

## Irish Dancing Groups

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ABSTRACT. There is so much symmetry involved in the arrangements of the dancers in traditional Irish céilí dancing that it seems natural to try to describe some of these dances using group theory. We describe a connection between progressive dances such as the Siege of Ennis and the calculation of the cosets of the dihedral group  $D_n$  in the symmetric group  $S_n$  and offer suggestions for using this application as a classroom activity. This article may be of interest to students taking a first course in group theory and their professors.

### 1. INTRODUCTION

Did you know that every time you dance the Siege of Ennis at a wedding, you are calculating cosets of a dihedral group as a subgroup of a symmetric group? We describe how to do this and offer suggestions for ways to include this as an engaging activity in a group theory course. Apart from the obvious purpose of illustrating the concept of group cosets, this example can be used to illustrate many of the standard theorems concerning cosets, as well as Lagrange's Theorem. It aids in a review of the symmetric groups and the dihedral groups and can serve as a starting point for the discussion of normal subgroups and group presentations.

### 2. PRELIMINARIES

Two families of groups we will see in this paper are the symmetric groups and the dihedral groups. We let  $S_n$  denote the symmetric group on  $n$  letters. It has a presentation with generators  $s_i$ ,  $1 \leq i \leq$

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$n - 1$  and relations

$$s_i^2 = 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \text{ for } |i - j| > 1.$$

The generators  $s_i$  can be identified with the transpositions  $(i, i + 1)$  for each  $1 \leq i \leq n - 1$ . We let  $D_n$  denote the dihedral group of order  $2n$ , the symmetry group of a regular  $n$ -gon. It has a presentation with generators  $a$  and  $b$  together with the relations

$$a^2 = b^2 = (ab)^n = 1.$$

We can identify the elements of  $D_n$  with permutations of the vertices of a regular  $n$ -gon, written using cycle notation. We will compose cycles from right to left so that, for example,  $(123)(13) = (23)$  and not  $(12)$ .

### 3. COSETS AND THE SIEGE OF ENNIS

To dance the Siege of Ennis, dancers line up in teams of four down the dance hall. The teams stand side by side holding hands and facing another team as in Figure 1.

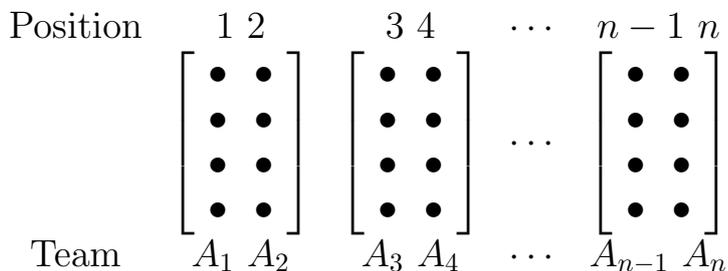


FIGURE 1. Siege of Ennis

See [7] for a list of the dance moves for the Siege of Ennis or [8] for a video of the dance. Here, we are not interested in the dance moves themselves, rather we will look solely at the progression of the dancers from one round of the dance to the next.

We will begin by considering  $n = 4$  teams and in Section 4 we will generalize to  $n$  teams. Suppose we have four teams,  $A$ ,  $B$ ,  $C$ , and  $D$ , each consisting of four people, dancing the Siege of Ennis. In the first round of the dance, team  $A$  is dancing with team  $B$  and team  $C$  is dancing with team  $D$ , as illustrated below in Figure 2. We number the positions of the teams 1 to 4 from left to right.

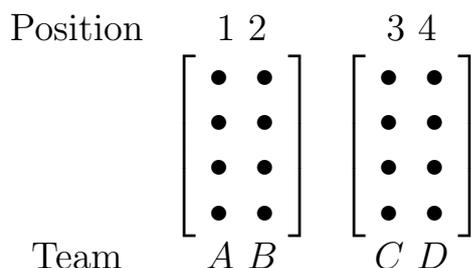


FIGURE 2. Round 1

At the end of the first round of the dance, (as you will remember from all the weddings you’ve been to), teams *A* and *B* switch places with one another: the dancers from team *A* lift their arms up to form arches, and the dancers from team *B* proceed through those arches to switch places with team *A*. Similarly, teams *C* and *D* switch places. Then in the second round of the dance, teams *A* and *D* dance with one other while teams *B* and *C* wait out the second round, as in the left-hand illustration of Figure 3.

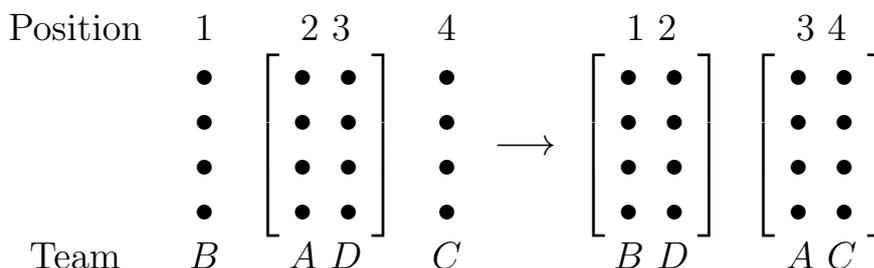


FIGURE 3. Round 2 to Round 3

When the second round of the dance has finished, teams *A* and *D* switch places again via arches and the dance starts all over again for round 3, now with teams *B* and *D* dancing together, and *A* and *C* dancing together, as in the right-hand side of Figure 3.

From here, the progressions continue similarly, with the outer two pairs of teams switching places according to the permutation  $a = (12)(34)$ , followed by the inner two teams switching places according to  $b = (23)$ , and so on. In the second column of Figure 4 we list the permutations of the teams at each round of the dance, stopping right before the teams arrive back to their starting positions whence the pattern would repeat itself if the dance were to continue.

We might, at this point, wonder what happens if we start again from a permutation of the teams we have not yet encountered,

say  $BACD$ . And then, once we have figured out the possible permutations that result from this starting position, we could try a third starting position, say  $CBAD$ . The resulting permutations are listed in Figure 4.

Round	Prog. 1	Prog. 2	Prog. 3
1	$ABCD$	$BACD$	$CBAD$
2	$BADC$	$ABDC$	$BCDA$
3	$BDAC$	$ADBC$	$BDCA$
4	$DBCA$	$DACB$	$DBAC$
5	$DCBA$	$DCAB$	$DABC$
6	$CDAB$	$CDBA$	$ADCB$
7	$CADB$	$CBDA$	$ACDB$
8	$ACBD$	$BCAD$	$CABD$

FIGURE 4. All permutations

Thus we have listed all  $24 = 4!$  possible permutations of the four teams. Each permutation corresponds to an element of the symmetric group  $S_4$ . We take  $ABCD$  to be the identity position. The list of permutations in the second column of Figure 4 is generated by the permutations  $a = (12)(34)$  and  $b = (23)$ . Since their product  $ab = (1243)$  is of order 4 this second column corresponds to the group  $D_4$  following the presentation of the dihedral group given in Section 2. Since  $BACD$  is obtained from  $ABCD$  by applying the permutation  $(12)$ , the third column corresponds to the right coset  $D_4(12)$ , of  $D_4$  in  $S_4$ . And since  $CBAD$  is obtained from the identity position by applying the permutation  $(13)$ , the fourth column is the right coset  $D_4(13)$  of  $D_4$  in  $S_4$ . See Section 5 for suggestions on how to use these observations in a classroom setting.

#### 4. GENERALIZATION TO $n$ TEAMS

Suppose now, for the more general case, that we have  $n$  teams of dancers dancing the Siege of Ennis. If  $n = 2m$  is even, we will start with the situation in Figure 5.

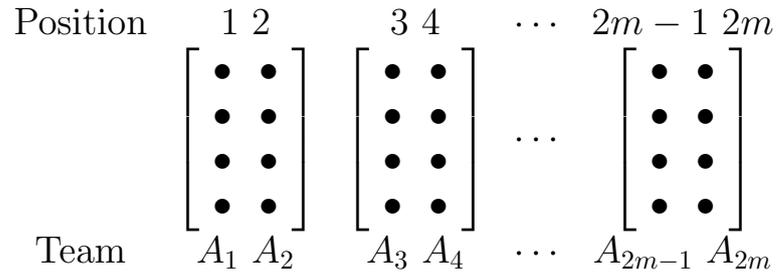


FIGURE 5.  $n$  even

And if  $n = 2m + 1$  is odd, we will start with the situation in Figure 6.

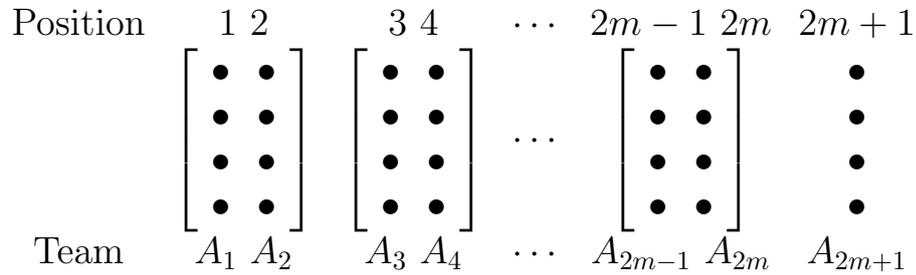


FIGURE 6.  $n$  odd

As before, the dancers progress from one round of the dance to the next via arches, exchanging places with the team facing them. The permutations

$$r = \prod_{i=1}^m (2i - 1, 2i) \text{ and } s = \prod_{j=1}^{m-1} (2j, 2j + 1)$$

generate the list of possible team permutations from the starting position when  $n = 2m$  is even and the permutations

$$t = \prod_{i=1}^m (2i - 1, 2i) \text{ and } u = \prod_{j=1}^m (2j, 2j + 1)$$

generate the list of permutations in the case where  $n = 2m + 1$  is odd.

We can see by a direct calculation that the products  $rs$  and  $tu$  can each be written as cycles of length  $n$ . We get

$$rs = (2, 4, 6, \dots, 2m - 2, 2m, 2m - 1, 2m - 3, \dots, 5, 3, 1)$$

and

$$tu = (2, 4, 6, \dots, 2m, 2m + 1, 2m - 1, \dots, 5, 3, 1).$$

Therefore, recalling the presentation of  $D_n$  from Section 2, the elements  $r$  and  $s$ , being two elements of order 2 such that their product  $rs$  has order  $n = 2m$  generate the dihedral group  $D_n$  in the case that  $n$  is even. And the elements  $t$  and  $u$ , being two elements of order 2 such that their product,  $tu$  has order  $n = 2m + 1$  generate the dihedral group  $D_n$  in the case that  $n$  is odd.

As in the  $n = 4$  case, we have an algorithm for calculating all of the right cosets of  $D_n$  in  $S_n$  via the Siege of Ennis. We choose a new starting position - one that we have not already encountered - and then apply the generators repeatedly to this new starting position. We then repeat this procedure, choosing another starting permutation we have not yet seen. We continue until we have listed all  $n!$  permutations of  $S_n$ .

## 5. CLASSROOM NOTES

We offer some suggestions for using these ideas as part of a classroom activity once the symmetric and dihedral groups have been studied and right after the notion of a coset has been introduced. The  $n = 4$  example is easy to work through in class, either having 8 students demonstrate the progressions (with 2 students in each team, for simplicity), or involving the whole class if that is feasible, perhaps for some extra credit! As the students are producing the permutations that are listed in Figure 4, one may wish to have the the students “translate” them into cycle notation – for example,  $CDBA$  would become  $(1423)$ . Here are some exercises that could be asked of the students at this point:

- (1) Are there any repeats in your table of permutations?
- (2) All 24 permutations form which group?
- (3) Taking  $ABCD$  to be the identity permutation, prove that the set of permutations in the second column of Figure 4 is isomorphic to the group  $D_4$ . (Either set up a mapping relating the permutations in the second column to the permutations of the vertices of a square, or tackle this via generators and relations.)
- (4) Neither the third nor the fourth columns are subgroups of  $S_4$ . Why not?
- (5) Show that the third and the fourth columns are right cosets of  $D_4$  in  $S_4$ . Are they also left cosets of  $D_4$  in  $S_4$ ?

- (6) Consider the way we produced the permutations in the second column of Figure 4. Identify the generators of  $D_4$  that are being used here.

It may be helpful for the students to keep this example in mind when proving some of the standard results about cosets: any element of a coset can be used as its representative, cosets are either identical or disjoint, cosets are all of the same size and partition the group, as well as Lagrange's Theorem. In fact, before these theorems are proved formally, students could be asked to explain why each of these statements holds in the context of the Siege of Ennis, perhaps as homework exercises following this activity. In addition, Exercise (5) above could be used as a set-up for discussing normal subgroups, and Exercises (3) and (6) could be used to introduce group presentations.

## 6. FURTHER READING

This is but one example of group theory in Irish dance and there are many more if we care to look for them. The observation of group theory in dance is not new. For example, [1], [2], [5], and [6] all discuss the symmetries of contra dancing. And at least one other author has written about group theory in Irish dance: Andrea Hawksley, in [4], discusses the appearance of braid groups in the céilí dance *The Waves of Tory*.

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