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Composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions

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Abstract. An analytic self-map ϕ of the open unit disk $\mathbb{D}$ in the complex plane induces the so-called composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$, $f \mapsto f \circ \phi$, where $H(\mathbb{D})$ denotes the set of all analytic functions on $\mathbb{D}$. Motivated by [5] we analyze under which conditions on the weight v all composition operators C_ϕ acting between the weighted Bergman space and the weighted Banach space of holomorphic functions both generated by v are bounded.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ endowed with the compact-open topology *co*. Moreover, let ϕ be an analytic self-map of $\mathbb{D}$. Such a map induces through composition the classical composition operator

$$C_\phi : H(\mathbb{D}) \to H(\mathbb{D}), \ f \mapsto f \circ \phi.$$

Composition operators acting on various spaces of analytic functions have been studied by many authors, since this kind of operator appears naturally in a variety of problems, such as e.g. in the study of commutants of multiplication operators or the study of dynamical systems, see the excellent monographs [8] and [17]. For a deep insight in the recent research on (weighted) composition operators we refer the reader to the following papers as well as the references therein: [4], [5], [6], [7], [11], [13], [14], [15], [16].

Let us now explain the setting in which we are interested. We say that a function $v : \mathbb{D} \to (0, \infty)$ is a *weight* if it is bounded and continuous. For a weight v we consider the following spaces:

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(1) The *weighted Banach spaces of holomorphic functions* defined by

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}.$$

Endowed with the weighted sup-norm $\|.\|_v$ this is a Banach space. These spaces arise naturally in several problems related to e.g. complex analysis, spectral theory, Fourier analysis, partial differential and convolution equations. Concrete examples may be found in [3]. Weighted Banach spaces of holomorphic functions have been studied deeply in [2] and also in [1].

(2) The *weighted Bergman spaces* given by

$$A_v^2 := \left\{ f \in H(\mathbb{D});\ \|f\|_{v,2} := \left(\int_{\mathbb{D}} |f(z)|^2 v(z)\, dA(z)\right)^{\frac{1}{2}} < \infty \right\},$$

where $dA(z)$ is the normalized area measure such that area of $\mathbb{D}$ is 1. Endowed with norm $\|.\|_{v,2}$ this is a Hilbert space. An introduction to Bergman spaces is given in [10] and [9].

In [19] we characterized the boundedness of composition operators acting between weighted Bergman spaces and weighted Banach spaces of holomorphic functions in terms of the involved weights as well as the symbols. In this article we are interested in the question, for which weights v all composition operators $C_\phi : A_v^2 \to H_v^\infty$ are bounded.

2. Background and basics

2.1. Theory of weights. In this part of the article we want to give some background information on the involved weights. A very important role play the so-called *radial* weights, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. If additionally $\lim_{|z|\to 1} v(z) = 0$ holds, we refer to them as *typical* weights. Examples include all the famous and popular weights, such as

(a) the *standard weights* $v(z) = (1-|z|)^\alpha$, $\alpha \geq 1$,
(b) the *logarithmic weights* $v(z) = (1-\log(1-|z|))^\beta$, $\beta > 0$,
(c) the *exponential weights* $v(z) = e^{-\frac{1}{(1-|z|)^\alpha}}$, $\alpha \geq 1$.

In [12] Lusky studied typical weights satisfying the following two conditions

$$(L1) \qquad \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0$$

and

$$(L2) \qquad \limsup_{n\to\infty} \frac{v(1-2^{-n-j})}{v(1-2^{-n})} < 1 \text{ for some } j \in \mathbb{N}.$$

In fact, weights having $(L1)$ and $(L2)$ are *normal* weights in the sense of Shields and Williams, see [18]. The standard weights are normal weights, the logarithmic weights satisfy $(L1)$, but not $(L2)$ and the exponential weights satisfy neither $(L1)$ nor $(L2)$. In our context $(L2)$ is not of interest while $(L1)$ will play a secondary role. The formulation of results on weighted spaces often requires the so-called associated weights. For a weight v its associated weight is given by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H(\mathbb{D}),\ \|f\|_v \le 1\}}, \ z \in \mathbb{D}.$$

See e.g. [2] and the references therein. Associated weights are continuous, $\tilde{v} \ge v > 0$ and for every $z \in \mathbb{D}$ there is $f_z \in H(\mathbb{D})$ with $\|f_z\|_v \le 1$ such that $f_z(z) = \frac{1}{\tilde{v}(z)}$. Since it is quite difficult to really calculate the associated weight we are interested in simple conditions on the weight that ensure that v and $\tilde{v}$ are equivalent weights, i.e. there is a constant $C > 0$ such that

$$v(z) \le \tilde{v}(z) \le Cv(z) \text{ for every } z \in \mathbb{D}.$$

If v and $\tilde{v}$ are equivalent, we say that v is an *essential* weight. By [5] condition $(L1)$ implies the essentiality of v.

2.2. **Setting.** This section is devoted to the description of the setting we are working in. In the sequel we will consider weighted Bergman spaces generated by the following class of weights. Let ν be a holomorphic function on $\mathbb{D}$ that does not vanish and is decreasing as well as strictly positive on $[0,1)$. Moreover, we assume that $\lim_{r\to 1} \nu(r) = 0$. Now, we define the weight as follows:

$$v(z) := \nu(|z|) \text{ for every } z \in \mathbb{D}. \tag{1}$$

Obviously such weights are bounded, i.e. for every weight v of this type we can find a constant $C > 0$ such that $\sup_{z\in\mathbb{D}} v(z) \le C$. Moreover, we assume additionally that $|\nu(z)| \ge \nu(|z|)$ for every

$z \in \mathbb{D}$.
Now, we can write the weight v in the following way

$$v(z) = \min\{|g(\lambda z)|,\ |\lambda| = 1\},$$

where g is a holomorphic function on $\mathbb{D}$. Since ν is a holomorphic function, we obviously can choose $g = \nu$. Then we arrive at

$$\begin{aligned}\min\{|\nu(\lambda z)|,\ |\lambda| = 1\} &= \min\{|\nu(\lambda r e^{i\Theta})|,\ |\lambda| = 1\} \\ &\leq |\nu(e^{-i\Theta} r e^{i\Theta})| = |\nu(r)| = |\nu(|z|)| = v(z)\end{aligned}$$

for every $z \in \mathbb{D}$. Conversely, by hypothesis, for every $\lambda \in \partial\mathbb{D}$ we obtain for every $z \in \mathbb{D}$

$$|\nu(\lambda z)| \geq \nu(|\lambda z|) \geq \nu(|z|) = v(z).$$

Thus, the claim follows. The standard, logarithmic and exponential weights can all be defined like that.

2.3. **Composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions.** In the setting of weighted Banach spaces of holomorphic functions the classical composition operator has been studied by Bonet, Domański, Lindström and Taskinen in [4] and [5]. Among other things they proved that in case that v and w are arbitrary weights the boundedness of the operator $C_\phi : H_v^\infty \to H_w^\infty$ is equivalent to

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\phi(z))} < \infty.$$

Moreover, they showed that v satisfies condition $(L1)$ if and only if every composition operator $C_\phi : H_v^\infty \to H_v^\infty$ is bounded.

This was the motivation to study the boundedness composition operators acting between weighted Bergman spaces and weighted Banach spaces of holomorphic functions. Doing this we obtain the following results which we need in the sequel. For the sake of understanding and completeness we give the proof here.

For $p \in \mathbb{D}$ let

$$\alpha_p(z) := \frac{p - z}{1 - \overline{p}z},\ z \in \mathbb{D},$$

be the Möbius transformation that interchanges p and 0.

Lemma 2.1 ([20], Lemma 1)**.** *Let $v(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ with $\nu \in H(\mathbb{D})$ be a weight as defined in Section 2.2. Then there is a constant $M > 0$ such that*

$$|f(z)| \leq M \frac{\|f\|_{v,2}}{(1-|z|^2)v(z)^{\frac{1}{2}}}$$

for every $f \in A_v^2$.

Proof. As we have seen in Section 2.2 a weight as defined above may be written as

$$v(z) := \min\{|g(\lambda z)|;\ |\lambda| = 1\} \text{ for every } z \in \mathbb{D},$$

where g is a holomorphic function on $\mathbb{D}$. In the sequel we will write $g_\lambda(z) := g(\lambda z)$ for every $z \in \mathbb{D}$. Now, fix $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Moreover, let $p \in \mathbb{D}$ be arbitrary. Then, we consider the map

$$T_{p,\lambda} : A_v^2 \to A_v^2,\ T_{p,\lambda}f(z) = f(\alpha_p(z))\alpha_p'(z)g_\lambda(\alpha_p(z))^{\frac{1}{2}}.$$

Let $f \in A_v^2$. Then a change of variables yields

$$\begin{aligned}
\|T_{p,\lambda}f\|_{v,2}^2 &= \int_{\mathbb{D}} v(z)|f(\alpha_p(z))|^2|\alpha_p'(z)|^2|g_\lambda(\alpha_p(z))|\,dA(z)\\
&\leq \int_{\mathbb{D}} v(z)|f(\alpha_p(z))|^2|\alpha_p'(z)|^2|v(\alpha_p(z))|\,dA(z)\\
&\leq \sup_{z\in\mathbb{D}} v(z)\int_{\mathbb{D}} |f(\alpha_p(z))|^2|\alpha_p'(z)|^2|v(\alpha_p(z))|\,dA(z)\\
&\leq C\int_{\mathbb{D}} v(t)|f(t)|^2\,dA(t) = C\|f\|_{v,2}^2.
\end{aligned}$$

Next, put $h_{p,\lambda}(z) := T_{p,\lambda}f(z)$ for every $z \in \mathbb{D}$. By the Mean Value Property we obtain

$$v(0)|h_{p,\lambda}(0)|^2 \leq \int_{\mathbb{D}} v(z)|h_{p,\lambda}(z)|^2\,dA(z) \leq \|h_{p,\lambda}\|_{v,2}^2 \leq C\|f\|_{v,2}^2.$$

Since λ was arbitrary, we obtain

$$v(0)|f(p)|^2(1-|p|^2)v(p)^{\frac{1}{2}} \leq C\|f\|_{v,2}^2.$$

Finally,

$$|f(p)| \leq M\frac{\|f\|_{v,2}}{(1-|p|^2)v(p)^{\frac{1}{2}}} < \infty.$$

$\square$

The following result is obtained by using the previous lemma and following exactly the proof of [19] Theorem 2.2. Again, for a better understanding we give the proof.

Theorem 2.2. *Let $v(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ with $\nu \in H(\mathbb{D})$ be a weight as defined in Section 2.2. Then the operator $C_\phi : A_v^2 \to H_v^\infty$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\phi(z)|^2)v(\phi(z))^{\frac{1}{2}}} < \infty. \tag{2}$$

Proof. First, we assume that (2) holds. Applying Lemma 2.1 for every $f \in A_v^2$ we have

$$|f(z)| \leq C \frac{\|f\|_{v,2}}{(1-|z|^2)v(z)^{\frac{1}{2}}}$$

for every $z \in \mathbb{D}$. Thus, for every $f \in A_v^2$:

$$\|C_\phi f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(\phi(z))| \leq C \sup_{z\in\mathbb{D}} \frac{v(z)\|f\|_{v,2}}{(1-|\phi(z)|^2)v(\phi(z))^{\frac{1}{2}}} < \infty.$$

Hence the operator must be bounded.
Conversely, let $p \in \mathbb{D}$ be fixed. Then there is $f_p^2 \in H_v^\infty$, $\|f_p^2\|_v \leq 1$ with $|f_p^2(p)| = \frac{1}{\tilde{v}(p)}$. Now, put

$$g_p(z) := f_p(z)\alpha_p'(z) \text{ for every } z \in \mathbb{D}.$$

Changing variables we obtain

$$\|g_p\|_{v,2}^2 = \int_{\mathbb{D}} |g_p(z)|^2 v(z)\, dA(z) = \int_{\mathbb{D}} |f_p(z)|^2 |\alpha_p'(z)|^2 v(z)\, dA(z)$$
$$\leq \sup_{z\in\mathbb{D}} v(z)|f_p(z)|^2 \int_{\mathbb{D}} |\alpha_p'(z)|^2\, dA(z) = 1$$

Next, we assume to the contrary that there is a sequence $(z_n)_n \subset \mathbb{D}$ such that $|\phi(z_n)| \to 1$ and

$$\frac{v(z_n)}{(1-|\phi(z_n)|^2)v(\phi(z_n))^{\frac{1}{2}}} \geq n \text{ for every } n \in \mathbb{N}.$$

Now, we consider

$$g_n(z) := g_{\phi(z_n)}(z) \text{ for every } z \in \mathbb{D} \text{ and every } n \in \mathbb{N}$$

as defined above. Then $(g_n)_n$ is contained in the closed unit ball of A_v^2 and we can find a constant $c > 0$ such that

$$c \geq v(z_n)|g_n(\phi(z_n))| = \frac{v(z_n)}{(1-|\phi(z_n)|^2)v(\phi(z_n))^{\frac{1}{2}}} \geq n$$

for every $n \in \mathbb{N}$. Since we know that under the given assumptions we have $v = \tilde{v}$ this is a contradiction. $\square$

Having now characterized the boundedness of the composition operator acting between A_v^2 and H_v^∞ we take the second result of Bonet, Domański, Lindström and Taskinen as a motivation to ask the question: For which weights v are all operators $C_\phi : A_v^2 \to H_v^\infty$ bounded?

3. Results

Lemma 3.1. *Let $v(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ with $\nu \in H(\mathbb{D})$ be a weight as defined in Section 2.2. Moreover, let $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} < \infty$ and $C_{\alpha_p} : A_v^2 \to H_v^\infty$ be bounded for every $p \in \mathbb{D}$. Then all composition operators $C_\phi : A_v^2 \to H_v^\infty$ are bounded.*

Proof. Let $\phi : \mathbb{D} \to \mathbb{D}$ be an arbitrary analytic function. We have to show that $C_\phi : A_v^2 \to H_v^\infty$ is bounded. Now, $\phi = \alpha_p \circ \psi$ where $p = \phi(0)$, $\psi = \alpha_p \circ \phi$ and $\psi(0) = 0$. Since $\psi(0) = 0$, by the Schwarz Lemma we obtain that $|\psi(z)| \leq |z|$. Hence we get

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\psi(z)|^2)v(\psi(z))^{\frac{1}{2}}} \leq \sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|z|^2)v(z)^{\frac{1}{2}}} = \sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)}$$

$< \infty$. Thus, C_ψ is bounded. Finally, we can conclude that C_ϕ is bounded since it is a composition of bounded operators. $\square$

The proof of the following theorem is inspired by [5].

Theorem 3.2. *Let $v(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ with $\nu \in H(\mathbb{D})$ be a weight as defined in Section 2.2. Moreover, let $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} < \infty$. Then the composition operator $C_\phi : A_v^2 \to H_v^\infty$ is bounded for every analytic self-map ϕ of $\mathbb{D}$ if and only if*

$$\inf_{n\in\mathbb{N}} \frac{(2^{-n} - 2^{-2n-2})v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} > 0. \tag{3}$$

Proof. By Lemma 3.1 we have to show that condition (3) holds if and only if $C_{\alpha_p} : A_v^2 \to H_v^\infty$ is bounded for every $p \in \mathbb{D}$.

First, let each $C_{\alpha_p} : A_v^2 \to H_v^\infty$ be bounded. Then we have that for every $p \in \mathbb{D}$ there is $M_p > 0$ such that

$$v(z) \leq M_p v(\alpha_p(z))^{\frac{1}{2}}(1 - |\alpha_p(z)|^2) \text{ for every } z \in \mathbb{D}.$$

Since $\sup_{|z|=r} |\alpha_p(z)| = \frac{|p|+r}{1+|p|r}$ it follows that $v(z) \leq M_p v\left(\frac{|p|+r}{1+|p|r}\right)^{\frac{1}{2}}\left(1 - \left(\frac{|p|+r}{1+|p|r}\right)^2\right)$ for all $|z| = r$. Let $l(r) = v(1-r)^{\frac{1}{2}}(1-(1-r)^2)$ and $s = 1-r$. Now, since $1 - \frac{|p|+1-s}{1+|p|(1-s)} = \frac{s(1-|p|)}{1+|p|-|p|s}$, for $s < \frac{1}{2}$ we obtain

$$l\left(s\frac{1-|p|}{1+|p|}\right) \leq l\left(1 - \frac{|p|+1-s}{1+|p|(1-s)}\right) \leq l\left(s\frac{1-|p|}{1-\frac{|p|}{2}}\right) \tag{4}$$

Next, choose $p = \frac{2}{5}$ and find $M > 0$ and $s_0 > 0$ such that

$$v(1-s) \leq Ml\left(\frac{s}{2}\right) = Mv\left(1-\frac{s}{2}\right)^{\frac{1}{2}}\left(1-\left(1-\frac{s}{2}\right)^2\right) \text{ for all } s \in]0, s_0[.$$

Hence the claim follows.

Conversely we assume that (3) holds. Then l as defined above has the property that there are $M > 0$ and $t_0 \in]0,1[$ with

$$v(1-t) \leq Ml\left(\frac{t}{2}\right) \text{ for all } t \geq t_0.$$

Hence, for any $c < \infty$ we find $n \in \mathbb{N}$ such that $c < 2^n$ and thus $l(t) \leq M^n l\left(\frac{t}{c}\right)$. We take $c = \frac{1+|p|}{1-|p|}$. By the first inequality in (4) for all $p \in \mathbb{D}$ there is $M_p > 0$ such that

$$v(1-t) \leq M_p l\left(1 - \frac{|p|+1-t}{1+|p|(1-t)}\right)$$

for all $t > t_0$. Clearly this implies that for all $p \in \mathbb{D}$ there exists $M_p > 0$ such that for every $|z| = r$ we have that $v(z) < M_p v(\alpha_p(z))^{\frac{1}{2}}(1 - |\alpha_p(z)|^2)$. □

Example 3.3. (a) Let $v(z) = (1-|z|)^n$, $n \geq 2$. Then all composition operators $C_\phi : A_v^2 \to H_v^\infty$ are bounded. To prove this we have to show that the weight v satisfies the following conditions

(1) $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} < \infty$,

(2) $\inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} > 0$.

Indeed,

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\frac{n}{2}}}{(1-|z|)(1+|z|)} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\frac{n-2}{2}}}{(1+|z|)}$$

$$\leq \sup_{z\in\mathbb{D}} (1-|z|)^{\frac{n-2}{2}} < \infty \text{ since } n \geq 2. \text{ Moreover,}$$

$$\begin{aligned}
\inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} &= \inf_{k\in\mathbb{N}} (2^{-k}-2^{-2k-2})2^{\frac{1}{2}(kn-n)} \\
&= \inf_{k\in\mathbb{N}} 2^{-\frac{n}{2}}(2^{k(\frac{n}{2}-1)} - 2^{k(-2+\frac{n}{2})-2}) \\
&> 0 \text{ for every } n.
\end{aligned}$$

(b) The weight $v(z) = 1-|z|$ satisfies neither (1) nor (2). We obtain

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} = \sup_{z\in\mathbb{D}} \frac{1}{(1-|z|)^{\frac{1}{2}}(1+|z|)} \geq \frac{1}{2}\sup_{z\in\mathbb{D}} \frac{1}{(1-|z|)^{\frac{1}{2}}} = \infty.$$

Furthermore easy calculations show

$$\begin{aligned}
\inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} &= \inf_{k\in\mathbb{N}} (2^{-k}-2^{-2k-2})2^{\frac{1}{2}(k-1)} \\
&= \frac{1}{\sqrt{2}} \inf_{k\in\mathbb{N}} (2^{-\frac{k}{2}} - 2^{\frac{-3}{2}k-2}) = 0.
\end{aligned}$$

Hence, in this case there exists a composition operator $C_\phi : A_v^2 \to H_v^\infty$ that is not bounded. For example, the operator C_ϕ generated by the map $\phi(z) = z$ for every $z \in \mathbb{D}$ is not bounded, since

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\phi(z)|^2)v(\phi(z))^{\frac{1}{2}}} = \sup_{z\in\mathbb{D}} \frac{1}{(1+|z|)(1-|z|)^{\frac{1}{2}}} = \infty.$$

(c) The exponential weights $v(z) = e^{-\frac{1}{(1-|z|)^n}}$, $n > 0$, satisfy condition (1), but not condition (2). First, we get

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} = \sup_{z\in\mathbb{D}} \frac{e^{-\frac{1}{2(1-|z|)^n}}}{1-|z|} < \infty.$$

It follows, that (1) is fulfilled. Now,

$$\inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} = \inf_{k\in\mathbb{N}} (2^{-k}-2^{-2k-2})\frac{e^{-2^k n}}{e^{-2^k n}} = 0.$$

Thus, (2) is not satisfied. There must be a composition operator $C_\phi : A_v^2 \to H_v^\infty$ that is not bounded.

(d) The logarithmic weights $v(z) = \frac{1}{(1-\log(1-|z|))^n}$, $n > 0$, neither satisfy (1) nor (2). Indeed,

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} = \sup_{z\in\mathbb{D}} \frac{1}{(1-|z|^2)(1-\log(1-|z|))^{\frac{n}{2}}} = \infty$$

and

$$\inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} = \inf_{k\in\mathbb{N}} (2^{-k}-2^{-2k-2}) \frac{(1-\log 2^{-k})^n}{(1-\log 2^{-k-1})^{\frac{n}{2}}}$$

$= 0$. With the criteria above we cannot decide, whether all composition operators $C_\phi : A_v^2 \to H_v^\infty$ are bounded or not. But, again selecting $\phi(z) = z$ we see that

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\phi(z)|^2)v(\phi(z))^{\frac{1}{2}}} = \sup_{z\in\mathbb{D}} \frac{1}{(1-|z|^2)(1-\log(1-|z|))^{\frac{n}{2}}} = \infty.$$

Hence the corresponding composition operator is not bounded.

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