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A Mean Value Inequality for Euler's Beta Function

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Abstract. Let $B(x,y)$ be Euler's beta function. We prove that the inequalities

$$0 < \frac{B\Big(\sqrt{xy}, \dfrac{x+y}{2}\Big)}{B(x,y)} < 1$$

hold for all $x, y > 0$ with $x \neq y$. The given constant bounds are best possible. This result is extended to the case when the beta function in the numerator has arguments given by the weighted geometric and arithmetic means.

1. Introduction

The beta function, also known as the Eulerian integral of the first kind, is defined for positive real numbers x and y by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{1}$$

where Γ denotes the classical gamma function. From the product representation of $1/\Gamma(x)$ [6, Eq. (1.1.9)], it follows that

$$B(x,y) = \frac{x+y}{xy} \prod_{n=1}^{\infty} \Big(1 + \frac{xy}{n(x+y+n)}\Big)^{-1}. \tag{2}$$

The beta function plays an important role in the theory of special functions and it also has remarkable applications in physics, stochastic processes and other fields. A collection of the main properties of $B(x,y)$ as well as interesting historical comments on this subject can be found, for instance, in [6].

In the recent past, several research papers have appeared providing various inequalities for the beta function and its relatives. We refer to [1-5], [7, 9, 10, 11] and the references cited therein. For example,

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for all $x, y > 0$ we have that $B(x, y)$ separates the geometric and arithmetic means of $B(x, x)$ and $B(y, y)$, that is,

$$\sqrt{B(x,x)B(y,y)} \le B(x,y) \le \frac{B(x,x)+B(y,y)}{2}.$$

The first inequality is given in [9], whereas a proof for the second one can be found in [3]. In this note we present a new mean value inequality for the ratio of two beta functions.

Theorem. *For all positive real numbers x and y with $x \neq y$ we have*

$$0 < \frac{B\Big(\sqrt{xy}, \dfrac{x+y}{2}\Big)}{B(x,y)} < 1. \tag{3}$$

Both constant bounds are sharp.

2. A Lemma

In order to prove the right-hand side of (3) we apply a lemma which offers an upper bound for the ratio given in (3) in terms of geometric and arithmetic means.

Let $x, y > 0$ and $w \in (0,1)$. The weighted geometric and arithmetic means are defined by

$$G_w(x,y) = x^w y^{1-w} \quad \text{and} \quad A_w(x,y) = wx + (1-w)y.$$

Moreover, we set

$$G = G_{1/2}(x,y) = \sqrt{xy} \quad \text{and} \quad A = A_{1/2}(x,y) = \frac{x+y}{2}.$$

Lemma. *Let $v, w \in (0,1)$. The inequality*

$$\frac{B\big(G_v(x,y), A_w(x,y)\big)}{B(x,y)} \le \frac{1}{2}\left[\Big(\frac{G_v(x,y)}{A_w(x,y)}\Big)^2 + \frac{G_v(x,y)}{A_w(x,y)}\right] \tag{4}$$

holds for all $x, y > 0$ if and only if $v = w = 1/2$.

Proof. First, we assume that (4) is valid for all $x, y > 0$. Let

$$F_{v,w}(x,y) = 2\,\frac{B\big(G_v(x,y), A_w(x,y)\big)}{B(x,y)}\left[\Big(\frac{G_v(x,y)}{A_w(x,y)}\Big)^2 + \frac{G_v(x,y)}{A_w(x,y)}\right]^{-1}.$$

Then, we have for $x, y > 0$:

$$F_{v,w}(x,y) \le 1 = F_{v,w}(y,y).$$

Use of

$$\frac{\partial}{\partial x}B(x,y) = B(x,y)\big[\psi(x) - \psi(x+y)\big],$$

where $\psi = \Gamma'/\Gamma$ is the logarithmic derivative of the gamma function, yields

$$0 = 2y\frac{\partial}{\partial x}F_{v,w}(x,y)\Big|_{x=y} = 3(w-v)+2(v+w-1)y\big(\psi(y)-\psi(2y)\big). \quad (5)$$

We denote the expression on the right-hand side of (5) by $H_{v,w}(y)$. Since

$$\lim_{t\to 0+} t\psi(t) = -1 \quad \text{and} \quad \psi(1)-\psi(2) = -1,$$

we obtain

$$\lim_{y\to 0+} H_{v,w}(y) = 1 - 4v + 2w = 0$$

and

$$H_{v,w}(1) = 2 - 5v + w = 0.$$

This leads to $v = w = 1/2$.

Next, we prove (4) with $v = w = 1/2$. Application of (2) leads to

$$B(x,y) = \frac{2A}{G^2}\prod_{n=1}^{\infty}\Big(1+\frac{G^2}{n(2A+n)}\Big)^{-1}$$

and

$$\frac{B(G,A)}{B(x,y)} = \frac{G(G+A)}{2A^2}\prod_{n=1}^{\infty} f_n \quad (6)$$

with

$$f_n = \Big(1+\frac{G^2}{n(2A+n)}\Big)\Big(1+\frac{GA}{n(G+A+n)}\Big)^{-1}. \quad (7)$$

Since $A - G \geq 0$, we obtain

$$f_n = 1 - \frac{G(A-G)(G+2A+n)}{(G+n)(A+n)(2A+n)} \leq 1 \quad \text{for} \quad n \geq 1. \quad (8)$$

Therefore,

$$\frac{B(G,A)}{B(x,y)} \leq \frac{G(G+A)}{2A^2} = \frac{1}{2}\Big[\Big(\frac{G}{A}\Big)^2 + \frac{G}{A}\Big]. \quad (9)$$

This establishes (4) with $v = w = 1/2$. □

Remark 2.1. If $x \neq y$, then $A - G > 0$, so that (8) gives $f_n < 1$ for $n \geq 1$. This implies that (9) holds with "$<$" instead of "$\leq$". Thus, if $v = w = 1/2$, then the sign of equality is valid in (4) if and only if $x = y$.

We are now in a position to establish our main result.

3. Proof of Theorem

An application of (4) with $v = w = 1/2$ yields for $x, y > 0$ with $x \neq y$:

$$\frac{B(G,A)}{B(x,y)} < 1 - \frac{(A-G)(A+G/2)}{A^2} < 1.$$

It remains to show that the bounds 0 and 1 are best possible. We denote the ratio in (3) by $R(x,y)$. Then,

$$R(x,x) = 1. \tag{10}$$

Use of the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ and (1) gives

$$R(x,1) = \sqrt{x}\,\frac{\Gamma(\sqrt{x}+1)\Gamma\big((x+1)/2\big)}{\Gamma\big(\sqrt{x}+(x+1)/2\big)}.$$

It follows that

$$\lim_{x\to 0^+} R(x,1) = 0. \tag{11}$$

From (10) and (11) we conclude that the constant upper and lower bounds given in (3) cannot be improved. □

Remark 3.1. Inequality (4) with $v = w = 1/2$ reveals that

$$\frac{A}{G} \leq \frac{B(x,y)}{B(G,A)}$$

is valid for all $x, y > 0$. This is a converse of the well-known arithmetic mean - geometric mean inequality $A/G \geq 1$. Many additional inequalities for arithmetic and geometric means as well as for numerous other mean values are given in the monograph [8].

It is natural to ask whether the right-hand side of (3) is valid for geometric and arithmetic means with a weight different from 1/2. The following remark reveals that if both means have the same weight, then the answer is "no".

Remark 3.2. Let $w \in (0,1)$. The inequality

$$B\big(G_w(x,y), A_w(x,y)\big) \leq B(x,y) \tag{12}$$

holds for all $x, y > 0$ if and only if $w = 1/2$. We define

$$I_w(x) = B(x,1) - B\big(G_w(x,1), A_w(x,1)\big).$$

If (12) is valid for all $x, y > 0$, then we obtain

$$I_w(x) \geq 0 = I_w(1) \quad \text{and} \quad I'_w(1) = 2w - 1 = 0.$$

Thus, $w = 1/2$.

Remark 3.3. If $w \in (0,1)$, then (12) holds for $x, y > 0$ satisfying $G^2 \le G_w(x,y)A_w(x,y)$ or, equivalently, $G_w(y,x) \le A_w(x,y)$. To see this we observe that an extension of (6) and (7) (we omit to display the x, y dependence of G_w and A_w) shows that

$$\frac{B(G_w, A_w)}{B(x,y)} = Q_w \prod_{n=1}^{\infty} g_n(w), \quad Q_w = \frac{G^2(G_w + A_w)}{2AG_wA_w}$$

where

$$g_n(w) = \Big(1 + \frac{G^2}{n(2A+n)}\Big)\Big(1 + \frac{G_wA_w}{n(G_w + A_w + n)}\Big)^{-1}$$

$$= 1 - \frac{n(G_wA_w - G^2) + 2AG_wA_w(1 - Q_w)}{(2A+n)(G_w+n)(A_w+n)}.$$

Since

$$\frac{G^2(G_w + A_w)}{G_w} = G^2 + G_{1-w}A_w \le A_w(G_w + G_{1-w})$$

$$\le A_w(A_w + A_{1-w}) = 2AA_w,$$

where we have employed the arithmetic mean - geometric mean inequality $G_w \le A_w$, we see that $Q_w \le 1$ and $g_n(w) \le 1$ for $n \ge 1$, whence the result follows.

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References

[1] P. Åhag, R. Czyż: *An inequality for the beta function with application to pluripotential theory*, J. Inequal. Appl. 2009, Article ID 901397, 8 p.
[2] H. Alzer: *Sharp inequalities for the beta function*, Indag. Math. 12 (2001), 15-21.
[3] H. Alzer: *Some beta-function inequalities*, Proc. Roy. Soc. Edinb. 133A (2003), 731-745.
[4] H. Alzer: *Inequalities for the beta function of n variables*, ANZIAM 44 (2003), 609-623.
[5] H. Alzer: *Inequalities for the beta function*, Anal. Math. 40 (2014), 1-11.
[6] G.E. Andrews, R. Askey, R. Roy: *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.
[7] B.A. Bhayo, J. Sándor: *On the inequalities for beta function*, Notes Numb. Th. Disc. Math. 21 (2015), 1-7.
[8] P.S. Bullen, D.S. Mitrinović, P.M. Vasić: *Means and Their Inequalities*, Reidel, Dordrecht, 1988.

[9] S.S. Dragomir, R.P. Agarwal, N.S. Barnett: *Inequalities for beta and gamma functions via some classical and new integral inequalities*, J. Inequal. Appl. 5 (2000), 103-165.
[10] P. Ivády: *On a beta function inequality*, J. Math. Ineq. 6 (2012), 333-341.
[11] P. Ivády: *On a beta function inequality II*, J. Math. Sci. Adv. Appl. 22 (2013), 27-37.

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