.

Martin Kerin, University of Bonn and University of Muenster, Germany

Non-negative curvature on exotic spheres

Since their discovery, there has been much interest in the question of precisely which exotic spheres admit a metric with non-negative sectional curvature. In dimension 7, Gromoll and Meyer found the first such example. It was subsequently shown by Grove and Ziller that all of the Milnor spheres admit non-negative curvature. In this talk, it will be demonstrated that the remaining exotic 7-spheres also admit non-negative curvature. This is joint work with K. Shankar and S. Goette.

Martin Kilian, U.C.C.

On finite gap curves

A curve in a 3-dimensional space form is a finite gap curve if it is stationary under some evolution in the self-focusing non-linear Schroedinger hierarchy. I will survey some recent results about such curves and will explain techniques from the theory of integrable systems for this setting.

Adib Makrooni, N.U.I.G.

Compact homogeneous spaces with positive Euler characteristic and their 'strange formulae'.

I will describe a generalisation of the 'strange formula' of Freudenthal and de Vries for compact homogeneous spaces with positive Euler characteristic. I will apply the results to computing a topological invariant used to study hyper-Kaehler structures. In my talk I will make use of a sharpened version of the Borel and de Siebenthal Theorem, describing the isotropy representation of K on the tangent space to G/K, where K denotes a maximal connected subgroup of maximal rank in a compact simple Lie group G. This is joint work with J. Burns.

Carlos Olmos, National University of Cordoba, Argentina

The nullity of homogeneous Riemannian manifolds

the nullity distribution of the curvature tensor of a Riemannian space was defined by Chern and Kuiper in 1952. This distribution turns out to be autoparallel, around the points where the dimension is locally constant. Nevertheless, nothing was known about the nullity distribution in homogeneous spaces. In this talk we will mainly refer to some recent results obtained jointly with Antonio J. Di Scala and Francisco Vittone, that will motivate the presentation of some interesting points of view in homogeneous geometry. Let M be a locally irreducible homogeneous Riemannian manifold. We prove that if M is either compact, or Kahler, or more generally nearly Kaehler, then the distribution of nullity is trivial. We will present also a general structure theory for homogenous manifolds with non-trivial nullity that predicts the existence of a transvection (i.e. a Killing field which is parallel at some point) with null Jacobi operator and not in the nullity. With the aid of this result we are able to find a one parameter family irreducible homogeneous spaces of dimension 4 with non-trivial distribution of nullity (as far as we know these are the first known examples). By making use of the above mentioned structure theorem, we also show that any homogeneous space with a transitive semisimple subgroup of isometries, has trivial nullity distribution.

Thomas Schick, University of Goettingen, Germany
'Poison' submanifolds prevent positive scalar curvature
The Gauss-Bonnet theorem implies that the sphere is the only compact oriented surface admitting a metric with positive scalar curvature. In higher dimensions, the role of the Gauss-Bonnet theorem is taken by various index-invariants of the Dirac operator.

A particularly intricate way to employ this goes back to Gromov and Lawson: they identify certain types of submanifolds whose existence prevents the existence of a metric of positive scalar curvature. We present this and a generalization (joint work with Hanke and Pape). Then we discuss recent constructions which put these obstructions into the wider context of higher index theory (relating this also to conjectures like the Baum-Connes isomorphism conjecture or the strong Novikov conjecture).

Finally (if time permits), on non-compact manifolds we will comment on the difference between the non-existence of complete metrics with positive scalar curvature versus "scalar curvature bounded below by a positive constant" (joint with Cecchini).

Report by Prof David Wraith, Maynooth University
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Third Irish Linear Algebra and Matrix Theory Meeting September 8th, 2017, Maynooth University Department of Mathematics & Statistics

The 3rd Irish Linear Algebra and Matrix Theory Meeting took place at the Maynooth Univeristy Department of Mathematics &

Statistics on September 8th, 2017. In the spirit of the 2 previous and successful meetings in this series, hosted by NUI Galway (2012) and UCD (2014), the event's principal aim was to provide a relaxed environment in which researchers interested in Linear Algebra and related applications could meet and discuss a range of topics and results from the field. The venue for the meeting was MS2 in Logic House on Maynooth's historic South Campus, and there were numerous opportunities for participants to hold informal discussions over coffee and lunch during the day. The total number of participants at the event was 16, of whom 5 were research students.

Linear Algebra and Matrix Theory Meeting

The meeting consisted of 7 talks covering a wide range of topics across the spectrum of pure and applied linear algebra. The speakers at the meeting and the talk titles are listed below.

- Thomas J. Laffey (UCD), *Generating sets for full matrix algebras.*
- Niall Madden (NUI Galway), *Parameter robust linear solvers for singularly perturbed differential equations.*
- Ollie Mason (Maynooth University), *The joint spectral radius and extremal norms in the max algebra.*
- Cian O'Brien (NUI Galway), *Alternating signed bipartite graphs.*
- Rachel Quinlan (NUI Galway), *Rank problems for entry pattern matrices.*

- Sergey Sergeev (University of Birmingham), *Reachability, circulants and interval analysis in max algebra.*
- Helena Šmigoc (UCD), *Companion type realisations in the nonnegative inverse eigenvalue problem.*

We would like to thank all speakers and participants for contributing to what was a most enjoyable day.

The organizers are very grateful for the support received from the Department of Mathematics & Statistics at Maynooth University and the Irish Mathematical Society.

Report by Oliver Mason, Maynooth University
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Generalised Hopficity and Products of the Integers

BRENDAN GOLDSMITH

In Memoriam: Eoin Coleman (Oren Kolman) 1959-2015.

Abstract. Hopfian groups have been a topic of interest in algebraic settings for many years. In this work a natural generalization of the notion, so-called R-Hopficity is introduced. Basic properties of R-Hopfian groups are developed and the question of whether or not infinite direct products of copies of the integers are R-Hopfian is considered. An unexpected result is that the answer to this purely algebraic question depends on the set theory assumed.

1. Introduction

The finiteness of a set S can be expressed in two equivalent ways in terms of functions from $S \to S$. Thus the set S is finite if, and only if, every one-one function $S \to S$ is invertible, if, and only if, every onto function $S \to S$ is invertible. The comparable statements in the category of all groups $\mathcal{G}$ fail to be true: multiplication by the prime p in the additive group of integers is a one-one homomorphism (or monomorphism) which is not an invertible homomorphism (or automorphism) and the same multiplication in the quasi-cyclic group $\mathbb{Z}(p^\infty)$ is an onto homomorphism (or epimorphism) which is not an automorphism. Nonetheless these statements about homomorphisms may be used to define certain classes of groups which will contain all finite groups. Specifically we shall say that a (possibly non-commutative) group G is

(i) *co-Hopfian* if every monic endomorphism of G is an automorphism;

(ii) *Hopfian* if every epic endomorphism of G is an automorphism.

The terminologies arise from the fact that groups satisfying condition (ii) arose in work of the topologist H. Hopf on fundamental groups of closed two-dimensional orientable surfaces, while (i) is in

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a certain weak sense a notion dual to (ii). Co-Hopfian and Hopfian groups were investigated under the names S-groups and Q-groups respectively by Baer in [1]. The terminology using S, Q reflects the fact that it is easy to establish that (i) and (ii) are respectively equivalent to the following:

(S) G cannot have a proper isomorphic subgroup;

(Q) G cannot have a proper isomorphic quotient group.

There is an extensive literature on both Hopfian and co-Hopfian groups, the papers [1, 2, 3, 5, 9, 10, 12, 13, 16, 17] and the references therein give a small cross-section of the literature.

To simplify our presentation we are going to consider only additively written Abelian groups but many of our results can be extended without difficulty to both the category of (all) groups and the category of modules over a commutative ring. Furthermore, we will focus attention on a weaker version of Hopficity introduced below, even though there are many results of a dual nature that can be established for the corresponding generalisation of co-Hopficity.

If one interprets the defining condition (ii) of Hopficity as saying that a group G is Hopfian if every surjective endomorphism has a two-sided inverse, then it is possible to weaken the definition in two natural ways, where in an obvious notation the letters 'R, L' stand for 'right' and 'left' respectively. Note that in this paper, maps are always written on the left.

Definition 1.1. A group G is said to be R-Hopfian [L-Hopfian] if for every surjection $\phi \in \mathrm{End}(G)$, there is an endomorphism ψ of G such that $\phi\psi = 1_G$ [$\psi\phi = 1_G$].

Observe firstly that if G is Hopfian, then certainly G is both R-Hopfian and L-Hopfian. Moreover, if G is L-Hopfian and ϕ is a surjection, then the equation $\psi\phi = 1_G$ implies that ϕ is also an injection, so that ϕ is an automorphism of G. Consequently the class of L-Hopfian groups coincides with the class of Hopfian groups. Our focus will therefore be on the class of R-Hopfian groups.

The paper is organised into a further three sections: in Section 2 we will explore some elementary properties of our new class and how it relates to the original concept of Hopficity. The rather simple Example 2.4 is important for subsequent developments because it is well known - see [18] or [7, Lemma 112.1] - that a group has regular endomorphism ring if, and only if, it is both (Ker)-direct and (Im)-Direct in the sense that the kernel (image) of every endomorphism is a direct summand. Recently the notions of (Ker)-directness and (Im)-directness of modules have been investigated in [14, 15], where the concepts are called *Rickart modules* and *dual Rickart modules*; a weakening of the idea of (Ker)-directness will be key to understanding R-Hopficity.

In the third section we investigate the behaviour of the class under the formation of direct sums. This is a natural consideration in light of the rather surprising examples given by Corner [5] in 1965:

(C1) the direct sum of two torsion-free Hopfian groups need not be Hopfian

(C2) there exist torsion-free Hopfian groups A such the direct sum $A \oplus A$ is not Hopfian.

We shall show that a result analogous to (C1) holds for R-Hopfian groups but the analogue of (C2) seems, on the surface at least, to be extremely difficult.

Sections 2 and 3 are reasonably straightforward and do not require a great deal of specialised knowledge.

The final section on the Baer-Specker groups, i.e., direct products of the group of integers, $P_\kappa = \mathbb{Z}^\kappa$, makes use of much deeper results and introduces a number of ideas relating to the interaction of set theory with group theory. The main result of that section is the rather surprising fact that the seemingly totally algebraic notion of R-Hopficity leads one to natural questions whose answers are independent of the usual Zermelo-Fraenkel axioms of set theory along with the axiom of choice, ZFC.

We finish off this introduction by noting that notation in the paper is standard as in the two volumes of Fuchs [7, 8]; in particular mapping are consistently written on the left and for an Abelian group G, the ring of endomorphisms of G shall be denoted by $\mathrm{End}(G)$ and the group of all homomorphisms:$A \to B$ shall be denoted $\mathrm{Hom}(A, B)$. We shall frequently make use of the well-known representation of endomorphisms of the direct sum of groups A, B by a matrix: if ψ is an endomorphism of $A \oplus B$, then ψ has a matrix representations of the form $\Delta = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ where $\alpha \in \mathrm{End}(A), \beta \in \mathrm{End}(B), \gamma \in \mathrm{Hom}(B, A)$ and $\delta \in \mathrm{Hom}(A, B)$. The additive group of integers is denoted by $\mathbb{Z}$, while the additive group of rationals is denoted by $\mathbb{Q}$.

All groups will be additively written Abelian groups.

2. Elementary Results

The notion of direct finiteness provides the connection between Hopficity and R-Hopficity. Recall that a group G is said to be *directly finite* if, given endomorphisms ϕ, ψ of G with $\phi\psi = 1_G$, the identity map on G, then $\psi\phi = 1_G$ also.

It is easy to show for an Abelian group G, that direct finiteness is equivalent to the statement that G cannot have a proper isomorphic direct summand.

Proposition 2.1. A group G is Hopfian if, and only if, it is R-Hopfian and directly finite; in particular, if $\mathrm{End}(G)$ is commutative, then G is R-Hopfian if, and only if, it is Hopfian.

Proof. If G is Hopfian then every surjection has an inverse, so G is certainly R-Hopfian. However, if $\alpha\beta = 1_G$ for some $\alpha, \beta \in \mathrm{End}(G)$, then α is surjective and so, by the Hopficity of G, it has an inverse α^{-1}. It follows immediately that $\beta = \alpha^{-1}$ and so $\beta\alpha = 1_G$, whence G is directly finite.

Conversely, given any surjection $\phi \in \mathrm{End}(G)$, R-Hopficity ensures the existence of an endomorphism ψ such that $\phi\psi = 1_G$. By direct finiteness, we have that $\psi\phi$ is also equal to 1_G and so ϕ is invertible with inverse ψ. Since ϕ was arbitrary, we have that G is Hopfian.

The particular case in which $\mathrm{End}(G)$ is commutative is now immediate. □

Corollary 2.2. *An indecomposable group is Hopfian if, and only if, it is R-Hopfian. In particular, the group $\mathbb{Z}(p^\infty)$ is not R-Hopfian for any prime p.*

Proof. The necessity is immediate and doesn't require the indecomposability. Conversely suppose that G is R-Hopfian. It suffices by Proposition 2.1 to show that G is directly finite. Suppose then that $fg = 1_G$ for endomorphisms f, g of G. Then gf is an idempotent endomorphism which cannot be the zero map and so the indecomposability of G forces $gf = 1_G$, as required. □

Given the close connection between Hopfian and R-Hopfian groups just established, we would expect R-Hopfian groups to share some properties known for Hopfian groups. Our first result is an analogue of such a property of Hopfian groups.

Proposition 2.3. *A direct summand of an R-Hopfian group G is again R-Hopfian.*

Proof. Suppose then that $G = H \oplus S$ and let α be an arbitrary surjection in End(H). Then $\psi = \alpha \oplus 1_S$ is a surjection in End(G) and so there is a $\phi \in \text{End}(G)$ such that $\psi\phi = 1_G$. Using the standard matrix representation of endomorphisms of a direct sum, this means that

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1_S \end{pmatrix} \cdot \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix} = \begin{pmatrix} 1_H & 0 \\ 0 & 1_S \end{pmatrix}, \quad \text{where } \phi = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix}.$$

Thus $\alpha\mu = 1_H$, and so, since α was an arbitrary surjection in End(H), H is R-Hopfian. □

Recall that a ring R is said to be *(von Neumann) regular* if, given any $a \in R$ there exists a $b \in R$ such that $aba = a$. We can then deduce a simple result which will provide some motivation for a somewhat deeper result that gives us a classification of R-Hopfian groups.

Example 2.4. If G is a group with a regular endomorphism ring, then G is R-Hopfian.

Proof. Suppose that α is an arbitrary surjection in End(G), then by regularity, there is a $\beta \in \text{End}(G)$ such that $\alpha\beta\alpha = \alpha$. If $x \in G$, then $x = \alpha(y)$ for some $y \in G$ and so

$$x = \alpha(y) = \alpha\beta\alpha(y) = \alpha\beta(x) \text{ for all } x \in G.$$

Hence the composition $\alpha\beta$ is the identity 1_G on G and G is R-Hopfian. □

Note that it follows easily from Example 2.4 that the class of R-Hopfian groups properly contains the class of Hopfian groups: it is well known that the ring of linear transformations of an infinite dimensional rational vector space is regular and hence, for example, the group $G = \bigoplus_{\aleph_0} \mathbb{Q}$ is R-Hopfian but it is clearly not Hopfian since it contains a proper isomorphic direct summand.

As mentioned in the Introduction, a group with regular endomorphism ring has the property that the kernel of every endomorphism is a direct summand of the group; the group is then said to be (Ker)-direct. A natural weakening of this concept is the following:

Definition 2.5. A group G is said to be *(sKer)-direct* if the kernel of each surjective endomorphism of G is a direct summand of G.

The proof of the next result is reasonably well known but we give a detailed proof for completeness.

Theorem 2.6. *A group G is R-Hopfian if, and only if, it is (sKer)-direct.*

Proof. Assume that G is R-Hopfian and that ϕ is an arbitrary surjective endomorphism of G. Then there exists a (necessarily monic) endomorphism α such that $\phi\alpha = 1_G$. Since $\alpha\phi$ is then an idempotent endomorphism of G, its kernel is a summand of G. However, the fact that α is monic implies that $\mathrm{Ker}\phi = \mathrm{Ker}\alpha\phi$ and so $\mathrm{Ker}\phi$ is a summand of G. Since ϕ was arbitrary, we have that G is (sKer)-direct.

Conversely suppose that G is (sKer)-direct and let σ be an arbitrary surjective endomorphism of G. Then $G = \mathrm{Ker}(\sigma) \oplus T$ for some complement T. Now $\sigma \restriction T : T \to G$ has the property that $\sigma \restriction T(T) = G$ since the surjectivity of σ means that an arbitrary $x \in G$ has the form $x = \sigma(y)$ for some $y \in G$; but $y = k + t$ for some $k \in \mathrm{Ker}(\sigma), t \in T$ and so $x = \sigma(k+t) = \sigma(t)$. Since $\mathrm{Ker}(\sigma \restriction T) = T \cap \mathrm{Ker}(\sigma) = \{0\}$, we conclude that $\sigma \restriction T$ is an isomorphism $T \twoheadrightarrow G$. So there is an endomorphism $\eta : G \to T$ such that $(\sigma \restriction T)\eta = 1_G$. Hence $\sigma\eta = 1_G$ and G is R-Hopfian. □

It is clear from Theorem 2.6 that the class of R-Hopfian groups is large: free and divisible torsion-free groups of arbitrary rank, elementary p-groups of arbitrary dimension and torsion-free reduced algebraically compact groups are all R-Hopfian. (The first three classes are easy to see while the final one results from the fact that in this class, the kernel of any endomorphism is both complete and pure and hence a summand - see, for example [7, Corollary 39.3].)

Further results including a detailed discussion of R-Hopficity in the context of Abelian p-groups, may be found in [11].

3. Direct Sums

Our first result, which has been been proved in an outline form in [11, Proposition 3.5], is the simple:

Proposition 3.1. *If A is an R-Hopfian group, B a Hopfian group and*
$\mathrm{Hom}(A, B) = 0$*, then $A \oplus B$ is R-Hopfian.*

Proof. In the standard matrix representation of an endomorphism of $G = A \oplus B$, the entry in the (2,1) position must be 0 since $\mathrm{Hom}(A, B) = 0$. Let $\Delta = \left(\begin{smallmatrix} \mu & \nu \\ 0 & \sigma \end{smallmatrix}\right)$ be an arbitrary surjection and observe that this forces σ to be a surjection of B. Since B is Hopfian this implies that σ is an automorphism of B. We claim that μ is also a surjection. To see this pre-multiply Δ by the invertible matrix $\left(\begin{smallmatrix} 1 & -\nu\sigma^{-1} \\ 0 & \sigma^{-1} \end{smallmatrix}\right)$ which corresponds to performing two standard elementary row operations in the normal diagonalizing process, so that we again obtain a surjection which is equal to the diagonal matrix $\left(\begin{smallmatrix} \mu & 0 \\ 0 & 1 \end{smallmatrix}\right)$; consequently μ is a surjection as claimed and since A is R-Hopfian, there is an endomorphism α of A with $\mu\alpha = 1_A$. The proof is completed by observing that post-multiplying Δ with the matrix $\left(\begin{smallmatrix} \alpha & -\alpha\nu\sigma^{-1} \\ 0 & \sigma^{-1} \end{smallmatrix}\right)$ - this is the standard technique for inverting a matrix but here we are producing just a right inverse - yields the identity matrix.

□

We now show that Proposition 3.1 fails if we replace the condition $\mathrm{Hom}(A, B) = 0$ with $\mathrm{Hom}(B, A) = 0$. First we need a technical lemma.

Lemma 3.2. *Let A be a group having a surjection α whose kernel is a summand of A and suppose that there is a surjection γ from* Ker α *onto the group B. Then there is a mapping $\delta : A \to B$ extending γ such that the group $A \oplus B$ has a surjection with matrix representation* $\Delta = \begin{pmatrix} \alpha & 0 \\ \delta & 0 \end{pmatrix}$.

Proof. Let $A = K \oplus A_1$ where $K = \mathrm{Ker}\ \alpha$ and define the mapping $\delta : A \to B$ by
$$\delta(z) = \begin{cases} \gamma(z) & : \quad z \in K \\ 0 & : \quad z \in A_1 \end{cases}$$
. Now let b be an arbitrary element of B. Then there is an element $x \in K$ such that $\delta(x) = \gamma(x) = b$. Direct computation shows that Δ maps the element $(x, 0)$ of $A \oplus B$ onto $(0, b)$. Furthermore, if a is an arbitrary element of A, then there is an element $y \in A$ with $\alpha(y) = a$; note that we may assume that $y \in A_1$. Again direct computation, noting that $\delta(y) = 0$ since $y \in A_1$, gives us that Δ maps $(y, 0)$ onto $(a, 0)$. It follows that Δ is a surjection of $A \oplus B$, as claimed. □

Theorem 3.3. *If A is free of infinite rank κ and B is a group of cardinality $\leq \kappa$, then the group $G = A \oplus B$ is R-Hopfian if, and only if, B (or equivalently G) is free.*

Proof. The sufficiency is clear: free groups are always R-Hopfian.

For the necessity, decompose A as $A_1 \oplus A_2$ with $A \cong A_1 \cong A_2$ and define α to map A_1 isomorphically onto A and to act as 0 on A_2; α is then a surjection of A having kernel isomorphic to A. Furthermore, since Kerα is free of rank κ, there is a surjection γ : Ker$\alpha \twoheadrightarrow B$. Applying Lemma 3.2, we obtain a surjection Δ of $A \oplus B$ with matrix representation $\begin{pmatrix} \alpha & 0 \\ \delta & 0 \end{pmatrix}$ for some $\delta : A \to B$ which extends γ. Now every endomorphism of $A \oplus B$ has a matrix representation of the form $\begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix}$ and so if G is R-Hopfian, there is a product $\begin{pmatrix} \alpha & 0 \\ \delta & 0 \end{pmatrix} \cdot \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix}$ equal to the identity matrix for some choice of μ, ν, ρ, σ.

Hence we deduce that $\alpha\nu = 0$ and $\delta\nu = 1_B$; it follows immediately from the latter equality that Ker$\nu = 0$. Furthermore, the first equality forces $\nu(B) \leq$ Kerα, so that $\nu(B)$ is free, being a subgroup of the free group A. Since ν has trivial kernel, we have that $B \cong \nu(B)$, so that B is necessarily free. □

The choice of $B = \mathbb{Q}$ in the above theorem yields the desired analogue of Corner's example (C1): the direct sum of two R-Hopfian groups need not be R-Hopfian. We record this as:

Corollary 3.4. *The direct sum of two R-Hopfian groups need not be R-Hopfian, even when one of the groups is Hopfian.*

Corollary 3.5. *If A is free of rank κ and B has a non-free summand X of cardinality $\leq \kappa$, the group $A \oplus B$ is not R-Hopfian. In particular, an R-Hopfian group having a free summand of infinite rank is necessarily torsion-free.*

Proof. Suppose that X is a summand of B and $| X | \leq \kappa$. If $A \oplus B$ were R-Hopfian, then $A \oplus X$ would also be R-Hopfian, contrary to Theorem 3.3. In particular, if G is R-Hopfian of the form $G = F \oplus H$, where F is free of infinite rank, then H must be torsion-free since otherwise G would have an R-Hopfian summand of the form $F \oplus C$, where C is either finite or of the form $\mathbb{Z}(p^\infty)$ for some prime p, both of which are impossible. □

Note a consequence of the above result: unlike the situation for Hopfian groups where the direct sum of a Hopfian group and a cyclic group is necessarily Hopfian (see [13] or [9]), the direct sum of a free (and hence R-Hopfian) group of infinite rank and a finite cyclic group is never R-Hopfian. In fact, this example shows that the class of R-Hopfian groups does *not* satisfy the weak closure property of Hopfian groups (see, for example, [10, Proposition 2.3]): if $0 \to H \to G \to K \to 0$ is an exact sequence and H, K are both Hopfian and if H is left invariant by each surjection $\phi : G \to G$, then G is Hopfian. In particular, extensions of Hopfian torsion groups by torsion-free Hopfian groups are again Hopfian but this is no longer true if we replace Hopfian by R-Hopfian.

If we apply our arguments to countable groups we can say a little more:

Theorem 3.6. *A countable group G is R-Hopfian if, and only if, G is free or G has the form $G = F \oplus N$, where F is free of finite rank and N is R-Hopfian with* $\mathrm{Hom}(N, \mathbb{Z}) = 0$.

Proof. The sufficiency is straightforward: if G is free then it is (Ker)-direct and hence R-Hopfian, while if it has the given form $G = F \oplus N$, a direct application of Proposition 3.1 (i) shows that G is R-Hopfian.

Conversely, suppose that G is R-Hopfian and not free. Since G is countable, we may make use of a standard result on countable Abelian groups due to Stein - see, for example, [7, Corollary 19.3] - that $G = F \oplus N$, where F is free and $\mathrm{Hom}(N, \mathbb{Z}) = 0$; note that N is also R-Hopfian as a summand of an R-Hopfian group. It remains only to show that F has finite rank. However, if F were of infinite rank, then it would follow from Theorem 3.3 that G is not R-Hopfian – contradiction. □

Corollary 3.7. *An R-Hopfian group G which is not reduced is of the form $G = \mathbb{Q}^{(\kappa)} \oplus X$, where $\kappa \neq 0$ is a cardinal, X is reduced R-Hopfian and X does not have a free summand of infinite rank.*

Proof. As G is not reduced it is of the form $G = D \oplus X$, where D is divisible and X is a reduced R-Hopfian group. Since for all primes p, the group $\mathbb{Z}(p^\infty)$ is directly finite but not Hopfian, it is not R-Hopfian. Thus D must be torsion-free divisible, $D = \mathbb{Q}^{(\kappa)}$ for some cardinal $\kappa \neq 0$. However, if X has a free summand F of infinite rank, then G has a summand (necessarily R-Hopfian) of the form $\mathbb{Q} \oplus F$, contrary to Theorem 3.3. Thus X does not have a free summand of infinite rank. □

The requirement in Theorem 3.3 that $|B| \leq |A|$ cannot be omitted. The proof of the following result is based on modern realization theorems which utilise sophisticated arguments deriving from the combinatorial arguments of Shelah's Black Box. We state it without proof.

Theorem 3.8. *For each infinite cardinal κ there is a non-free torsion-free R-Hopfian group G having a free summand of rank κ.*

We finish this section by posing the analogue of Corner's result (C2):

Problem: Does there exist an R-Hopfian group A such that the square $A \oplus A$ is not R-Hopfian?

We remark that it is not easy to just adapt Corner's original argument since the key point in his solution is that the group $A \oplus A$ is not directly finite and hence not Hopfian. However, failure to be directly finite does not imply failure to be R-Hopfian.

4. Baer-Specker Groups

If κ is an infinite cardinal then the groups $P_\kappa = \mathbb{Z}^\kappa$ are usually referred to as *higher Baer-Specker groups*; when $\kappa = \aleph_0$ we normally write P rather than $P_{\aleph_0}$ and P is the familiar Baer-Specker group. (For an informative introduction with an excellent list of references, see the article by Eoin Coleman in the IMS Bulletin [4].) In this section we investigate the groups P_κ in relation to the property of being R-Hopfian. Many properties of such products derive from fundamental work of Nunke but we shall find the more modern exposition in Eklof-Mekkler [6, Chapter IX] more useful for our purposes; in particular [6] contains the necessary set-theoretic background required and a detailed discussion of the so-called *Whitehead Problem*: recall that a group G is said to be a Whitehead group if every extension of the group $\mathbb{Z}$ by G splits, i.e., the group $\mathrm{Ext}(G, \mathbb{Z}) = 0$. Whitehead's problem had asked if every Whitehead group is necessarily free but Shelah showed in 1974 that the question is undecidable in the sense that it depends on the set-theoretic assumptions made. In some models of set theory all Whitehead groups are free but in other theories which are relatively consistent with ZFC, non-free Whitehead groups exist. Inevitably, discussions of large products of groups lead to set-theoretic issues concerning the existence of so-called ω-measurable cardinals. A detailed discussion of these is not appropriate for this type of paper and so we restrict ourselves to the naive approach which considers these cardinals as being extraordinarily large. Further details may be found in the discussion on slender modules in [6, Chapter III] and in the section on the Axiom of Constructibility [6, Chapter VI].

First we consider the situation in relation to the group P. Recall that if A is any subgroup of P, then adopting standard terminology from the theory of vector spaces, the annihilator $A^\perp$ of A, is given by $A^\perp = \{f \in \mathrm{Hom}(P, \mathbb{Z}) : f(a) = 0 \text{ for all } a \in A\}$ and the second annihilator $A^{\perp\perp}$ of A, i.e., the subgroup of P given by $\{x \in P : f(x) = 0 \text{ for all homomorphisms } f \in A^\perp\}$ is always a summand of P – see, for example, [6, Chapter IX, Proposition 1.3]. However, if ϕ is an arbitrary surjection : $P \to P$ and $A = \mathrm{Ker}\phi$, then the group $P/A \cong P$ is certainly torsionless (it is even reflexive) and so it follows from [6, Chapter IX, Lemma 1.1] that $A = A^{\perp\perp}$. Thus $\mathrm{Ker}\phi$ is a summand of P, and since ϕ was an arbitrary surjection, we have that P is (sKer)-direct and hence R-Hopfian by Theorem 2.6.

The situation becomes more complicated when we move to the higher Baer-Specker groups P_κ with $\kappa > \aleph_0$. However, it is possible to use the argument above in another, more general, situation. If we make the set-theoretic assumptions that every Whitehead group of cardinality $\leq \kappa$ is free and that κ is not too large in the technical sense that it is not ω-measurable, then, as above, we can deduce that the subgroup A is a summand using [6, Chapter IX, Theorem 1.5] and thus it easily follows that:

Theorem 4.1. *(i) The Baer-Specker group P is R-Hopfian;*

(ii) if κ is non-ω-measurable and if every Whitehead group of cardinality $\leq \kappa$ is free, then P_κ is R-Hopfian.

Our next result shows that if we make even stronger set-theoretical assumptions then the groups P_κ are always R-Hopfian. We have outlined below an algebraic approach using standard results in [6], but it is possible to deduce the result by using the more technical approach due originally to Scott which shows that ω-measurable cardinals cannot exist under the Axiom of Constructibility (V = L).

Theorem 4.2. *If (V = L), then the group P_κ is R-Hopfian for every infinite cardinal κ.*

Proof. It follows from [6, Chapter IX, Corollary 1.6] that for all infinite cardinals κ, a subgroup A of P_κ is a direct summand of P_κ

if, and only if, the quotient P_κ/A is a product. In particular, if $\phi : P_\kappa \to P_\kappa$ is a surjection and $A = \mathrm{Ker}\phi$, then $\mathrm{Ker}\phi$ is a summand of P_κ. Hence the group P_κ is (sKer)-direct and thus it is R-Hopfian by Theorem 2.6. □

Our next result is based upon Example 1.7 in [6, Chapter IX].

Theorem 4.3. If there exists a non-free Whitehead group B of non-ω-measurable cardinality κ, then the group P_λ is not R-Hopfian for any cardinal $\lambda \geq \kappa$.

Proof. Choose a free resolution $0 \to G \to F \to B \to 0$ in which both G and F are free of rank κ and apply the the functor $\mathrm{Hom}(-, \mathbb{Z})$ to get a short exact sequence

$$0 \to B^* \to F^* \xrightarrow{\eta} G^* \to 0 \qquad (*).$$

Note that $G^* \cong F^* \cong P_\kappa$ and fix an isomorphism $j : G^* \to F^*$. Define $\phi : F^* \to F^*$ by $\phi = j\eta$. Then $\mathrm{Ker}\phi = \mathrm{Ker}\eta$ since j is monic, so $\mathrm{Ker}\phi = B^*$. Note also that $\mathrm{Im}\phi = j\eta(F^*) = j(G^*) = F^*$, so that ϕ is onto. Claim that B^* is not a summand of F^*; it follows from Theorem 2.6 that this will suffice to show that $F^* \cong P_\kappa$ is not R-Hopfian. Since a direct summand of an R-Hopfian group is again R-Hopfian (Proposition 2.3), this will ensure that P_λ is not R-Hopfian for any $\lambda \geq \kappa$.

If B^* were a summand of F^*, then the sequence $(*)$ would be splitting exact, giving that G^* is a summand of F^*. Taking second duals and using the fact that products of $\mathbb{Z}$ over non-ω-measurable indexing sets are reflexive, this would lead to the conclusion that the resolution $0 \to G \to F \to B \to 0$ splits, hence B is free – contradiction. □

Since the existence of a non-free Whitehead group of cardinality $\aleph_1$ can be established under the set-theoretic assumption (MA + $\neg$ CH), our last result shows that Theorem 4.2 cannot be proved in ZFC, and that it is independent of ZFC whether every higher Baer-Specker group P_κ ($\kappa \geq \aleph_1$) is R-Hopfian.

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Modular Metric Spaces

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Abstract. We give a short introduction to the theory of modular metric spaces, including some fixed point theorems due to Chistyakov. We give new proofs of these theorems, inspired by Palais's approach to the Banach fixed point theorem

1. Introduction

We begin with a simple motivating example. Let X be the set of all points above water on the earth's surface. For two points x, y, let us denote by $v_t(x, y)$ the average speed required to travel directly over land from x to y in a time t. What are the properties of the function $v_t(x, y)$?

Clearly, if we fix x and y, then $v_t(x, y)$ takes nonnegative values and is a non-increasing function of t. And of course, this function is symmetric in x and y. But there is an issue we have glossed over — what if x and y lie in different landmasses? Since we are required to travel by land, it is impossible to get from x to y in a time t, no matter how fast we travel. As we would like our speed function to be defined in all circumstances, it is reasonable to allow extended real values and to assign the value $v_t(x, y) = \infty$ in this case.

To summarize, we now have a function taking $t > 0$ and x, $y \in X$ to $v_t(x, y) \in [0, \infty]$ that is symmetric in x, y and non-increasing in t. There is one further property worthy of note. A simple calculation with average speeds shows that

$$v_{s+t}(x, y) \leq v_s(x, z) + v_t(z, y)$$

for all $s, t > 0$ and all $x, y, z \in X$.

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In 2006, Vyacheslav Chistyakov [4, 2] introduced the concept of a **metric modular** on a set, inspired partly by the classical linear modulars on function spaces employed by Nakano and other in the 1950s. Our average speed function is an example of a modular in the sense of Chistyakov.

The concept of a modular on a vector space was introduced by Nakano in 1950 [10] and refined by Musielak and Orlicz in 1959 [9]. In the formulation given by Kowzslowski [7, 6], a **modular** on a vector space X is a function $m\colon X \to [0, \infty]$ satisfying

(1) $m(x) = 0$ if and only if $x = 0$;
(2) $m(ax) = m(x)$ for every $a \in \mathbb{R}$ with $|a| = 1$;
(3) $m(ax + by) \leq m(x) + m(y)$ if $a, b \geq 0$ and $a + b = 1$.

A modular m is said to be **convex** if, instead of (3), it satisfies the stronger property

(3') $m(ax + by) \leq am(x) + bm(y)$ if $a, b \geq 0$ and $a + b = 1$.

Given a modular m on X, the **modular space** is defined by

$$X_m = \{x \in X : m(ax) \to 0 \text{ as } a \to 0\}$$

It is possible to define a corresponding F-norm (or a norm when m is convex) on the modular space. The Orlicz spaces L^{φ} are examples of this construction [12].

The notion of a metric modular on an arbitrary set was introduced in 2006 by Chistyakov as a generalization of these ideas. Chistyakov's formulation provides a good framework in which to study fixed point phenomena.

2. Modular metric spaces

We start with the definition given by Chistyakov.

Definition 1. Let X be a nonempty set. A **metric modular** on X is a function

$$w\colon (0, \infty) \times X \times X \to [0, \infty],$$

written as $(\lambda, x, y) \mapsto w_\lambda(x, y)$, that satisfies the following axioms:

(1) $w_\lambda(x, y) = 0$ if and only if $x = y$ for all $\lambda > 0$ and $x, y \in X$
(2) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$
(3) $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(y, z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If the context is clear, we refer to a metric modular simply as a **modular**. A modular w is said to be **strict** if it has the following property: given $x, y \in X$ with $x \neq y$, we have $w_\lambda(x, y) > 0$ for all $\lambda > 0$.

A modular w on X is said to be **convex** if, instead if (3), it satisfies the stronger inequality

$$w_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu} w_\lambda(x,z) + \frac{\mu}{\mu+\lambda} w_\mu(z,y), \tag{1}$$

for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Let (X, d) be a metric space with at least two points. There are several ways to define a metric modular on X.

Example 2.1.

$$w_\lambda(x,y) = d(x,y)$$

In this case, property (3) in the definition of a modular is just the triangle inequality for the metric. This modular is not convex - just take $z = y$ and $\mu = \lambda$ in (1).

Example 2.2.

$$w_\lambda(x,y) = \frac{d(x,y)}{\lambda}$$

In this case, we can think of $w_\lambda(x, y)$ as the average velocity required to travel from x to y in time λ. A simple calculation with the triangle inequality shows that this modular is convex.

Example 2.3.

$$w_\lambda(x,y) = \begin{cases} \infty & \text{if} \quad \lambda < d(x,y), \\ 0 & \text{if} \quad \lambda \geq d(x,y) \end{cases}$$

This simple example could be seen as an extreme case of the velocity metaphor — if the time available is less than $d(x, y)$, then it is impossible to travel from x to y, but if the time is at least $d(x, y)$ then we can travel instantaneously. This modular is also convex.

We now look at some of the basic properties of modulars.

Proposition 2.4. *Let w be a modular on the set X.*

(a) *For every $x, y \in X$, the function $\lambda \mapsto w_\lambda(x, y)$ is non-increasing.*

(b) *Let w be a convex modular. For $x, y \in X$, if $w_\lambda(x, y)$ is finite for at least one value of λ, then $w_\lambda(x, y) \to 0$ as $\lambda \to \infty$ and $w_\lambda(x, y) \to \infty$ as $\lambda \to 0+$.*

(c) *If w is a convex modular, then the function $v_\lambda(x, y) = \dfrac{w_\lambda(x, y)}{\lambda}$ is a modular on X.*

Proof.
(a) follows from property (3) of modulars, taking $z = y$.
(b) Taking $z = y$ in equation (1) gives

$$w_{\lambda'}(x, y) \leq \frac{\lambda}{\lambda'} w_\lambda(x, y)$$

whenever $\lambda' > \lambda$. Choosing λ for which $w_\lambda(x, y)$ is finite, we see that $w_{\lambda'}(x, y) \to 0$ as $\lambda' \to \infty$. And if we fix λ' such that $w_{\lambda'}(x, y) < \infty$, we get $w_\lambda(x, y) \to \infty$ as $\lambda \to 0+$.
(c) It is obvious that v satisfies the first two properties of a modular Property (3) for v follows easily from the convexity condition on w_λ. □

3. Modular sets and modular convergence

Given a modular w on X and a point x_0 in X, the two sets

$$X_w(x_0) = \{x \in X : w_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X_w^*(x_0) = \{x \in X : \exists \lambda > 0 \text{ such that } \; w_\lambda(x, x_0) < \infty\}$$

are each known as **Modular Sets** around x_0. These sets can be thought of as comprising all the points that are "accessible" in some sense from x_0. In both cases, the modular sets form a partition of X. In our motivating example, the modular sets are the individual land masses.

It is clear that $X_w(x_0) \subset X_w^*(x_0)$ in general and Example 1 shows that this inclusion may be strict. While there is some ambiguity in using the same term for these two types of sets, in the sequel we shall only be working with modular sets of the form $X_w^*(x_0)$.

Proposition 3.1. *If w is convex then*

$$X_w(x_0) = X_w^*(x_0)\,.$$

Proof. This follows immediately from Proposition 2.4 (b). □

We now turn our attention to some notions of convergence.

Definition 2. Let w be a modular on X. A sequence (x_n) in X is said to be w**-convergent** (or **modular convergent**) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on (x_n) and x, such that $\lim_{n\to\infty} w_\lambda(x_n, x) = 0$ A sequence (x_n) in X is said to be w–**Cauchy** if there exists $\lambda > 0$, possibly depending on the sequence, such that $w_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$. X is said to be w–**complete** if every w–Cauchy sequence is w–convergent.

Metrics on the modular set

Let w be a modular on X and let X_w be any one of the modular sets defined by w. Then the formula

$$d_w(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \geq \lambda\}, \quad \forall x, y \in X_w$$

defines a metric on X_w [4].

If the modular w is convex, then the modular space can be endowed with another a metric d_w^* given by

$$d_w^*(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq 1\}\,.$$

These metrics on the modular set are strongly equivalent:

$$d_w \leq d_w^* \leq 2d_w\,.$$

We refer to [4] for details.

The following result shows the relationship between modular and metric convergence. The proof can be found in [4].

Proposition 3.2. *Let w be a convex modular on X. Given a sequence x_n for X_w^* and an element $x \in X_w^*$, we have:*

$$\lim_{n\to\infty} d_w^*(x_n, x) = 0 \iff \lim_{n\to\infty} w_\lambda(x_n, x) = 0 \text{ for every } \lambda > 0.$$

Chistyakov gives an example to show that modular convergence is strictly weaker than metric convergence in general [3]

4. Fixed Point Theorems

In 2011, Chistyakov generalized the Banach fixed point theorem to the setting of modular metric spaces. Consider the definition of a contraction on a metric space: $d(Tx, Ty) \leq kd(x, y)$ for all x, y, where k is some constant between 0 and 1. Looking at Examples 1 and 2, we see that there are at least two ways to generalise this to modular metric spaces. Chistyakov gives two definitions:

Definition 3. Let w be a modular on a set X and let X_w^* be a modular set. A mapping $T\colon X_w^* \to X_w^*$ is said to be **modular contractive** (or a *w-contraction*) if there exist numbers $k \in (0,1)$ and $\lambda_0 > 0$ such that

$$w_{k\lambda}(Tx, Ty) \leq w_\lambda(x, y)$$

for all $0 < \lambda \leq \lambda_0$ and all $x, y \in X_w^*$.

His second definition is stronger:

Definition 4. A mapping $T\colon X_w^* \to X_w^*$ is said to be **strongly modular contractive** (or a *strong w-contraction*)if there exist numbers $0 < k < 1$ and $\lambda_0 > 0$ such that

$$w_{k\lambda}(Tx, Ty) \leq k w_\lambda(x, y)$$

for all $0 < \lambda \leq \lambda_0$ and all $x, y \in X_w^*$.

Chistyakov proved fixed point theorems for modular contractive and strongly modular contractive mappings. Rather than follow his proofs, our approach to these results is inspired by Richard Palais's proof of the Banach fixed point theorem [11].

Suppose that T is a contraction on a metric space (X, d) with contraction constant k. Thus, we have $d(Tx, Ty) \leq kd(x, y)$ for all x, $y \in X$. Combining this with an application of the triangle inequality to the points x, y, Tx and Ty, we get the inequality

$$d(x, y) \leq \frac{d(x, Tx) + d(y, Ty)}{1 - k}\,.$$

Palais called this the **Fundamental Contraction Inequality**. It is a key ingredient in his proof, where it is used to establish the Cauchy property for the sequence generated by iterating the mapping T on some initial point.

We begin with a variant of Palais's inequality for modular contractive mappings.

Proposition 4.1 (Fundamental Modular Contraction Inequality)**.** *Let w be a convex modular in X, let $T\colon X_w^* \to X_w^*$ be a modular contractive mapping, with $w_{k\lambda}(Tx, Ty) \leq w_\lambda(x, y)$ for $0 < \lambda \leq \lambda_0$. Let $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = (1 - k)\lambda$, where $0 < \lambda < \lambda_0$. Then*

$$w_\lambda(x, y) \leq \frac{\lambda_1 w_{\lambda_1}(x, Tx) + \lambda_2 w_{\lambda_2}(y, Ty)}{\lambda(1 - k)} \tag{2}$$

for every $x, y \in X_w^*$

Proof. By the convex property, taking $\lambda = \lambda_1 + k\lambda + \lambda_2$, we get

$$w_{\lambda_1+k\lambda+\lambda_2}(x,y) \leq \frac{\lambda_1}{\lambda} w_{\lambda_1}(x,Tx) + kw_{k\lambda}(Tx,Ty) + \frac{\lambda_2}{\lambda} w_{\lambda_2}(y,Ty)$$

since $\lambda_1 + \lambda_2 = (1-k)\lambda$ and $w_{k\lambda}(Tx,Ty) \leq w_\lambda(x,y)$.

Therefore,

$$w_\lambda(x,y) \leq \frac{\lambda_1}{\lambda} w_{\lambda_1}(x,Tx) + kw_\lambda(x,y) + \frac{\lambda_2}{\lambda} w_{\lambda_2}(y,Ty)\,.$$

Hence

$$w_\lambda(x,y) = \frac{\lambda_1 w_{\lambda_1}(x,Tx) + \lambda_2 w_{\lambda_2}(y,Ty)}{\lambda(1-k)}\,.$$

□

We now give the first fixed point theorem on modular metric spaces by Chistyakov, but our proof uses the Fundamental Modular Contraction Inequality given in the preceding proposition.

Theorem 4.2 ([3])**.** *Let w be a strict convex modular on X such that the modular space X_w^* is w–complete and let $T\colon X_w^* \to X_w^*$ be a w-contractive map such that for each $\lambda > 0$ there exists an $x = x(\lambda) \in X_w^*$ such that $w_\lambda(x,Tx) < \infty$.Then T has a fixed point x_* in X_w^*. If the modular w assumes only finite values on X_w^*, then the condition $w_\lambda(x,Tx) < \infty$ is redundant, the fixed point x_* of T is unique and for each $x_0 \in X_w^*$ the sequence of iterates $T^n x_0$ is modular convergent to x_*.*

Proof. If x, y are both fixed point then $w_\lambda(x,y) = 0$ so $x = y$. Hence a contraction mapping can have at most one fixed point. Its remains to show that for any x_0 in X the sequence $T^n(x_0)$ is Cauchy. Taking $x = T(x_0)$ and $y = T^n(x_0)$ and using the Main Contraction Modular Inequality, we get

$$\begin{aligned} &w_\lambda(T^n(x_0), T^m(x_0)) \leq \\ &\quad \frac{\lambda_1 w_{\lambda_1}(T(T^n(x_0), T^n(x_0)) + \lambda_2 w_{\lambda_2}(T(T^m(x_0), T^m(x_0))}{1-k} \\ &\quad = \frac{\lambda_1 w_{\lambda_1}(T^n(T(x_0), T^n(x_0)) + \lambda_2 w_{\lambda_2}(T^m(T(x_0), T^m(x_0))}{1-k} \\ &\quad \leq \frac{\lambda_1 w_{k^{-n}\lambda_1}(T(x_0,x_0)) + \lambda_2 w_{k^{-m}\lambda_2} T(x_0,x_0)}{1-k}\,. \end{aligned}$$

Since $k^{-n}\lambda_1 \to \infty$ as $n \to \infty$ then $k^{-n}\lambda_1 > \lambda$ if n big enough. Similar $k^{-m}\lambda_2 > \lambda$ if n big enough. Since the function $\lambda \mapsto w_\lambda(x,y)$ is non-increasing, and $0 < k < 1$ we have

$$w_{k^{-m}\lambda_2}(Tx_0, x_0), w_{k^{-n}\lambda 1}(Tx_0, x_0) \le w_\lambda(Tx_0, x_0) < \infty\,.$$

Hence

$$w_\lambda(T^n(x_0), T^m(x_0)) \to 0\,.$$

Therefore $T^n(x_0)$ is Cauchy. □

Chistyakov's second fixed point theorem drops the convexity assumption on the modular, replacing it with the requirement that the mapping be strongly contractive. We start with another variation of Palais's inequality for these mappings.

Proposition 4.3 (Fundamental Strong Modular Contraction Inequality)**.** *Let w be a modular on X and let $T\colon X_w^* \to X_w^*$ be strongly modular contractive mapping, with $w_{k\lambda}(Tx, Ty) \le kw_\lambda(x,y)$ for $0 < \lambda \le \lambda_0$. Let $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 = (1-k)\lambda$, $0 < \lambda < \lambda_0$. Then*

$$w_\lambda(x,y) \le \frac{w_{\lambda_1}(x,Tx) + w_{\lambda_2}(y,Ty)}{1-k} \tag{3}$$

for all $x, y \in X_w^*$.

Proof. By property (3) in the definition of a modular, we get

$$w_{\lambda_1 + k\lambda + \lambda_2}(x,y) \le w_{\lambda_1}(x,Tx) + w_{k\lambda}(Tx,Ty) + w_{\lambda_2}(y,Ty)$$

Using $\lambda_1 + \lambda_2 = (1-k)\lambda$ with the strong contractive property of T,

$$w_\lambda(x,y) \le w_{\lambda_1}(x,Tx) + kw_\lambda(x,y) + w_{\lambda_2}(y,Ty)$$

and so

$$w_\lambda(x,y) \le \frac{w_{\lambda_1}(x,Tx) + w_{\lambda_2}(y,Ty)}{1-k}$$

for all $x, y \in X_w^*$. □

We the Fundamental Strong Contraction Modular Inequality (3) to prove the second fixed point theorem.

Theorem 4.4 ([3])**.** *Let w be a strict modular on X such that X_w is w-complete, let $T\colon X_w^* \to X_w^*$ be a strongly w-contractive map such that for each $\lambda > 0$ there exists an $x = x(\lambda) \in X_w^*$ such that $w_\lambda(x,Tx) < \infty$ holds. Then T has a fixed point x_* in X_w^*. If, in addition, w is finite valued on X_w^*, then the condition $w_\lambda(x,Tx) < \infty$ is redundant, the fixed point x_* of T is unique and for each $x_0 \in X_w^*$ the sequence of iterates $T^n x_0$ is modular convergent to x_** [3].

Proof. If x, y are both fixed points then $w_\lambda(x,y) = 0$ so $x = y$. Hence a contraction mapping can have at most one fixed point. Its remains to show that for any x_0 in X the sequence $T^n(x_0)$ is Cauchy. Taking $x = T(x_0)$ and $y = T^n(x_0)$,

$$\begin{aligned} w_\lambda(T^n(x_0), T^m(x_0)) \leq \\ &\frac{w_{\lambda_1}(T(T^n(x_0), T^n(x_0)) + w_{\lambda_2}(T(T^m(x_0), T^m(x_0))}{1-k} \\ &= \frac{w_{\lambda_1}(T^n(T(x_0), T^n(x_0)) + w_{\lambda_2}(T^m(T(x_0), T^m(x_0))}{1-k} \\ &\leq \frac{k^n w_{k^{-n}\lambda 1}(T(x_0, x_0)) + k^m w_{k^{-m}\lambda_2}(T(x_0, x_0))}{1-k} . \end{aligned}$$

If $k^{-n}\lambda_1 \to \infty$ then $k^{-n}\lambda_1 > \lambda$ if n is big enough. Similarly, $k^{-m}\lambda_2 > \lambda$ if n also is big enough.

Since the function $\lambda \mapsto w_\lambda(x,y)$ is non-increasing, and $0 < k < 1$ we have

$$w_{k^{-m}\lambda_2}(Tx_0, x_0), w_{k^{-n}\lambda 1}(Tx_0, x_0) \leq w_\lambda(Tx_0, x_0) < \infty .$$

Hence $w_\lambda(T^n(x_0), T^m(x_0)) \to 0$. Therefore $T^n(x_0)$ is Cauchy. □

5. Applications

Electrorheological fluids are liquids that rapidly solidify in the presence of an electric field. They are often studied using Sobolev spaces with a variable exponent [8]. It has been suggested that modular metric spaces may be useful in modelling them [1].

Recent work indicates that modular metric space fixed point results are well adapted to certain types of differential equations [4]. Finally, we refer to [5] for a detailed study of nonlinear superposition operators on modular metric spaces of functions.

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Aspects of Positive Scalar Curvature and Topology I

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Abstract. Whether or not a smooth manifold admits a Riemannian metric whose scalar curvature function is strictly positive is a problem which has been extensively studied by geometers and topologists alike. More recently, attention has shifted to another intriguing problem. Given a smooth manifold which admits metrics of positive scalar curvature, what can we say about the topology of the *space* of such metrics? We provide a brief survey, aimed at the non-expert, which is intended to provide a gentle introduction to some of the work done on these deep questions.

1. Introduction

A central problem in modern geometry concerns the relationship between curvature and topology. A topological space may take many geometric forms. For example, while a sphere is usually thought of as round, one may alter its shape in various ways and still maintain the topological condition of being a sphere; see Fig.1. A large part of modern geometry therefore concerns the problem of finding a "good" geometry for some topological form, given a plethora of possibilities. By a good geometry, one may mean geometries with a particular property, concerning symmetry or curvature perhaps. Given such a geometric constraint, say constant curvature, the problem is to find examples of topological shapes which admit such geometries and to understand what the topological obstructions are in the ones that do not. For example, the round geometry is a constant curvature geometry on the sphere. It is also an example of a positive curvature geometry, the curvature itself being simply the reciprocal of the radius squared. A torus on the other hand can never be made to

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have everywhere positive curvature no matter how we deform it. This latter fact follows from the classic theorem of Gauss-Bonnet, a theorem we will return to very shortly.

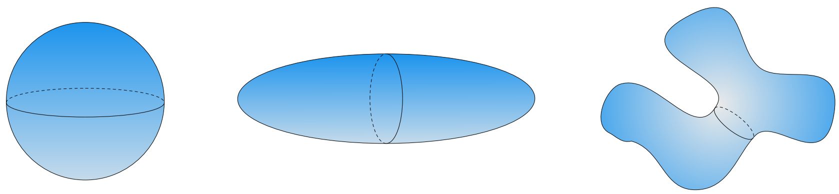

FIGURE 1. A selection of geometric structures on the sphere

Although there are a great many topological spaces one might consider, we will focus here on smooth compact manifolds. Usually, we are interested only in closed manifolds (those without boundary) although later we will discuss the case of manifolds with boundary. So what of the geometry? Given a smooth manifold M, of dimension n, the smooth structure on M assigns to each point $x \in M$, an n-dimensional real vector space, T_xM, known as the tangent space to M at x. These tangent spaces vary smoothly on M and their disjoint union, $TM := \bigsqcup_{x\in M} T_xM$, forms a vector bundle over M, known as the tangent bundle to M. It is possible to assign to each $x \in M$, a smoothly varying inner product $\langle \cdot, \cdot \rangle_x$ on T_xM. Such a choice of smoothly varying inner product is known as a Riemannian metric on M. It gives rise to a distance function (in the metric space sense) on M, but also allows for the measurement of angles and in particular, curvature. Any smooth manifold admits a multitude of possible Riemannian metrics each one determining a geometric structure on M. Once equipped with such a structure, the smooth manifold M is said to be a *Riemannian manifold.* Although there are other types of metric beyond the Riemannian and other types of geometric structure beyond this, in this article we will regard a geometric structure on M to be a choice of Riemannian metric.

In this context, the initial problem we mentioned can be restated as follows: given a smooth compact manifold M and a geometric constraint, can we find a Riemannian metric on M which satisfies this constraint? This is a very broad question and one which has motivated an enormous amount of research over the years. In this article, we will discuss this problem for one particular geometric constraint: *positive scalar curvature.* Thus, we consider the following question.

Question 1. Given a smooth compact manifold M, can we find a Riemannian metric, g, whose associated scalar curvature function, $s_g : M \to \mathbb{R}$, is strictly positive at all points on M? In other words, does M admit a metric of positive scalar curvature (psc-metric)?

Before defining the scalar curvature function, or providing any justification as to why this geometric constraint is even interesting, let us consider a further question which is of significant current interest.

Question 2. In the case when M admits a psc-metric, what can we say about the topology of the space of all psc-metrics on M? In particular, is this space path-connected? What about its higher homotopy or homology groups?

The first of these questions, the existence problem, has been studied extensively for several decades and a great deal is now understood on this matter. Much less is known about the second problem; however in recent years there have been some significant breakthroughs. The purpose of this article is to provide a brief survey on these problems which lie in an intriguing overlap of geometry, topology and analysis. This survey is not intended to be comprehensive. Such an undertaking would require the combined efforts of many experts and result in a voluminous article. Instead, it is intended to give the non-expert a taste of what is happening in a very interesting area of mathematics.

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2. Why positive scalar curvature?

We begin with the case of a 2-dimensional Riemannian manifold M. The scalar curvature, $s : M \to \mathbb{R}$, is a smooth function which agrees up to a multiple of 2 with the classical Gaussian curvature, $K : M \to \mathbb{R}$. More precisely, $s = 2K$. Recall that, in the case when

M is obtained as an embedded submanifold of $\mathbb{R}^3$, the Gaussian curvature at any point $x \in M$, $K(x)$, is the product of the pair of principal curvatures at x. At this level, scalar curvature is a fairly intuitive concept. Round spheres have positive scalar curvature as principal curvatures have the same sign. Planes and cylinders have zero scalar curvature as at least one of the principal curvatures is zero. Finally, a saddle surface displays scalar curvature which is negative as principal curvatures in this case have opposite signs; see Fig.2. Importantly, this type of curvature is *intrinsic* to the surface itself and does not depend on the way the surface is embedded in Euclidean space. Hence, the flat plane and the round cylinder, both have the same flat scalar curvature, even though extrinsically they curve differently in $\mathbb{R}^3$.

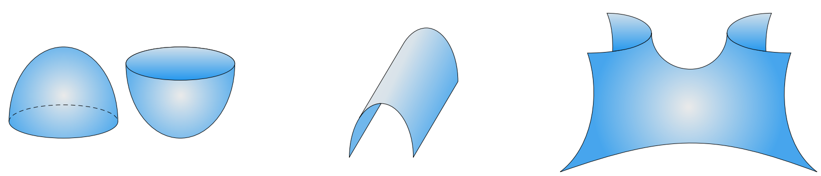

FIGURE 2. Two dimensional regions of positive, zero and negative scalar curvature

Geometrically then, positive scalar curvature can be thought to make a surface close in on itself whereas negative scalar curvature causes it to spread out. The topological consequences of this are evident from the classical theorem of Gauss-Bonnet. This theorem relates the scalar curvature s, of a compact oriented Riemannian 2-manifold M, with its Euler characteristic $\chi(M)$ by the formula

$$\frac{1}{4\pi}\int_M s = \chi(M).$$

Recall that $\chi(M)$ is an integer obtained by the formula $\chi(M) = v - e + f$ where v, e and f are respectively the numbers of vertices, edges and faces of a triangulation of M. That this formula is independent of the triangulation and gives a well-defined topological invariant is well-known. It follows that a closed surface with zero Euler characteristic, such as a torus, does not admit a metric of strictly positive (or negative) scalar curvature. Similarly, a surface with positive Euler characteristic such as a sphere cannot have scalar curvature which is everywhere non-positive. From the Uniformisation Theorem we know that every closed surface admits a metric of constant scalar curvature; see theorem 1.7 of [58]. This implies the following classification result: *a closed surface admits a metric of positive, zero or negative scalar curvature if and only if its Euler characteristic is respectively positive, zero or negative.*

In higher dimensions, the relationship between curvature and topology is much more complicated. The scalar curvature is one of three intrinsic curvatures which are commonly studied, the others being the Ricci and sectional curvatures. These curvatures vary greatly in the amount of geometric information they carry. The sectional curvature is the strongest and contains the most geometric information. At each point $x \in M$, where M is a Riemannian manifold, the sectional curvature K_x is a smooth real valued function on the space of all 2-dimensional subspaces of the tangent space T_xM. Each 2-dimensional subspace $V \subset T_xM$ is tangent to a locally specified 2-dimensional Riemannian submanifold of M. The sectional curvature $K_x(V)$ is simply the classical Gaussian curvature of this 2-dimensional Riemannian submanifold at x. The Ricci and scalar curvatures are successively weaker curvature notions and are obtained as follows. Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis for T_xM. Assuming $i \neq j$, we define $K_x(e_i, e_j)$ to be the sectional curvature at the point x of the plane spanned by e_i and e_j. For each i, we define $\mathrm{Ric}_x(e_i)$ by the formula

$$\mathrm{Ric}_x(e_i) = \sum_{j, j \neq i} K_x(e_i, e_j).$$

This extends linearly to specify a quadratic form on T_xM. Thus for some $v \in T_xM$, we define the Ricci curvature at x in the direction v to be $\mathrm{Ric}_x(v)$. Finally, we obtain the scalar curvature at x, $s(x)$ by the formula

$$s(x) = \sum_i \mathrm{Ric}_x(e_i) = 2 \sum_{i<j} K_x(e_i, e_j).$$

When we say, for example, that a Riemannian manifold M has positive (zero, negative etc) sectional curvature, we mean that for every point $x \in M$ and every 2-plane $V \subset T_xM$, the number $K_x(V)$ is positive (zero, negative etc). The analogous statements for Ricci and scalar curvature are defined similarly. It follows from the successive averaging in the formulae above that conditions such as positivity or negativity of the sectional curvature necessarily hold for the subsequent Ricci and then scalar curvatures. The converse however is not true.

It is important to understand that conditions such as strict positivity or negativity of the sectional or even Ricci curvatures impose severe topological restrictions on the underlying manifold. For example, it follows from the theorem of Bonnet-Meyers that a Riemannian manifold whose Ricci curvature is bounded below by a positive constant, is necessarily compact and has finite fundamental group; see theorem 11.7 of [58]. The scalar curvature however, is a substantially weaker notion than the sectional or even Ricci curvatures. In particular, topological invariants such as the size of the fundamental group do not by themselves prevent positivity of the scalar curvature, unlike in the Ricci and sectional cases. In fact, given *any* compact manifold M, the product manifold $M \times S^2$, where S^2 is the two dimensional sphere, admits a psc-metric. This is because of the way the scalar curvature function splits over a product metric as a sum of the scalar curvatures on the individual factor metrics. By equipping the sphere with a round metric of sufficiently small radius, we can increase the curvature on the sphere factor to compensate for any negativity arising from M.

Given the extent of the averaging process in obtaining the scalar curvature, especially in high dimensions, it seems surprising that any geometric information survives at all. Interestingly, there is one piece of geometric information the scalar curvature does carry, concerning the volume growth of geodesic balls. In particular, the scalar curvature $s(x)$ at a point x of an n-dimensional Riemannian manifold M, appears as a constant in an expansion

$$\frac{Vol(B_M(x,\epsilon))}{Vol(B_{\mathbb{R}^n}(0,\epsilon))} = 1 - \frac{s(x)}{6(n+2)}\epsilon^2 + \cdots ,$$

comparing the volume of a geodesic ball in M with the corresponding ball in Euclidean space; see [34]. Thus, positive scalar curvature implies that **small** geodesic balls have less volume than their Euclidean counterparts while for negative scalar curvature this inequality is reversed. This, at least at the local level, coincides with our

2-dimensional intuitions about the way surfaces respectively close in or spread out under positive or negative curvature.

We now return to the question which formed the title of this section: why positive scalar curvature? At this point, the reader may well ask: given its geometric weakness, why care about scalar curvature at all? And why positivity? Why do we not consider metrics of negative, non-negative or zero scalar curvature? Regarding the first question, in lower dimensions such as $2, 3$ and 4, the scalar curvature is still geometrically quite significant. In particular, it plays an important role in general relativity; see for example chapter 3 of [7]. As for the second and third questions, a partial justification is that there are no obstructions to the existence of metrics of negative scalar curvature in dimensions $\geqslant 3$; see [60]. Furthermore, any closed manifold which admits a metric whose scalar curvature is non-negative and not identically zero, always admits a metric of positive scalar curvature. This follows from the Trichotomy Theorem of Kazdan and Warner; see [52] and [53] (for a more thorough discussion of this matter, see section 2 of [76]). In the case of *positive* scalar curvature however, it is a fact that there are obstructions in dimensions $\geqslant 3$. Not all of these smooth compact manifolds admit Riemannian metrics of positive scalar curvature. This remarkable observation is discussed in the next section.

3. The Existence Question

We now focus on the first of the two questions posed in the introduction. When does a given smooth compact n-dimensional manifold admit a metric of positive scalar curvature? The case when the dimension $n = 2$ is the classical situation described above and so the question is completely answered. Note that when $n = 1$, the scalar curvature is not defined as one-dimensional manifolds have no intrinsic curvature. We therefore focus on the case when $n \geqslant 3$.

3.1. **The Obstructive Side.** In the early 1960s, André Lichnerowicz discovered an obstruction to the existence of positive scalar curvature metrics in the case of certain manifolds; see [59]. These manifolds, among other things, satisfy the condition of being *spin*. As the reader may gather from this term, the notion of spin manifold has strong connections to Physics. In fact, the fundamental formula used by Lichnerowicz to demonstrate this obstruction was actually derived independently by Erwin Schrödinger in 1932; see [82] (also section 3.3 of [29]).[1] We will state a version of this formula (in a case suitable for our purposes) shortly. Before doing so, we should say a few words about spin manifolds.

Spin, in the case of manifolds, is rather a technical notion to define. Here, we will provide only a casual description of one interpretation; for a detailed account the reader should consult [57]. Spin is essentially a strengthening of the condition of orientability and so we will begin by recalling what it means to say that a smooth manifold is orientable. Suppose on a smooth manifold M, we are given a point $x \in M$, a basis for T_xM and a directed loop starting and ending at x. Let's assume for simplicity that the loop is an embedded circle. It is possible to continuously move the basis around the loop to obtain a new basis at T_xM by a process called parallel translation. This is depicted in Fig.3 below where the finishing basis is depicted with dashed lines. There are many ways to do this although a particularly nice one arises from a choice of Riemannian metric. We say that the manifold M is *orientable,* if for any such quadruple of point, basis, loop and translation, the starting and finishing bases at T_xM have the same orientation. Thus, the linear transformation of T_xM which moves the starting basis to the finishing basis has positive determinant. Notice how this is not possible for certain loops on a Möbius band; see Fig4.

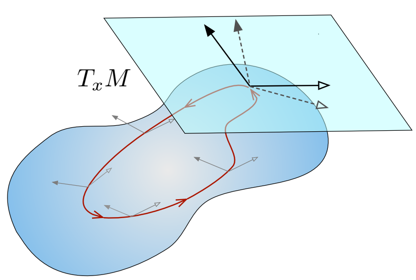

FIGURE 3. Translating a basis around a loop on an orientable manifold to obtain a new basis.

An equivalent formulation of orientability is as follows: the tangent bundle over any loop is *trivial.* That is, there is an isomorphism between TM (restricted to the loop) and a product of the loop with $\mathbb{R}^n$. In particular, this necessitates that the normal bundle, N, the

[1] I am grateful to Thomas Schick for bringing to my attention Schrödinger's role in this story.

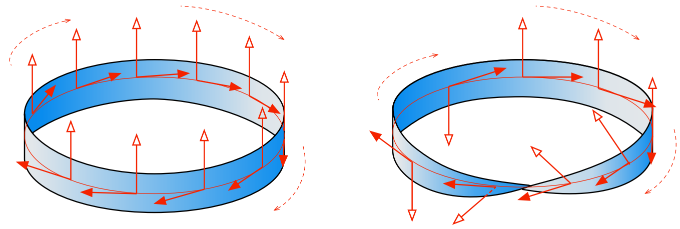

FIGURE 4. Comparing the translation of a basis around a loop on a cylinder and a Möbius band.

subbundle of TM consisting of $(n-1)$-dimensional subspaces orthogonal to the loop, is also trivial (isomorphic to a product of the loop with $\mathbb{R}^{n-1}$). It is possible to identify this normal bundle with a small "tubular" neighbourhood of the loop. In the case of a loop with trivial normal bundle, the tubular neighbourhood is topologically just a product of the loop with the $(n-1)$-dimensional disk, $S^1 \times D^{n-1}$. Thus, in the case of a two dimensional surface, a trivial normal bundle would correspond to a tubular neighbourhood which is, topologically, a regular cylindrical band of the type shown on the left of Fig.4. Loops whose tubular neighborhoods are Möbius bands correspond to loops with non-trivial normal bundles. A surface with such loops is therefore non-orientable. In fact, non-orientable surfaces are simply surfaces which contain Möbius bands.

In that it concerns loops, orientability can be thought of as a 1-dimensional notion. Spin is in a sense a 2-dimensional strengthening of this. To simplify matters, let us assume that M is a simply connected manifold with dimension $n \geqslant 5$. Simply connected manifolds are necessarily orientable. We now suppose we have an embedded two-dimensional sphere $S^2 \subset M$, with $x \in S^2$. This embedded sphere can be thought of as a sort of "2-dimensional loop." The manifold M is said to be *spin* if given any basis for T_xM, this basis can be extended continuously to obtain a family of bases over the embedded sphere. This determines a trivialisation of the tangent bundle restricted to the embedded sphere. Another interpretation (which is more useful for our purposes) concerns the normal bundle to the embedded sphere. A spin manifold is one for which *every* *embedded 2-dimensional sphere has trivial normal bundle.*[2] This interpretation will be particularly relevant later on when we discuss the role of surgery.

Just as we may specify an orientation on an orientable manifold, on a spin manifold we may specify something called a *spin structure*. Let us suppose now that M is a Riemannian spin manifold (with a specified spin structure). Such a manifold contains a wealth of interesting algebraic and geometric information. In particular, there exists a certain vector bundle over M known as the *spinor bundle*.[3]We will not describe the construction of the spinor bundle here, except to say that its construction involves, for each $x \in M$, replacing each tangent space T_xM with a certain finite-dimensional complex vector space S_x which is also a module over an algebra, known as the Clifford algebra of T_xM; see Part 1 of [3] for a readable introduction to Clifford algebras/modules. The construction of a Clifford algebra allows one to specify a geometrically significant multiplicative (algebra) structure on certain inner product spaces. When applied, for example, to $\mathbb{R}^2$ with the usual Euclidean dot product, it produces the quaternion algebra. In our case, Clifford algebras are associated, for each $x \in M$, to the tangent space T_xM with multiplication arising ultimately from the inner product on T_xM given by the Riemannian structure. This Clifford multiplication captures a great deal of information about the algebraic behaviour and symmetries of the curvature associated to the Riemannian metric. Put bluntly, it allows us to see things about the curvature which we would otherwise miss. We will return to this point in a moment.

Sections of the spinor bundle are known as *spinor fields* and so, denoting by $S \to M$ the spinor bundle itself, we consider the vector space ΓS of all sections of this bundle. This is an infinite dimensional vector space, the construction of which is described in detail in [57]. On this vector space, one can define a certain first order linear differential operator, $D : \Gamma S \to \Gamma S$, known as the Dirac operator. Importantly, this operator is elliptic and self-adjoint. As the name suggests it was invented by the physicist Paul Dirac. One of the motivations behind the construction of this operator was a

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[2] Without the hypothesis that the manifold is simply connected and of dimension at least 5, these interpretations do not quite work. See chapter 2 of [57] for a more thorough description.

[3] Spinors themselves are objects which are of great interest in Physics, describing intrinsic angular momentum or "spin" of subatomic particles.

need to find a first order linear differential operator which squares as the Laplacian. In regular Euclidean space, this is exactly what happens. In our case, things are a little more complicated. The regular Laplacian is now replaced by an object known as the *connection Laplacian* on the spinor bundle $S \to M$. A connection on a smooth vector bundle is a means of differentiating sections of this bundle along tangent vector fields, something which is actually equivalent to being able to perform the parallel transport of vectors mentioned earlier. There are many choices of connection but in the case of a Riemannian manifold there is a particularly good choice of connection on the tangent bundle $TM \to M$, which best reflects the geometry of the underlying metric. This is the so-called Levi-Civita connection. In our case this leads, given the choice of spin structure, to a spinor bundle connection which allows us to sensibly differentiate spinor fields. Denoting this connection ∇ and its adjoint (with respect to a certain global inner product on ΓS arising from the Riemannian metric) by ∇^*, we obtain the connection Laplacian $\nabla^*\nabla$. The Schrödinger-Lichnerowicz formula can now be stated as follows:

$$D^2 = \nabla^*\nabla + \frac{1}{4}s,$$

where s is the scalar curvature function on M. Notice in this case that the scalar curvature is the obstruction to whether or not D is truly a "square root of the Laplacian." Importantly, it is the role of the aforementioned Clifford multiplication which arises in the definition of D that allows us to untangle the curvature information and see the right hand side of this equation.

So what does the Lichnerowicz formula tell us? If we assume that the scalar curvature $s : M \to \mathbb{R}$ is a strictly positive function, a fairly straightforward argument shows that the operator D has trivial kernel. Thus, the existence of a non-trivial element of $\ker D$ (a so-called harmonic spinor field) implies that the Riemannian metric in question does not have positive scalar curvature. But this is only the beginning. It is a fact (although not a trivial one), that the kernel and cokernel of D are both finite dimensional subspaces. In the case when the dimension of M is a multiple of 4, the spinor bundle splits into a certain pair of "even" and "odd" sub bundles S_+ and S_- with a corresponding splitting of D into D_+ and D_-. Although D is self-adjoint, implying that its kernel and cokernel have the same dimension, this is no longer the case with D_+ or D_-.

We now define the *index of* D, denoted $\mathrm{ind}D$ as

$$\mathrm{ind}D = \dim \ker D_{+} - \dim \mathrm{coker} D_{+}.$$

It then follows that a necessary condition for the scalar curvature function to be positive is for the integer $\mathrm{ind}D$ to be zero.

Notice however that we are still only dealing with a single Riemannian metric, the one used in the construction of D in the first place. It is now we employ one of the great theorems of twentieth century mathematics: the Atiyah-Singer Index Theorem; see chapter III of [57] for a comprehensive discussion. This theorem equates the above index (which is an analytic index arising among other things from an individual choice of Riemannian metric) with a topological index for M. This topological index is known as $\hat{A}(M)$, the "A-hat"-genus of M. It depends only on the topology of M and so is independent of the individual choice of Riemannian metric. This has a powerful consequence. If for any one Riemannian metric, the index $\mathrm{ind}D$ above is computed to be non-zero, then it is non-zero for all choices of Riemannian metric. This is summarised in the following theorem.

Theorem 3.1. (Lichnerowicz [59])
Suppose M is a closed spin manifold of dimension $n = 4k$ which admits a metric of positive scalar curvature. Then $\hat{A}(M) = 0$.

The topological index $\hat{A}$ is therefore, in the case of closed, spin manfolds of dimension $4k$, an obstruction to the existence of positive scalar curvature metrics. There are a great many closed spin manifolds for which this index is non-zero and so many examples of manifolds which do not admit psc-metrics. Consequently, such manifolds admit no metrics of positive Ricci or sectional curvature either.

In the 1970s, Hitchin showed how to generalise this index obstruction to other dimensions (beyond those divisible by 4); see [43]. Before saying anything further on this, it is worth briefly digressing to introduce a concept which plays a vital role in this story: cobordism. Two closed n-dimensional manifolds, M_0 and M_1, are said to be *cobordant* if there exists an $(n+1)$-dimensional manifold, W, with boundary ∂W so that $\partial W = M_0 \sqcup M_1$. Thus, the boundary of W is a disjoint union of M_0 and M_1; see Fig.5. An elementary example of a pair of cobordant manifolds is the 2-dimensional sphere and torus. To see this, imagine a solid bagel with a round bubble trapped inside. The inner boundary is the sphere while the outer-boundary is the torus. Cobordism is an equivalence relation on closed n-dimensional manifolds and the set of equivalence classes actually forms a group, denoted Ω_n, under the operation of disjoint union. In fact the collection, $\Omega_* = \{\Omega_n\}_{n\in\mathbb{N}\cup\{0\}}$, forms a graded ring under the operation of cartesian product of manifolds. There are more refined versions of cobordism, such as oriented cobordism, denoted Ω_*^{SO}, and spin cobordism, denoted Ω_*^{Spin}. In each case, two closed oriented (spin) manifolds are said to be *oriented (spin) cobordant* if their disjoint union forms the boundary of an oriented (spin) manifold with a consistent orientation (spin structure).

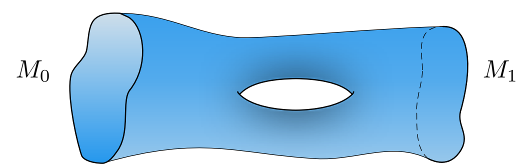

FIGURE 5. Two cobordant manifolds, M_0 and M_1

There is a graded ring homomorphism, defined by Milnor in [68], which takes the following form:

$$\alpha : \Omega_*^{\mathrm{Spin}} \longrightarrow KO_*.$$

The target of this homomorphism is the real K-homology of a point, something we will not define here, except to say that it satisfies the following *periodic* isomorphism conditions.

$$KO_n \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \mod 4, \\ \mathbb{Z}_2 & \text{if } n \equiv 1, 2 \mod 8, \\ 0 & \text{otherwise.} \end{cases}$$

This periodicity is an example of a phenomenon known as Bott periodicity, named after Raoul Bott, who first observed this behaviour in his study of the (stable) higher homotopy groups of certain classical Lie groups; see [9]. The homomorphism itself is surjective and moreover, when $n = 4k$, α and $\hat{A}$ coincide (at least up to a multiple of 2). Hitchin showed, in [43], that if M is a closed spin manifold admitting a psc-metric, then $\alpha([M]) = 0$. Thus, a necessary condition for a closed spin manifold to admit a psc-metric is that it lies in a spin cobordism class which is in the kernel of α. Establishing circumstances under which this is a sufficient condition (in general it is not) is the subject of the next section.

One fascinating consequence of Hitchin's work concerns exotic spheres. An exotic sphere is a smooth manifold which is homeomorphic to but not diffeomorphic to a standard sphere. Thus, an exotic sphere is topologically the same as, but smoothly different from, a standard sphere. Such objects were first constructed by Milnor in 1953, causing quite a stir in the mathematical community; see [66]. Hitchin showed that, starting in dimension 9, there are exotic spheres which admit no psc-metrics. In other words, the smooth structure on such a sphere is sufficiently exotic as to prevent positivity of even the weakest curvature. Given that topologically the sphere is the manifold most suited to positive curvature, this is an extraordinary fact.

We close this section by pointing out that the α-invariant is not the only known obstruction to positive scalar curvature. Using certain minimal hypersurface methods of Schoen and Yau, described in [80], Thomas Schick constructed examples of spin manifolds in dimensions 5 through 8 with trivial α-invariant and yet admitting no psc-metrics. These manifolds all have non-trivial fundamental group. Hence, the fundamental group of a manifold can, when coupled with other restrictions, be an obstruction. A much simpler example of this, following from work of Schoen and Yau in [81], is the fact that any 3-dimensional manifold whose fundamental group contains the fundamental group of a surface with positive genus, admits no psc-metric. Thus, the 3-torus, $T^3 = S^1 \times S^1 \times S^1$ admits no psc-metric, despite being a spin manifold and satisfying $\alpha([T^3]) = 0$. Indeed, the n-torus (the n-fold product of circles), $T^n = S^1 \times \cdots \times S^1$ admits no psc-metric; see [36]. The situation in dimension 3 is now completely understood, since Perelman's proof of Thurston's Geometrisation Conjecture. The classification, which is discussed in the introduction to [63], is as follows: *a closed 3-dimensional manifold admits a metric of positive scalar curvature if and only if it takes the form of a connected sum (see the next section for a definition) of spherical space forms (certain quotients of the round 3-sphere by actions of particular groups of isometries) and copies of $S^2 \times S^1$.* There is also a specific obstruction in the case of manifolds of dimension 4 arising in Seiberg-Witten theory; see [85]. Interestingly, it follows that there are examples of 4-dimensional manifolds which are simply-connected, spin and with $\hat{A} = 0$, but which admit no psc-metric.

There is still a great deal we do not know about obstructions to positive scalar curvature. Consider for example, the case of non-simply connected *totally non-spin manifolds* of dimension at least five. A *totally non-spin* manifold is one whose universal cover is non-spin and is, therefore, itself a non-spin manifold. This is a very large class of manifolds and it seems that there should be some obstructions here to positive scalar curvature. There are some individual examples of such manifolds which do not admit psc-metrics. In particular, Schoen and Yau have shown that the totally non-spin manifold obtained by taking a connected sum of T^6 with the product $\mathbb{C}P^2 \times S^2$ (of 2-dimensional complex projective space with the 2-sphere), admits no such metric. However, although some experts have made conjectures about this problem (see for example Stanley Chang's discussion of this matter in [20]), we have as yet nothing remotely analogous to the α-invariant here.

3.2. **The Constructive Side.** The other side of the existence question concerns the problem of constructing examples of psc-metrics on manifolds where no known obstructions exist. The principle tool for doing this is known as the Surgery Theorem. This theorem was proved in the late 1970s by Gromov and Lawson [37] and, independently, by Schoen and Yau [80], and provides an especially powerful device for building positive scalar curvature metrics. A p-surgery (or codimension $q+1$-surgery) on a manifold M of dimension n is a process which involves removing an embedded sphere-disk product $S^p \times D^{q+1}$ and replacing it with $D^{p+1} \times S^q$, where $p+q+1 = n$. The result of this is a new n-dimensional manifold M' whose topology is usually different from that of M; see Fig.6. For example, the 2-dimensional torus can be obtained from the sphere via a 0-surgery. Importantly, surgery preserves the cobordism type of the original manifold. Thus, if M' is obtained from M by surgery, then M and M' are cobordant and there is a complementary surgery which returns M' to M. The Surgery Theorem for psc-metrics can now be stated as follows.

Theorem 3.2. (Gromov-Lawson [36], Schoen-Yau [80])
Suppose M is a smooth manifold which admits a metric of positive scalar curvature. If M' is a smooth manifold obtained from M by

a surgery in codimension at least three, then M' admits a metric of positive scalar curvature.

Note that a well-known example of a 0-surgery is the connected sum construction. Here, we remove a product $S^0 \times D^n \cong D_1^n \sqcup D_2^n$ (a pair of disks) from a pair of disjoint n-manifolds M_1 and M_2, where $D_1^n \subset M_1$ and $D_2^n \subset M_2$. We then connect the resulting boundaries via a tube $[0,1] \times S^{n-1}$ (as suggested in Fig.6) to obtain the connected manifold $M_1 \# M_2$, the *connected sum* of M_1 and M_2. With this in mind it is worth stating the following easy corollary of the Surgery Theorem.

Corollary 3.3. *Suppose that M_1 and M_2 are n-dimensional manifolds, each admitting a metric of positive scalar curvature, and with $n \geqslant 3$. Then the manifold $M_1 \# M_2$ obtained as a connected sum of M_1 and M_2 also admits a metric of positive scalar curvature.*

In a moment we will discuss the implications of the Surgery Theorem. But first we will make a few comments about its proof. The proofs of this theorem by Gromov-Lawson and Schoen-Yau are quite different. The latter authors make use of PDE and certain minimal hypersurface methods to demonstrate the existence of psc-metrics on M'. The Gromov-Lawson method is more obviously constructive.[4] They begin with an arbitrary psc-metric, g on M, and perform an explicit geometric construction of a new psc-metric, g' on M'. This construction involves showing that, in a neighbourhood of the sphere to be removed by surgery, the psc-metric g can be replaced by a psc-metric on $S^p \times D^{q+1}$ which is standard near $S^p \times \{0\}$. By this we mean the standard product of a round sphere with a "torpedo" shaped hemisphere. The difficulty is in adjusting the metric near the boundary of the region $S^p \times D^{q+1}$ to ensure a smooth transition back to the original metric while maintaining positive scalar curvature. This is where the codimension $\geqslant 3$ hypothesis comes in to play. It means that $q \geqslant 2$ and so the radial spheres making up the disk factor have dimension at least 2, and therefore have some scalar curvature. Provided we are sufficiently close to $S^p \times \{0\}$, these radial spheres behave like round spheres and so, for small radius, have

[4] It was pointed out by Bernhard Hanke at a recent Oberwolfach workshop that many of us working in this field do not pay enough attention to the methods employed by Schoen and Yau here and work almost exclusively with the Gromov-Lawson version of this theorem. This may result in some missed opportunities.

large positive scalar curvature. This is then used to compensate for any negative curvature arising from the standardising adjustment. Once this standardised psc-metric is obtained, it is trivially easy to replace its standard torpedo shaped ends with a standard "handle" and complete the surgery, as shown in Fig.6.

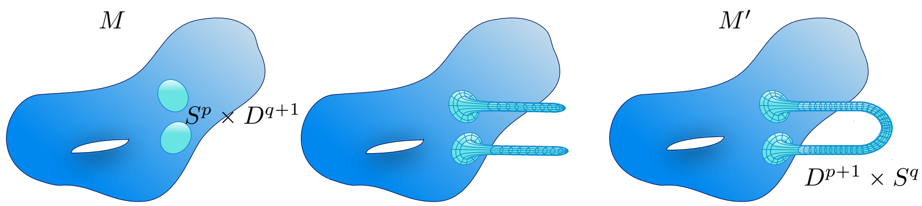

FIGURE 6. The original metric on M (left), the metric after standardization on the region $S^p \times D^{q+1} \subset M$ (middle) and the post-surgery metric on M', after attachment of the handle $D^{p+1} \times S^q$ (right)

The effect of this theorem was to increase enormously the number of known examples of manifolds admitting psc-metrics. Suppose M is a manifold which admits a psc-metric. It now follows that every manifold obtained by an *appropriate* surgery from M also admits a psc-metric. It is worth pointing out that this theorem does not hold for positive Ricci or sectional curvature, as these curvatures do not exhibit the same flexibility as the scalar curvature. Positive scalar curvature however is sufficiently resilient as to be able to withstand significant topological adjustment.

To better understand this implication, let us consider the non-spin and spin cases separately. To avoid some other possible obstructions, let us further restrict our search to manifolds which are simply connected and of dimension at least 5. In the non-spin case, no other obstructions were known. In [36], Gromov and Lawson went on to show that in fact there is no obstruction here. Every simply connected *non-spin* manifold of dimension $n \geqslant 5$ admits a psc-metric. This was done by considering the ring of oriented cobordism classes, Ω_*^{SO} and using the fact that this ring is generated by classes containing representatives built from complex projective space. These generating manifolds are non-spin and known to admit psc-metrics. Now, simply-connected manifolds are all orientable and every simply connected manifold is oriented cobordant to one of these known examples. Thus, given an arbitrary simply-connected manifold, there is a finite sequence of surgeries (though not necessarily in codimension $\geqslant 3$) which turn one of these known examples into our arbitrary manifold. Provided the arbitrary manifold is of dimension at least 5 and also non-spin, a topological argument shows such surgeries can be assumed to be in codimension at least three. The Surgery Theorem then allows us to use the psc-metric on the known example to construct one on the arbitrary manifold.

So what of the spin manifolds? We know from the work of Hitchin, that in looking for spin manifolds which admit psc-metrics we should restrict our search to spin cobordism classes of manifolds in the kernel of α. But is there another obstruction here? Without added restrictions on dimension and the fundamental group the answer is yes. However, in analogous fashion to the non-spin case, Gromov and Lawson show in [36] that, provided we restrict to simply connected manifolds of dimension at least 5, it is possible that no such obstruction exists. In particular, they show that (in these dimensions) it is enough to construct a collection of manifolds admitting psc-metrics which represent generating classes for the kernel of α, since each simply connected spin manifold of dimension at least 5 which lies in such a class is obtainable from such a representative by codimension $\geqslant 3$ surgeries. An important step here is the removal from a spin manifold, by surgery, of certain topologically significant embedded 2-dimensional spheres. The spin condition, as interpreted in the previous section, implies that such embedded spheres have trivial normal bundles and so surgery is possible.

The task of constructing such a collection of representative manifolds, whose spin cobordism classes generate the kernel of α, was finally completed by Stolz in [83]. Stolz showed that each generating class could be represented by a manifold which is the total space of a fibre bundle with quaternionic projective space fibres. The fibre of such a bundle admits a standard psc-metric and it is possible to construct a metric on the total space which restricts on fibres to this standard metric. Using well known curvature formulae due to O'Neill (see chapter 9 of [7]), it follows that the total space metric can be made to have positive scalar curvature by appropriately scaling the fibre metric. In summary we have the following theorem, known as *the classification of simply connected manifolds of positive scalar curvature.*

Theorem 3.4. (Gromov-Lawson [36], Stolz [83]) *Let M be a compact simply connected manifold of dimension $n \geqslant 5$. Then M admits a metric of positive scalar curvature if and only if M is either not spin or M is spin with $\alpha([M]) = 0$.*

In the non-simply connected case, the problem of deciding which manifolds admit psc-metrics is ongoing. For certain types of fundamental group, this is the subject of a conjecture known as the Gromov-Lawson-Rosenberg Conjecture. Various analogues of this conjecture exist, which due to the discovery of counter-examples (most famously by Thomas Schick in [78]), have been reformulated a number of times. The statement of this conjecture is quite complicated and so we refer the reader to the following thoughtful surveys on these matters: [76], [79].

3.3. **The Existence Question for Manifolds with Boundary.** So far we have considered only the case of closed manifolds. It is important to mention that there are analogous problems for manifolds with boundary. Suppose W is a smooth compact $(n+1)$-dimensional manifold with boundary, $\partial W = M$, an n-dimensional closed manifold. Without any further restrictions, the question of whether or not W admits any psc-metrics is actually not such an interesting question. It turns out that, without some condition on the boundary, W will not only always admit a metric of positive scalar curvature, but will in fact admit a metric of positive sectional curvature! This is a result of Gromov, see Theorem 4.5.1 of [35]. However, with appropriate boundary conditions, the question becomes extremely interesting. We consider only Riemannian metrics on W which, near the boundary of W, take the form of a cylindrical product metric $(M \times [0,1], g + dt^2)$, where g is some Riemannian metric on M and dt^2 is the standard metric on the interval. We then consider the following question.

Question 3. When does a given psc-metric on M extend to a psc-metric on W which takes a product structure near the boundary?

Such extensions are not always possible. Indeed, as we will see in the next section, a better understanding of this problem would greatly help in our efforts to answer the second of the motivational questions posed in the introduction.

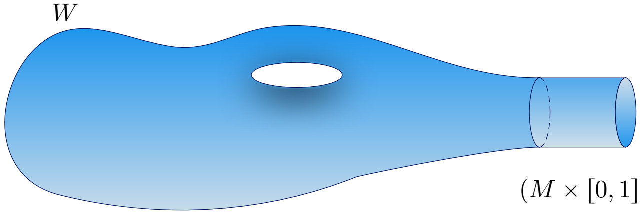

Figure 7. A metric on W which takes the form of a cylindrical product $(M \times [0,1], g + dt^2)$ near the boundary

This survey will be concluded in the next issue of the Bulletin.

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Mircea Pitici (Editor): The Best Writing on Mathematics 2015, Princeton University Press, 2016. ISBN:9-78-069-1169-65-1, USD 24.95, 392 pp.

REVIEWED BY CIARÁN MAC AN BHAIRD

The 2015 edition of the *Best Writing on Mathematics* is the 6th in the series, and contains 29 essays which cover a wide range of intriguing topics. All readers, whether professional mathematicians or statisticians, teachers or those with a general interest in these subjects and their applications, will find engaging articles to consider and discuss with their peers.

In this review, I will provide a very general overview rather than an in-depth analysis of all the essays, and give further details on specific papers which grabbed my attention. I am aware that some may not share my preferences, so for added insight, I would first direct the reader to the excellent introduction from Mircea Pitici, the book editor. This contains his thoughts on the *Best Writing on Mathematics* series, this specific edition of the series and an overview of the chapters. Furthermore, he gives an extensive list of new books on mathematics which is a very useful resource for the reader, as is the provision of a webpage which gives an extended introduction with supplementary online materials (`http://assets.press.princeton.edu/chapters/s10558.pdf`).

One of several recurring themes in this book is that of mathematical games and puzzles, for example, there is Mutalik's paper on the importance of puzzles or mathematical problems to our development as humans, Mulcahy and Richards on the impact of Martin Gardner's mathematical puzzles, Walsh on the mathematics behind the game *Candy Crush*, Freiberger on the chaos of mathematical billiards and several more. It is refreshing to see the historical development of certain topics included in several essays, for example Rougetet's work on the earliest written material found on the game of Nim, and Rittaud and Heeffer on the Pigeonhole Principle. Many

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more areas are considered, with papers which can be loosely characterised as covering topics in geometry, statistics, philosophy, art, and synthetic biology. Several of these papers are described in more detail in other extant reviews of the book, and also in Pitici's introduction.

The contributions I found most absorbing in this book were those which are related, albeit sometimes tenuously, to mathematics instruction and education. A prime example is the first essay, *A Dusty Discipline* by Barany and MacKenzie. It is particularly striking, considering the struggle that many mathematics instructors at third level in Ireland have had in recent years to hold onto blackboards in our classrooms. This essay may not convince those who are trying to remove blackboards. However, the focus of the essay on the role of the blackboard in teaching and research, and its influence on how we present and think about our work should resonate with everyone who uses them.

In light of the recent substantial changes made to second level mathematics in Ireland, readers may find many points of common ground with Zhang and Padilla in their observations on the differences between US and Chinese mathematics education. They present on why Chinese students consistently outperform their US counterparts in mathematics, and give a list of recommendations on how to address this gap. The much shorter *The Future of High School Mathematics* by multiple authors is also very interesting from an Irish context. While the paper does not reveal anything new in terms of the issues it discusses, it re-emphasises five key points for the reform of high school mathematics, which are not just applicable to the US. They state that *Despite understandable controversy . . . the Common Core standards provide a useful framework for further efforts, provided they are viewed as a living document to be modified as recommended by experience.* The reference to *a living document* is perhaps a point that all involved in curriculum and associated changes, including those who comment on such changes, should consider carefully.

Hanna and Mason's *Key Ideas and Memorability in Proof* was an absorbing read. They consider mathematical proof and, in particular, discuss Tim Gower's 2007 paper (Mathematics, Memory and mental arithmetic). They dissect Gower's observations on the *width* and *memorability* of proof and place this in the context of several other works in the area of proof. The word *beauty* is mentioned in this essay, a word which is often overused in mathematics and this is expanded upon by Cellucci in his paper *Mathematical Beauty, Understanding and Discovery*, one of the longest articles in the book. This essay was an interesting journey through many different understandings and misunderstandings, from ancient through to modern times, of what mathematical beauty actually is (Cellucci does not provide a definition).

Perhaps one of the biggest challenges that faces the mathematical community is trying to explain the *beauty* or significance of their work to others. Strogatz's paper, based largely on his *Elements of Math* series in the New York Times, gives a very interesting perspective. He discusses three groups (he contends) that most people in a general audience will fall into in relation to mathematics, i.e. the traumatized, the perplexed and the naturals. He maintains that when someone is teaching or writing about maths, they need to give the first two groups special consideration or attention. He stresses the importance of *providing illumination, making connections and being friendly* and expands on these using examples from the writings and teaching skills of the great Richard Feynman, Stephen Gould and Lewis Thomas. I found his position particularly compelling, in an environment where the promotion and effective communication of mathematics, statistics and science to students and the public is becoming an increasingly important part of our mission as researchers and instructors.

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Anthony Lo Bello: Origins of Mathematical Words, Johns Hopkins University Press, 2013. ISBN:978-1-4214-1098-2, USD 51.95, 350+xv pp.

REVIEWED BY ANTHONY G. O'FARRELL

We are entertained, rather than surprised, by occasional reports that document the ignorance, provincialism, and cultural poverty found in Americans selected at random. But even one-half percent of 300 million is a large number, and those of us who have profited from time spent in that great transatlantic republic know that millions of its citizens are far from typical. Anthony Lo Bello is one of those other Americans. He profited from a fine liberal education at Kenyon College in Ohio, and went on to graduate studies at Yale. He studied Mathematics under Kakutani, and along the way added a mastery of Arabic to his grasp of Latin and Greek. He now works as Professor of Mathematics at Allegheny College in north-western Pennsylvania. His main scholarly work concerns the late mediaeval period, particularly the transmission of mathematical knowledge and skill from the Islamic world to the Christian world.

Alfonso VI of Leon captured Toledo in 1085. The libraries, the best in Europe, were not pillaged, and the Muslim and Jewish scholars were not expelled. This provided a unique opportunity, and Gerard of Cremona was one of those who exploited the situation, and made Latin translations of the material. In particular, he translated the text of Euclid as given in Arabic, with commentary, by Abu'l Abbas al-Fadl ibn Hatim al-Nayrizi (Arabic:النيريزي, Latin: Anaritius, 865?-922?) who worked in Baghdad. Lo Bello has published annotated translations to English of al-Nayrizi's work, and of Gerard's translation, and of the subsequent commentary of St Albert the Great[2, 3], the patron saint of natural scientists (among other accomplishments).

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In 2008, Trevor Lipscombe of Johns Hopkins University Press invited Lo Bello to write 'a discursive etymological dictionary of mathematical words whose origins are in Greek, Latin, or Arabic'. This was an inspired idea, because it gave Lo Bello scope to share with a wider public some of the contents of his well-stocked mind, in a way that is hardly possible using any other format.

The entries range in length from a few words to an extended essay, reminiscent of the pronouncements one heard at high tables in days gone by. Facts mingle with opinions. An opinion, arrived at after a thorough examination of any relevant facts, is better called a *judgement*. Lo Bello is opinionated, but he has put in the work.

The essays are highly entertaining, delightful, surprising, enlightening, provocative, and amusing. A few are impassioned rants — *American spelling*, 3pp; *Mathematics*, 8pp; *cant*, 5pp; *(teaching) evaluation*, 5pp — and others are concise, and well-written summaries — *Timaeus*, 13pp; *Philosophy*, 5pp; *Cartesian*, 7pp; *algebra*, 3pp; *Euclid*, 2pp; *calendar mathematics*, 4pp; and *Archimedes*, 5pp. I noted many interesting details, and found some accounts thoroughly illuminating. For instance, I had not realized that the grand scheme of the Elements — building up to the regular solids in Book XIII — was (according to Proclus) inspired by the 'Physics' expounded in the Timaeus dialogue, which sees four of these solids as the reality behind all matter.

He has a bibliography of 39 items, mostly linguistic; just five are mathematics books (three of them are curve dictionaries), but he also quotes many other sources in the text. The book is laced with well-chosen quotations from writers ancient and modern, some given at length. Q. H. Flaccus gets pride of place, and I was touched by the line about the great poet's visit to Athens to study Mathematics (and other matters),

> *Atque inter silvas Academi quarere verum.*
> (and seek truth among the Academy's groves.)

This could serve as a motto for us all. Occasionally, there is mathematical content, and Lo Bello writes with some authority about Probability and Statistics.

He rages against the decline in the quality of liberal education. In the entry *innovation*, (pp 179-82) he quotes, in full, a final 'honors' examination paper for mathematics majors set in his *alma mater* in the sixties, and says: "If this examination were given today, after fifty years of innovation, there would be no honors mathematics graduates in the United States." Under *Euclid*, he notes that in addition to his translation of Heiberg's definitive text of Euclid, Sir Thomas Heath published his own edition in Greek of Book I for students of mathematics who could read Greek. Lo Bello says that such students are 'a type of scholar that does not exist anymore'. I feel that he is unduly pessimistic.

Universal education tends to disappoint the hopes of the reformers, as did the BBC, Radio Éireann, the gaelic revival, the great radio orchestras, and the national theatres. The worthy blogs of Fields' medallists, the Ted Talks, and Project Maths may fare no better. Gresham's Law operates across the whole spectrum of value. Most people decide by the time they reach adolescence that they no longer need to learn whatever is on offer, and thereafter learn on a self-motivated basis. It happens that they learn things, because they become interested, that they were told, but did not hear, many years before. This is nothing new. The fact is that Lo Bello was an exceptional student, even in his day. It is better to look on the bright side, as did Max Dehn. When asked whether he had been frustrated by the paucity of serious mathematics students at Black Mountain, he said: "Not at all. In fact, I have been very fortunate. In my sixty years of teaching I have had at least fifteen real students". My own experience of observing students gives me ground for hope that a positive proportion of them have the makings of real scholars. When I was at Brown, a respectable number of students took up Greek every year, and a few studied Akkadian. It is still possible to study the languages and classics of the ancient world in many US colleges and universities, and people do. I have been heartened to meet people more than thirty years younger than me who can read cuneiform tablets or hieroglyphic inscriptions as I would read the daily newspaper. The scholars are few, but it does not matter. They are probably more numerous than ever.

Lo Bello admits that *vox populi* is the voice of God, yet rails against its lack of culture. He has strong and exact views about the proper way to coin new words and he frequently mocks the efforts of those who coined our mathematical vocabulary, condemning various words as 'macaronic', 'comical', 'low', 'a sign of illiteracy', or 'pleonastic'. Right at the start, in the preface and again in the entry for *a-,an-,in-,im-,un-*, he lays down the law about the proper way to go about inventing new words. The main idea is that we usually build on roots that are Greek, Latin, or Teutonic, and we should not combine prefixes or suffixes from one source with words from another.

For example, *normalization* is 'macaronic'; all attempts at the plural of *Latus rectum* apart from the correct *latera recta* are 'comical'; *sector* and *subgroup* are 'mistakes'; *septagon* is *'vox nullius'*, a 'learnèd mistake'; *subset* and *superset* are 'low', as are *subharmonic* and *superharmonic* (they 'should be hypoharmonic and hyperharmonic'); *tautochrone* was 'coined by someone who did not know what he was doing' and 'should be isochrone'. *multifoil*, *cinquefoil*, *quatrefoil*, etc show 'inconsistency of language'.

Occasionally, he gives the thumbs up: *ethnomathematics* is 'correctly formed'. Once, he exhibits actual enthusiasm: Napier's 1614 word *logarithm* was a 'happy' choice.

In wider discourse, beyond mathematics, he rails against *droid* and *software*: 'uncultured'; *cosmetology*: 'contemptible'; *automobile*, *homosexual*, *neuroscience*, *sociopath*, and *television*: 'absurdities'; *virtual*: 'hideous computer lingo' *professor emerita*: 'comical to those who know Latin'; *triangulate*: 'misused. Two respectable meanings are now swallowed up and lost in new vernacular'; *the virgule (/)*: 'is poor style and should never be used'.

In all this, I have sympathy with Lo Bello, when he says that 'our ears are assaulted with the most ugly concoctions and constructions.' But he swims against the current. Under *glottochronology*, he quotes the two 'main theorems' of the subject: (1) With k=0.217, and t in millennia, $N(t) = N(0)\exp(-kt)$, where $N(0)$ is the number of words in a basic list at time 0, and $N(t)$ is the number of these words still present at time t. (Thus a language loses about one-fifth of its basic words per millennium.)
(2) If languages L_1 and L_2, derived from a common root share M words in the basic list $N(0)$, then the time now is

$$T = -2.30 \ln\left(\frac{M}{N_0}\right).$$

I suppose that he is being ironical in giving the constants to three significant figures, and perhaps also in referring to these hypotheses as theorems. But the fact is clear: languages change, and they do so at uneven rates, but always rather rapidly. Lo Bello remarks that the spelling *mirror* came about 'a millennium ago, in an age careless of detail'. It is precisely at times of conflict and mayhem that language mutates most rapidly. We live now in such an age. England lost control of English two centuries ago. In the exuberance of the melting-pot, the American language adopted new patterns of word-formation, and now America has lost control of American. Besicovitch said to Hayman[1, p38]: "Fifty million Englishmen speak English you speak. Five hundred million persons speak English I speak." Today, half the human population speak broken English, and they break it a little more every day. We just have to live with that.

Lo Bello makes an interesting point, in a rant about the poor quality of English prose written by American academics. He attributes it to the fact that English-speaking people do not trouble to study foreign languages because the language is used in books intended for a universal audience (replacing Latin). It would be interesting to check how many great writers were monoglot.

Lo Bello is not unreasonable, and says that he does not intend "to exhibit the kind of inflexible behaviour criticized by Voltaire" in a passage quoted from his *Lettres anglaises* that was aimed at mathematicians:

> *...animé...par cette inflexibilité d'esprit que donne d'ordinaire l'étude opiniâtre des science de calcul.*
>
> (gripped by the inflexibility of spirit that results from the mulish study of methods of calculation.)

He does not presume to cavil at terms adopted by Newton, the Bernoullis, Leibniz, Euler, or Gauss, "from whose authority no appeal is possible", but I fear it too late to complain about the practice of using the title *references* for a bibliography, or to criticise the use of citations in the form [LoB], instead of [2].

He is quite correct to rail at the barbarous use of ℵ or И for N, and Λ for A. He labels these, and the construct *numb3rs* as "illiterate fontese". It is a pity that he could not overcome his distaste for the *spellings* TeX and LaTeX to a sufficient extent to allow him to make use of Knuth's peerless system for typesetting mathematics. Indeed, not only the mathematical formulas, but even the quoted Arabic words and phrases are poorly typeset. There is no excuse these days for printing such things as $1 + \frac{1}{4} + 1/9 + 1/16 + \cdots.$ instead of $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$

For me, these facts take nothing from the interest in the etymology of our mathematical language, and I derived great pleasure and much instruction from Lo Bello's account. For instance, it was a delight to realize that a *test* was a pot and a *text* a woven thing; that *transitive (transitivus)* goes back to Priscian of Caesarea (about 500 AD,), who also gave us *numeral (numeralis)*; that the Incas had the parabolic spiral; that the ovals of Cassini are the plane sections of a torus; and how to construct the trisectrix of Maclaurin. There is a small essay about renaissance architecture under the heading *vault*. The entry under *numerals* is a mine of information. Best of all, the steps by which a word mutates in form and meaning are just fascinating. For instance, the Greek word ἄλογος, meaning *irrational*, also has the literal meaning *lacking word*, so Islamic translators used أصمّ ('asamm), meaning *deaf*, and this became Latin *surdus*, giving us the English *surd* and *absurd*.

The book is full of such nuggets. Every library should have a copy, and it would make a fine gift, if someone you love loves words.

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Patrick D. Barry: Geometry with Trigonometry, 2nd Edition, Woodhead Publishing, 2015. ISBN:978-1-898563-69-3, USD 84.96, 280pp.

REVIEWED BY ANCA MUSTAŢĂ

The second edition of *Geometry with Trigonometry* by Patrick D. Barry is both timely and important. It provides an accurate diagnostic of difficulties encountered in the teaching of geometry at school and undergraduate level; it offers a thoughtful, rigorous and balanced response to this problem; in the context of the new secondary school curriculum Project Maths, it provided solid axiomatic foundations for the development of the geometry component; with its wealth of appealing material, it's a valuable reference for secondary school teachers across Ireland and beyond. Last but not least, this book serves as the basis for a substantial follow-up work – *Some Generalization in Geometry* (see [3]) – which develops a coherent and unitary exposition on conic sections in Projective geometry via computational methods based on sensed-areas and rotors.

The author Patrick D. Barry is Professor Emeritus at University College Cork, where he has started his undergraduate mathematics education by earning first place in the Entrance Scholarship Examination in 1952. After a PhD in Complex Analysis at the Imperial College of Science and Technology, London, and an instructorship at Stanford University, he returned to Ireland in 1964 to become Head of the Mathematics Department at his alma mater. Throughout an extensive practice in research and teaching, Professor Barry has kept in touch with overall developments in mathematics education including at post-primary level. The accumulated experience has led him to provide this rigorous and modern treatment of Euclidean geometry.

Geometry is one of the most ancient areas of mathematical inquiry. Long before the algebraic formulation of equations in the 16-17th centuries, solutions for simple polynomial equations were known to Egyptian, Greek and Chinese mathematicians as early as

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600 BC, based on geometric constructions with ruler and compass. Euclid's elements (ca. 300 BC) record the first attempts at building solid axiomatic foundations, and have determined the general approach to geometry for two millennia. To this day, Euclidean geometry is taught in schools mostly as a means to develop logical deductive thinking, while relying on students' visual observations and an informal understanding of basic facts derived from Euclid's axioms. However, the diminishing role of classical synthetic geometry in school curricula worldwide is undeniable. Inquiring into the causes behind this development is certainly worthwhile as it can help develop comprehensive solutions.

From ancient times, geometry developed under a dual set of methods and motivations: on the one hand, the quantitative geometry of measurements involving lengths, areas and volumes; on the other hand, the qualitative part focusing on structures and the interrelationships between them: incidence, generation, transformations, variations in families and related invariants. For a while, these two aspects have complemented and enriched each other. Nowadays, there is no smooth transition from the qualitative aspects of synthetic geometry introduced in secondary schools to the quantitative content of undergraduate university courses. Indeed, geometry as a subject is missing altogether in the first year of college courses in Ireland, even though geometric problems underpin much of their core content: from calculations of areas and volumes in integral calculus to linear transformations and intersections of linear spaces in linear algebra, as well as applications in classical mechanics. By the time students meet metric spaces and differential geometry in their third/fourth years of study, the break with synthetic methods and motives is complete and profound. No wonder students may question the utility of learning Euclidean geometry in school.

In *Geometry with Trigonometry*, Professor Barry proposes a third level course which revisits Euclidean geometry from a richer perspective: on one hand, setting up solid axiomatic bases, clearly delineating between primary assumptions and their logical consequences; on the other hand, reworking a multitude of classical results quantitatively, by a variety of means: Euclidean coordinates, complex numbers, position vectors, areal coordinates and "mobile coordinates", an ingenious way of moving the reference axes to suit the geometry of the structures studied.

With this book, Professor Barry offers third level students – in particular future teachers of mathematics – an opportunity to ponder more deeply and carefully on the foundations of the subject. Here, the main imperative is rigour and completeness. Although Euclid's Elements attempted to set up logical foundations for geometry, the list of premises provided therein was far from sufficient to support all the theorems derived. On various occasions, Euclid's text relied on observations which seemed "visually obvious", but which had not been spelled out among the original set of postulates. Such "common sense" unstated assumptions included existence of rigid transformations (superposition) on which the treatment of congruences was based; assumptions about relative positions of lines and points e.g. concerning boundaries, sides, "betweenness"; and assumptions on completeness which would insure for example the existence of an expected number of intersection points between circles and lines. At the same time, Euclid attempted to provide definitions for primitive terms like points, lines, planes, which are the basic building blocks of Euclidean geometry, but which cannot be intrinsically described by a list of properties. Hilbert's foundational work in 1890's [10] provided the first logically sound and complete system of axioms; starting from undefined notions like point, line, plane, congruent, and describing the relationships between these in no less than 20 axioms. The long list of axioms seems off-putting for any student aiming to learn about plane geometry. Other equivalent systems have been proposed since, the most notable by Birkhoff ([4], [5]) which uses the theory of real numbers to simplify the exposition, thus making it more amenable to use by teachers. It is this approach that Patrick Barry favours in his book. This choice also facilitates the transition to coordinate geometry in the second part of the book.

After a review of the history, basic concepts and pre-requisites in Chapter 1, there follows a systematic presentation of axioms and basic properties. The exposition is thoughtfully divided into different chapters or sections for each axiom and its immediate consequences: Chapter 2 discusses *incidence*, *order*, and *separation* by half-planes. Chapter 3 introduces axioms of *distance* and *angle measurements*, and discusses midpoints, midlines, ratios and perpendicularity. Chapter 4 discusses congruences of triangles, while Chapter 5 deals with *the parallel axiom* and its consequences including similar triangles, Pythagoras' theorem, harmonic ranges and areas. Results are carefully proven by synthetic methods. At this fundamental stage when only basic facts are given, proofs often require elegant solutions based on imaginative additions of auxiliary lines and points to the diagrams. These are presented beautifully and efficiently. While such constructions may hardly seem obvious to beginners in geometry, it is hoped that they are fully appreciated by teachers and 3rd level students looking for an approach that is both complete and coherent.

Chapter 6 signals a change in approach, beginning a coordinate treatment of the main objects developed earlier: Cartesian and parametric equations of a line, criteria for parallelism and perpendicularity, areas of triangles and a thorough discussion of harmonic ranges. In Chapter 7 basic properties of circles are developed, including angles standing on arcs, harmonic ranges and the polar line of a point exterior to a circle; the power of a point with respect to a circle, and the radical axis of two circles (the locus of points having the same "power" with respect to both circles). Synthetic discussions of these topics would necessarily have to distinguish cases according to the relative positions of points and circles. By contrast, a unitary treatment is granted here by the notions of *sensed distance, sensed product* and *sensed ratios* (depending on a chosen order).

Chapter 8 starts a brief discussion of isometries (namely translations and reflections); a treatment of rotations is postponed till the following chapters, when more suitable computational tools have been developed. Chapter 9 consists of a very careful introduction to trigonometric functions and their properties. To deal with angles and rotations, complex coordinates are introduced, as is the homomorphism $\theta \to \text{cis}\theta := e^{i\theta}$ which relates angles to unitary complex numbers. The notion of *sensed angles* distinguishes between angle orientations (clockwise or anticlockwise), and correspondingly, a notion of *sensed area* is defined. These tools are then employed to develop the criterion for 4 concyclic points via a real cross-ratio, and the resulting proof of the famous Ptolemy (c 200 AD) formula involving the lengths of sides and products of diagonals. We are also treated to a special case of Pascal's theorem (1640), involving a cyclic hexagon made of three pairs of parallel sides.

The extended Chapter 11 discusses a variety of coordinate methods, starting with vectors and areal coordinates. We revisit in this context classical results like Menelaus' theorem (c.100 A.D.) about a line cutting a triangle, the dual theorem of Ceva (1678 A.D.), Desargues' perspective theorem (1648 A.D.), and Pappus' theorem (c.300 A.D.). The original concept of *mobile coordinates* employs complex numbers to suitably describe points on perpendicular lines, leading to economical descriptions of special points for a triangle: the centroid, orthocentre, incentre and circumcentre, along with proofs for the existence of Euler's line and the 9 point circle. Other theorems displaying pleasing structural symmetries – a trademark of Euclidean geometry – include the following: *On the sides of a triangle are erected three similar triangles (with the same orientation). Then the circumcentres of these triangles forms a fourth triangle similar to the previous three.* Computational proofs of other well loved theorems, like Feuerbach's, Wallace-Simpson, and Miquel's theorem are provided. The concurrence of isogonal conjugate lines in a triangle (and the particular case of symmedians) complete the delightful collection of classical results.

Chapter 12 develops proofs for the differential formulas of sine and cosine functions, a must-see for every student. Hence the length of an arc of circle can be calculated by integration, and a radian measure can be properly defined.

I warmly recommend this book for those who wish to deepen their understanding of Euclidean geometry. The first part sets up solid comprehensive bases, through carefully formulated axioms and elegant efficient proofs. The second part involves considerable computational sophistication and ingenuity in the choice of coordinates.

Geometry with Trigonometry is a perfect resource for post-primary teaching. The axiomatic part is covered at a level of rigour and sophistication not required (or indeed productive) when working with school children, but useful for teachers' own understanding. The major classical theorems present in this book have a perennial aesthetic appeal, and I hope that teachers will feel tempted to include a good part of them in their teaching, either during class time or for maths clubs and maths circles. When planning their own teaching methodologies though, it is left to the teachers to compare and select between synthetic and coordinate strategies, on a case by case basis. When applying synthetic methods, the main difficulties spring from a number of factors:

- The need for casework – depending on the relative positions of points, lines and circles. This is more a case of awkwardness than difficulty, as whenever casework is required, closely similar strategies tend to work for all cases. The problem is mostly pedagogical, as systematic working of cases may try the patience of young minds; this can be mitigated by applying discretion in the level of detail and rigour in school geometry proofs.
- The need for an ability to navigate complex diagrams, focusing on their useful configurations, ignoring elements irrelevant to the problem at hand, and sometimes drawing new lines and points to bring a new perspective. Modern graphic design tools like Geogebra [11] are quite helpful, as they produce crisp, exact diagrams which can be zoomed in or varied continuously while preserving - and thus highlighting - the relevant features in each configuration.
- The need to combine a variety of measurement methods within one proof, e.g. making transitions from angle-chasing to computations of lengths and areas and vice-versa via congruences, similarities, or trigonometric functions. At first sight, the coordinate method does away with this difficulty. However, as seen in this book, one needs flair in choosing the most efficient coordinate setting. In particular, simple angle-chasing arguments can become quite complicated in Cartesian coordinates; only the use of complex numbers can offer comparable simplicity, but these are only taught in more advanced classes. An alternative way to deal with this difficulty would be to organize the teaching of school synthetic geometry into distinct stages:
  (1) The *foundational stage*, involving primary objects, notations, definitions, axioms and immediate consequences.
  (2) the *isometry stage*, including the use of congruent triangles in deriving properties of parallelograms and other special quadrilaterals; as well as properties of rotations and reflections;
  (3) the *angle-chasing stage*, including the theorems on sums of angles in triangles and other convex polygons; cyclic quadrilaterals; the concurrency of altitudes; orthocentre

as incentre of the orthic triangle; and many other beautiful theorems like Miquel's point, Miquel and Steiner's quadrilateral theorem, Miquel's pentagon and six circles theorems;

(4) the *similarity stage*, making the transition from angle-chasing to more complex computations involving lenghts and ratios: including Pythagoras' theorem, and topics like Ceva's and Menelaus' theorems, power of the point, Simson's line, Ptolemy's theorem, harmonic ranges;

(5) the *geometric loci stage*, including all important lines in the triangle and their *universal properties*, as well as radical axes for circles; leading to the existence of incentre, circumcentre, centroid, orthocentre, radical centre, Euler's line, the 9 point circle, the Newton-Gauss line.

This book inspires the reader to think more deeply about ways to develop a coherent approach to teaching geometry from school to university level: including seamless transitions in the development of topics, methods and themes from classical plane geometry to advances undergraduate and graduate courses. Here are a few of reviewer's thoughts on this issue:

- Offer 3rd level courses that lead to a deeper perspective on Euclidean geometry, for example based on professor Barry's book.
- Study conic sections by both synthetic and coordinate approaches (e.g. defining focal points as tangency points of the plane section with certain spheres, and using synthetic methods to deduce the reflection properties applied in optics).
- Transition from plane Euclidean geometry to projective geometry over division rings or fields following Hilbert's ideas (defining operations on the abstract line using translations and similarities; rehashing Desargues' and Pappus' configurations as associativity and commutativity of the base field). Thus setting basic premises for Algebraic Geometry.
- Use the "power of the point" to study inversion in a circle and build Poincare's model of Hyperbolic Geometry.
- Transition from study of triangles and triangulations to simplices and simplicial complexes in early courses on Algebraic Topology.

- Show how the theme of geometric loci from Euclidean geometry is continued with the study of parameter or moduli spaces and universal families in Algebraic, Complex and Differential Geometry.
- Transition from the study of invariants in transformation geometry to Representation Theory.

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PROBLEMS

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Problems

The first problem this issue was contributed by Finbarr Holland of University College Cork.

Problem 80.1. Let $x_0, x_1, x_2, \ldots$ be a null sequence generated by the recurrence relation

$$(n+1)(x_{n+1}+x_n) = 1, \quad n = 0, 1, 2, \ldots .$$

Prove that the series

$$\sum_{n=0}^{\infty} (-1)^n x_n$$

converges, and determine its sum.

The next problem was suggested by J.P. McCarthy of Cork Institute of Technology.

Problem 80.2. Let

$$S(\sigma) = \sum_{i=1}^{n} \frac{1}{\sqrt{n^{i+\sigma(i)}}},$$

where σ is a nonidentity permutation of $\{1, 2, \ldots, n\}$. Find the maximum of S over all such permutations.

We finish with an easy problem, quoted verbatim from *Lectures and Problems: A Gift to Young Mathematicians*, by V.I. Arnold. The book gives an appealing backstory to the problem, which we will pass on with the solution, in *Bulletin* Number 82.

Problem 80.3. Two volumes of Pushkin, the first and the second, are side-by-side on a bookshelf. The pages of each volume are 2cm thick, and the front and back covers are each 2mm thick. A bookworm has gnawed through (perpendicular to the pages) from the

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first page of volume 1 to the last page of volume 2. How long is the bookworm's track?

Solutions

Here are solutions to the problems from *Bulletin* Number 78. The first problem was solved by Prithwijit De of Mumbai, India, Henry Ricardo of the Westchester Area Math Circle, New York, USA, the North Kildare Mathematics Problem Club, as well as the proposer, Ángel Plaza of Universidad de Las Palmas de Gran Canaria, Spain. We present a solution, which is equivalent to that of De, Ricardo and Plaza.

Problem 78.1. Given positive real numbers a, b, c, u and v, prove that

$$\frac{a}{bu+cv}+\frac{b}{cu+av}+\frac{c}{au+bv}\geqslant\frac{3}{u+v}.$$

Solution 78.1. Let L be the expression on the left. Then

$$L=\frac{a^2}{abu+cav}+\frac{b^2}{bcu+abv}+\frac{c^2}{cau+bcv}.$$

By the Cauchy–Schwarz inequality,

$$L\big((abu+cav)+(bcu+abv)+(cau+bcv)\big)\geqslant(a+b+c)^2.$$

Hence

$$\begin{aligned}L(ab+bc+ca)(u+v)&=L\big((abu+cav)+(bcu+abv)+(cau+bcv)\big)\\&\geqslant(a+b+c)^2\\&\geqslant 3(ab+bc+ca),\end{aligned}$$

where, to obtain the last line, we have expanded the brackets and applied inequalities of the type $a^2+b^2\geqslant 2ab$. Dividing throughout by $(ab+bc+ca)(u+v)$ gives the result we require. □

The second problem from *Bulletin* Number 78 originated from an exercise found in an Open University complex analysis module. The exercise asks students to prove, in the case $n=2$, that if p is a polynomial of degree n with n distinct fixed points $a_1,\dots,a_n$, then at least one of these fixed points a_j satisfies $|p'(a_j)|\geq 1$ (it is a repelling fixed point). Assuming that p is monic, and $n>1$, we can write

$$p(z)=z+\prod_{i=1}^{n}(z-a_i),$$

and hence

$$p'(a_j) = 1 + \prod_{\substack{i=1 \\ i\neq j}}^{n} (a_j - a_i).$$

Problem 78.2 is thereby equivalent to the general form of the student exercise.

The problem can be solved by complex dynamics, arguing that each component of the Fatou set of p can have at most one attracting fixed point, and each such component contains a critical point of p, of which there are at most $n-1$. However, we favour the solutions submitted (jointly) by Prithwijit De and Finbarr Holland, and also by the North Kildare Mathematics Problem Club. The two solutions were much the same, and we present them here.

Problem 78.2. Let $a_1, \dots, a_n$ be distinct complex numbers. Prove that

$$\left|1 + \prod_{\substack{i=1 \\ i\neq j}}^{n} (a_j - a_i)\right| \geqslant 1$$

for at least one of the integers $j = 1, \dots, n$.

Solution 78.2. The inequality is trivial if $n = 1$, so suppose that $n \geqslant 2$. Define

$$f(z) = \prod_{i=1}^{n} (z - a_i).$$

Observe that

$$f'(a_j) = \prod_{\substack{i=1 \\ i\neq j}}^{n} (a_j - a_i), \quad j = 1, \dots, n.$$

Observe also that the residue of $1/f$ at the simple pole a_j is $1/f'(a_j)$. Now let Γ_R be the circular path of radius R centred at 0, where R is chosen to be greater than $|a_j|$, for $j = 1, \dots, n$. Then, by the Residue Theorem,

$$\frac{1}{2\pi i} \int_{\Gamma_R} f(z)\, dz = \sum_{j=1}^{n} \frac{1}{f'(a_j)}.$$

Since $|f(z)| \geq \prod_{i=1}^{n}(|z|-|a_i|)$, we see that the integral converges to 0 as $R\to\infty$. Therefore

$$\sum_{j=1}^{n}\frac{1}{f'(a_j)}=0.$$

It follows that

$$\sum_{j=1}^{n}\frac{2\mathrm{Re}\,(f'(a_j))}{|f'(a_j)|^2}=0,$$

so

$$\sum_{j=1}^{n}\frac{|1+f'(a_j)|^2}{|f'(a_j)|^2}=n+\sum_{j=1}^{n}\frac{1}{|f'(a_j)|^2}.$$

Consequently $|1+f'(a_j)|>1$ for at least one integer j. □

The third problem was solved by Ángel Plaza, the North Kildare Mathematics Problem Club, and the proposer, Finbarr Holland. All solutions were more or less equivalent. Finbarr points out that the problem is in fact much the same as Problem 11946 from the December 2016 issue of *The American Mathematical Monthly*.

Problem 78.3. Suppose that the continuous function $f:[0,1]\to\mathbb{R}$ is twice differentiable on $(0,1)$ and the second derivative f'' is square integrable on $[0,1]$. Suppose also that $f(0)+f(1)=0$. Prove that

$$120\left|\int_0^1 f(t)\,dt\right|^2 \leqslant \int_0^1 |f''(t)|^2\,dt,$$

and show that 120 is the best-possible constant in this inequality.

Solution 78.3. Integrating by parts twice, we see that

$$\begin{aligned}\int_0^1 t(1-t)f''(t)\,dt &= -\int_0^1 (1-2t)f'(t)\,dt\\ &= -2\int_0^1 f(t)\,dt.\end{aligned}$$

Since

$$\int_0^1 t^2(1-t)^2\,dt=\frac{1}{30},$$

we can apply the Cauchy–Schwarz inequality, to give

$$\frac{1}{30}\int_0^1 |f''(t)|^2\,dt \geq \left|\int_0^1 t(1-t)f''(t)\,dt\right|^2 = 4\left|\int_0^1 f(t)\,dt\right|^2.$$

The result follows on multiplying throughout by 30.

Equality is attained if and only if $f''(t) = kt(1-t)$, for some constant k, which is so if and only if $f(t) = a(t^4 - 2t^3) + bt + c$, for constants a, b and c. The condition $f(0) + f(1) = 0$ implies that $-a + b + 2c = 0$, so equality is attained for any function of the form

$$f(t) = a(t^4 - 2t^3 + t) - c(2t - 1). \qquad \square$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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